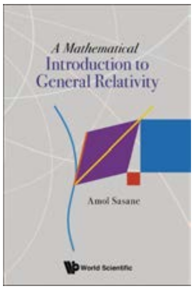
Book reviews

A Mathematical Introduction to General Relativity
by Amol Sasane

Reviewed by José Natário



Although general relativity is a highly mathematical theory, and arguably one of the main drivers behind the development of Riemannian geometry in the last 100 years, there are relatively few introductory books on this subject that specifically target mathematicians. The book under review is a welcome addition to this scant literature, aiming to introduce Einstein's theory, as well as the needed differential geometry, in a fully rigorous manner. It is interesting to note that the author, a professor of mathematics at the London School of Economics, is not an expert in general relativity, and so is in an ideal position to connect with mathematicians who are encountering the theory for the first time.

The book starts by developing the main ideas of differential geometry, and then goes on to discuss general relativity. It is carefully written, containing numerous appealing figures, and averaging more than ten exercises per chapter (with full solutions provided in an appendix, which is ideal for autonomous study). Moreover, many of the examples and exercises in the differential geometry part are calculations in general relativity (where the author supplies the relevant metrics to be derived in later chapters), which no doubt will appeal to the reader eager to learn general relativity. The level is more elementary than that of other books written in the same mathematical vein, such as "General Relativity for Mathematicians" by Sachs and Wu (which already assumes the differential geometry background), or "Semi-Riemannian Geometry" by O'Neill, and is well suited for mathematics or mathematically inclined physics undergraduate or beginning graduate students.

The detailed plan of the book is as follows: smooth manifolds and smooth maps are introduced in Chapter 1, without assuming point set topology (indeed the prerequisites of the book are simply the usual linear algebra, multivariate calculus and differential equations courses common to most degrees in mathematics, physics or engineering). Chapter 2 discusses tangent vectors, and Chapter 3 studies vector fields. General (mixed) tensor fields are defined in Chapter 4, and semi-Riemannian (in particular Lorentzian) manifolds are introduced in Chapter 5. The Levi-Civita connection, parallel transport and geodesics are discussed in Chapters 6, 7 and 8, respectively, and the notion of curvature is addressed in Chapter 9. Chapters 10 and 11 constitute a digression into differential forms and integration, including the Hodge star (later used to formulate the Maxwell equations); this is a subject not covered in many introductory general relativity books (e.g. O'Neill's "Semi-Riemannian Geometry"). The relativity part of the book starts in Chapter 12 with a discussion of physics in Minkowski spacetime, including a detailed analysis of relativistic velocity addition and electromagnetism. Chapter 13 gives a geometric reformulation of Newtonian gravity and defines the relativistic energy momentum tensor, motivating the introduction of the Einstein field equation in Chapter 14. This chapter also contains a derivation of the Schwarzschild metric and the calculation of the perihelion precession. Chapter 15 introduces black holes, including the Kruskal extension of the Schwarzschild solution, and Chapter 16 briefly discusses cosmology.

On the whole, the book does a good job of introducing differential geometry and general relativity in a mathematically rigorous fashion. It can be used as the textbook for a course on either differential geometry or general relativity (or both) for undergraduate or beginning graduate mathematics or physics students, and is also well suited for autonomous study. My one criticism of the book would be that, after making it through the differential geometry part, the reader should perhaps be rewarded with more general relativity. For example, the discussion of differential forms and electromagnetism in chapters 10, 11 and 12 is nicely followed up by a discussion of the Reissner–Nordström charged black hole solution in Chapter 15, but only as an exercise, with no further exploration of the rich geometry of this spacetime. Other topics of current mathematical and physical interest, such as the linearized Einstein equations, gravitational waves, or the ΛCDM cosmological model for our universe, are likewise only addressed in the exercises, and some other topics, such as the singularity theorems or the

Cauchy problem for the Einstein equations, are not addressed at all. While it is of course unrealistic to ask for a detailed treatment of all these subjects, especially in a book for undergraduates, more steps in that direction could perhaps have been taken. Nevertheless, these small quibbles should not take away from the fact that this book is a valuable addition to the general relativity literature for mathematicians, and one which I highly recommend.

Amol Sasane, *A Mathematical Introduction to General Relativity.* World Scientific, 2021, 500 pages, Hardback ISBN 978-981-124-377-6, eBook ISBN 978-981-12-4379-0.

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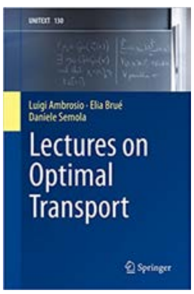
Lectures on Optimal Transport
by Luigi Ambrosio, Elia Brué and Daniele Semola

Reviewed by Filippo Santambrogio



This is the first of the two books that I am reviewing for this issue of the EMS Magazine. It is a textbook on optimal transport (in the same spirit of a book I published in 2015 [9], or of the two books by Cédric Villani [11, 12]), meant yo be used by graduate students. The first author is one of the leading experts on the topic, who has been giving lectures on it for decades at SNS Pisa (by the way, it is in the course that he gave exactly 20 years ago that I started learning about optimal transport). The second and third authors are two of the brilliant students who attended these courses in Pisa.

The book is organized into 19 chapters, each meant to correspond to a single lecture. The duration of a single lecture is not suggested explicitly, but I find the rhythm a little bit slow for graduate students, as I usually cover the material of the first 6 or 7 lectures in approximately 6 hours. Regardless, the idea of organizing the presentation according to teaching time is a very useful pedagogical tool.

The 19 lectures can be roughly divided into four series. Lectures 1 to 7 are essentially devoted to the main theory of the Monge and Kantorovich problems, where two measures are fixed and one looks for the optimal plans or maps to transport the first measure onto the second at minimal cost. At the beginning the cost function is as general as possible, which allows to develop the whole Kantorovich theory, including existence of optimal plans and duality. Only in the last of these lectures the focus is on some precise Euclidean examples, and in particular on the quadratic cost, together with its connections with the Monge–Ampère equation (whose name is spelled correctly all along the book, except for the title of the corresponding lecture where, unfortunately, we can see an acute accent). Another very natural cost, the distance cost originally studied by Monge, is deliberately discussed for only a single page, since it is clearly the goal of the authors to move on to some notions, in connection with PDEs and differential geometry, that are more related to the quadratic cost. Some choices in the proofs or in the presentation could be debatable, for instance regarding duality: the authors do present, shortly, a proof based on rather general convex analysis (the Fenchel–Rockafellar theorem), but devote more space to a full and self-contained proof based on the c-cyclical monotonicity of the support, arguing that it is more constructive, which is absolutely true. On the other hand, this approach might suggest the wrong idea that each optimizer in the Kantorovich problem is associated with a specific maximizer of the dual (the one built from the support of this very optimizer) and this can be seen in the (absolutely classical) proof of uniqueness of optimal transport maps. This proof is based on the clever statement that if every optimal plan is induced by a map, then it is unique, but does not exploit the fact that the map corresponding to a plan can be chosen to be the same for all plans.

After the general presentation of the optimal transport problem, a second series of lectures (8–10) on the Wasserstein distances and Wasserstein spaces follows. Here the authors do a remarkable work by systematically analysing which metric properties of a metric space (X, d) are inherited by the corresponding Wasserstein space $(\mathcal{P}(X), W_2)$ (we see that the focus is explicitly on the case $p = 2$, in order to pave the way for the next part of the book): compactness, completeness, geodesics, … Some parts require the introduction of suitable tools from analysis in metric spaces, in particular the notion of metric derivative, which are independent of optimal transport, but not always well known among graduate students in analysis.

Similarly, the next series of lectures (11–14) is not specifically related to optimal transport: it is devoted to a detailed analysis of gradient flows in Hilbert spaces, paying attention to those notions which can be extended to metric spaces, and in particular the EVI (Evolution Variational Inequality) and the EDI (Energy Dissipation Inequality) formulations. The role played by convexity or λ-convexity is emphasized from the very beginning. A full chapter is devoted to the study of the heat flow as a gradient flow with different choices of the functional and of the Hilbert norm (the heat flow is, for instance, the gradient flow of the Dirichlet energy $u \mapsto \frac{1}{2}\int|\nabla u|^2$ in the L^2 space, but also of the simplest functional $u \mapsto \frac{1}{2}\int u^2$ in the