Categorical smooth compactifications and neighborhoods of infinity

Alexander I. Efimov

In this note we give a short overview of some of our results on derived categories of coherent sheaves, in particular on smooth categorical compactifications and on the formal punctured neighborhoods of infinity.

Introduction

This note is devoted to a short overview of some results on derived categories of coherent sheaves concerning smooth categorical compactifications and the formal punctured neighborhoods of infinity.

In Section 1, we discuss the conjecture of Bondal and Orlov about the categorical properties of the resolution of singularities of an algebraic variety with rational singularities (Conjecture 1.1). This conjecture states that the derived pushforward functor on the derived categories of coherent sheaves is a quotient functor (that is, a localization). The conjecture is difficult and still open in general. It turns out that it is possible (Theorem 1.2) to prove a version of such statement for an arbitrary separated scheme of finite type over a field of characteristic zero (the reader may safely assume that we are dealing with quasi-projective schemes). The methods make it possible to prove Conjecture 1.1 for a cone over a projective embedding of a smooth Fano variety (that is, a smooth projective variety with an ample anti-canonical line bundle).

In Section 2, we consider DG categorical smooth compactifications. Here DG stands for "differential-graded". This is a straightforward generalization of the usual algebro-geometric smooth compactification. The following natural question was formulated by B. Toën (Question 2.3 below): is it true that any smooth DG category "of finite type" admits a smooth categorical compactification? The question was considered to be difficult, but most experts expected that the answer should be "yes". However, in [4] we gave a negative answer, obtained by disproving a closely related conjecture of Kontsevich (Conjecture 2.5 below) on the generalized version of the degeneration of the Hodge-to-de-Rham spectral sequence. We also obtained a dual version of these results, in which smooth DG categories are replaced by proper DG categories, and a smooth compactification is replaced by a categorical resolution of singularities.

In Section 3 we outline a certain construction called a "categorical formal punctured neighborhood of infinity". For a smooth algebraic variety X this is obtained as follows: take some smooth compactification $\bar{X}$, consider the formal completion at the infinity locus $\bar{X} - X$, and then take the corresponding punctured formal scheme. The resulting object $X_{\hat{\infty}}$ (considered for example as an adic space) is independent of the compactification, as is the category of perfect complexes on it. In [3] we give a purely categorical construction of $\mathrm{Perf}(X_{\hat{\infty}})$ which generalizes to arbitrary smooth DG algebras and DG categories. A curious special case is the algebra of rational functions on a smooth projective curve. There, our construction gives exactly the ring of adeles.

1 Rational singularities and a conjecture of Bondal and Orlov

Let X be an algebraic variety over a field of characteristic zero. Recall that X has rational singularities if for some (and then any) resolution of singularities $\pi : Y \to X$ we have $\mathbf{R}\pi_*\mathcal{O}_Y \cong \mathcal{O}_X$. Equivalently, the pullback functor $\mathbb{L}\pi^* : D_{\mathrm{perf}}(X) \to D_{\mathrm{perf}}(Y)$ is fully faithful. The following conjecture is still open.

Conjecture 1.1 ([1]). *With the above notation, the functor $\mathbf{R}\pi_* : D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$ is a localization. That is, the induced functor $D^b_{\mathrm{coh}}(Y)/\ker(\mathbf{R}\pi_*) \to D^b_{\mathrm{coh}}(X)$ is an equivalence.*

The following result is a version of such statement which holds in a much more general framework.

Theorem 1.2 ([5]). *Let X be a separated scheme of finite type over a field* k *of characteristic zero. Then there exist a smooth projective variety Y and a functor $\Phi : D^b_{\mathrm{coh}}(Y) \to D^b_{\mathrm{coh}}(X)$ such that the induced functor $D^b_{\mathrm{coh}}(Y)/\ker(\Phi) \to D^b_{\mathrm{coh}}(X)$ is an equivalence. Moreover, the triangulated category* $\ker(\Phi)$ *is generated by a single object.*

 EMS MAGAZINE 120 (2021) — DOI 10.4171/MAG-34

This theorem in particular confirms a conjecture of Kontsevich on the homotopy finiteness of the DG category $D^b_{\mathrm{coh}}(X)$. The proof is based on a certain construction of a categorical resolution of singularities, due to Kuznetsov and Lunts [8].

The methods developed to prove Theorem 1.2 actually also work to prove Conjecture 1.1 in a certain class of cases. In particular, the following result holds.

Theorem 1.3 ([5]). *Let $X \subset \mathbb{A}^n$ be a cone over a smooth Fano variety in $\mathbb{P}^{n-1}$. Let $\pi : Y \to X$ be the resolution given by the blow-up of the origin point. Then the induced functor $D^b_{\mathrm{coh}}(Y)/\ker(\mathbf{R}\pi_*) \to D^b_{\mathrm{coh}}(X)$ is an equivalence.*

2 Categorical smooth compactifications

Theorem 1.2 deals with a special case of a categorical smooth compactification. We first recall some basic definitions.

Definition 2.1 ([7]). 1. A small DG category $\mathcal{C}$ over k is smooth if the diagonal $\mathcal{C}$-$\mathcal{C}$-bimodule is perfect.
2. $\mathcal{C}$ is called proper if for $X, Y \in \mathcal{C}$ the complex $\mathcal{C}(X,Y)$ is perfect over k.

In particular, we have the notions of smoothness and properness for DG algebras (a DG algebra can be considered as a DG category with a single object). When X is a separated scheme of finite type over a field k, then X is smooth (resp. proper) if and only if the DG category $\mathrm{Perf}(X)$ is smooth (resp. proper) ([11, Proposition 3.30], [10, Proposition 3.13]). Hence, these basic geometric properties of X are reflected by the DG category $\mathrm{Perf}(X)$.

We recall the following definition.

Definition 2.2. For a pre-triangulated DG category $\mathcal{A}$, a categorical smooth compactification is a DG functor $F : \mathcal{C} \to \mathcal{A}$, such that:

1. $\mathcal{C}$ is a smooth and proper pre-triangulated DG category;
2. the induced functor $\mathcal{C}/\ker(F) \to \mathcal{A}$ is fully faithful;
3. every object $x \in \mathcal{A}$ is a direct summand of some $F(y)$, $y \in \mathcal{C}$.

The basic geometric example of a categorical smooth compactification is given by the usual one. Namely, let X be a smooth algebraic variety over k, and let $j : X \hookrightarrow \bar{X}$ be an open embedding, where $\bar{X}$ is smooth and proper. Then the restriction functor $j^* : \mathrm{Perf}(\bar{X}) \to \mathrm{Perf}(X)$ is a categorical smooth compactification.

Theorem 1.2 provides a categorical smooth compactification of the DG categories of the form $D^b_{\mathrm{coh}}(X)$, where X is a separated scheme of finite type over a field of characteristic zero.

There is a notion of a homotopically finitely presented (hfp) DG category which should be thought of as a smooth DG category "of finite type" (we refer to [14] for the precise definition). The following general question was formulated by Bertrand Toën.

Question 2.3 (Toën). Is it true that any homotopically finitely presented DG category over a field of characteristic zero has a smooth compactification?

The question is difficult, but the general consensus was that the answer should be "yes". However, in [4] the author gave a negative answer to this question. Here we explain the rough idea of the results of [4].

It turns out that Question 2.3 is closely related with the noncommutative (categorical) Hodge-to-de Rham degeneration. Recall that the classical Hodge theory implies (via GAGA) the following algebraic statement: for any smooth algebraic variety X over a field k of characteristic zero the spectral sequence

$$E_2^{pq} = H^q(X, \Omega_X^p) \Rightarrow H_{DR}^{p+q}(X)$$

degenerates.

The following categorical generalization was conjectured by Kontsevich and Soibelman [7], and proved by Kaledin [6].

Theorem 2.4 ([6, Theorem 5.4]). *Let A be a smooth and proper DG algebra over a field of characteristic zero. Then the Hochschild-to-cyclic spectral sequence degenerates, so that we have an isomorphism $HP_\bullet(A) \cong HH_\bullet(A)((u))$.*

In the special case when $\mathrm{Perf}(A) \simeq \mathrm{Perf}(X)$ for a smooth and proper variety X, Theorem 2.4 gives exactly the usual (commutative) Hodge-to-de Rham degeneration.

The following two conjectures were formulated by Kontsevich for smooth and for proper DG algebras.

Conjecture 2.5 (Kontsevich). *Let A be a smooth DG algebra over a field of characteristic zero. Then the composition*

$$\begin{aligned} K_0(A \otimes A^{\mathrm{op}}) &\xrightarrow{\mathrm{ch}} \big(HH_\bullet(A) \otimes HH_\bullet(A^{\mathrm{op}})\big)_0 \\ &\xrightarrow{\mathrm{id}\otimes\delta^-} \big(HH_\bullet(A) \otimes HC_\bullet^-(A^{\mathrm{op}})\big)_1 \end{aligned}$$

vanishes on the class $[A]$ of the diagonal bimodule.

Here $\delta^- : HH_\bullet(A^{\mathrm{op}}) \to HC_\bullet^-(A^{\mathrm{op}})[-1]$ denotes the boundary map, see [2, Section 3].

Conjecture 2.6 (Kontsevich). *Let B be a proper DG algebra over a field* k *of characteristic zero. Then the composition map*

$$\big(HH_\bullet(B) \otimes HC_\bullet(B^{\mathrm{op}})\big)[1] \xrightarrow{\mathrm{id}\otimes\delta^+} HH_\bullet(B) \otimes HH_\bullet(B^{\mathrm{op}}) \to \mathrm{k}$$

is zero.

Here $\delta^+ : HC_\bullet(B^{\mathrm{op}})[1] \to HH_\bullet(B^{\mathrm{op}})$ denotes the boundary map, see [9, Section 2.2].

Both conjectures 2.5 and 2.6 hold, roughly speaking, for all DG categories coming from (commutative) algebraic geometry.

Conjecture 2.5 is related to Question 2.3 as follows. Suppose that we have a smooth compactification $\mathcal{C} \to \mathcal{A}$ (hence $\mathcal{A}$ is smooth). Then we have the following commutative diagram:

$$\begin{tikzcd} HH_\bullet(\mathcal{C}) \otimes HH_\bullet(\mathcal{C}^{\mathrm{op}}) \arrow[r, "\mathrm{id}\otimes\delta^-"] \arrow[d] & HH_\bullet(\mathcal{C}) \otimes HC^-_\bullet(\mathcal{C}^{\mathrm{op}})[-1] \arrow[d] \\ HH_\bullet(\mathcal{A}) \otimes HH_\bullet(\mathcal{A}^{\mathrm{op}}) \arrow[r, "\mathrm{id}\otimes\delta^-"] & HH_\bullet(\mathcal{A}) \otimes HC^-_\bullet(\mathcal{A}^{\mathrm{op}})[1]. \end{tikzcd}$$

The left vertical map sends $\mathrm{ch}(I_{\mathcal{C}})$ to $\mathrm{ch}(I_{\mathcal{A}})$. Hence, applying Kaledin's Theorem 2.4, we obtain that Conjecture 2.5 holds for $\mathcal{A}$.

A dual argument implies that Conjecture 2.6 holds for proper DG categories which can be fully faithfully embedded into a smooth and proper DG category (such an embedding is called a *categorical resolution* in the terminology of Kuznetsov and Lunts [8]).

However, in [4] we disproved both conjectures.

Theorem 2.7 ([4, Theorem 4.5, Theorem 5.4]). 1. *There exists a homotopically finitely presented DG algebra A for which Conjecture* 2.5 *does not hold. In particular, A gives a negative answer to Question* 2.3*: the DG category* Perf(A) *does not have a smooth categorical compactification.*

2. *There exists a proper DG algebra B for which Conjecture* 2.6 *does not hold. In particular, the category* Perf(B) *does not have a categorical resolution of singularities.*

The DG algebra B from part 2 is quasi-isomorphic to a certain explicit 10-dimensional A_∞-algebra for which the supertrace of m_3 on the second argument is non-zero.

3 Categorical formal punctured neighborhood of infinity

Another subject related to the notion of a smooth categorical compactification is that of a formal punctured neighborhood of infinity. Suppose that we have a usual smooth compactification $j : X \hookrightarrow \bar{X}$ of a smooth algebraic variety X. Then one can take the formal neighborhood $\bar{X}_{\hat{Z}}$, and then "remove" Z. The resulting object $\bar{X}_{\hat{Z}} - Z$ (the so-called generic fiber, considered as an adic space) does not depend on the choice of the compactification $\bar{X}$. Let us set $X_{\hat{\infty}} := \bar{X}_{\hat{Z}} - Z$. The corresponding category of perfect complexes $\mathrm{Perf}(X_{\hat{\infty}})$ also does not depend on Z and it is therefore an invariant of X.

The natural question arises: can we describe the category $\mathrm{Perf}(X_{\hat{\infty}})$ purely in terms of $\mathrm{Perf}(X)$? This question is partially motivated by mirror symmetry since an analogue of $\mathrm{Perf}(X_{\hat{\infty}})$ exists in symplectic geometry in the framework of Fukaya categories. It turns out that the purely categorical construction is possible, and it was described by the author in [3]. Here we give an outline.

First, we describe a "non-derived" version of the construction. Let A be an associative algebra over a field k. Then one can describe the algebra $H^0(A_{\hat{\infty}})$ as follows.

$$H^0(A_{\hat{\infty}}) = \{\varphi \in \mathrm{End}_{\mathrm{k}}(A) \mid \forall a \in A,\ \mathrm{rk}[\varphi, R_a] < \infty\}/(A^* \otimes A).$$

Here $\mathrm{End}_{\mathrm{k}}(A)$ is the algebra of k-linear endomorphisms of A (as a vector space) and $A^* \otimes A \subset \mathrm{End}_{\mathrm{k}(A)}$ is the two-sided ideal of operators of finite rank. The commutator is the additive one (the Lie algebra bracket) and $R_a : A \to A$, $R_a(b) = ba$, is the operator of right multiplication by a.

Example 3.1. It is a pleasant exercise to check that for $A = \mathrm{k}[t]$ we have $H^0(A_{\hat{\infty}}) \cong \mathrm{k}((t^{-1}))$. A similar computation shows that $H^0(\mathrm{k}[x^{\pm}]_{\hat{\infty}}) \cong \mathrm{k}((t)) \times \mathrm{k}((t^{-1}))$.

Example 3.2. A less trivial example is the following: let X be a smooth projective connected curve over k. Then we have $H^0(\mathrm{k}(X)_{\hat{\infty}}) \cong \mathbb{A}_X$, where $\mathbb{A}_X$ is the ring of adeles on X. Recall that $\mathbb{A}_X \subset \prod_{x \in X^{\mathrm{cl}}} \hat{K}_x$ is the subring of the product of complete local fields, consisting of elements $(a_x)_{x \in X^{\mathrm{cl}}}$ such that $a_x \in \hat{\mathcal{O}}_x$ for all but finitely many x.

Now let A be a smooth DG algebra. The DG algebra $A_{\hat{\infty}}$ is defined by the formula

$$A_{\hat{\infty}} := C^\bullet(A, \mathrm{End}_{\mathrm{k}}(A)/A^* \otimes A).$$

Here $C^\bullet(A, -)$ denotes the Hochschild cochain complex. The product on $A_{\hat{\infty}}$ comes from the product on $\mathrm{End}_{\mathrm{k}}(A)/A^* \otimes A$.

To describe the DG algebra $A_{\hat{\infty}}$ more conceptually, we recall the following notion.

Definition 3.3. 1. Let k be a field. The Calkin (DG) category $\mathrm{Calk}_{\mathrm{k}}$ is defined as the quotient Mod-k/ Perf(k). More explicitly, the objects of the DG category $\mathrm{Calk}_{\mathrm{k}}$ are complexes of k-vector spaces, and the morphisms are given by $\mathrm{Calk}_{\mathrm{k}}(V, W) = \mathrm{Hom}_{\mathrm{k}}(V, W)/V^* \otimes W$.

2. More generally, for a DG algebra A the Calkin category Calk_A is defined as the quotient Mod-A/ Perf(A).

We can consider A (and any other right A-module) as an object of $\mathrm{Rep}(A^{\mathrm{op}}, \mathrm{Calk}_{\mathrm{k}})$ – suitably defined category of representations of A^{op} in $\mathrm{Calk}_{\mathrm{k}}$. Note that

$$A_{\hat{\infty}} \simeq \mathrm{End}_{\mathrm{Rep}(A^{\mathrm{op}}, \mathrm{Calk}_{\mathrm{k}})}(A).$$

The DG category of topological perfect complexes over $A_{\hat{\infty}}$ is defined as follows.

Definition 3.4. For a smooth DG algebra A we define

$$\mathrm{Perf}_{\mathrm{top}}(A_{\hat{\infty}}) \simeq \ker\big(\mathrm{Rep}(A^{\mathrm{op}}, \mathrm{Calk}_{\mathrm{k}}) \to \mathrm{Perf}(A \otimes \mathrm{Calk}_{\mathrm{k}}) \to \mathrm{Calk}_A\big).$$

Here the embedding $\mathrm{Rep}(A^{\mathrm{op}}, \mathrm{Calk}_{\mathrm{k}}) \hookrightarrow \mathrm{Perf}(A \otimes \mathrm{Calk}_{\mathrm{k}})$ comes from the assumption that A is smooth. The functor $\mathrm{Perf}(A \otimes \mathrm{Calk}_{\mathrm{k}}) \to \mathrm{Calk}_A$ is given by the tensor product: $(A, V) \mapsto V \otimes A$ for $V \in \mathrm{Calk}_{\mathrm{k}}$.

Theorem 3.5 ([3]). *Let X be a smooth algebraic variety over a field* k, *and assume that X has a smooth compactification. Let A be a DG algebra such that* $\mathrm{Perf}(A) \simeq \mathrm{Perf}(X)$. *Then we have an equivalence* $\mathrm{Perf}(X_{\hat{\infty}}) \simeq \mathrm{Perf}_{\mathrm{top}}(A_{\hat{\infty}})$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Perf}(X) \arrow[r, "\sim"] \arrow[d] & \mathrm{Perf}(A) \arrow[d] \\
\mathrm{Perf}(X_{\hat{\infty}}) \arrow[r, "\sim"] & \mathrm{Perf}_{\mathrm{top}}(A_{\hat{\infty}}).
\end{tikzcd}$$

Remark 3.6. It is possible to obtain an extended version of Theorem 3.5 where the category $\mathrm{Perf}(X_{\hat{\infty}})$ is replaced by the category of nuclear modules in the sense of Clausen and Scholze [12, Definition 13.10]). This is more involved (and unpublished), and we will not cover this in the present note.

Remark 3.7. The construction of the DG algebra $A_{\hat{\infty}}$ and the DG category $\mathrm{Perf}_{\mathrm{top}}(A_{\hat{\infty}})$ is very much in the spirit of Tate's paper on residues of differential on curves [13].

References

[1] A. Bondal and D. Orlov, Derived categories of coherent sheaves. In *Proceedings of the International Congress of Mathematicians*, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 47–56 (2002)

[2] A. I. Efimov, Generalized non-commutative degeneration conjecture. *Proc. Steklov Inst. Math.* **290**, 1–10 (2015)

[3] A. I. Efimov, Categorical formal punctured neighborhood of infinity, I. arXiv:1711.00756 (2017)

[4] A. I. Efimov, Categorical smooth compactifications and generalized Hodge-to-de Rham degeneration. *Invent. Math.* **222**, 667–694 (2020)

[5] A. I. Efimov, Homotopy finiteness of some DG categories from algebraic geometry. *J. Eur. Math. Soc. (JEMS)* **22**, 2879–2942 (2020)

[6] D. Kaledin, Spectral sequences for cyclic homology. In *Algebra, Geometry, and Physics in the 21st Century*, Progr. Math. 324, Birkhäuser/Springer, Cham, 99–129 (2017)

[7] M. Kontsevich and Y. Soibelman, Notes on A_∞-algebras, A_∞-categories and non-commutative geometry. In *Homological Mirror Symmetry*, Lecture Notes in Phys. 757, Springer, Berlin, 153–219 (2009)

[8] A. Kuznetsov and V. A. Lunts, Categorical resolutions of irrational singularities. *Int. Math. Res. Not. IMRN*, 4536–4625 (2015)

[9] J.-L. Loday, *Cyclic Homology*. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] 301, Springer, Berlin (1992)

[10] V. A. Lunts, Categorical resolution of singularities. *J. Algebra* **323**, 2977–3003 (2010)

[11] D. Orlov, Smooth and proper noncommutative schemes and gluing of DG categories. *Adv. Math.* **302**, 59–105 (2016)

[12] P. Scholze, Lectures on analytic geometry. Available at: www.math.uni-bonn.de/people/scholze/Analytic.pdf (2020)

[13] J. Tate, Residues of differentials on curves. *Ann. Sci. École Norm. Sup. (4)* **1**, 149–159 (1968)

[14] B. Toën and M. Vaquié, Moduli of objects in DG-categories. *Ann. Sci. École Norm. Sup. (4)* **40**, 387–444 (2007)

Alexander I. Efimov is a mathematician at the Steklov Mathematical Institute of RAS and also a member of the International Laboratory of Mirror Symmetry and Automorphic Forms, Higher School of Economics, Moscow, Russia. He works in algebraic geometry and is primarily interested in its categorical aspects, with relations to mirror symmetry, noncommutative geometry and K-theory.

efimov@mccme.ru