Sergey P. Novikov (1938–2024)

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A great Soviet and Russian mathematician, Sergey Petrovich Novikov, passed away a year ago, on June 6, 2024, in Moscow.

He was born in a family of outstanding mathematicians on March 20, 1938, in Nizhnii Novgorod, called at that time Gorky, the city to which his parents moved for a short period from Moscow.

His father, Pyotr Sergeyevich Novikov (1901–1975) is well known for his proof of the existence of a finitely presented group for which the word problem is undecidable, and for the negative solution (jointly with his former student S. Adian) of the Burnside problem, which in turn follows from the proof that the free Burnside groups $B(m,n)$ are infinite for sufficiently large odd exponents n. In 1960 P. Novikov was elected full member of the Academy of Sciences of the Soviet Union.

His mother, Lyudmila Keldysh (1904–1976) worked in the Steklov Mathematical Institute and authored outstanding papers on set theory and set-theoretical topology.

In 1955, Sergey Novikov entered the Moscow State University. After a couple of years, Sergey chose as scientific advisor Mikhail Postnikov, whose impressive course on Galois theory he attended as a second-year student. This is how Sergey Novikov became a student in the Department of Higher Algebra.

At the beginning, Postnikov gave Novikov a preprint of the article "On the structure and applications of the Steenrod algebra" by J. F. Adams, and left for a year to teach in China.

Novikov always told that unlike other branches of mathematics, algebraic and differential topology have to be studied not from textbooks, but from the original articles, whose expositional level is hardly achieved by graduate texts. In the 2000s, when he together with the author of this text organised the publication of the three-volume set of such articles, titled "Topological Library," we included Adams' article in which the Adams spectral sequence was introduced.

In 1959, Novikov published his first paper, "Cohomology of the Steenrod algebra," *Dokl. Akad. Nauk SSSR* **128**, 893–895, in which he generalized certain results from Adams' article and, in particular, showed that there are arbitrarily long compositions of maps that realize nontrivial elements of stable homotopy groups of spheres:

$$S^{n_1} \to S^{n_2} \to \cdots \to S^{n_k}.$$

Sergey P. Novikov. (© Steklov Mathematical Institute, 2005)

The basic tool came from his observation that the analogue of the Steenrod operation Sq^0, which acts on the cohomology of the Steenrod algebra, is not the identity map, but is given by the Frobenius automorphism. Novikov liked his first article very much, he wrote it completely by himself and thanked his advisor for the interest in the work. This was his start.

The next year, again by using the Adams spectral sequence, Novikov computed the ring Ω^U of U-bordisms (the Milnor–Novikov theorem) and obtained particular results on the rings of SO-, SU-, and Sp-bordisms.

Later, in 1966–67, Novikov returned to the Adams spectral sequence, proposing to consider its analogues for extraordinary cohomology theories, introducing the so-called Adams–Novikov spectral sequence. In particular, he considered in detail such a sequence corresponding to U-cobordisms and described for them the analogue of the Steenrod algebra, the Landweber–Novikov algebra. In Novikov's fundamental article "Methods of algebraic topology from the point of view of the cobordism theory," *Izv. Akad. Nauk SSSR Ser. Mat.* **31**, 855–951 (1967), he discovered that the analogue of the Adams spectral sequence for unitary cobordisms is applicable to the classical problem of computing stable homotopy groups of spheres

$$\pi_k^S = \pi_{n+k}(S^n), \quad n \geq k+1$$

(for these values of n the homotopy groups $\pi_{n+k}(S^n)$ depend on k only).

We want to mention that Novikov stressed that whenever he is doing some work, he always wants to solve some explicit problem and never produce generalizations or new constructions just for their own sake. When listening to talks delivered at his seminars, he used to ask sometimes: "Please tell us which problem would you like to solve."

The problem of determining the stable homotopy groups of spheres π_k^S is still unsolved in general – these groups are known only for $k \leq 90$. The Adams–Novikov spectral sequence is a strong and important tool not only in the study of stable homotopy groups, but also in other problems of algebraic topology.

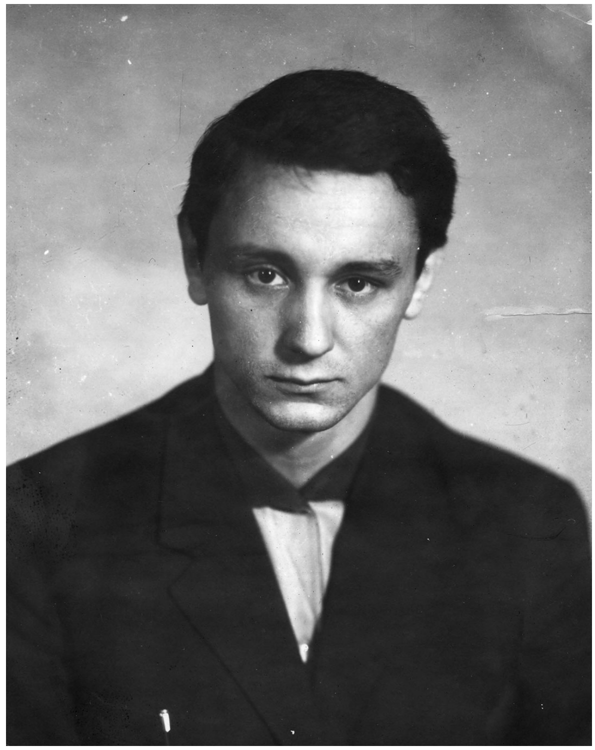
In "Methods of algebraic topology…," formal groups appeared for the first time in algebraic topology via a theorem by Novikov's student A. S. Mishchenko, who refused to publish it by himself and agreed that Novikov, who understood well the importance of this construction, will include it in an appendix to his article. Novikov had the intuition that the formal group of geometric cobordisms has rather special properties, and later Quillen established that this group is a universal group.

In 1960, Novikov graduated from Moscow University and became a PhD student at the Steklov Mathematical Institute.

In 1961, he solved the problem of the classification of simply-connected homotopically equivalent smooth manifolds of dimension > 4 up to diffeomorphisms. By a similar technique, in 1962 Browder characterized the homotopy types of smooth manifolds among simply-connected complexes. These results gave rise to the Browder–Novikov theory.

Once Novikov told that the great majority of results concern simply-connected manifolds, which form "the zero-measure set" and the wealth of interesting examples relate to nontrivial fundamental groups and their actions on higher homotopy groups.

In the mid-1960s, motivated by Grothendieck's construction of étale cohomology, Novikov came to his famous toric trick for proving the topological invariance of rational Pontryagin classes. For its realization it was necessary to apply the technique of the Browder–Novikov theory to certain manifolds with free Abelian fundamental groups. At first there were doubts that this was possible, however,



Sergey P. Novikov in the middle of the 1960s.

the trick "came through" and the topological invariance of rational Pontryagin classes was established. Novikov considered this his best topological result. The methods and ideas he introduced in this proof were later widely used in topology.

His interest in non-simply-connected manifolds led to the formulation of the Novikov conjecture on higher signatures, in the study of which he involved Kasparov and Mishchenko. The conjecture is still open, though it is confirmed for many interesting cases.

We roughly attribute Novikov's results obtained during the first ten years of his career (1959–1968) and mentioned above to two subjects: (1) the Adams spectral sequence and cobordism theory; (2) surgery theory and its applications. However, we have to speak about two more: (3) foliation theory, and (4) algorithmic problems in topology.

In the middle of the 1960s, Novikov became interested in the study of Haefliger's work on foliations in the Anosov and Arnold's seminars and proved the now-famous theorem on the existence of a compact leaf in every codimension-one foliation of the three-sphere.

Novikov's theorem on the algorithmic unsolvability of the recognition problem for spheres of dimensions greater than four was proved in the early 1960s, but was never published by Novikov. It was presented much later in a special section of the paper: I. A. Volodin, V. E. Kuznetsov and A. T. Fomenko "The problem of discriminating algorithmically the standard three-dimensional sphere," *Uspekhi Mat. Nauk* **29**, 71–168 (1974). Now algorithmic and computational methods in algebraic and combinatorial topology are a very popular research subject.

These topological results brought Novikov worldwide fame.

In 1963, he finished his PhD study and became a research fellow in the Department of Algebra of the Steklov Institute. In 1964 Novikov received his PhD degree, and in 1966 he received the Doctor of Sciences degree and was elected a corresponding member of the Academy of Sciences of the Soviet Union. The next year Novikov received the Lenin Prize, the highest prize in the Soviet Union, and in 1970 he was awarded the Fields Medal, becoming the first mathematician from the Soviet Union who was awarded the medal.

In 1970–71, he did the well-known works on Hermitian K-theory and on symplectic cobordisms (joint with V. Buchstaber) where two-valued formal groups were introduced. However, at this time Novikov's interests moved in other directions. He also started to write articles with his students, many of whom later developed ideas from the joint texts.

At the beginning of 1971, Novikov left the Steklov Institute for the Landau Institute of Theoretical Physics.

He started to work in relativity theory, considering homogeneous cosmological models. In his joint article with O. Bogoyavlensky, for the purpose of studying the singularities of such models, a method of resolution of singularities motivated by ideas from algebraic geometry was introduced.

In 1973, Novikov became attracted by the spectacular progress in the theory of integrable systems. In this field he already applied his physical background to mathematical problems. In his famous paper on the periodic problem for the Korteweg–de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx},$$

Novikov introduced the so-called finite-zone potentials of the one-dimensional Schrödinger operator and proved that a potential has a finite number of zones in its spectrum if and only if it is a stationary solution of a higher Korteweg–de Vries equation, i.e.,

$$[L, c_{2n+1}A_{2n+1} + \cdots + c_1A_1] = 0.$$

These equations are expressed as Lax pairs of the form

$$\frac{dL}{dt_{2n+1}} = [L, A_{2n+1}], \quad L = -\frac{d^2}{dx^2} + u(x),$$

where A_{2n+1} are ordinary differential operators of order $2n+1$. Some results were shortly after and independently obtained by Lax, who worked with the classical eigenvalues of L.

As a crucial step, Novikov introduced an analytic continuation of the Bloch spectrum of the operator L with periodic coefficients. He considered the linear problem

$$L\psi = E\psi$$

for the Schrödinger operator $L = -\frac{d^2}{dx^2} + u(x)$ with a periodic potential $u(x) = u(x+T)$ for complex values of E, as well as the complex curve Γ which parameterizes all solutions of this linear problem with the property that $\psi(x+T) = \text{const} \cdot \psi(x)$ (called Floquet–Bloch functions), and he proved that Γ is a first integral of the KdV equations. Since the operator L is of second order, this complex curve Γ is a two-sheeted covering of the E-plane, i.e., it is a hyperelliptic Riemann surface, and it is of finite genus if and only if the potential $u(x)$ is finite-zone. This was the first appearance of spectral curves in the theory of integrable systems and one of the first applications of algebraic geometry in mathematical physics.

Later it was shown that the explicit finite-zone solutions of the KdV equations can be described in terms of the theta-function of Γ (by means of the Its–Matveev formula). The general method of finding algebraic-geometric solutions of soliton equations is now called the finite-gap integration method, and Novikov's former students B. Dubrovin and I. Krichever made crucial contributions to its development. However, Novikov did not agree with the term "finite gap" because, as he said, his original terminology came from solid state physics, where "zones of instability" in the Bloch spectrum have an important physical meaning.

In 1976, Krichever introduced the general Baker–Akhiezer functions and used them to construct algebraic-geometric solutions to the Kadomtsev–Petviashvili (KP) equation in the form

$$u(x,y,t) = -2\frac{d^2}{dx^2}\theta(Ux + Vy + Wt) + C,$$

where U, V, W are constant g-dimensional vectors, C is a constant, and θ is the theta function of the spectral curve, which is a Riemann surface of genus g. Novikov surprisingly proposed to use this formula for solving the Riemann–Schottky problem on characterizing Jacobi varieties. He conjectured that a principally polarized indecomposable Abelian variety is the Jacobi variety of a Riemann surface if and only if there are vectors U, V, W and a constant C such that such the above formula gives a solution to the KP equation. First, at the beginning of the 1980s, Dubrovin proved that the Novikov condition distinguishes the loci of Jacobian varieties up to irreducible components. Then in the middle of the 1980s Shiota proved the Novikov conjecture in full generality. Thus, a famous problem from algebraic geometry was solved by using an equation that originated in plasma physics.

Novikov made numerous other contributions to the "finite-zone theory," among which we mention here only two.

In 1976, he introduced in terms of algebraic-geometric spectral data the notion of a two-dimensional Schrödinger operator finite-zone on a single energy level (jointly with Dubrovin and Krichever) and later, in 1984, in joint articles with A. Veselov characterized among them the potential operators and introduced a new two-dimensional generalization of the KdV equation – the Novikov–Veselov equation – which describes isospectral deformations of such operators.

The theory of algebraic-geometric solutions to soliton equations revised the theory by Burchnall and Chaundy of commuting ordinary differential operators of rank 1. The latter means that for generic constants λ and μ the space of solutions to the equations $L\psi = \lambda\psi$ and $M\psi = \mu\psi$ is one-dimensional, i.e., has rank 1. Here L and M are commuting ordinary differential operators. In 1978–79 Novikov jointly with Krichever proposed a method for studying commuting operators of higher rank based on the theory of holomorphic bundles over Riemann surfaces. They found the first explicit example of such operators of rank 2 and with an elliptic spectral curve.

In 2005, Novikov was awarded the Wolf Prize in Mathematics "for his fundamental and pioneering contributions to algebraic and differential topology, and to mathematical physics, notably the introduction of algebraic-geometric methods."

Looking for problems in mathematical physics, Novikov was always interested in physically meaningful ones. In 1980, he published the monograph of "Theory of solitons. The Method of the Inverse Problem" in coauthorship with three physicists, Zakharov, Manakov, and Pitaevskiĭ, one of the first books on this subject.

The stormy development of integrability theory in the 1970s–1990s involved both infinite-dimensional (solitons, etc.) and finite-dimensional systems, in particular, the Euler equations on Lie algebras. In 1980, Novikov became interested in the Kirchhoff equations which describe the motion of a body in an ideal fluid which is stationary at infinity. In the 1970s these equations were interpreted as the Euler equations on the Lie algebra of the group $E(3)$ of motions of the three-dimensional Euclidean space. For this Lie algebra the generic orbits of the coadjoint representation are four-dimensional manifolds, diffeomorphic to the tangent bundle of the two-sphere. Novikov and Schmelzer discovered that the reductions of the Euler equations to these orbits can mathematically be interpreted as the Euler–Lagrange equations for the motion of a charged particle on the sphere in a monopole-type magnetic field. The magnetic field on the two-sphere is given by a closed 2-form F, which in the case of a monopole is not exact — there is no globally defined vector-potential, i.e., no 1-form A such that $dA = F$. The periodic trajectories of a particle on a given energy level E satisfy the Euler–Lagrange equations for the multivalued Lagrange function

$$L^{\alpha}(x, \dot{x}) = \sqrt{2E}\sqrt{g_{ik}\dot{x}^i\dot{x}^k} + A^{\alpha}_k\dot{x}^k,$$

defined for $x \in U_\alpha$, where $dA^\alpha = F$. This immediately leads to multivalued functionals on spaces of closed forms, given by

$$S(\gamma) = \int_\gamma L^\alpha(x, \dot{x})dt;$$

here the functional is defined for curves lying in domains U such that the equation $dA = F$ is solvable in U, but the variational derivative δS is globally well defined.

Starting from that, he concluded the following.

1. Novikov initiated the study of periodic problem for such trajectories, now called magnetic geodesics. This problem cannot be approached by means of the classical Morse theory for geodesics, because even for exact magnetic fields the functional S is not necessarily bounded from below. He proposed several approaches to the periodic problem, including the principle of throwing out cycles; however, this is still not understood as well as its analogue for closed geodesics.
2. In a short note he considered multivalued functions on finite-dimensional manifolds, derived the Morse–Novikov inequalities for the critical points of such functions formulated in terms of the ranks of homologies with coefficients in what are presently called Novikov rings; these rings later found many applications, for instance, in quantum cohomology.
3. In the same note he formulated the general approach to multivalued functionals on spaces of maps not of closed curves, but of closed manifolds, and derived, as an example, the Wess–Zumino–Novikov–Witten (WZNW) functional.
4. Looking for physical examples of multivalued functionals, Novikov formulated the problem of the intersections of the Fermi surfaces $\{\varepsilon(p) = \text{const}\} \subset T^3 = \mathbb{R}^3/\mathbb{Z}^3$ in the space of quasimomenta, the three-torus T^3, with the planes orthogonal to a constant vector in $\mathbb{R}^3$. Such intersections are trajectories of Hamiltonian systems on Fermi surfaces. This led to many interesting mathematical results by Tsarev, Zorich, and Dynnikov, and even to the discovery by Novikov and Maltsev of experimentally-verifiable physical phenomena related to normal metals.

In the middle of the 1980s, returning to the Morse theory on manifolds that are not simply connected, Novikov together with his colleague M. Shubin introduced the well-known invariants that nowadays bear their names.

At the same time, Novikov jointly with Dubrovin introduced the so-called Hamiltonian systems of hydrodynamic type, which have the form

$$u^i_t = v^{i\alpha}_j(u)u^j_\alpha, \quad u^j_\alpha = \frac{\partial u^j}{\partial x^\alpha}$$

and are Hamiltonian with respect to the Poisson brackets of Dubrovin–Novikov type

$$\{u^q(x), u^p(y)\} = g^{qp}(u(x))\delta'(x-y) + b^{qp}_s(u(x))u^s_x\delta(x-y).$$

Such systems enjoy very rich geometrical properties. Their study led Dubrovin to introducing the Frobenius manifolds. The Novikov conjecture that the diagonalizable systems, i.e., those for which v^j_α is diagonal, are integrable was confirmed by Tsarev, who introduced the generalized hodograph method.

During the period 1996–2017 Novikov was a professor in the department of mathematics and Institute of Physical Science and Technology of University of Maryland, where in 1997 he was promoted to distinguished university professor.

Sergey P. Novikov at a workshop in Dubna.
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In the 21st century, Novikov engaged in the development of discretization of complex analysis motivated by integrable systems (with Dynnikov) and the extension of the spectral theory of one-dimensional Schrödinger operators to potentials with special singularities (singular solitons) which appear in physics (jointly with Grinevich).

At that time Novikov returned to the Steklov Institute, where since 1982 he held the part-time position of the head of department of geometry and topology.

In addition to the already mentioned Fields Medal and the Lenin and Wolf Prizes, Novikov was awarded the Lobachevsky Prize of the Academy of Sciences of USSR and the Bogolyubov, Euler, and Lomonosov Gold Medals of the Russian Academy of Sciences. The last one is the top award of the Russian Academy.

Novikov was elected to the London Mathematical Society, the National Academy of Sciences (USA), Academia Europaea, the Accademia dei Lincei, the Pontificia Accademia delle Scienze, the Serbian Academy of Sciences and Arts, and the Montenegrin Academy of Sciences and Arts, and became doctor honoris causa of the University of Athens and of Tel Aviv University.

In 1978 Novikov gave a plenary talk at the ICM, and was an invited speaker at ICM-1966 and ICM-1970.

During the years 1985–96 he was president of Moscow Mathematical Society, and during the years 1988–2022, the chief editor of the journal Uspekhi Matematiceskikh Nauk (Russian Mathematical Surveys).

Since the middle of the 1960s, Novikov had been working at the Moscow University, where since 1983 he had been the head of the Chair of Higher Geometry and Topology. There he developed new courses, and that led to the book "Modern Geometry" by Dubrovin, Fomenko, and Novikov, published at the beginning of the 1980s in Russian and then as a three-volume set by Springer.

At the beginning of 2000s Novikov and the author of this text wrote a book for a graduate course on "Modern Geometrical Structures and Fields" which took into account the experience of the 1980–90s. It was published in Russian and then translated by the AMS.

The scientific school lead by Novikov was started very early, in 1960, when the third-year student V. Golo came to Novikov, who just became a PhD student at the Steklov Institute, and asked him to be his advisor. As Novikov told, that gave him a strong push. On his homepage he presented the names of his PhD students noting by double asterisks the ones who were invited speakers on ICM or plenary speakers on ECM or ICMP and by asterisks section speakers on ECM. We are reproducing this list (with later improvements) below.

The students who received ScD degrees:
V. Golo, V. Buchstaber,** A. Mishchenko,** G. Kasparov,** O. Bogoyavlensky,** F. Bogomolov,** S. Gusein-Zade, I. Krichever,** B. Dubrovin,** I. Taimanov,** A. Veselov,* I. Babenko, R. Nadiradze, V. Vedenyapin, M. Brodsky, S. Tsarev, O. Mokhov, R. Novikov, P. Grinevich, I. Dynnikov*, A. Maltsev, O. Musin*, D. Millionshchikov.

The students who received PhD degrees:
A. Brakhman, V. Peresetski, I. Volodin, A. Grigoryan, Th. Voronov, A. Zorich,** N. Panov, A. Lyskova, M. Pavlov, Thang T. Q. Le, L. Alania, S. Piunikhin, V. Sadov, A. Lazarev, R. De Leo, A. Giacobbe, K. Kaipa.

Sergey Novikov gave rise to many directions in modern mathematics and mathematical physics, and his ideas are developed by many researchers who, in particular, used his fundamental results in their studies.

Iskander Asanovich Taimanov works in geometry, topology, and integrable systems. Graduated in mathematics at Moscow State University in 1983 and defended his PhD there in 1987 (with S. P. Novikov as advisor). Since 1987 he has been working in the Sobolev Institute of Mathematics in Novosibirsk. In 2011, he was elected as a member of the Russian Academy of Sciences. Since 2024 he is the head of the Chair of Higher Geometry and Topology at Moscow State University.

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