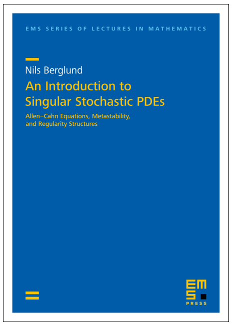
Book reviews

An Introduction to Singular Stochastic PDEs: Allen–Cahn Equations, Metastability, and Regularity Structures
by Nils Berglund

Reviewed by Martin Hairer



The past decade has seen fast paced progress in our understanding of stochastic partial differential equations (SPDEs), especially of the so-called *singular SPDEs*, and this nice little book provides a gentle introduction to the subject. The author wisely eschews the construction of a general theory and instead chooses to focus on the example of the stochastic Allen–Cahn equation, which allows to showcase increasing levels of complexity by varying the dimension of the underlying space.

The deterministic Allen–Cahn equation is the model for phase separation given by

$$\partial_t u = \Delta u + u - u^3, \tag{AC}$$

where u is a real-valued function of time and of d-dimensional space. It clearly admits $u = \pm 1$ as stable stationary states (assuming the spatial variable takes values in a domain without boundaries, like $\mathbf{R}^d$ or the torus $\mathbf{T}^d$, or that the equation is endowed with Neumann boundary conditions) and $u = 0$ as an unstable state.[1] The main subject of study of the book under review is then the behaviour of (AC) under the addition of random noise. More precisely, writing ξ for *space-time white noise*, namely a centred Gaussian random distribution with covariance formally given by $\mathbf{E}\,\xi(s,x)\xi(t,y) = \delta(t-s)\delta(x-y)$, where δ denotes the Dirac distribution, one considers the model

$$\partial_t u = \Delta u + u - u^3 + \sqrt{2T}\xi. \tag{SAC}$$

[1] Depending on the size of the domain, the dynamics can admit further non-trivial saddle points, but the author mostly assumes that the domain is small enough so that this doesn't happen.

Here, the parameter $T \geq 0$ is interpreted as the "temperature" of the system, which is justified in view of formula (BG) below.

The author then studies two types of questions. First, there are "local" questions around the existence and uniqueness of solutions. In the present case however, there is actually an even more basic question that arises, namely, what does (SAC) actually mean? The operation $u \mapsto u^3$ plainly makes sense if u is a (random) function but, since ξ is only a distribution, it is a priori not clear whether (SAC) admits function-valued solutions. In fact, it turns out that this is the case if and only if $d < 2$, so that, in higher dimensions, there is a non-trivial question as to how to even interpret (SAC). The second type of questions studied in this book are "global" questions regarding our solutions. This includes of course the question of global well-posedness, but also the question of the description of the invariant measure for the Markov process generated by (SAC).

Another global question that is being systematically addressed is that of the metastability of the ± 1 steady states. For this, one considers (SAC) at low temperature, namely with T very small. In this case, if one starts with the initial condition $u_0 = 1$, say, then one would expect the solution to remain within a small neighbourhood of 1 for a very long duration. A natural question then is how long it typically takes for the noise to kick the solution over to a neighbourhood of the other stable steady state -1. This question is being tackled using potential-theoretic methods and the book also serves as a nice introduction to this subject.

Regarding the structure of the book, it proceeds by increasing dimension of the underlying physical space, which neatly corresponds to an increase in sophistication of the methods required. Chapter 2 actually starts with "dimension 0", namely the case where the "space" is a finite set Λ of points and the linear operator Δ is a finite-difference operator. In this case, the local questions mentioned above are trivial and one focuses on the global questions. One of the main features of (AC) is that it is a gradient flow for the energy functional

$$V(u) = \int \Big(\frac{|\nabla u|^2}{2} + \frac{u^4}{4} - \frac{u^2}{2} \Big)\, dx, \tag{V}$$

which yields sufficient control on (SAC) to get global solutions. (In the discrete, "zero-dimensional" case, the integral is with respect to the counting measure on Λ and the gradient is a finite difference.) One furthermore shows that the Boltzmann–Gibbs measure

$$\mu_T(du) = Z^{-1} \exp(-V(u)/T)\, du, \tag{BG}$$

where Z is a normalisation constant and du denotes the Lebesgue measure on $\mathbf{R}^\Lambda$, is invariant for the dynamics. The *plat de résistance* of this chapter is a sketch of the proof of the Eyring–Kramers law: provided that 0 is the only saddle point for V, the expected time to go from $+1$ to -1 is asymptotically as $T \to 0$ of order

$$\frac{2\pi}{|\mu_0|}\sqrt{\left|\frac{\det \operatorname{Hess} V(0)}{\det \operatorname{Hess} V(1)}\right|}\, \exp\big((V(0)-V(1))/T\big)\big(1+\mathcal{O}(T)\big), \tag{EK}$$

with μ_0 the lowest eigenvalue of the Hessian $\operatorname{Hess} V(0)$.

Chapter 3 proceeds to the continuum one-dimensional case. In this case, while (V) still has an obvious meaning, interpreting (BG) and (EK) is a bit more tricky. In the case of the Boltzmann–Gibbs measure, the problem is that there is no Lebesgue measure in infinite dimensions, while the problem with (EK) is that $\operatorname{Hess} V$ is of the form "Laplacian plus constant", so that it is an unbounded operator. Both of these difficulties can be resolved in relatively straightforward ways, in particular the ratio of determinants in (EK) is nothing but the Fredholm determinant $\det(1 - 3(2-\Delta)^{-1})$, but this gives the author a good opportunity to introduce some of the basic concepts in the study of stochastic PDEs, including a solution theory for (SAC), Schauder theory, the description of space-time white noise, etc.

This lays a good foundation on which to build the study the two-dimensional case in Chapter 4. It is in this case that, for the first time, the word "singular" appearing in the title of the book takes its meaning. Indeed, considering solutions to the *linear* stochastic heat equation

$$\partial_t v = \Delta v + \sqrt{2T}\xi,$$

one already finds that these are no longer function-valued in dimension two, but instead do at best take values in some Besov spaces with strictly negative regularity index. As a consequence, it is unclear a priori what "being a solution to (SAC)" actually means in this case. The author gives a short introduction to Wick calculus, which permits to give a meaning to "renormalised" powers $v^{\diamond p}$ of v by means of a suitable approximation procedure. For example, one has $v^{\diamond 2} = \lim_{\varepsilon\to 0}(v_\varepsilon^2 - C_\varepsilon)$, where v_ε is some smooth approximation to v and C_ε is a suitable chosen (and typically diverging as $\varepsilon \to 0$) sequence of constants. It is then natural to *define* solutions to (SAC) by setting $u = v + w$ and looking for w solving

$$\partial_t w = v + w - v^{\diamond 3} - 3v^{\diamond 2}w - 3vw^2 - w^3. \tag{$\star$}$$

It turns out that this not only provides a well-defined solution theory, but u can be approximated by solutions to a version of (SAC) with smoothened noise, provided that the nonlinearity $-u^3$ is replaced by $3C_\varepsilon u - u^3$. A very interesting consequence discussed in Section 4.6 is that the effect of renormalisation is to turn the Fredholm determinant appearing in the Eyring–Kramers formula, which is no longer well-defined since $(2-\Delta)^{-1}$ is no longer trace class, into the well-defined Carleman–Fredholm determinant $\det_2$.

Chapter 5 finally deals with the three-dimensional case. There, while it is still possible to define $v^{\diamond 2}$ and $v^{\diamond 3}$ as random distributions, the equation ($\star$) for the remainder term is itself ill-posed. Dealing with this problem was one of the original motivations for the development of the theory of regularity structures. Building on the concepts introduced in the previous parts, the main goal of this last chapter is to provide an introduction to the various aspects of this theory (reconstruction theorem, lift of various operations, renormalisation, etc.) in the context of the problem of building a robust solution theory for (SAC). Note that in this case, while a Freidlin–Wentzell type large deviations result is still available and is briefly discussed in Section 5.7, the interpretation and justification of the Eyring–Kramers formula is still an open problem, to the best of the reviewer's knowledge.

As the reader may have come to suspect by now, a complete mathematical treatise of all the aspects mentioned here would take much more space than the roughly 200 pages of this short book. Instead, the style chosen by the author is to provide details for some of the simpler proofs and only rough sketches of the main steps for many of the more advanced statements. This strikes a nice balance between self-contained proofs and references to more advanced material and makes the book a must read for anyone with a graduate-level background in probability and analysis who is interested in a quick introduction to the modern tools used in the analysis of singular SPDEs.

Nils Berglund, *An Introduction to Singular Stochastic PDEs: Allen–Cahn Equations, Metastability, and Regularity Structures.* EMS Press, 2022, 230 pages, Softcover ISBN 978-3-98547-014-3, eBook ISBN 978-3-98547-514-8.

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Les Mathématiques comme Métaphore: Essais choisis
by Yuri Manin

Reviewed by Ulf Persson

At the ICM 2006 in Madrid I attended a lecture by Manin speaking about the different uses of mathematics, as models, theories, and metaphors. Of all the lectures I attended at that congress, this was the one that stuck out to me. It was obviously not a technical talk, but a philosophical one in the best sense of the term, namely, fuelled not by professional pedantry, but by a deep personal curiosity expressed in a very original and captivating way. A year later, a collection of Manin's essays had been translated into English and handsomely published by the AMS under the title of 'Mathematics as Metaphor.' I got the book, read it with delight, as I had read previous books by him as a young man, and in fact I wrote a review of it which was published in 2010 in the EMS Newsletter – incidentally, a fact I had already forgotten this spring. However, I was alerted to it and learned that I had at its end expressed my regret that not more of his essays were available to readers not knowing Russian. Now my wish has been granted. That a wider collection had recently been published, I actually found out from Manin himself in what would turn out to be my last communication with him. I immediately got the book, published by the small French firm, Les Belles Letters, and thus containing French translations of his texts. The AMS version was about 200 pages, while this edition runs well over 500 pages, so one surmises that it is a very significant extension. On the other hand, a mere page count is a bit misleading because the pages of the first edition are larger than those in the latter one and also the font size employed is somewhat smaller. I estimate that the American edition sports about 3500 characters a page, and the French edition about 2000 characters, but still we are talking about a significant extension. My first intention was to single out what was new in the extended edition and concentrate on that, but I have decided to abandon that and treat it as a whole, fully independent of my first review.

We are talking about essays, not scientific articles, and there is of course a significant difference between the two. An essay is, like the terminology indicates, an attempt. Namely, an attempt to come to grips with a subject in a non-technical way using a meta-perspective. You should not write a scientific article if you are not an expert, but anyone is welcome to write an essay on any subject that occurs to them (they need not be published). In fact, any such attempt reminds me of the American diplomat George Kennan, who during his career wrote many dispatches from his various postings with scant hope that they would ever be read, but justifying his activity by claiming that he wrote in order to discover what he thought. This points to a crucial aspect of essay writing, namely, exploration. Karl Popper did not, unlike his colleagues in the Vienna Circle (disparaged by posterity as positivists) reject metaphysics, instead he was thinking of it as proto-science, potentially developing into one.

As indicated, most essays in general may be ignored (which does not necessarily mean that writing them is a useless activity); what makes Manin's essays worth pondering is the originality of his mind and imagination, the precision of his formulations, all supported by his wide culture, and the boundless curiosity which made this culture possible. Essays should be classified as literature, and thus subjected to the demanding criteria such writing invites. Imagination requires obstacles to be circumvented in order to be properly stimulated; this is why, according to Hilbert, mathematics requires more imagination than poetry, or, as claimed by the biographer Peter Acroyd, the writing of a biography requires more imagination than the writing of a novel. But in this general frame there are different kinds of imaginations, the iron-clad laws of logic typically lead to frustration, while writing essays and fiction leaves you more liberty. Arguments need not to be watertight as long as they are exciting, and inconvenient facts can be ignored or simply made up, as typically in fiction; what matters are the ideas, which need not be technically developed. Thus, I cannot resist speculating that the writing of essays (and poetry?) gave Manin a relief from the rigors of mathematical work, but this does not necessarily mean that it should be thought of as a mere diversion – on the contrary, it was an essential component of his mathematical work, without which the latter may not have been possible. His essays are also more accessible to readers, provided they have the required temperament, than his purely mathematical work, although the charm of the latter derives much from being presented in an essayistic spirit (this is why the above-mentioned books made such an impression on my young mind).

The point of an essay is not only to profit the writer but also to inform and inspire the reader, this is why it is very hard for me not to elaborate on Manin's essays, and to just present sober resumes; but then again, they are published and available for everyone to read and engage with in their own ways, so I hope that my taking of liberties can be excused as a kind of homage.

First, what is the nature of mathematics? This is a question that cannot be treated mathematically, but nevertheless must at least to some degree engage every serious mathematician, and even influence the way and why they persist in their obsessions. Manin himself is puzzled why mathematics engages him so much, yet without this potential skepticism in any way dampening his enthusiasm for the subject. Now there is a vulgar idea of mathematics, prevalent not so much among the general public as among philosophers and physicists and other concerned academics. Mathematics is, according to this view, seen as a game; you set up some axioms as rules and then apply logic to it and grind away. From this it does not take much to conclude that mathematics is just a matter of symbolic manipulation, and although its concepts do not have any real meaning (like vertices in a graph), it can still amazingly serve as a useful language and even tool in the study of the real world. The idea that mathematics is applied logic goes at least as far back as Frege and was further developed by his successors Russell and Wittgenstein. On the other hand, the American philosopher Charles Peirce claimed that the integers were more basic than logic, and that mathematicians had no need to study logic, they were anyway able to instinctively draw the necessary conclusions, on which mathematics rests and develops. The emphasis on logic has led to the dictum that mathematics is but a sequence of tautologies, which has been taken to heart by many. Any idea that has spread successfully must have some truth to it, so it is admittedly true that a large part of a mathematician's everyday work may amount to a ceaseless manipulation of symbols. Manin cites, not without approval, the claim by Schopenhauer that when computation begins, thought ends. Mathematics is indeed a very special activity which delights in reasoning using long deductive chains and thereby coming up with true facts in a systematic way. We recall Leibniz's exhortation, stop arguing let us calculate, hoping there would be a verbal calculus which would resolve human problems as neatly as celestial ones (for which calculus was once invented). Manin insists that the logical straight-jacket that mathematics is forced into is necessary – without it, it would degenerate, as anything to remain solid has to be contained. It is the possibility of falsification, that allows things to grow purposefully by pruning off false leads. It is also this that leads to the frustrations of mathematicians, by the presence of what which cannot be willed away. But for the serious mathematicians there is also something else to mathematics without which they would never pursue it. Mathematics involves more than a random walk in a logical configuration space. It requires thinking in a natural language, a thinking that is not in the nature of a computation in some generalized sense, but is meta-thinking whose mission is not to produce new facts, but to distinguish between the interesting and the fruitful, of coming up with new ideas and strategies. Without this meta-thinking mathematics would be a sterile subject indeed. In fact, what the serious mathematician aims for is the elusive goal of understanding, of seeing different pieces coming together, something which cannot be conveyed by mere mathematical formulations, just as little as ideas can be precisely formalized and expressed, at best only conveyed obliquely, and in this elusive vagueness lies their power. One important difference between a natural language and a formal artificial one is that the latter is precise, while the former is vague; as a result, the latter can be treated as a mathematical object. Being vague, natural languages have a recourse to forming metaphors, which, I never tire of pointing out, should never be taken literally, as they then become merely silly; while metaphors in formal languages have no choice but be taken literally. In a natural language nothing stops you from imagining the set of all sets (or the wish to have all ones wishes granted), but in a formal, strict logical setting one is forced to make explicit the different notions of 'set' involved and be forced to adopt a new word for one of them, such as 'class.' The Russell paradox does not affect natural languages, as they thrive on contradictions – in fact, languages evolved socially, meaning in particular that expressing truth is not necessarily the main purpose, rather deception; which incidentally ties up with Manin's fascination with the 'Trickster.' Thus, the metaphorical idea of the diagonal argument when applied 'literally' (in the sense of rigidly logical) has interesting consequences. At the heart of Gödel's argument, as Manin points out, is this partial embedding of the meta-language into a formal one on which it comments. Incidentally, there is much hype connected with Gödel's theorem and Manin's excellent presentation of it has as a purpose to demystify it. As he notes, the theorem has had marginal influence on mathematics as practiced.

What is mathematical intuition? Mathematical and logical concepts are anchored in a physical and hence tangible reality in the human mind. Numbers are in particular associated with the counting of physical objects, such as buttons and shells. One may talk about small numbers such as billions and trillions when they can so be concretely represented; but with the advent of the positional system of representing numbers one was able and hence seduced to write down huge numbers with millions of digits, numbers that in no way can be represented by the counting of physical objects of any kind, only of imagined objects of the mind, such as all possible books in Borges' celebrated story. Let us call such numbers, numbers of the second kind, which for all practical matters can serve as (countable) infinities. Then of course there are numbers of the third kind, represented by those which need a number of second kind to count their digits, and we can proceed inductively, and the whole thing carries an uncanny analogue of Cantor's hierarchy of infinities, except there are of course no precise boundaries between them, but the idea remains (one could of course impose precise demarcations, but that would be artificial and pointless). We are in the realm of natural language after all, where precision is not required. Of course, they are all finite, but even finite numbers can be unbelievably large and induce a sense of vertigo our usual congress with infinity does not involve. What is easier than suggesting infinity by a sequence of dots $1, 2, 3, \ldots$ (you get the idea), but to really feel it, your imagination must be suitably stimulated by tangible intuition.

It is tempting to insert a slight digression here, touching upon Manin's interest in Kolmogorov complexity. It is trivial to write down numbers of any kind by using specialized notation (or more generalized inductively-defined functions), but the generic number of, say, the third kind cannot be physically represented in, say, decimal form, which is the type of form that in general is the most efficient. So in what sense can we get our hands on them? How many '7's are there in the decimal representation of a number of type $7^{7^{\cdot^{\cdot^{\cdot^{7}}}}}$? Can any solution to this problem be feasibly described in any other sense than by the question itself? Maybe an interesting example of a totally uninteresting question.

Metaphors are important for human thinking, and Manin brings up the notion of the Turing Machine and the influence it had on logic. Yes, machines are tangible objects of the imagination and embody themselves logic in a palpable way – after all, their parts are connected in long chains of causes and effects, like the deductive chains in logical reasoning. Classically, they were represented by the sophisticated machinery of a clockwork; nowadays, we have the computer, although its machinery is not so much exhibited in its hardware, of which most users are blissfully ignorant, but in its software when the old tinkering with cogwheels has been replaced by letting the fingers dance on the keyboard instead, through writing computer codes. As David Mumford has pointed out, a mathematical proof and a computer program have much in common. Indeed, the word 'mechanical' is what we use in describing a mindless manipulation of objects subjected to inexorable laws outside our control.

Set theory was created by Cantor by taking infinities very literally as objects to be mathematically handled (but one may argue that infinite convergent sums actually involve a literal, not only potential infinity, and go back to antiquity – just think of Zeno). The uncountability of the reals is something most of us encounter in our teens, and it is usually considered as something rather metaphysical, apart from mainstream mathematics. However, without the negative aspect of the uncountability of the reals modern measure theory with its countable additivity would be impossible. For it to work, the setting has to be uncountable, and that uncountability could indeed be seen as the metaphysical setting of all those manipulations. It stands to reason that such a theory would have been developed sooner or later and then the uncountability of the reals would have been staring in our faces. Cantor's hierarchy of infinities met a lot of resistance when it appeared, Manin reminds us, and also a lot of skepticism as it was developed. As it is based on human mathematical intuition involving the manipulation of physical objects, which has no longer any relevance, that ordinary expectations would come to grief is not surprising. What could be more natural than picking one object each from a collection of non-empty sets, but the Axiom of Choice has very counterintuitive consequences when applied in, say, an uncountable context, giving rise to the Banach–Tarski paradox, or the well-ordering of the reals. The fate of the continuum hypothesis is a case in point, the physical intuition was that here it was, a subset of the real line just in front of our eyes, it had to be true or not. But it turned out to be a question of mere convention, what rules are allowed or not in forming subsets. Thus, it degenerated to a formal game having no relation whatsoever to our conception of physical reality. The very notion of mathematical Platonism seemed to founder when exploring the transfinite world, where we seem at liberty to bend the rules at our discretion. 'What did the paradoxes and problems of set theory have to do with the solidity of a bridge?' – Ulam rhetorically asked, as reported by Rota. Our sense of the solidity of mathematics seems to be connected to tangible models, such as physical space to classical Euclidean geometry. The real line has for us an almost physical existence. But when it comes to models for set theory, the very notion of a set as a mental construct seems inextricable from a verbal description; but there is only a countable infinitude of such, and hence the existence of countable models even for uncountable sets (where there are two notions of cardinality, one extrinsic, and one intrinsic). Naively we think of all subsets existing of, say, the reals, but from a strict logical and formal point of view, only those which in principle can be described. This threatens, as noted, to indeed reduce mathematics to a game whose objects mean nothing (just like the chess pieces on a board). On the other hand, a piece of mathematics considered as a game has nevertheless some content as a game, and we can ask questions about it, such as its consistency, which we feel is a definite yes or no question, not contingent upon some axioms we introduce in the meta-game of investigation. According to Manin, it is as if we feel that the game itself, defined by its axiomatic rules, is a physical object, and systematically drawing all the conclusions is a physical activity anchored in the real world, no matter how unfeasible in practice; just as concluding that a Diophantine equation must have a solution or not by making an almost physical thought experiment of an infinite search. Manin's attitude to set theory is pragmatic, as that of most mathematicians. He does not seem engaged in the classical controversies and refers to intuitionists and constructivists as somewhat neurotic. Set theory for Manin, like for most mathematicians, provides a convenient language of mathematics, as famously exemplified by Bourbaki. On a more existential level, Manin's attitude to mathematical Platonism is ambivalent; he has described it as psychologically inescapable and intellectually indefensible. What is really meant by that can only be speculated upon. He stresses that his physically tangible intuition, especially when confirmed by mathematical applications to physics as a scientific discipline, makes him inclined to Platonism, an attitude made even more inescapable from his own experience as a mathematician, in particular when studying number theory; but as strong as those convictions may be, they are ultimately based on subjective experience. Of course intellectually Platonism is not amenable to any formal proof, as little as proofs of the existence of God pursued by the scholastics (the concerns of whom seem uncannily similar to those of set-theorists). But as Pascal famously noted 'Le cœur a ses raisons que la raison ne connaît point.'

I would like to conclude this mathematical section with a nice toy example of Manin. Consider a finite set X of m elements. The power set $P(X)$ is naturally an m-dimensional vector space over the field $\mathbb{Z}_2$ with $\varnothing$ corresponding to the 0. Its algebra of functions is given by the Boolean polynomials $\mathbb{Z}_2[x_1, x_2, \dots, x_m](x_1^2 + x_1, x_2^2 + x_2, \dots, x_m^2 + x_m)$, thus any such polynomials can be written as a sum of monomials which are naturally identified by the elements (vectors) $x \in P(X)$, where, say, $(1, 1, 0, 1)$ is identified with $x_1x_2x_4$ and 0 with the trivial (constant) monomial 1. Thus, given x we have $x(y) = 1$ iff $x \subset y$. The polynomials (P) are thus tautologically paired with the subsets S of $P(X)$ by $P = \sum_{x \in S} x$. But there is also another way of associating a subset to a polynomial, namely, to associate its zeroes. The fact is that every subset is given by the zeroes of a unique polynomial, so in particular 1 is the only polynomial with an empty set of zeroes. To see this, we have to introduce $I = X$ (and note that $x + I$ is the complement of x and the zeroes of $1 + P$ make up the complement of the zeroes of the polynomial of P). Consider now the polynomial

$$x(u)(x+I)(u+I) = \prod_{i \in x} x_i \prod_{j \notin x} (1 + x_j).$$

We have $x(u)(x+I)(u+I) = 1$ iff $u = x$; thus, for any set S the polynomial

$$\prod_{x \in S} (1 + x(u)(x+I)(u+I)) \quad \left(= \prod_{x \in S} \left(\prod_{i \in x} x_i \prod_{j \notin x} (1 + x_j) \right) \right)$$

vanishes exactly on S (if $S = \emptyset$, then of course the polynomial is 1). What is the point of this formal almost tautological game? Manin brings it up as a finite version of the Axiom of Choice: given a set of polynomials how do we pick an element in each of the sets they define, or show that the polynomial is 1? Given the polynomial in canonical form (or any random form), this is not so easy in general: do we have to check all the elements of the vector space? This also leads to a particular instance the P/NP problem, an instance which, according to Manin, is intractable at the time.

Now I have not touched upon the section of mathematics and physics, which is greatly expanded, nor upon the essays on general topics from linguistics, Jungian psychology (of which Manin was charmed with many references in his works), art and poetry. Had I done so, the review would have been far too long, not only too long as it already is. Having thus failed to do full justice to the book, I hope that I have at least inspired a few readers to consult the master himself.

Yuri Manin, *Les Mathématiques comme Métaphore: Essais choisis.* Les Belles Lettres, 2021, 600 pages, Softcover ISBN 978-2-251-45172-5.

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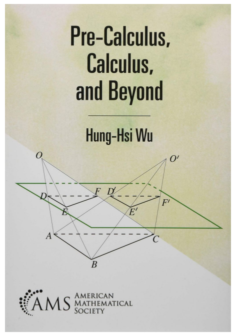
Pre-Calculus, Calculus, and Beyond by Hung-Hsi Wu

Reviewed by António de Bivar Weinholtz



This is the sixth and final book of a series covering the K-12 curriculum, as an instrument for the mathematical education of school teachers. It is the third and final volume of the series dedicated to high-school teachers. Unlike the two previous such volumes, which included topics that had already been treated in the series (to ensure that high-school teachers could have at their disposal a set of self-contained instruments for their mathematical education, expressly written for them, thus not neglecting the pre-requisites to what they have to teach), this final book is composed of entirely new topics.

The first chapter is dedicated to trigonometry and the definition of trigonometric functions with domain $\mathbb{R}$. It starts with the basic definitions, the general notion of extension of a function, then applied to extending trigonometric functions, with the use of the unit circle, to the interval $[-360, 360]$ and finally to $\mathbb{R}$. The laws of sines and cosines, as well as other basic trigonometric identities, such as the addition formulas, are proven in this general setting. It proceeds to the definitions of radian and the new trigonometric functions obtained by switching from degrees to radians, and to the definition of polar coordinates. Finally, trigonometric functions are put to use in the geometric interpretation of complex numbers and the derivation of the De Moivre and Euler formulas, with the exponential notation; applications are given to the study of n-th roots of unity, to a formulation of basic isometries in terms of complex numbers, and to the study of graphs of quadratic functions, with the use of rotations to eliminate the mixed term in general quadratic equations in two variables. The chapter concludes with the introduction of inverse trigonometric functions and a final section where the author analyzes the importance of these functions in the study of general periodic functions, which play a fundamental role in the physics of many phenomena, through Fourier series. More advanced treatments of trigonometric and general exponential functions are given a brief overview, which provides adequate complementary useful knowledge to the readers.

The following chapter proceeds with a rigorous treatment of real numbers. Thus it becomes finally possible to justify what had previously been called "FASM" (the "fundamental assumption of school mathematics") and enabled students to use real numbers, without betraying the basic principles of mathematical studies, from the moment it becomes mandatory for the development of their mathematical instruction, but before it is possible to include in the curriculum a rigorous treatment of the real line, due to the inner complexity of the subject. After an algebraic reformulation of the theory of rational numbers, the introduction of an extra axiom finally leads to the fundamental distinction of the sets of real numbers and rational numbers, that can then be identified with a dense subset of the real line. The concept of limit of a sequence of real numbers is defined and its basic properties are then presented, proved and applied to the rigorous treatment of some of the concepts and properties that had been previously accepted with the use of FASM, namely the existence and basic properties of positive n-th roots of positive real numbers and the fundamental theorem of similarity, followed by a whole chapter dedicated to a full study of the decimal expansion of a number, including repeating and non-repeating decimals, and using the concept and basic properties of infinite series.

A new chapter follows where the delicate concepts of length and area are treated as rigorously as possible at this stage, based on a list of fundamental principles for geometric measurements that are accepted as a guide to the foundation of those concepts, but the inherent difficulties of these topics are explained. In this framework, the author introduces the concept of rectifiable curve and identifies the problems one faces when trying to obtain a rigorous argument that leads to the formula for the circumference of a circle, postponing the final solution to the end of the volume, where a more advanced treatment is given of trigonometric functions, that is put to this use.

Some basic formulas for the area of elementary figures are revisited in this more general setting and obtained using the assumptions of this chapter; a famous proof of the Pythagorean theorem using the concept of area is finally given a proper formulation, whereas it is very often presented to students without the due care to observe that it depends on rather subtle and nontrivial concepts and properties of area and without some apparent geometrical properties being adequately proven. As the author explains in one of his illuminating pedagogical comments, this is another example of how misleading some rather common incoherences in the teaching of school mathematics can be.

After length and area, it is time for the introduction of some comments on three-dimensional geometry and the concept of volume. By the formulation of some elementary principles that, at this stage, have to be accepted without further foundation, the author proceeds to the proof of some basic facts on perpendicularity and parallelism of lines and planes in three-space and to the analysis of Cavalieri's principle, which leads to the formula for the volume of a sphere.

The two final chapters are dedicated to an introduction of derivatives and integrals of real-valued functions of one variable and their basic properties, and applications to trigonometric functions and to new formulations of the logarithmic and exponential functions; they start with the notions of limit of a function in a point and of continuity.

As in the previous volume, this one also contains a very helpful Appendix with a list of assumptions, definitions, theorems and lemmas from the companion volumes. I strongly recommend reading first the review of the first volume (António de Bivar Weinholtz, Book review, "Understanding numbers in elementary school mathematics" by Hung-Hsi Wu, Eur. Math. Soc. Mag. **122** (2021), pp. 66–67). There, one can find the reasons why I deem this set of books a milestone in the struggle for a sound mathematical education of youths. I shall not repeat here all the historical and scientific arguments that sustain this claim, but I have to restate, regarding this final volume, that although it is written for high-school teachers, as an instrument for their mathematical education (both during pre-service years and for their professional development), and to provide a resource for authors of textbooks, the set of its potential readers should not be restricted to those for which it was primarily intended; it should include anyone with the basic ability to appreciate the beauty of the use of human reasoning in our quest to understand the world and the capacity and will to make the necessary efforts, which are required here as for any worthwhile enterprise. Of course, as the content and presentation of the three last volumes of the series is of a more advanced nature, a wider mathematical background is required. This volume being the last of the series, we are now able to fully appreciate the magnitude of the enterprise undertaken by Prof. Wu and how it is indisputable, as I wrote before, that with this set of books at hand there is no excuse left for school (including high-school) teachers, textbook authors and government officials to persist in the unfortunate practice of trying to serve to school students mathematics in a way that is in fact unlearnable…

Like the previous two volumes, this one is punctuated with pedagogical comments that give extremely useful advice regarding what content details should be used in classrooms and which are essentially meant to teachers; mathematical comments are also added to the main text, in order to extend the views of the reader whenever it helps to clarify the subject in question. To the readers interested in the full scope of the pedagogical comments of this volume I also recommend the lecture of my preceding review (António de Bivar Weinholtz, Book review, "Teaching school mathematics: Algebra" by Hung-Hsi Wu, Eur. Math. Soc. Mag. **125** (2022), pp. 50–52), where a detailed description is made of what the author considers to be the main characteristics of mathematics and how they have been neglected in schools for such a long period of time and replaced by what he calls "Textbook School Mathematics" (TSM); the same concern is present in all the topics treated in the present volume.

As it is almost inevitable in any printed book, there are some minor misprints that can be easily detected and corrected by the reader. I just point out some details in formal definitions that deserve some attention.

The definition of the i-th term of a sequence (p. 118) as the value assigned by the function (that the sequence is, by definition) to i is commonly found in these same terms in many mathematical texts, but it can lead to some awkward consequences; for instance, it is then not strictly true that "every sequence has infinitely many terms", as the appreciation of this statement, with the above given definition, depends on the number of *values* of the sequence rather than on the fact that its *domain* (the set of natural numbers) is infinite. With this definition, a *constant* sequence would have only one term…. A formal definition that would allow us to state that the number of terms of a sequence is always infinite, one for each whole number, could be to identify the i-th term of the sequence (s_n) with the ordered pair (i, s_i).

In the definition of convergence of regions (p. 230), apart from the stated condition on the boundaries, one needs some extra condition, as, for instance, the coincidence of the approximating regions with the limit region R outside a "vanishing neighborhood" of the boundary of R. This condition is very easily verified in all the cases where Theorem 4.3 (convergence theorem for area) is applied in this book, and also in the graphical examples that are used in the treatment of area; this treatment, of course, has to rely on some intuitive assumptions at this stage.

Also, the definition of the limit in a point x_0 of a real-valued function defined in a subset I of $\mathbb{R}$ (p. 286) adopted in this book is what we can call the "exclusive" limit, inasmuch as, to "test" the limit of the function, one only considers sequences in I with limit x_0 that never assume the value x_0, as opposed to what we can call the "inclusive" limit definition, where we can also consider such sequences that can assume the value x_0; but in the case of this "exclusive" limit, to ensure that the limit is unique, when it exists, one has to assume as well that x_0 is the limit of a sequence in I that never assumes the value x_0 (x_0 is then usually called an *accumulation point* of I). It is not enough to ensure that it is just the limit of a sequence in I (a *limit point*); if x_0 is what is usually called an *isolated point* of I, i.e., if it is a limit point but not an accumulation point of the domain, with this "exclusive" definition of limit, the function would have every number as limit in x_0 (because to contradict this fact one would have to find a sequence in I that has x_0 as its limit, never assuming the value x_0; but this contradicts the definition of an isolated point). So, either one only considers domains with no isolated points, or one has to define this kind of limit only in accumulation points and not in general limit points of the domain, as the uniqueness of the limit is an essential feature of this concept. Strictly speaking, when considering the algebra of limits of functions, like in Lemma 6.2 (p. 290), one also has to be careful to consider only accumulation points of the domain of the functions obtained by performing each algebraic operation in the pair of functions, as it is not mandatory that if a point is an accumulation point of the domain of each function in the pair it will also have this property with respect to the intersection of domains.

Finally, the definition of continuity (p. 289) is not affected by these subtleties, as it is not dependent on the definition of limit of functions (only in the intuitive motivation of this concept a link is established with limits). In fact, with the alternative ("inclusive") definition of the limit of a function, to be continuous in a point of the domain could simply be defined as having a limit in that point; however, if one aimed to use the adopted "exclusive" limit definition of continuity, one would have to treat separately the isolated points of the domain. Nevertheless, this leads to the conclusion that in the proof of Lemma 6.3, on the "algebra of continuity", one cannot fully rely on Lemma 6.2; once again we could be spared all these subtleties either if one excluded domains with isolated points, or if one considered the "inclusive" definition of limit (in this case, however, with some care also with domains in the algebra of limits).

All these details should not, of course, be brought to a high school classroom, although they can be of some use to teachers.

As in the previous volumes of this series, on each topic the author provides the reader with numerous illuminating activities, and an excellent choice of a wide range of exercises.

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