

Interpreting convex integration results in hydrodynamics

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The aim of this article is to encourage debate of issues of the applications of modern methods of mathematical analysis in fluid dynamics. A recent surprising result derived by convex integration techniques shows non-uniqueness of weak solutions in initial value problems of the Navier–Stokes equations. The question of relevance of such a result to physical observed flows allows a variety of answers, some of which are discussed below.

1 Introduction

Convex integration techniques, introduced in the context of differential geometry by Nash [39] and Gromov [27], have been later applied to fluid dynamics by De Lellis and Székelyhidi in the path-breaking papers [14, 15]. Subsequently, these techniques have been used with great success by many researchers to prove *inter alia* non-uniqueness of weak solutions of Euler and Navier–Stokes equations. A recent review of the technique is provided in the monograph by Markfelder [37]. Below we refer to weak solutions obtained by convex integration techniques as *wild* solutions.

The question of present concern is what relevance these outstanding mathematical results have for the prediction and explanation of observed physical phenomena. For definiteness, consider the striking result proved by Buckmaster and Vicol [10]:

Theorem 1.1 ([10, Theorem 1.2]). *There exists $\theta > 0$ such that for any non-negative smooth function $e(t) : [0, T] \rightarrow \mathbb{R}_{\geq 0}$, there exists a weak solution of the Navier–Stokes equation in $C_t^0([0, T]; H_x^\theta(\mathbb{T}^3))$ such that its velocity $\mathbf{v}(x, t)$ satisfies*

$$\int_{\mathbb{T}^3} |\mathbf{v}(x, t)|^2 dx = e(t)$$

for all $t \in [0, T]$.

A remarkable feature of this result is that the function $e(\cdot)$ does not have to be decreasing in time. The Buckmaster–Vicol theorem implies non-uniqueness of weak solutions of the Navier–Stokes equations on the 3-torus \mathbb{T}^3 as $e(\cdot)$ can have compact support. These wild velocity fields exhibit a strong degree of irregularity.

Although this theorem is clearly a great achievement of 21st century analysis, its relevance to turbulence and other observed fluid behaviour is not obvious and indeed is disputed by some. It is a potentially rich topic for discussion.

Thus, with the aim of making available to the wider scientific community differing views on this important subject, we¹ approached for their opinion a representative cross-section of leading practitioners not only in mathematics and physics, but also in philosophy of physics and mathematics.

Accordingly, we invited answers to the following specific question:

How, in your opinion, does the theorem of Buckmaster and Vicol relate to the prediction of observed fluid flows?

Section 2 contains the replies we have received. Section 3 summarises the discussion and contributes some concluding remarks.

To give the reader some guidance, we now describe how the contributors see their scientific outlook. Gregory Eyink and Nigel Goldenfeld are theoretical physicists with a wide range of interests, who, to quote Eyink, try “to talk with everyone and understand everyone, including philosophers, mathematicians, physicists, chemists, engineers, etc.”; Ondřej Kreml is primarily a mathematician with a background in physics and mathematical modelling; Ilya Karlin and Florian Kogelbauer are applied mathematicians and mathematical physicists; Colin McLarty is a philosopher and a mathematician; Simon Markfelder defines himself as a mathematician, Mikhail Osipov is a theoretical physicist and an applied mathematician. We (understood in the sense of footnote 1) see ourselves as mathematicians who worry about the connections of mathematics with the natural world.

2 Expert opinions

This section reproduces in full the replies received to our invitation.

¹ Here and in Section 3 “we” means H. Gimperlein, M. Grinfeld, R. J. Knops and M. Slemrod.

2.1 Gregory Eyink and Nigel Goldenfeld

The question posed is whether there is any physical relevance of the convex integration techniques of Nash and Gromov, as successfully applied by De Lellis and Székelyhidi [14, 15] to the mathematics of the partial differential equations of continuum fluid mechanics. A particular focus is the remarkable result of Buckmaster and Vicol [10] proving non-uniqueness of incompressible Navier–Stokes solutions for fixed initial data. In fact, they proved the existence of weak Navier–Stokes solutions whose kinetic energy is any prescribed non-negative function. Here, as requested, we point out what aspects of this body of work are likely to be physically relevant to hydrodynamics, and which are of purely mathematical interest. We focus on turbulence in particular. Our conclusion is that the potentially important application of convex integration to hydrodynamic turbulence is to show the non-uniqueness of solutions of the Euler equations, not the Navier–Stokes equations! The reason is that it is the Euler solutions which describe the inertial-range cascade at asymptotically large Reynolds numbers [40], and the convex integration proofs of their non-uniqueness establish a prerequisite for the conjectured phenomenon of spontaneous stochasticity induced by thermal fluctuations [4, 21].

As physicists, our perspective is based on the physical relevance and structural stability of the starting assumptions, specifically the deterministic incompressible Navier–Stokes equations. These equations have been traditionally assumed in the fluid mechanics community to be valid at all scales down to the mean free path, where hydrodynamics must break down. However, molecular noise is required by the assumption of local thermal equilibrium and such noise produces many experimental effects in laminar flows that are missed by the deterministic equations [50]. It was realised already by Landau and Lifschitz [34] that hydrodynamic equations need to incorporate thermal fluctuations together with viscous dissipation, and they formulated a successful theory of “fluctuating hydrodynamics” consistent with the fluctuation-dissipation theorem of statistical physics. Using a shell model of these equations, we recently showed, in a high-Reynolds turbulent flow at parameters relevant to the atmospheric boundary layer, that thermal fluctuations become significant already at the Kolmogorov scale, orders of magnitude larger than the mean free path length [3]. Our result was subsequently confirmed by a numerical simulation at much lower Reynolds numbers, corresponding to a gently stirred vial of water, using the full Landau–Lifschitz equations [7]. Thus, we doubt the physical importance of the non-uniqueness results of Buckmaster and Vicol [10] because the deterministic Navier–Stokes equations which they study are based on an unphysical continuum approximation. The Landau–Lifschitz equations, on the contrary, are easily proved to have strong, pathwise-unique solutions (e.g., see [1]), because they are understood to be a mesoscopic effective field theory with an explicit high-wavenumber cutoff.

Our claims may appear in contradiction with rigorous mathematical derivations of incompressible Navier–Stokes as a deterministic

PDE by means of a hydrodynamic scaling limit, e.g., in stochastic lattice gas models [41]. These rigorous results support the so-called “macroscopic fluctuation theory” [8], according to which velocity fields evolve by deterministic Navier–Stokes, with probability going to one. This theory attempts to incorporate thermal fluctuations by probabilistic large deviations theory, in which spontaneous fluctuations arise in thermal equilibrium by “instanton solutions” governed by time-reversed Navier–Stokes equations with negative viscosity [8]. In that setting, the Buckmaster and Vicol wild solutions might be accorded physical significance, because, under time reversal, they could describe a very rare fluctuation in which the fluid spontaneously cools down and the lost thermal energy is converted into macroscopic kinetic energy. The Buckmaster–Vicol theorem suggests that such macroscopic thermal fluctuations could arise in a strictly finite time from the rest state of the fluid and likewise could decay back to rest in a finite time, whereas smooth Navier–Stokes solutions require infinite time [43]. See Gess et al. [24] for a recent discussion of the Buckmaster–Vicol theorem in the context of macroscopic fluctuation theory. However, this interpretation of the Buckmaster–Vicol theorem is also unphysical in our opinion, because the “hydrodynamic scaling limit” invoked in macroscopic fluctuation theory is essentially impossible to achieve in any realistic molecular fluid. Indeed, the numerical counterexamples of Bandak et al. [3] and Bell et al. [7] showed that the thermal fluctuation effects can never be removed in practice even if the flow were scaled by ridiculously large factors of millions or billions!

In our opinion, convex integration techniques have physical relevance for turbulent fluids not when applied to Navier–Stokes but instead to Euler equations. Unlike the Navier–Stokes equations, the inviscid Euler equations have a precise physical regime of validity, since their weak solutions plausibly describe the inertial range of turbulent flows at very high Reynolds numbers ([40]; for a detailed review, see [20]). The inertial range is described by continuum PDEs not because the size of the smallest turbulent eddies shrinks below the molecular scale, but instead because the Kolmogorov length is much smaller than the outer scale of energy injection in the limit of large Reynolds numbers. Of particular physical significance are the convex integration proofs of non-uniqueness of weak solutions of the incompressible Euler equations with fixed initial data, even after imposing reasonable admissibility conditions such as dissipation of kinetic energy [12, 13]. The non-uniqueness of the Euler solutions suggests an intrinsic unpredictability or “spontaneous stochasticity” of turbulent flow at high enough Reynolds number. In fact, in recent work we have provided strong numerical evidence in the context of a shell model of the turbulent cascade that such stochastic behaviour with fixed deterministic initial data may be triggered by vanishingly small thermal noise [4]. Furthermore, we have proved using the physical Landau–Lifschitz equations that such persistent stochasticity requires non-uniqueness of the Euler solutions [21], exactly the condition established by the breakthrough methods of convex integration theory.

2.2 Ilya Karlin and Florian Kogelbauer

Convex integration is a powerful mathematical tool that allows for the construction of exotic solutions to partial differential equations – solutions that defy classical intuition and exist even when standard methods fail or when uniqueness is lost [11]. Originally introduced by Nash in the study of isometric embeddings [39], the technique was later expanded by Gromov and Eliashberg [27] in the realm of geometric PDEs.

Its impact on fluid dynamics became evident with the groundbreaking work of De Lellis and Székelyhidi [14, 15], who applied convex integration to construct weak solutions of the unforced Euler equations. Building on their insights, Buckmaster and Vicol revealed something striking: the existence of weak solutions to the unforced Navier–Stokes equations with arbitrary energy growth, as described in Theorem 1.1 above.

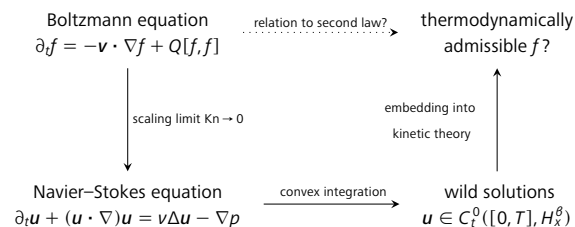
The existence of wild solutions ties deeply into the phenomenon of anomalous dissipation in turbulence. Onsager’s famous conjecture states that weak solutions of the Euler equations with regularity above the critical $C^{1/3}$ threshold conserve energy, while rougher solutions may dissipate it anomalously [40]. Could a similar Onsager-type condition emerge for the Navier–Stokes equations, delineating when turbulent energy cascades lead to dissipation [18]? Despite their mathematical validity, wild solutions pose another important question: Are they physically observable? Can their bizarre behaviour – such as unbounded kinetic energy growth – be reproduced in a real-world experiment? Or is there some physical obstruction that prevents their manifestation in nature?

It seems that the puzzling properties of wild solutions, especially the loss of uniqueness, are contradictory to physically observable fluid flows and thus require further restrictions on the fluid initial conditions, i.e., a selection criterion similar to stability for dynamically observable states [46]. A recent variational admissibility principle proposed for the Euler equations [25] may serve as a step toward addressing this issue in the Navier–Stokes setting as well.

From a broader perspective, the Navier–Stokes equations arise as a scaling limit of the Boltzmann equation in the regime of vanishing Knudsen number – the ratio of the mean free path to a system’s characteristic length scale [5, 26, 35, 36]. At the kinetic level, entropy translates into kinetic energy at the fluid scale, meaning that unrestricted energy growth in the Navier–Stokes framework would be in conflict with the second law of thermodynamics [47].

The previously raised question can thus be reformulated as: How do wild solutions of the Navier–Stokes equations relate to solutions of the Boltzmann equation? We know from the iterative structure of convex integration that these solutions involve extreme oscillations at very fine spatial scales [11]. However, at any finite Knudsen number, the hydrodynamic description is constrained by a critical wave number beyond which the Navier–Stokes model may break down [32].

Building on these insights, we propose the following conjecture: There exists a critical Onsager-type regularity threshold for



Schematic depiction of the relation of the Navier–Stokes equation to the Boltzmann equation: How do wild initial conditions translate to thermodynamically admissible kinetic states?

Navier–Stokes solutions beyond which anomalous dissipation is no longer possible. Moreover, wild solutions of the Navier–Stokes equations cannot be derived from thermodynamically admissible kinetic states within the Boltzmann framework. In other words, the extreme irregularity and energy growth of these solutions may be fundamentally incompatible with the underlying kinetic theory governing physically observable fluids.

2.3 Ondřej Kreml

Recent convex integration results in the analysis of systems of partial differential equations describing fluid flows remind us that the equations we work with are merely simplified models of reality, and scientists should treat them as such. These models are highly useful in some scenarios, while in others, they may simply be unsuitable. This is evident in the case of the Euler equations, as physical fluids always have some nonzero viscosity, but to a certain extent, it also applies to the Navier–Stokes system.

Theorems stating the existence of infinitely many solutions to systems of equations used as models of reality suggest that the very definition of a solution may need careful reconsideration. It is well known that in the case of hyperbolic conservation laws, merely satisfying the weak formulation of the equations is insufficient to single out a unique solution. Additional admissibility criteria must be considered to determine what can be regarded as a physically meaningful solution. For the compressible Euler system in multiple space dimensions, identifying the correct notion of a solution that possesses the desired properties remains an open problem.

It may well be that, similar to hyperbolic conservation laws, the theorem by Buckmaster and Vicol for the Navier–Stokes equations establishes the existence of solutions that are not physically relevant. Only one – or perhaps none – of these solutions may accurately describe the behaviour of real fluids. While the theorem is a fascinating mathematical result with significant implications for the analysis of PDEs, its most important contribution to predicting real fluid behaviour may lie in reminding us that further work may be needed to identify the correct notion of a physical solution for some models describing physical processes.

2.4 Colin McLarty

It is an understatement to say convex integration gives beautifully motivated proofs of surprising geometric results. Beyond surprising, these results have been called “wild,” and “paradoxical,” and a Quanta magazine headline says they mark “where shapes give way” [10, 42, 48]. So the question arises: have these results any physical relevance? In particular, De Lellis and Székelyhidi [14], and then Buckmaster and Vicol [10], use convex integration for striking, mathematically revealing, physically shocking solutions to the Euler and Navier–Stokes equations. A fluid completely stationary for some time, and subject to no external force, suddenly bursts into rapid motion, and after some time moving with however high kinetic energy you choose, it returns to complete rest. No one expects to see this in real fluids. Does that make the results irrelevant to real fluid dynamics?

History and philosophy of science have developed terms for discussing this kind of novelty, as for example in Thomas Kuhn’s *Structure of Scientific Revolutions* [33]. This book is not all about revolutions. To the contrary, Kuhn says a successful science *normally* has a widely accepted theory, together with recognized means of interpreting that theory in observational data, and specific research problems, accepted research methods, and landmark successes. He calls the bundle of a theory, problems, methods, and exemplary achievements a *paradigm*.² A productive science normally faces some empirical problems applying the theory to observed facts, and also conceptual difficulties extending the theory and saying exactly how the theoretical terms are meant to connect with observations. Kuhn [33, Chapter IV] compares these problems to *puzzles* in this sense: Facing a new crossword or jigsaw puzzle you cannot immediately solve it, but you assume it has a solution of known form, which can be found by skilful use of the known rules. Most often in a successful science that is what happens. Every once in a while a problem, which might not even seem like a big problem at first, grows in prominence to a point where practitioners feel it cannot be ignored, yet neither can it be settled by any of the standard means. It becomes a *crisis* and may eventuate in a *revolution* where the previously normal theory, methods, and questions give way to fundamentally new ones.

For our purposes the theory of fluid motion would be the Euler and Navier–Stokes equations [23, p. 3]:

Till a few decades ago (early 1930s), there was unanimous opinion that the Navier–Stokes equations were useful (in agreement with the experiments, that is) only at “low” velocity regimes. It is also thanks to the efforts of outstanding mathematicians such as Jean Leray, Eberhard Hopf, Olga Ladyzhenskaya, and Robert Finn that they are nowadays regarded as the universal foundation of fluid mechanics.

In the vast literature on this theory the top questions are: Do the equations have smooth global solutions? Can this framework model turbulence? Convex integration points away from smoothness, but bears directly on turbulence [10, 14, 42, 44, 48].

The theory is not the whole paradigm. The Euler and Navier–Stokes equations per se do not specify what kind of functions count as physically meaningful (weak) solutions [38]. This is wide open today. Restricting to smooth functions is too narrow for applications. Taking all Schwartz distributions is too broad. Moderately rough functions produce the paradoxical flows.

Villani notes the most direct way to rule out these paradoxical solutions to the Euler equation would be to explicitly require kinetic energy be conserved. Or more likely just require it not to increase, since a long tradition suggests turbulence dissipates kinetic energy. This is related to requiring solutions to be at least moderately smooth. Villani discusses the problems at length [48, pp. 105, 127f., 130]. These have got sharper with the recent work on Navier–Stokes, where viscosity also dissipates energy [6, 10].

Shnirelman offers an opposite approach: Instead of requiring moderately smooth solutions, allow even rougher initial conditions than Schwartz distributions. He suggests finer analytic tools (e.g., Young measures) might make the known paradoxical solutions less paradoxical, by revealing the fluid in them is not really at rest initially, but is just so “infinitely-fast oscillating in space” that even Schwartz distributions cannot resolve the motion [44, p. 3].

From our Kuhnian point of view, then, yes, the paradoxical uses of convex integration are an integral part of current fluid dynamics. They are at least a *puzzle*. A given researcher in either pure theory or applications might not be interested in the wild solutions—if they do not need any theory of turbulence. But the decisive points are: First, we have so far no articulable paradigm for fluid mechanics doing justice to current applications while either ruling out the paradoxical ones or clearing them of the air of paradox. And second, the paradoxical solutions fit within the current paradigm in the sense that they are publicly discussed by the normal means in the subject. This note will not speculate on whether this puzzle could one day grow to a *crisis* requiring a revolution in fluid dynamics.

2.5 Simon Markfelder

The question of the physical relevance of the theorem by Buckmaster–Vicol is strongly related to the question what is the right solution concept for the PDEs of mathematical fluid mechanics. The same holds for all the other results achieved by convex integration in the context of fluid dynamics.

Since the equations are supposed to model the real world (i.e., they should describe and predict what happens in nature), one at least expects existence and uniqueness of solutions. For the 2-D incompressible Euler and Navier–Stokes equations, strong solutions

² Kuhn refined his terminology in later editions, but we can use the original.

(i.e., solutions which are sufficiently differentiable) exist globally in time. Consequently, one may regard strong solutions as a good solution concept in this case. However, for the compressible Euler and Navier–Stokes equations, as well as the 3-D incompressible Euler equations, strong solutions only exist for small times and may blow up afterwards, see e.g., [45], [49] and [19], respectively. Thus, in these cases the study of weak solutions (or even weaker solution concepts, see below) is unavoidable. Note, that for the 3-D incompressible Navier–Stokes equations whether smooth solutions exist globally in time or smooth solutions may blow up in finite time is an outstanding open question and is one of the famous millennium prize problems.

When considering weak solutions, an energy or entropy inequality has to be imposed in order to rule out unphysical solutions. This is a well-known fact in the context of systems of hyperbolic conservation laws like the compressible Euler equations. Moreover, energy/entropy inequalities can be justified from physics by thermodynamical explanations. Interestingly, both for the incompressible and the compressible Euler equations, the resulting concept of an admissible weak solution (i.e., a weak solution which complies with the corresponding energy/entropy inequality) does not lead to a unique solution, as shown by convex integration, see e.g., [14, 15]. The result by Buckmaster and Vicol on the incompressible Navier–Stokes equations is of similar type, with the difference that their solutions do not satisfy the usual energy inequality (i.e., they are not Leray–Hopf weak solutions). This is due to the fact, that in the context of the Navier–Stokes equations, one needs some minimal regularity in order to make sense of the energy inequality, as it involves terms on the gradient of the solution. And the solutions obtained by Buckmaster and Vicol do not have this minimal regularity. Still they comply with any prescribed energy profile.

So, strictly speaking, there is still hope that the non-uniqueness issue raised by the Buckmaster–Vicol theorem does not have great relevance. Namely, if it turns out that strong solutions exist globally in time, then one may question the importance of weak solutions. Moreover, Leray–Hopf solutions might still be unique. And in this case, one may argue that the solutions obtained by Buckmaster and Vicol are just too weak, and thus irrelevant. On the other hand, in view of the aforementioned results, one should better not bet on either of the two possibilities. In addition to that, for the Euler equations we know already that neither the consideration of strong solutions, nor the study of admissible weak solutions leads to existence and uniqueness. So in my opinion, one has to discard the classical notion of admissible weak solutions, and look for novel solution concepts instead. I will discuss some of them in the sequel.

In the context of the Euler equations, people have studied additional criteria with the goal to select one among the infinitely many admissible weak solutions, e.g., one has considered the solution whose energy dissipation is maximal. So far, such additional criteria have however not led to a satisfactory solution concept.

What people find quite promising in this direction, is the vanishing viscosity method. This would however require well-posedness for the corresponding viscous model, e.g., the Navier–Stokes equations. Another problem is that sequences of approximate solutions (like vanishing-viscosity sequences) usually exhibit oscillations and concentrations. Thus, the sequences in general only converge with respect to weak topologies. Moreover, by understanding the (weak) limit as a conventional function, one forgets the oscillatory and concentrative behaviour of the corresponding approximate sequence. A nice way to capture oscillations and concentration is to understand the limit as a (generalised) Young measure, which yields a probability distribution at each point in space-time instead of a particular value. This leads to the notion of a measure-valued solution, see e.g., [2, 17]. It is in many situations not difficult to prove that approximate sequences converge to measure-valued solutions, which yields existence of the latter. This is in contrast to admissible weak solutions, which are—at least for the Euler equations—not known to exist. Moreover, the generalisation of weak solutions to measure-valued solutions also seems more plausible when turbulent flows are considered, since the statements of turbulence theory are of statistical nature.

However, the consideration of measure-valued instead of weak solutions does not solve the non-uniqueness problem. Quite the contrary is true, namely there are even more measure-valued solutions than weak solutions. For this reason, one has tried to impose selection criteria similar to the ones mentioned above in the context of weak solutions, in order to identify the relevant solutions among the possibly many measure-valued solutions, see e.g., [9, 22]. Even though a satisfactory selection criterion has not been found yet, I feel that this approach is the most promising in order to find a good solution concept.

2.6 Mikhail Osipov

From the theoretical physics point of view, the theorem of Buckmaster and Vicol does not take into consideration that the Navier–Stokes equations are not a law of nature, but are approximate equations which are derived from the more general framework of molecular-statistical theory (e.g., from the BBGY chain of equations) using some important assumptions. In particular, these equations (as all classical hydrodynamics) are formally derived in the limit $k \rightarrow 0$, where k is the wave number. More subtle assumptions involve a properly imposed thermodynamic limit as discussed in classical works of Khinchin [30, 31]. In practice, such equations are used at finite k to describe real physical flows, but they are most likely incorrect even qualitatively in the opposite limit of large k . One notes also that the macroscopic fluid velocity is not an average over some small volume but is actually a statistical average of molecular velocity with a one-particle distribution function which is generally smooth. Thus, the average velocity is also smooth at small scales and can behave violently only on the macroscopic scale.

Therefore, any solutions of these equations that vary strongly on smaller and smaller scale are simply outside the limit of applicability of these macroscopic equations and are of purely mathematical significance. They do not have any direct relation to real physical flows.

This situation can be illustrated by a simple example from liquid crystals (my field of expertise; in this example this is elasticity but in the hydrodynamics the situation is similar). Indeed, continuum equations of elasticity of nematic liquid crystals yield point singularities in the orientational director field which correspond to point defects observed in real systems. However, continuum theory cannot describe the radius of the defect core as there is no length scale. In fact, the defect core has finite radius which is calculated approximately from a statistical theory.

It is possible to obtain formal mathematical solutions of the continuum equations for the director distribution sufficiently close to the point discontinuity. However, these solutions have no relation to the real physical defect because in this domain we are inside the finite defect core with a completely different structure. The macroscopic theory simply breaks down because of large gradients which increase on approach to the singularity and the mathematically correct solutions in this domain lose their physical validity because they are obtained outside the range of validity of the continuum equations. One has to describe the system in this domain within a different framework of a molecular statistical theory which yields the results which correspond to the experiment.

3 Discussion

We suggest that the Buckmaster–Vicol theorem can be viewed in three main ways.

The first is that the result is solely of mathematical interest and whether or not it has physical relevance is of no concern. It is unnecessary to consider the requirements of fluid dynamics. It is only essential that the result is rigorously proved.

The second way is to accept that the result is a contribution to fluid dynamics but the implications, especially in relation to turbulence, are not yet fully understood. What, for example, is the connection to the work of Kolmogorov and Onsager? A discussion of this aspect is presented by De Lellis and Székelyhidi in [16].

The third point of view accepts that the Buckmaster–Vicol results might have no physical relevance. The difficulty here is that there are many potential reasons for this lack of physical relevance, by which again we mean observability in nature or in experiments, and some of these are discussed below. Being able to rule out some of the reasons and deciding between the ones that still remain (if any), appears to be an important task for mathematics, physics, and the philosophy of science. In particular, we assert that this debate is a fruitful one for philosophy of science in its rôle as the

logic inspector of the scientific enterprise, and that issues raised in this particular debate have a broader domain of application.³

Below we shall briefly review what seem to us as, for now, legitimate reasons to doubt physical relevance of the Buckmaster–Vicol result. The list is as exhaustive as we could make it; we can envisage it becoming both longer and shorter in the future: longer due to the failure of our imagination, and shorter as arguments become available to rule out some of the possible grounds for criticism.

3.1 Classification of reasons

The possible reasons for lack of physical relevance are of different levels of depth. At the “shallowest” (not used pejoratively) end of the spectrum are grounds that agree with the approach and its use of the object of enquiry (the Navier–Stokes equations), but find fault with the result itself in the sense of not accepting the wild solutions obtained as being of physical relevance. At the next level of depth lie objections that have to do with the applicability of convex integration to the specific object of interest. The deepest objections have to do with the structure of matter (fluids in particular) at very short scales that the convex integration technique as a technique of analysis, does not respect *a priori*.

3.2 Result-level reasons

Here we are dealing with arguments that accept that the convex integration techniques can be applied to the Navier–Stokes equations but still find the results (such as the Buckmaster–Vicol theorem) wanting in physical plausibility.

As an example of such an argument, let us consider admissibility criteria. While there certainly are wild solutions that satisfy the entropy admissibility criterion [37], there is not one known instance of a situation where any admissibility criterion yields a *unique* preferred wild solution. This is trivially true as wild solutions are created in physically indistinguishable (uncountable) equivalence classes. Hence, if one accepts that there should exist a mechanism that selects a unique solution of the Cauchy problem, such a mechanism will never select a wild solution; see for example the discussion in [25].

However, first of all, this suggestion invites the rejoinder that there is no *a priori* binding principle that requires such a property of solutions in order to force uniqueness; perhaps “God plays dice” at the macro-level. At the same time, the question arises *why* in all known cases even equivalence classes of wild solutions do not satisfy admissibility criteria. In other words, at this level objections are consequences of objections at a deeper level.

³ An example would be a discussion of the gelation phenomenon in coagulation-fragmentation equations derived under the assumption of solute diluteness.

Another point of view, articulated in the contributions by Kreml and by Markfelder, is that the concept of weak solution is not appropriate here. Again, the questions arise why this framework, so often productively used in PDEs, is inappropriate in the particular case of Navier–Stokes equations and what other framework (measure value solutions, or considering ensembles of weak solutions as opposed to working with individual ones) should come to replace it.

3.3 Object-level reasons

In this class of reasons one claims that the possible physical implausibility of results such as the Buckmaster–Vicol theorem follows from flaws in the Navier–Stokes equations and/or the application of convex integration techniques to it. This class naturally splits in two.

For the first subclass, one could claim that convex integration is applicable but that the flaw lies in the Navier–Stokes equations themselves, and that there exists another version of the equations, e.g., one involving thermal noise modifications as advocated by Eyink and Goldenfeld, or incorporating higher-order spatial terms or nonlocal/peridynamics ones, in which results would be more intuitively realistic.

In the rest of this subsection and in the next one we discuss arguments based on the conviction that the convex integration techniques should not be in principle applied to *inter alia* the Navier–Stokes equations or any other continuum formulation of laws of fluid motion; see also the contributions by Karlin and Kogelbauer and by Osipov.

To do that we need a one-sentence summary of the convex integration technique; the reader should consult [37] for all the necessary definitions and details. It is this: starting with a suitable initial function (a sub-solution of an equation), we iteratively obtain a (family of) weak solutions to the equation in question in the limit by corrugating the sub-solution at ever decreasing scales.

It is therefore debatable if modifications at every scale can be made to a solution representing the velocity of a fluid, i.e., one may ask what it means for two such solutions to differ at the sub-Planck length scale. Observe that the incompressibility condition that assumes a well-defined macro-variable (density) and the definition of viscosity (that involves density as well) do not make sense at too short scales [29].

The main question that arises here is *how an equation derived by making the assumption of continuum can be manipulated at scales where the assumption no longer holds*.

Note that the formal object of enquiry bears no indication of the cut-off scales beyond which it cannot be manipulated.

We also observe that any analysis, such as the fundamental work of Onsager [40], that assumes that the velocity field contains information about the flow at *all* scales, appears open to the same criticism. See [20, 21] for a defence against this criticism.

This argument may harbour a hint to what concept of solution is appropriate for matter made of atoms and voids that is being analysed using a continuum description.

3.4 Structure of matter level reasons

The real numbers \mathbb{R} provide a suitable setting for differential geometry, which is scale-free. However, it is arguable that a description of matter needs a fundamentally discrete setting at very short length scales (Planck length is of order 10^{-35} m and Planck time is of order 10^{-42} s). Without entering into details, we quote C. J. Isham [28, p. 189]:

The general assumption is that something “dramatic” happens to the nature of space and time at these fundamental scales. Precisely what that dramatic change might be, has been the source of endless speculation and conjecture. However, there is a fairly widespread anticipation that insofar as spatiotemporal concepts have any meaning at all in the “deep” quantum-gravity regime, the appropriate mathematical model will not be based on standard, continuum differential geometry.

3.5 Final remarks

We are aware that not all the possible views on the subject have been presented. Hence, reader comments and opinions are welcome, for example, on computational aspects of wild solutions. These comments can be sent to m.grinfeld@strath.ac.uk. If a lively debate ensues, a continuation paper on the subject will be considered by the EMS Magazine.

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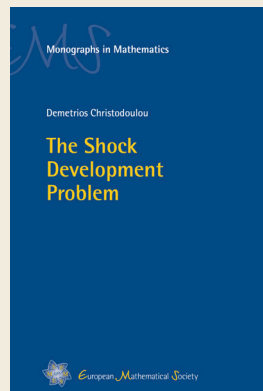
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