Traces and trajectories: in memory of Edi Zehnder (November 10, 1940–November 22, 2024)

Paul Biran and Felix Schlenk

Edi Zehnder began as a physicist, then turned to analytical dynamics studying small denominators, and eventually became one of the founding figures of (global) symplectic geometry and topology. He was born in Leuggern, a Swiss village in the canton of Aargau, where the Aare River merges with the Rhine. His father decided that the best education for him would be in a convent, so he attended high school at the boarding school of the Benedictine convent Einsiedeln. The education focused on ancient languages (Greek and Latin), and the only way to leave the convent during the week was to play in a local football club, which Edi did with enthusiasm. It was also at Einsiedeln that Edi met Jeannette (he 18 and she 15 at the time), who came with her class for a visit to the convent and was very impressed by Edi's pencil drawings and the fact that he smoked a pipe against the rules. From then on, they were together for 66 years, until Jeannette died three months before Edi.

Edi studied mathematics and physics from 1960 until 1965 at ETH Zürich. The curricula of these two disciplines were nearly identical at that time. Heinz Hopf was no longer accepting PhD students, so Edi became a PhD student in theoretical physics, working under the supervision of Res Jost.

Jürg Fröhlich, Edi's close friend and colleague at ETH, recalls:

> I became a PhD student in theoretical physics in the autumn of 1969. At the time, the theory institute of ETH was located in a very large apartment of an old villa on Hochstrasse, above the main building of ETH. As the pictures show, Edi enjoyed smoking the pipe.
>
> Since this was not appreciated by everyone, he was exiled into the bathroom of that apartment, not a very friendly place. I was a cigarette smoker and I did not mind the smell of pipe smoke. It was therefore decided that Edi and I should share a large, comfortable room of the institute. Edi was in the process of completing his PhD thesis on a subject in celestial mechanics: *On the restricted three-body problem*. The aim of his work was to understand the stability of orbits of the asteroids in the Hilda group. Res Jost, who was quite a heavy smoker too, often came to our office to discuss various questions related to Edi's thesis. He would argue that the orbits of the Hilda group must be stable, because his wife Hilde was a stable person.
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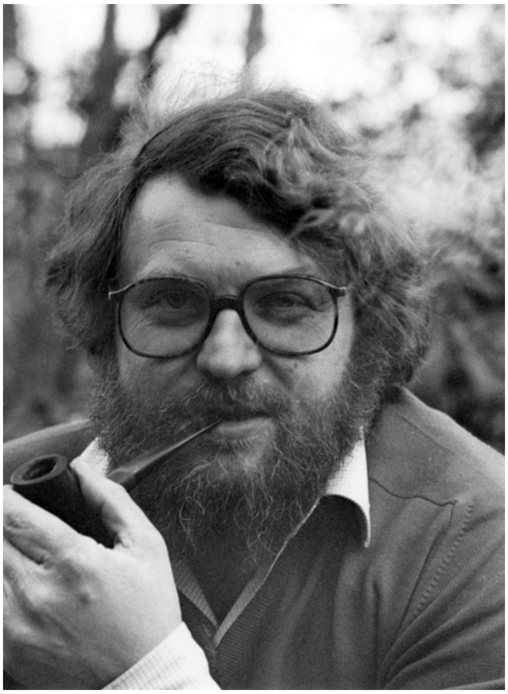
> Res Jost admired Jürgen Moser's work on Hamiltonian mechanics, in particular his contribution to the KAM theorem, and he passed his admiration on to Edi. The topic of Edi's PhD thesis and his erudition in celestial mechanics made it natural for him to apply for a postdoctoral position in Moser's group at the Courant Institute. In 1971, the Zehnders (and their dog) moved to New York and, after a year, on to the Institute for Advanced Study (IAS), where Freeman Dyson was Edi's host.



Edi, 1985. (© Gerd Fischer, München; source: Archives of the Mathematisches Forschungsinstitut Oberwolfach)

In these years, Edi worked on small divisor problems (Nash–Moser method and hard implicit function theorems), clarifying and simplifying the proof of the KAM theorem.[1] His stay at IAS led to a lifelong friendship with Freeman Dyson (Dyson invited Edi again for the academic year 1979/80).

In 1974, Edi moved to Germany and obtained his habilitation at the University of Erlangen–Nuremberg. He became a full professor at the University of Bochum in 1976.

In 1977 one of the great mathematical stories begins to develop, which ultimately culminates in Floer theory—it is a story of random interactions and mathematical curiosity across fields; a story of opportunities one needs to sense and grab, even when mathematical orthodoxy is dismissive. Edi plays a pivotal role in this story.

The context was as follows: In an important paper, Jürgen Moser proves an existence result about periodic orbits of Hamiltonian systems near an equilibrium point. For this, he uses the Hamiltonian action principle, which was generally considered useless for mathematical purposes. He carries out a local finite-dimensional reduction, so that the periodic orbits of interest can be found as critical points of a function defined locally on a finite-dimensional space by critical point theory (reminiscent of ideas from Morse theory). Moser, very knowledgeable about variational problems, also gives a short explanation of why the Hamiltonian action principle is useless for global questions—his local use of it is a breakthrough, but there is no way to use it globally. Paraphrased from today's perspective, he just said (pre-Floer) that there will never be a Floer theory. The error is short-lived. His former student Paul Rabinowitz shortly afterward submits a paper to the Communications of Pure and Applied Mathematics (CPAM) in which he finds global periodic orbits of an infinite-dimensional Hamiltonian system, a wave equation problem. Moser, who is an editor of CPAM, immediately sees his mistake and asks Rabinowitz if he can adapt his method to prove a global version of his local result, and Rabinowitz indeed succeeds in doing so.

Edi recognizes at once that Rabinowitz's ideas have broader ramifications, which he then explores with Herbert Amann.[2] While previous finite-dimensional reductions only approximated solutions to Hamilton's equation, the Amann–Zehnder reduction produces exact solutions, making it possible to count them. In the same year, Charles Conley publishes his influential book, which can be viewed as an extension of Morse theory to topological flows (Conley index theory). Together with Conley, Edi develops the idea of generalizing the two-dimensional Poincaré–Birkhoff fixed point theorem to higher dimensions. They introduce the so-called Conley–Zehnder index and use Conley's index theorem in the study of asymptotically linear Hamiltonian systems.[3]

Edi and Jeannette, around 1970. (Courtesy: Jürg Fröhlich)

While writing this paper, Conley is on a sabbatical at Bochum. At the beginning of December 1982, on an invitation by Jürgen Moser they visit the Forschungsinstitut für Mathematik (FIM) of the ETH Zürich. An unexpected twist happens, which is summarized by Helmut Hofer:

> John Mather from Princeton arrived in early December at the FIM to spend part of his sabbatical. He had just visited Michel Herman in Paris. According to Edi, Herman was one of the sharpest minds in the theory of dynamical systems, and Conley and Zehnder queried Mather about what Herman was up to. They learned that Herman and other members of the so-called Orsay topology group had tried to read a paper by the then-unknown Russian mathematician Yakov Eliashberg but struggled to understand it. The paper contained a complicated proof of the Arnold conjecture on symplectic fixed points in the case of surfaces.[4] To be polite, Conley and Zehnder ask Mather what the Arnold conjecture is all about. Mather explains the conjecture, and more or

[1] E. Zehnder, Generalized implicit function theorems with applications to some small divisor problems. I. *Comm. Pure Appl. Math.* **28**, 91–140 (1975), and E. Zehnder, Generalized implicit function theorems with applications to some small divisor problems. II. *Comm. Pure Appl. Math.* **29**, 49–111 (1976).

[2] H. Amann and E. Zehnder, Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **7**, 539–603 (1980), and H. Amann and E. Zehnder, Periodic solutions of asymptotically linear Hamiltonian systems. *Manuscripta Math.* **32**, 149–189 (1980).

[3] C. Conley and E. Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations. *Comm. Pure Appl. Math.* **37**, 207–253 (1984).

[4] The interesting story of Eliashberg's paper on the Arnold conjecture and the way it arrived to the "west" is retold in detail in the forthcoming book by Siobhan Roberts and Helmut Hofer.

less instantaneously, after looking at each other, Conley and Zehnder pronounce: "We can do this, give us a couple of days." Indeed, three days later they explain their proof to Mather. And two weeks later Edi gave the last colloquium talk of the year at the University of Zürich, describing the proof of the Arnold conjecture for tori.

It turned out that the Conley–Zehnder ideas to prove an extension of the Poincaré–Birkhoff theorem were more than what was actually needed to prove the Arnold conjecture for the standard $2n$-dimensional tori:

Theorem.[5] *Every Hamiltonian diffeomorphism on the torus* $\mathbb{T}^{2n}$ *must have at least as many fixed points as a smooth function must have, namely* $2n+1$. *If the fixed points are non-degenerate there must be at least* 2^{2n}.

This was the first higher-dimensional global theorem in symplectic geometry. It showed that Hamiltonian flows are fundamentally different from volume-preserving flows. The method of proof shocked the specialists in the field, since it had little to do with symplectic geometry as understood at the time. Arnold invited Edi to Moscow, Gromov invited him and Conley for a talk in Paris, and Edi became an invited speaker at the 1986 ICM.

The novel use of Morse theory in their proof of the Arnold conjecture opened up a new field of research at the intersection of dynamical systems, geometry, and topology. The finite-dimensional reduction of the action functional inspired a great deal of work, in particular from the French school by Chaperon, Laudenbach, Sikorav, and Viterbo on generating functions. In a different direction, Edi's student Andreas Floer intertwined the variational methods of Conley and Zehnder with Gromov's pseudo-holomorphic curves to what is now called Floer homology, a key tool in symplectic geometry and topology, that has many applications to Hamiltonian systems, mirror symmetry, and low-dimensional topology. We again refer to the forthcoming book by Hofer and Roberts for more on Edi's role in the creation of Floer homology. After proving the Arnold conjecture for tori, Conley and Zehnder tried to prove it for more general symplectic manifolds, but were not successful. However, in their proof of the torus case, which used a global finite-dimensional reduction, one could take a formal limit to infinite dimensions to obtain a perturbed version of Gromov's pseudo-holomorphic curve equation about two years before Gromov introduced that equation. Edi had a gut feeling that one should be able to study this partial differential equation directly, but had no idea how to do this. However, his student Andreas Floer succeeded. After his undergraduate studies at Bochum, Floer began graduate studies at Berkeley in fall 1982. Edi had never consciously met him before the following moment in early 1983.

> One day, a student knocked on my office door on the 7th floor. I had never seen him before in my analysis lectures. He did not introduce himself but just asked, "Do you have an interesting topic for my thesis?"

Edi explained the Arnold conjecture and described his proof with Conley. For Floer's thesis, he proposed: Analyze the structure of bounded solutions of the classical gradient flow, using Conley's index theory.

> Floer was immediately attracted to the challenging problem. I suggested that he begin with the Arnold conjectures for surfaces of higher genus, in order to verify the claims of Eliashberg.

While Alan Weinstein thought that Floer was a student of Cliff Taubes and vice versa, Floer was in regular communication with Edi, usually calling to his dismay at midnight in Bochum. Eventually, Floer received a PhD from Bochum. While still a graduate student, he started working in the direction of Edi's suggestion, leading to his fundamental work during the period 1986–1989.

After ten years at Bochum, Edi moved to Aachen in 1986. Claude Viterbo recalls:

> I vividly remember the first time I met Edi. He invited Andreas Floer and me—both of us postdocs at Courant at the time—for a week in Aachen. One morning, I went to knock on Edi's office door. Inside a room that felt like a tennis court, I saw a dense cloud. Through the cloud, I could barely see Edi sitting at his desk, smoking his pipe, and greeting us in his cheerful, unmistakable manner. During our stay, we were invited to his home. I was struck by Jeannette's remarkable ability to put two socially awkward mathematicians at ease—but we were likely not the first ones!

Edi returned to ETH Zürich in 1987. The same year, he discovered with Hofer[6] the following phenomenon: *the* C^0*-size of a Hamiltonian function (with support in a given domain) will force the existence of non-constant periodic orbits.* From this viewpoint the authors then define the Hofer–Zehnder capacity, prove that it is finite in a number of cases (e.g., bounded domains in $\mathbb{R}^{2n}$) and then promote as an important question the proof of

[5] C. Conley and E. Zehnder, The Birkhoff–Lewis fixed point theorem and a conjecture of V. I. Arnold. *Invent. Math.* **73**, 33–49 (1983).

[6] H. Hofer and E. Zehnder, Periodic solutions on hypersurfaces and a result by C. Viterbo. *Invent. Math.* **90**, 1–9 (1987), and H. Hofer and E. Zehnder, A new capacity for symplectic manifolds. In *Analysis, et cetera: Research papers published in honor of Jürgen Moser's 60th birthday*, pp. 405–427, Academic Press, Boston (1990).

its finiteness on other manifolds. (This has inspired a lot of both old and recent work, see "Hofer–Zehnder capacity" on Zentralblatt or MathSciNet.) The Hofer–Zehnder capacity is a so-called symplectic capacity. The existence of this capacity immediately implies Gromov's famous non-squeezing theorem as well as the C^0-rigidity of symplectomorphisms discovered by Eliashberg and Gromov.

Edi continued to produce brilliant mathematics up to and beyond his retirement from ETH in 2005. With Dietmar Salamon[7] he worked on the Conley conjecture, and they proved that every Hamiltonian diffeomorphism on the torus having only nondegenerate periodic points has infinitely many of them, paving the way for the resolution of the Conley conjecture by Hingston, later generalized by Ginzburg. With Helmut Hofer and Kris Wysocki, Edi systematically studied punctured holomorphic curves in noncompact symplectic manifolds, with the aim of capturing dynamical properties of Reeb flows (a special class of Hamiltonian flows).[8] They used these curves to construct global sections for these flows (in the sense of Poincaré). As a result, they proved that on every convex energy level of an autonomous Hamiltonian flow on $\mathbb{R}^4$ there must be either exactly two closed orbits or infinitely many.[9] Only recently, several groups of researchers have proved this dichotomy for any Reeb flow on a closed three-manifold. At the same time, the work of Hofer–Wysocki–Zehnder on punctured holomorphic curves builds the foundations of symplectic field theory, whose analytic underpinnings have been developed by the same authors during Edi's last 15 research years.[10]

After their return to Switzerland, the Zehnders continued to enjoy the company of friends. Claude Viterbo remembers:

> Edi and Jeannette's hospitality was boundless—first on Gloriastrasse, and later in Greifensee. There was always wine, delicious food, and plenty of sharp humor. Their deep and enduring love was immediately apparent and deeply moving. On one of my last visits, Jeannette sweetly recalled the first time she saw Edi, dressed like a "young monk." Jeannette, a psychologist who had worked for many years in clinical practice, was deeply engaged in Edi's mathematical world (for example, she translated Constance Reid's book about Courant into German) and often joined Edi to conferences and extended visits.

Working out and formulating his mathematical findings "with loving care" gave Edi a lot of joy. His textbooks, which are striking in their clarity and elegance, testify to this. They teach valuable methods using well-chosen problems. In Jürg Fröhlich's words:

> Edi was a very hard-working researcher with an excellent taste for important problems and a clear focus. His contributions to Hamiltonian dynamics and symplectic topology have an everlasting impact on these fields and beyond. He was a very devoted teacher and advisor to students. Edi had strong opinions and was willing to defend them very forcefully if the situation demanded it. He always remained modest and did not mind acting, sometimes decisively, in the background, letting other colleagues occupy the limelight. He was a wonderful colleague and friend.—I will miss him.

And Claude Viterbo writes:

> Edi was very close to Jürgen Moser, and both were strongly skeptical of overly formal mathematics. They believed that mathematics is not about simply generalizing known results. One important principle that Edi taught me is: *"Mathematics is about discovering new phenomena."* He was undoubtedly a master at that. I will miss him and Jeannette terribly—personally, and like everyone else, mathematically.

[7] D. Salamon and E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure Appl. Math.* **45**, 1303–1360 (1992).

[8] Among this long series of fundamental works are F. Bourgeois, Ya. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder, Compactness results in symplectic field theory. *Geom. Topol.* **7**, 799–888 (2003), and H. Hofer, K. Wysocki and E. Zehnder, Finite energy foliations of tight three-spheres and Hamiltonian dynamics. *Ann. of Math.* **157**, 125–255 (2003).

[9] H. Hofer, K. Wysocki and E. Zehnder, The dynamics on three-dimensional strictly convex energy surfaces. *Ann. of Math.* **148**, 197–289 (1998).

[10] H. Hofer, K. Wysocki and E. Zehnder, *Polyfold and Fredholm theory.* Ergeb. Math. Grenzgeb. (3) **72**, Springer, Cham (2021).

Paul Biran studied mathematics at Tel Aviv University as an undergraduate, and then did his PhD there with Leonid Polterovich from 1994 to 1997, working on symplectic packing problems. Through regular visits to ETH he got to know Edi in person and received from him a lot of inspiration for mathematics and also in other directions. Since 2009, he is a professor of mathematics at ETH.

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Felix Schlenk studied at ETH Zürich and did his PhD there with Edi Zehnder from 1996 to 2001, working on symplectic embedding problems. Ever since, he has learned a great deal from Edi—not only about mathematics. Since 2008, he has held the Chair of Dynamical Systems at the University of Neuchâtel.

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