number, and the promised deep dive into the category-theoretical foundations.

Entropy and Diversity is a thorough presentation of the mathematics of measuring diversity, including many new results of uniqueness, unification and utility. Beyond the practical value, it is also a display of mathematical art. Beautiful patterns, at deeper and deeper levels of abstraction, are exhibited to clarify the simplicity of what at first might appear to be abstruse formulas. Leinster approaches the subject like a craftsman, paying attention to every detail. The book is over 450 pages long, but it is so nicely organized and readable that I felt immediately drawn in rather than intimidated. The book is directly accessible to a general audience comfortable with mathematical reasoning. It will be a valuable reference for both mathematicians and mathematical ecologists. The new material has already engendered a lot of discussion on future directions, as can be seen in some recent online conversations:

- johncarlosbaez.wordpress.com/2011/11/07/measuring-biodiversity
- golem.ph.utexas.edu/category/2020/12/entropy_and_diversity_the_axio.html

Tom Leinster, *Entropy and Diversity: The Axiomatic Approach.* Cambridge University Press, 2021, 458 pages, Paperback ISBN 978-1-108-96557-6, eBook ISBN 978-1-108-96217-9.

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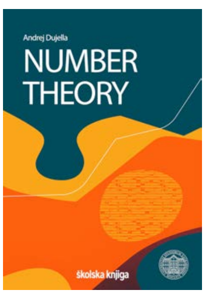
Number Theory by Andrej Dujella

Reviewed by Jean-Paul Allouche



As a student of Number Theory, I really appreciated the famous book of G. H. Hardy and E. M. Wright, while some of my friends frequently mentioned the book of Z. I. Borevich and I. R. Shafarevich. Of course, even these two books did not cover the whole (huge) field of Number Theory, and several other excellent books could be cited as well. More recently, many books have been devoted to (parts of) this vast field whose characteristic is to be both primary and not primary (pun intended). The meaning of the expression "Number Theory" itself has changed over time – partly because the domain has exploded – to the point that some contemporary authors now refer to "Modern Number Theory" …

A very recent book, entitled *Number Theory* and based on teaching materials, has been written by A. Dujella. Devoted to several subfields of this domain, this book is both extremely nice to read and to work from. It starts from primary results given in the first three chapters, ranging from the Peano axioms to the principle of induction, from the Fibonacci numbers to Euclid's algorithm, from prime numbers to congruences, and so on. Chapter 3 ends with primitive roots, decimal representation of rationals, and pseudoprimes. Chapters 4 and 5 then deal with quadratic residues (including the computation of square roots modulo a prime number) and quadratic forms (including the representation of integers as sums of 2, 4, or 3 squares). Chapters 6 and 7 are devoted to arithmetic functions (in particular, multiplicative functions, asymptotic behaviour of the summatory function of classical arithmetic functions, and the Dirichlet product), and to the distribution of primes (elementary estimates for the number of primes less than a given number, the Riemann function, Dirichlet characters, and a proof that an infinite number of primes are congruent to ℓ modulo k when $\gcd(\ell, k) = 1$). Chapter 8 deals with first results on Diophantine approximation, from continued fractions to Newton approximations and the LLL algorithm, while Chapter 9 studies applications of Diophantine approximation to cryptography (RSA, attacks on RSA, etc.). Actually two more chapters are devoted to Diophantine approximation, Chapter 10 (linear Diophantine approximation, Pythagorean triangles, Pellian equations, the Local-global principle, …) and Chapter 14 (Thue equations, the method of Tzanakis, linear forms in logarithms, Baker–Davenport reduction, …). Chapters 11, 12, and 13 deal with polynomials, algebraic numbers, and approximation of algebraic numbers. The book ends with Chapters 15 and 16 which cover elliptic curves and Diophantine problems.

This quick and largely incomplete description clearly shows that this book addresses many jewels of number theory. This is done in a particularly appealing way, mostly elementary when possible, with many well-chosen examples and attractive exercises. I arbitrarily choose two delightful examples, the kind of "elementary" statements that a beginner could attack, but whose proofs require some ingenuity, namely the unexpected statements 4.6 and 4.7:

Example 4.6. Let $p > 5$ be a prime number. Prove that there are two consecutive positive integers that are both quadratic residues and two consecutive positive integers that are both quadratic nonresidues modulo p.

Example 4.7. Let n be an integer of the form $16k + 12$ and let $\{b_1, b_2, b_3, b_4\}$ be a set of integers such that $b_i \cdot b_j + n$ is a perfect square for all i, j such that $i \neq j$. Prove that all numbers b_i are even.

The book also comprises short historical indications and 426 references. It really made me think of my first reading of Hardy and Wright, and I almost felt regret that I cannot start studying Number Theory again from scratch, but using this book! I highly recommend it not only to neophytes, but also to more "established" scientists who would like to start learning Number Theory, or to refresh and increase their knowledge of the field in an entertaining and subtle way.

Andrej Dujella, *Number Theory.* Textbook of the University of Zagreb, Školska knjiga, Zagreb, 2021, 621 pages, translated by Petra Švob, ISBN 978-953-0-30897-8.

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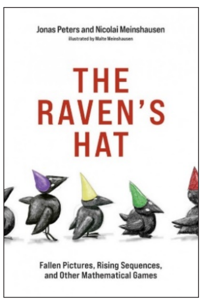
The Raven's Hat by Jonas Peters and Nicolai Meinshausen

Reviewed by Adhemar Bultheel



This book introduces, with some variations, eight mathematically flavoured games or puzzles. As the authors accurately explain in their preface, the type of problems they present look at first sight almost impossible to solve. It is only after a careful analysis, reducing it to a formal (say mathematical) reformulation, that it becomes clear that a solution strategy can be designed that is in some sense even optimal. Each time, the discussion of the solution to a problem is taken as an excellent pretext to explain some piece of mathematics. A reader with a minimal mathematical background will learn what a Hamming code is, what a cyclic group is, and some elements of linear algebra, probability, and even broader topics such as information theory, projective geometry, and algebraic topology. What starts as a playful game with a seemingly impossible solution becomes, after placing it in an appropriate mathematical context, relatively easy to solve. Moreover, by isolation and abstraction of the essentials, it becomes simple to consider more general situations. A better advertisement for the power of mathematics and a stronger motivation to study mathematical formalism and mathematical structures can hardly be found.

The book opens with a classic game, which is also for the title of the book. So let's take this as an illustration of the concept used by the authors throughout. Consider three players (in this book the players are adorable ravens illustrated in graphics by Malte Meinhausen). Each player has a blue or red hat on his head. They see the other players but they cannot see the colour of their own hat. All players have to guess their own hat's colour (red, blue, or don't know) without communicating with each other. The players, as a team, win the game if at least one player is correct and none is wrong (where the don't know answer is not considered wrong). There exist innumerable variations of this type of game, generally called "hat-problems". The present problem, which asks for a successful collective strategy, was originally formulated by Todd Ebert in 1998. It was Elwyn Berlekamp who later connected the solution to coding theory and solved it for n players when n is of the form $2^m - 1$. The solution for general n is still an open problem today. This kind of (brief) historical background of the problem is also given for the problems presented in the other chapters, which is a nice feature of the book. To work towards solving the problem, the authors first propose some guessing strategies or naive trials, possibly introducing a first formalism such as coding the red-blue colours as 0-1 and the configurations of three hats as binary numbers from 000 to 111, with the first bit for player 1, the second for player 2, and the last for player 3. This shows that there are a total of 8 possible states, and each player knows 2 of the 3 bits in the true number and must guess the bit of her own position. The probability of winning for the 3 players is maximized if each player chooses her own colour (0-1 bit) so that the "distance" to one of the $8 = 2^3$ possibilities is minimal. The crux is to define this distance as the Hamming distance, which is the mathematical contribution to the solution in the form of coding theory. Once the principle is clear, it is easy to generalize the solution to $n = 2^m - 1$ players.

This problem involves some elementary probability, and probability is also an ingredient for several of the other problems discussed in this book (both of its authors are professors of statistics). Several variants of the game correspond to the following description: a set of players needs to guess something on the basis of partial information that is available to them, and the goal is to agree on a strategy that will maximize their chance of winning the game as a team. For example, in the second game of this book, the n players have their name hidden in n boxes, and they have to find the box with their own name in a minimal number of trials for the whole team. Here the mathematical tool is the factorization of a permutation into cycles. In the existing literature, the players are often presented as prisoners that are collectively freed if they win. In this book, the stories vary, but all the illustrations portray ravens with hats.

Let me skim more quickly over the other chapters. Somewhat related to the two hat-problems mentioned above is a problem where there are more colours for the hats and where the players are lined up in such a way that each player can only see the other players (and their hats) who are positioned in front of them. This