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EUROPEAN MATHEMATICAL SOCIETY

André Lichnerowicz Prizes for Poisson Geometry 2020/2021

We are glad to announce the winners of the André Lichnerowicz Prizes for Poisson Geometry 2020/2021 which were awarded at the Global Poisson webinar on the 27th of May 2021.

The prize is named in memory of André Lichnerowicz whose works have been fundamental in establishing Poisson geometry as a branch of modern mathematics and is traditionally assigned during the International Conference on Poisson Geometry in Mathematics and Physics. It has been attributed every two years since 2008, to researchers who obtained their doctorate eight years prior to the Conference.

The laureates of the 2020/2021 edition are:

Pavel Safronov (University of Edinburgh)

Pavel Safronov completed his PhD degree in 2014 at the University of Texas at Austin under the supervision of David Ben-Zvi. After postdoctoral positions in Oxford, Bonn and Geneva, and a lectureship at the University of Zurich, he joined the University of Edinburgh as a Lecturer. Safronov is awarded an André Lichnerowicz Prize in Poisson Geometry 2020/2021 for his fundamental contributions in shifted Poisson geometry and in deformation quantization theory. He advanced the understanding of classical notions of symplectic reduction and of Poisson-Lie groups within the framework of shifted Poisson geometry. His results on deformation quantization led to applications to the Bonahon-Wong conjecture on Azumaya locus of the Kauffman bracket and to Witten's conjecture on finiteness of skein modules in quantum topology.

Xiaomeng Xu (Peking University)

Xiaomeng Xu completed his PhD degree in 2016 at the University of Geneva, under the supervision of Anton Alekseev. After a postdoctoral position at MIT, he joined Peking University as an Assistant Professor. In his work, Xu constructed explicit Ginzburg-Weinstein linearizations of Poisson-Lie groups and their quantization. His results on the relationships between Stokes phenomena, Yang-Baxter equations, and Frobenius manifolds uncovered deep connections between the theory of meromorphic ODE's with higher order poles and the theory of quantum groups. Xu also used classical integrable systems on Lie-Poisson spaces to study the structure of Stokes matrices, which advanced the understanding of Stokes phenomena and isomonodromy deformations. In earlier work, Xu has contributed to the theory of Courant algebroids, string principal bundles, and homotopy Poisson manifolds as objects in higher structure aspects of Poisson geometry.

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The cover illustration, by António B. Araújo, is a digital composition made from a single pencil portrait of Reuben Hersh. The original portrait is graphite and white pencil on toned paper, measuring 15 × 12 cm. The composition is a fragmentation/repetition of this portrait upon three faces of a cropped flattening of a dodecahedron, alluding to a passage in the interview presented in the present issue, where Hersh mentions his dissatisfaction with presenting students with a view of the foundations of mathematics made up of "three viewpoints – logicist, intuitionist, and formalist, (…) all three (…) inadequate, unsatisfactory, failures."

Dear EMS members,

Let me focus my editorial on some reflections on the virtual European Congress 8ECM that took place in Portorož. Despite the difficult circumstances of which we all know, the congress was extremely well-run; the organizers did a wonderful job, and the number of registered participants was record-breaking. Thus we may truly speak of the great success of the congress, even though sadly some major aspects were lacking (or available only to the few people who were physically present): personal meetings with colleagues, lively exchanges of ideas, initiation or continuation of scientific cooperations, and the networking that is so important, in particular for the younger generation. I was hoping that the virtual congress would be able to provide at least some help with these social aspects, but unfortunately this did not work out in a fully satisfactory way.

This brings me to some general observations on the congress that I would like to share, and in the course of which I would also like to ask some **(possibly provocative)** questions, with the goal of starting a discussion.

The number of people that were logged in for the plenary, invited and prize talks was very disappointing. Most of these talks, with only three or four exceptions, were attended by less than 100 participants. This observation also holds for the 7ECM that I organized in Berlin, which was a real and not a virtual congress. Therefore it seems natural to discuss whether such a big congress, with almost 50 major talks, is the right format.

Do we need to organize such a big congress (with a major CO_2 footprint when real and not virtual) if these major talks are not sufficiently interesting to enough people? I definitely believe that we need to reduce the number of talks. Indeed, I sometimes have the feeling that the main aspect of these talks is to provide the speaker with a quality stamp of being a leader in the field. Is this what we want as a community? Is the mathematical community perhaps so large, and its spectrum so broad, that the idea of having a joint congress for all mathematicians may actually be obsolete?

Did all of those invited speakers truly realize their ambassador role? Having listened to more than 20 of the talks, which were of very mixed quality, I doubt this. Speakers should be told much more clearly that they are speaking to a broad mathematical audience and not to a small number of specialists. They should then reflect carefully upon whether they really want to give the talk if they are not able or willing to make this effort. If the invitation to give the talk is nothing more than a quality stamp, then maybe it should not be extended at all.

We should also discuss the inflation concerning the number of prizes that are now awarded in the academic community. Within the EMS, we present 12 prizes, of which 11 are for young scientists. In itself, this is reasonable. However, we do need to ask whether these prizes reflect the strength of the research or the lobbying talents of the various groups that make the nominations? Then again, if only very small numbers of people are sufficiently interested to listen to the prize talks, are we doing the right thing at all, or are we again merely offering the prizewinners subjective quality stamps? **Do we really need so many prizes?**

One aspect of the congress that surprised me in a very positive way was the special sessions that I attended. The quality of these was really great, and there were many lively discussions. This seems to be an aspect that is worth strengthening, which is why we want to start a series of EMS topical conferences.

My goal in asking these questions is to start a discussion about the way in which we, the mathematical community of Europe and the world, want to proceed into the future. Please feel free to send me your opinions on all this. If there is sufficient interest, then I will initiate a committee to discuss the next steps.

Volker Mehrmann
President of the EMS

Almost impossible E_8 and Leech lattices

Maryna Viazovska

We start this short note by introducing two remarkable mathematical objects: the E_8 root lattice Λ_8 in 8-dimensional Euclidean space and the Leech lattice Λ_{24} in 24-dimensional space. These two lattices stand out among their lattice sisters for several reasons.

The first reason is that these both lattices are related to other unique and exceptional mathematical objects. The E_8 lattice is the root lattice of the semisimple exceptional Lie algebra E_8. The quotient of Λ_8 by a suitable sublattice is isomorphic to the Hamming binary code of dimension 8 and minimum distance 4, which in its turn is an optimal error-correcting binary code with these parameters. The Leech lattice is famously connected to the exceptional finite simple groups, monstrous moonshine [7] *and the monster vertex algebra* [1]*.*

Another reason is that Λ_8 and Λ_{24} are solutions to a number of optimization problems. The E_8 and Leech lattice provide optimal sphere packings in their respective dimensions [5, 23]*. Also both lattices are* universally optimal*, which means that among all point configurations of the same density, the Λ_8 and Λ_{24} have the smallest possible Gaussian energy* [6]*.*

The third reason for our interest in these lattices is less obvious. The optimality of the E_8 and Leech lattices can be proven in a rather short way, while the solutions of analogous problems in other dimensions, even dimensions much smaller than 8 and 24, is still wide open. Finally, this last property seems to be inherited by other geometric objects obtained from Λ_8 and Λ_{24}, such as Hamming code, Golay code and the sets of shortest vectors of both lattices.

1 E_8 and Leech lattices

The E_8 lattice Λ_8 is the unique (up to an isomorphism) even unimodular lattice in the Euclidean space $\mathbb{R}^8$. We recall that a lattice $\Lambda \subset \mathbb{R}^d$ is *even* if for every lattice vector $\ell = (\ell_1, \dots, \ell_d)$ its Euclidean length squared $|\ell|^2 = \ell_1^2 + \dots + \ell_d^2$ is an even integer. A lattice $\Lambda \subset \mathbb{R}^d$ is *unimodular* if the volume of the quotient $\mathbb{R}^d/\Lambda$ is 1. Equivalently, the average number of lattice points per unit of volume is 1.

The existence of an even unimodular lattice in $\mathbb{R}^8$ was first proven non-constructively by H. J. S. Smith in 1867 and followed from his newly discovered mass formula for lattices. The mass formula for even unimodular lattices in dimension d divisible by 8 states that

$$\sum_{\Lambda} \frac{1}{|\mathrm{Aut}(\Lambda)|} = \frac{|B_{d/2}|}{d} \prod_{1 \le j < d/2} \frac{|B_{2j}|}{4j},$$

where the left-hand sum is taken over all isomorphism classes of lattices Λ, $|\mathrm{Aut}(\Lambda)|$ denotes the size of the group of orthogonal transformations acting on Λ, and B_k are the Bernoulli numbers. Note that even unimodular lattices exist only in dimensions divisible by 8. In the smallest possible dimension 8, the right-hand side of the mass formula becomes

$$\frac{|B_4|}{8}\frac{|B_2|}{4}\frac{|B_4|}{8}\frac{|B_6|}{12} = \frac{|-\frac{1}{30}|}{8}\frac{|\frac{1}{6}|}{4}\frac{|-\frac{1}{30}|}{8}\frac{|\frac{1}{42}|}{12} = \frac{1}{696729600}.$$

The mass is non-zero, and therefore there exists at least one even unimodular lattice in dimension 8. Moreover, the formula shows that such a lattice is highly symmetrical. The explicit Gram matrix of the E_8 lattice was first given by Korkin and Zolotarev in 1873 [14].

One remarkable property of the E_8 lattice is that the corresponding sphere packing has very high density. The E_8-lattice sphere packing $\mathcal{P}_{E_8}$ is the union of open Euclidean balls with centers at the lattice points and radius $\frac{1}{\sqrt{2}}$. These non-intersecting congruent balls cover $\Delta_{E_8} := \frac{\pi^4}{384} \approx 0.25367$ of the volume of $\mathbb{R}^8$. In 2016 the author showed that this density cannot be improved.

Theorem 1. *No packing of unit balls in Euclidean space $\mathbb{R}^8$ has density greater than that of the E_8-lattice packing.*

The Leech lattice Λ_{24} was constructed by J. Leech in 1967 [15]. This lattice is an even unimodular lattice of rank 24. There exist 24 isomorphism classes of such lattices. Among these 24, the Leech lattice is the unique one having the shortest non-zero vector of length 2 (in the other 23 classes, the shortest vector has minimal possible for even lattices length $\sqrt{2}$). As the minimal distance between two points in Λ_{24} is 2, it is a good candidate for a dense sphere packing. The Λ_{24}-lattice sphere packing is the packing of unit balls with centers at the points of Λ_{24}. This packing has density $\Delta_{\Lambda_{24}} := \frac{\pi^{12}}{12!} \approx 0.00193$. In joint work with H. Cohn, A. Kumar, S. Miller and D. Radchenko, we proved the following.

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Theorem 2. *No packing of unit balls in Euclidean space $\mathbb{R}^{24}$ has density greater than that of the Λ_{24}-lattice packing.*

In the next section, we explain how these results fit into a more general framework.

2 The sphere packing problem

The sphere packing problem asks for the maximal portion of Euclidean space that can be covered with non-overlapping congruent balls. This natural geometric question is interesting from many points of view. The sphere packing problem is a toy model for many physical systems [17] and a mathematical framework for error correcting codes in communication theory [20]. The known and putative solutions of the sphere packing problem are geometrically intriguing configurations, and in many cases possess other extremal properties and unexpected symmetries.

The recorded modern history of the sphere packing problem goes back to the sixteenth century and is documented in the correspondence between a statesman, Sir Walter Raleigh, and a scientist, Thomas Harriot. Harriot was asked by Raleigh to find the most efficient way to stack cannonballs on the deck of the ship. Harriot studied various stacking patterns, computed the number of cannonballs in a triangular pyramid and in a pyramid with square base, and constructed face-centered cubic and hexagonal closed packings. In 1591, he wrote a letter to Raleigh explaining some of these findings. At the beginning of the seventeenth century, Harriot exchanged letters with Johannes Kepler and shared his ideas on sphere packings. In 1611, Kepler wrote an essay "Strena Seu de Nive Sexangula", in which he described face-centered cubic and hexagonal close packings and asserted that "the packing will be the tightest possible, so that in no other arrangement could more pellets be stuffed into the same container". This assertion became famously known as Kepler's conjecture.

The quest to solve Kepler's conjecture lasted for almost three centuries. We briefly recall the most important landmarks on the way to the solution. In 1863, Carl Friedrich Gauss [12] showed that the densest *lattice* packings in $\mathbb{R}^3$ are the face-centered cubic and hexagonal closed lattices. For a long time, the proof of the conjecture in the general case remained beyond the reach. Even much simpler geometric questions created serious debates, for example the so-called *sphere kissing problem*. The sphere kissing problem asks for the maximal number of non-intersecting unit balls that can simultaneously touch one unit ball. This question can be seen as a weak local version of the sphere packing problem. The kissing number in dimension 3 is 12. Another important step was the rigorous solution of the packing problem for unit disks in dimension 2 [11, 21]. The final solution of Kepler's conjecture was famously given by Thomas Hales [13].

The sphere packing problem and the sphere kissing problem are easily generalized to Euclidean spaces of other dimensions. At the moment the sphere packing problem has been completely solved in dimensions 1, 2, 3, 8 and 24. Conjectural solutions to the sphere packing problem in dimensions from 4 to 10 are listed in [8]. Analogs of the packing problem can be formulated in other metric spaces. A subset X of a metric space $(\mathcal{M}, \rho)$ is called an r_0-code if the distance between any two distinct points of X is greater than or equal to r_0. One interesting example of a metric space is the *Hamming space*. The binary Hamming space of dimension d is the vector space $\mathbb{F}_2^d$ over the finite field $\mathbb{F}_2$ equipped with the following metric: the distance between the vectors $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ is the number of indices i between 1 and d such that $x_i \neq y_i$. A subset $X \subset \mathbb{F}_2^d$ is a code of length d, dimension n and distance r if X is a vector subspace over $\mathbb{F}_2$ of dimension n and an r-code with respect to Hamming distance. Then we say that X is a (d, n, r) code. Codes in Hamming spaces are particularly interesting for us because of their connection to the lattices in Euclidean spaces. There are several ways to produce a Euclidean lattice from a code in Hamming space; some of them are described in [9, Chapter 5]. For example, the E_8 lattice can be constructed from the binary Hamming code $(8, 4, 4)$ by applying the so-called "construction A", and the Leech lattice can be obtained in a more complicated way from the binary Golay code $(24, 12, 8)$.

3 Energy minimization

A natural generalization of the sphere packing problem is the question of minimizing the energy of pairwise interactions between points. In this case, we consider configurations with a fixed number of points on a compact metric space, or configurations with fixed *point density* in the non-compact case.

Let C_f be a finite subset in $\mathbb{R}^d$. Fix a *potential function*

$$p\colon (0, \infty) \to \mathbb{R}.$$

The *potential p-energy* of C_f is

$$\frac{1}{|C_f|} \sum_{\substack{x, y \in C_f \\ x \neq y}} p(|x - y|).$$

We would like to extend this definition to infinite discrete subsets of Euclidean space.

Let C be a discrete closed subset of $\mathbb{R}^d$. We say C has *density ρ* if

$$\lim_{r \to \infty} \frac{|C \cap B_d(0, r)|}{\operatorname{vol}(B_d(0, r))} = \rho.$$

The *lower p-energy* of C is

$$E_p(C) \coloneqq \liminf_{r \to \infty} \frac{1}{|C \cap B_d(0, r)|} \sum_{\substack{x, y \in C \cap B_d(0, r) \\ x \neq y}} p(|x - y|).$$

If the limit exists, we call $E_p(C)$ the *p-energy* of C.

4 Universal optimality

We rephrase a famous saying: "An optimal configuration is optimal everywhere". Is it possible that one configuration is optimal for all potentials? The answer is obviously no; however, some configurations provide an optimal solution for a wide family of potential functions p.

One important family of potentials in Euclidean space are Gaussian functions $p_\alpha(r) = e^{-\alpha r^2}$, where α is a positive real number. The convex cone spanned by all real Gaussians is the cone of completely monotonic functions of squared distance. In [3], H. Cohn and A. Kumar introduced the following definition.

Definition 3. Let C be a discrete subset of $\mathbb{R}^d$ with density ρ, where $\rho > 0$. We say that C is *universally optimal* if it minimizes p-energy whenever $p\colon (0,\infty) \to \mathbb{R}$ is a completely monotonic function of squared distance.

The following result was established in [22] back in 1979.

Theorem 4. *The lattice $\mathbb{Z}$ is universally optimal.*

This result is also proven in [3] with the help of linear programming, the proof technique which will be explained in the next section. Moreover, in the same paper, Cohn and Kumar made the following conjecture.

Conjecture 5. *The lattices A_2, Λ_8 and Λ_{24} are universally optimal.*

In joint work with H. Cohn, A. Kumar, S. Miller and D. Radchenko [6], we have proved the following.

Theorem 6. *The lattices Λ_8 and Λ_{24} are universally optimal.*

Not much is known about universally optimal configurations in Euclidean space, and in particular whether the lattices in Theorem 4 and Conjecture 5 give the complete list of all universally optimal lattices. In [4], the authors provide numerical evidence that the root lattice D_4 and the configuration D_9^+ (the definition of this configuration is given in [4]) might be universally optimal.

5 Magic functions for geometric optimization problems

In this section, we will talk about the proof techniques used in Theorems 1, 2 and 6. Curiously, similar methods were used to prove the optimality of the binary Hamming code $(8,4,4)$, the binary Golay code $(24,12,8)$, and the optimality of the shortest vectors of the E_8 and Leech lattices as kissing configurations in their respective dimensions. This method is often referred to as *linear programming*. The key idea is to reduce a geometric optimization problem on a space $\mathcal{M}$ to minimizing a linear functional on a certain suitably constructed cone of functions on $\mathcal{M}$.

For packing and energy minimization problems, the following two cones of functions play an important role. Let $(\mathcal{M},\rho)$ be a metric space. We denote by $\operatorname{Spec}(\rho)$ the set of values taken by $\rho\colon \mathcal{M}\times\mathcal{M} \to \mathbb{R}_{\geq 0}$. A function $f\colon \operatorname{Spec}(\rho) \to \mathbb{C}$ is *copositive* if for all finite subsets $X \subset \mathcal{M}$ we have

$$\sum_{x,y\in X} f(\rho(x,y)) \geq 0.$$

A function $f\colon \operatorname{Spec}(\rho) \to \mathbb{C}$ is *positive definite* if for all finite subsets $X \subset \mathcal{M}$ and all complex weights $(w_x)_{x\in X}$ we have

$$\sum_{x,y\in X} w_x \overline{w}_y f(\rho(x,y)) \geq 0.$$

The cone of copositive functions is extremely powerful, and the possibility of effectively optimizing over it would lead to solutions of many geometric questions. Unfortunately, this cone is very complex and to our knowledge there is no easy way to work with it directly. However, the cone of copositive functions contains a much simpler one, namely the cone of positive definite functions. This cone has a simple description in terms of harmonic analysis on $\mathcal{M}$. We refer the reader to [9, Chapter 9] for details.

The following theorem is a simple yet powerful tool for bounding the size of codes in compact metric spaces.

Theorem 7. *Let $(\mathcal{M},\rho)$ be a metric space. Suppose that*

$$f\colon \operatorname{Spec}(\rho) \to \mathbb{R}$$

is a copositive function such that

$$f(0) = 1, \tag{1}$$

$$f(r) \leq -\frac{1}{N-1} \quad \text{for } r \geq r_0. \tag{2}$$

Then an r_0 code in $\mathcal{M}$ contains at most N points.

Proof. The proof of this theorem is very simple. Suppose that $X \subset \mathcal{M}$ is an r_0 code. Then the copositivity of f implies

$$\sum_{x,y\in X} f(\rho(x,y)) \geq 0. \tag{3}$$

On the other hand, by conditions (1) and (2), we estimate

$$\begin{aligned}\sum_{x,y\in X} f(\rho(x,y)) &= \sum_{x\in X} f(\rho(x,x)) + \sum_{\substack{x,y\in X\\ x\neq y}} f(\rho(x,y)) \\ &\leq |X| - \frac{|X|(|X|-1)}{N-1}.\end{aligned} \tag{4}$$

The two equations above imply that $|X| \leq N$. ■

We are interested in the examples when the upper bound provided by Theorem 7 is sharp, in particular, the cases when the auxiliary function f is positive definite. We have already mentioned several configurations which are "LP-sharp". For instance, the optimality of the Hamming binary code $(8, 4, 4)$ follows from the fact that the polynomial

$$p_{H_8}(t) := \frac{1}{30}(t-4)(t-8) - \frac{1}{15}$$

is positive definite with respect to Hamming distance on $\mathbb{F}_2^8$, see [10] and [9, Chapter 9]. A positive definite auxiliary function proving the optimality of the binary Golay code $(24, 12, 8)$ is also given in [9, Chapter 9]. The 240 shortest vectors of the E_8-lattice and the 196 560 shortest vectors of the Leech lattice are the optimal kissing configurations in their respective dimensions. In 1979, Odlyzko and Sloane [18] and V. Levenstein [16] independently constructed positive definite polynomials on the sphere proving the optimality. A survey of these results and the polynomials can be found in [9, Chapter 9] and in [19]. Moreover, by similar techniques, Cohn and Kumar showed that the shortest vectors of E_8 and Leech lattices are universally optimal configurations on the sphere [3].

H. Cohn and N. Elkies [2] applied the ideas of linear programming to the sphere packing problem in Euclidean space. Before we explain their method, let us introduce some notation. The *Fourier transform* of an L^1 function $f\colon \mathbb{R}^d \to \mathbb{C}$ is defined as

$$\mathcal{F}(f)(y) = \hat{f}(y) := \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot y}\, dx, \quad y \in \mathbb{R}^d,$$

where $x \cdot y = \frac{1}{2}|x|^2 + \frac{1}{2}|y|^2 - \frac{1}{2}|x-y|^2$ is the standard scalar product in $\mathbb{R}^d$. A C^∞ function $f\colon \mathbb{R}^d \to \mathbb{C}$ is called a *Schwartz function* if it tends to zero as $|x| \to \infty$ faster than any inverse power of $|x|$, and the same holds for all partial derivatives of f. The following theorem is the key result of [2].

Theorem 8. *Suppose that $f\colon \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function, $r_0 \in \mathbb{R}_{>0}$, and they satisfy*

$$f(x) \le 0 \qquad \text{for } |x| \ge r_0, \tag{5}$$
$$\hat{f}(x) \ge 0 \qquad \text{for all } x \in \mathbb{R}^d, \tag{6}$$
$$f(0) = \hat{f}(0) = 1. \tag{7}$$

Then the density of d-dimensional sphere packings is bounded above by

$$\frac{\pi^{\frac{d}{2}} r_0^d}{2^d \Gamma(\frac{d}{2}+1)}.$$

Note that this number is the volume of a ball of radius $\frac{r_0}{2}$ in $\mathbb{R}^d$.

This theorem produces an upper bound for the density of a sphere packing in every dimension. However, this bound is not expected to be sharp in general. A surprising discovery made by Cohn and Elkies was that they were able to obtain bounds numerically extremely close to the sharp ones in dimensions 1, 2, 8 and 24.

In [23], the author showed that the linear programming bound is indeed sharp in dimension 8.

Theorem 9. *There exists a radial Schwartz function $f_{E_8}\colon \mathbb{R}^8 \to \mathbb{R}$ which satisfies*

$$f_{E_8}(x) \le 0 \qquad \text{for } |x| \ge \sqrt{2},$$
$$\hat{f}_{E_8}(x) \ge 0 \qquad \text{for all } x \in \mathbb{R}^8,$$
$$f_{E_8}(0) = \hat{f}_{E_8}(0) = 1.$$

Furthermore, in joint work with H. Cohn, A. Kumar, S. D. Miller and D. Radchenko [5], we proved the sharpness of the linear programming bound in dimension 24.

Theorem 10. *There exists a radial Schwartz function $f_{\Lambda_{24}}\colon \mathbb{R}^{24} \to \mathbb{R}$ which satisfies*

$$f_{\Lambda_{24}}(x) \le 0 \qquad \text{for } |x| \ge 2,$$
$$\hat{f}_{\Lambda_{24}}(x) \ge 0 \qquad \text{for all } x \in \mathbb{R}^{24},$$
$$f_{\Lambda_{24}}(0) = \hat{f}_{\Lambda_{24}}(0) = 1.$$

The energy minimization problem also can be addressed by linear programming. The following bound was introduced by H. Cohn and A. Kumar.

Theorem 11. *Let $p\colon (0,\infty) \to \mathbb{R}$ be any function, and suppose $f\colon \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function. If $f(x) \le p(|x|)$ for all $x \in \mathbb{R}^d \setminus \{0\}$ and $\hat{f}(y) \ge 0$ for all $y \in \mathbb{R}^d$, then every subset of $\mathbb{R}^d$ with density ρ has lower p-energy at least $\rho\hat{f}(0) - f(0)$.*

In [5], we construct functions $f_{\Lambda_d,\alpha}$ for all $d \in 8, 24$ and real positive α such that $f_{\Lambda_d,\alpha}(x) \le e^{-\alpha|x|^2}$ for all $x \in \mathbb{R}^d \setminus \{0\}$ and $\hat{f}(y) \ge 0$ for all $y \in \mathbb{R}^d$, and also $\rho\hat{f}_{\Lambda_d,\alpha}(0) - f_{\Lambda_d,\alpha}(0) = E_{r\mapsto e^{-\alpha r^2}}(\Lambda_d)$. The construction of these functions implies Theorem 6. Informally, we call the auxiliary functions $f_{E_8}, f_{\Lambda_{24}}, f_{\Lambda_d,\alpha}$ the "magic functions" as they magically prove difficult geometric statements.

6 Fourier interpolation and sharp bounds

In this final section, we briefly explain the strategy for finding magic functions. Let us first consider the case of compact spaces. Suppose that $X \subset \mathcal{M}$ is an optimal r_0 code in the metric space $(\mathcal{M}, \rho)$, and a copositive function $f\colon \mathrm{Spec}(M) \to \mathbb{R}$ satisfies the conditions of Theorem 7 for $N = |X|$; in other words, f proves the sharp bound on the size of r_0-codes. In this case, inequalities (3) and (4) imply that $f(\rho(x,y)) = \frac{-1}{N-1}$ for all pairs of distinct points $x, y \in X$ and $\sum_{x,y\in X} f(\rho(x,y)) = 0$. Moreover, if we represent f as a sum of copositive functions $f = f_1 + \dots + f_k$, then $\sum_{x,y\in X} f_i(\rho(x,y)) = 0$ for $i = 1, \dots, k$. In many cases, these linear conditions are sufficient to find the function f.

Similar ideas work in the case of Euclidean space and can be applied to magic functions for the Cohn–Elkies bound of Theorem 8 and the Cohn–Kumar bound of Theorem 11. Suppose that $\Lambda_d \subset \mathbb{R}^d$ is a unimodular lattice and f_{Λ_d} is a magic function satisfying the conditions of Theorem 8 and thus proving the optimality of the Λ_d lattice sphere packing. Without loss of generality, we may assume that f_{Λ_d} is radial and the value $f_{\Lambda_d}(x)$ depends only on the Euclidean length $|x|$. Combining the Poisson summation formula

$$\sum_{x \in \Lambda_d} f_{\Lambda_d}(x) = \sum_{y \in \Lambda_d^*} \hat{f}_{\Lambda_d}(y)$$

with conditions (5)–(7) of Theorem 8, we deduce that $f_{\Lambda_d}(x) = 0$ for all $x \in \Lambda_d \setminus \{0\}$ and $\hat{f}_{\Lambda_d}(y) = 0$ for all $y \in \Lambda_d^* \setminus \{0\}$ (here Λ_d^* is the lattice dual to Λ_d). Moreover, since f_{Λ_d} is smooth, these equalities hold up to second order.

It turns out that we can recover the whole function f_{Λ_d} from this information on its values at lattice points. In [6], we proved the following Fourier interpolation formula.

Theorem 12. *Let (d, n_0) be $(8,1)$ or $(24,2)$. There exists a collection of radial Schwartz functions $a_n, b_n, \tilde{a}_n, \tilde{b}_n \colon \mathbb{R}^d \to \mathbb{R}$ such that for every $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned} f(x) = &\sum_{n=n_0}^{\infty} f(\sqrt{2n}) a_n(x) + \sum_{n=n_0}^{\infty} f'(\sqrt{2n}) b_n(x) \\ &+ \sum_{n=n_0}^{\infty} \hat{f}(\sqrt{2n}) \tilde{a}_n(x) + \sum_{n=n_0}^{\infty} \hat{f}'(\sqrt{2n}) \tilde{b}_n(x), \end{aligned}$$

and these series converge absolutely.

The above interpolation formula allowed us to find magic functions $f_{E_8}, f_{\Lambda_{24}}, f_{E_8,\alpha}$ and $f_{\Lambda_{24},\alpha}$ as explicit contour integrals, and based on these integral representations prove the inequalities posed on these functions by Theorems 8 and 11, respectively.

Finally, the Fourier interpolation formulas of this type seem to be very intriguing objects in their own right, and it would be worth searching for more such examples and more geometric applications.

References

[1] R. E. Borcherds, Monstrous moonshine and monstrous Lie superalgebras. *Invent. Math.* **109**, 405–444 (1992)

[2] H. Cohn and N. Elkies, New upper bounds on sphere packings. I. *Ann. of Math. (2)* **157**, 689–714 (2003)

[3] H. Cohn and A. Kumar, Universally optimal distribution of points on spheres. *J. Amer. Math. Soc.* **20**, 99–148 (2007)

[4] H. Cohn, A. Kumar, A. Schürmann, Ground states and formal duality relations in the Gaussian core model, *Phys. Rev. E* **80**, 061116 (2009)

[5] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko and M. Viazovska, The sphere packing problem in dimension 24. *Ann. of Math. (2)* **185**, 1017–1033 (2017)

[6] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko and M. S. Viazovska, Universal optimality of the E_8 and Leech lattices and interpolation formulas, preprint, arXiv:1902.05438 (2019)

[7] J. H. Conway and S. P. Norton, Monstrous moonshine. *Bull. London Math. Soc.* **11**, 308–339 (1979)

[8] J. H. Conway and N. J. A. Sloane, What are all the best sphere packings in low dimensions? *Discrete Comput. Geom.* **13**, 383–403 (1995)

[9] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*. 3rd ed., Grundlehren Math. Wiss. 290, Springer, New York (1999)

[10] P. Delsarte, Bounds for unrestricted codes, by linear programming. *Philips Res. Rep.* **27**, 272–289 (1972)

[11] L. Fejes, Über die dichteste Kugellagerung. *Math. Z.* **48**, 676–684 (1942)

[12] C. F. Gauss, Recension der "Untersuchungen über die Eigenschaften der positiven ternären quadratischen Formen von Ludwig August Seeber", *Göttingische Gelehrte Anzeigen*, July 9, 1831. Reprinted in *Werke*, Vol. 2, Königliche Gesellschaft der Wissenschaften, Göttingen, 188–196 (1863); available at http://gdz.sub.uni-goettingen.de

[13] T. C. Hales, A proof of the Kepler conjecture. *Ann. of Math. (2)* **162**, 1065–1185 (2005)

[14] A. Korkine and G. Zolotareff, Sur les formes quadratiques. *Math. Ann.* **6**, 366–389 (1873)

[15] J. Leech, Notes on sphere packings. *Canadian J. Math.* **19**, 251–267 (1967)

[16] V. I. Levenshtein, Boundaries for packings in *n*-dimensional Euclidean space. *Dokl. Akad. Nauk SSSR* **245**, 1299–1303 (1979)

[17] H. Löwen, Fun with hard spheres. In *Statistical physics and spatial statistics (Wuppertal, 1999)*, Lecture Notes in Phys. 554, Springer, Berlin, 295–331 (2000)

[18] A. M. Odlyzko and N. J. A. Sloane, New bounds on the number of unit spheres that can touch a unit sphere in *n* dimensions. *J. Combin. Theory Ser. A* **26**, 210–214 (1979)

[19] F. Pfender and G. M. Ziegler, Kissing numbers, sphere packings, and some unexpected proofs. *Notices Amer. Math. Soc.* **51**, 873–883 (2004)

[20] C. E. Shannon, A mathematical theory of communication. *Bell System Tech. J.* **27**, 379–423, 623–656 (1948)

[21] A. Thue, Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene, *Norske Vid. Selsk. Skr.* **1**, 1–9 (1910)

[22] W. J. Ventevogel and B. R. A. Nijboer, On the configuration of systems of interacting particle with minimum potential energy per particle. *Physica* **98A**, 274–288 (1979)

[23] M. S. Viazovska, The sphere packing problem in dimension 8. *Ann. of Math. (2)* **185**, 991–1015 (2017)

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At the crossroads of simulation and data analytics

Patrice Hauret

For more than three years, the EU-MATHS-IN Industrial Core Team[1] has been developing transverse influences to advertise the role of Digital Twins at the service of European industry, as a key enabler for advanced products and connected services. Digital Twins has also spread to growing areas like health and climate monitoring. Behind this buzzword, the complementarity of first principle modelling and data analytics plays an instrumental role. In the present article, we depict the context and the opportunities, the European environment with its funded programs, some open software platforms, and most importantly a mathematical toolbox to address the underlying challenges, all of which testify to the tremendous vitality of this field.

1 Introduction

When identifying, within EU-MATHS-IN, a subject that would rally a majority of companies, from banks to aeronautics, from the health sector to the need of renewal in the energy business, Digital Twins emerged as a rather natural choice [17].

The concept was first created by NASA. Throughout its entire life cycle, a product or process can be accompanied by a virtual representation, called its Digital Twin. Digital Twins allow novel digital assistance for design optimisation, process control, lifecycle management, predictive maintenance, or risk analysis. Digital Twins have become so important to business today that they were identified as one of Gartner's Top 10 Strategic Technology Trends [18, 27, 47]. They are becoming a business imperative, covering the entire lifecycle of an asset or process and forming the foundation for connected products and services. New business opportunities will emerge, benefitting from the cooperation between large companies, SMEs, startups and academia. To turn this vision into reality, novel mathematical and computer science technologies are required to describe, structure, integrate and interpret across many engineering disciplines.

Supporting this development requires a combination of efforts. High fidelity modelling is key to account for multi-physics and multiscale systems, and to identify new design levers at the smallest scales. It is often derived from first principle approaches and relies on the power provided by High Performance Computing to deliver the expected prediction in a reasonable time frame. On the other hand, reduced order modelling must provide real-time estimates to enable system optimisation, or in combination with statistical learning to achieve efficient modelling compliant with available data in real time during operation. An example of that kind of need is given by autonomous transport. In particular, corresponding solutions must be realised on the edge to provide sufficiently fast and robust interactions with the real process. The fields of application are growing rapidly in the health sector (e.g., www.digitwin.org) and in climate monitoring [3].

Key to digital twinning is a joint use of data and first principle approaches. In the frequent cases where large amounts of data prove not to be available, this complementarity makes it possible to realise Smart Data concepts, fostering the efficiency and the robustness of predictions and enabling the quantification of associated uncertainties and risks.

The economic impact of related applications is estimated to cover a worldwide market of 90 billion euros per year by 2025. These opportunities are clearly reinforced by the high concentration of simulation firms in Europe, the highest in the world. Europe also benefits from a long-standing proficiency in mathematics. Several reports [11, 14, 15] have additionally shown the economic impact of mathematical sciences in the US, the Netherlands and France. Furthermore, US studies [41] in the past have also highlighted the European strength in the field.

This being said, the combined use of statistical learning and first principle modelling is not new. Model calibration is an old topic. The Kalman filter lay at the core of space conquest, and has been essential in the efficient control of remote systems. There remain, however, several challenges for achieving a clever way of combining simulation methods for complex systems, data collection, correction procedures and uncertainty quantification. Besides this, Digital Twins are expected to become self-learning objects that can actually provide guidance and high level services in op-

[1] ATOS, Bosch, Dassault Aviation, EcoMT, EY, Michelin, NORS Group, Shell, Siemens

eration, digesting sensor data with the help of reduced models. A significant part of these ambitions relies on a set of mathematical methods as well as integrated environments.

2 A favourable environment

2.1 European Commission Programs

Horizon Europe[2] is a Research and Innovation funding program that is in place until 2027, with a total budget of 95.5 billion euros. Its three pillars are (I) Excellent Science, (II) Global Challenges and European Industrial Competitiveness, (III) Innovative Europe. The European Research Council (President: J.-P. Bourguignon) supports pillar I, while the DG Connect Directorate (Dir: K. Rouhana) specifically supports the present topic, as well as High Performance or Quantum Computing, as part of pillar II. Calls are the mechanisms through which individual projects get funded.

Of key importance is the EuroHPC Joint Undertaking. This is a joint initiative between the EU, European countries and private partners to develop a World Class Supercomputing Ecosystem in Europe. The initiative has two major partners onboard: ETP4HPC and the Big Data Value Association, which reflects the ambition to establish synergies between highly accurate simulations and data analytics. It is also oriented towards large companies and SMEs, in order to boost innovation potential and competitiveness, while widening the use of HPC in Europe. EU-MATHS-IN is involved in the TransContinuum Initiative, as a binding effort between ETP4HPC and BDVA, and contributes to a Strategic Research Agenda coordinated by Zoltán Horváth.

2.2 Some open platforms and funded projects

Open software enables new actors including SMEs, potentially with the help of a proper knowledgeable accompaniment, to access highly technological solutions, develop new ideas and start new businesses depending on associated exploitation rights. It clearly shifts development effort from low-value work to value creation. Open platforms can also act as binding environments to interface commercial products with clear added value. Sharing software components between academia and industry may be a way to reinforce the European momentum on the development of new mathematical algorithms in order to, e.g., take advantage of new European HPC architectures or data-simulation hybridation. The development of better interoperability is expected to accelerate innovation and boost European leadership.

Open software has been developed ranging from simulation tools to data analysis with applications like mechanics or biology. Furthermore, it has clearly evolved to provide integrated environments, especially relying on Python interfacing which is a key element to foster collaboration, to integrate various expertises and to make content widely accessible. Let us mention here two projects supported by the European Community that clearly support this ambition.

MSO4SC

Societal challenges are increasing in complexity, and contributing to their resolution requires a holistic approach. It is necessary to provide decision-makers with tools that allow long-term risk analysis, improvements or even optimisation and control. One of the key technologies in this process is the use of mathematical Modelling, Simulation and Optimisation (MSO) methods, which have proven to be effective tools for solving problems such as the realistic prediction of wind fields, solar radiation, air pollution and forest fires, prediction of climate change, improving the filtration process for drinking water treatment and optimisation methods for intensity-modulated radiation therapy. These methods are highly complex and are typically processed via the most modern tools of ICT, including high performance computing and access to big data bases; they usually require the support of skilled experts, who are often not available, in particular in small and medium-sized businesses. The main goal of this project is to construct an e-infrastructure that provides, in a user-driven, integrative way, tailored access to the necessary services, resources and even tools for fast prototyping, also providing the service producers with the mathematical framework. The e-infrastructure consists of an integrated MSO application catalogue containing models, software, validation and benchmark and the MSO cloud, a user-friendly cloud infrastructure for selected MSO applications and developing frameworks from the catalogue. This will reduce the 'time-to-market' for consultants working on the above-mentioned societal challenges.

Open Dream Kit

OpenDreamKit is a project that brings together a range of projects and associated software to create and strengthen virtual research environments. The most widely-used research environment is the Jupyter Notebook from which computational research and data processing can be directed. The OpenDreamKit project provides interfaces to well-established research codes and tools so that they can be used seamlessly and combined from within a Jupyter Notebook. OpenDreamKit also supports open source research codes directly by investing into structural improvements and new features to not only connect all of these tools, but also enrich them and make them more sustainable.

[2] ec.europa.eu/info/horizon-europe_en

3 A mathematical toolbox

We emphasise here four complementary perspectives on the joint exploitation of simulation and data: (i) techniques coming from optimal command, ranging from calibration to filtering, that allow identification of hidden parameters, model correction and handling noisy forcing terms, (ii) solution space reduction that enables fast solving and efficient correction, (iii) multi-fidelity co-kriging in order to merge and prioritise the feedback issuing from simulations and measurements on given observables, (iv) physics-inspired neural networks that directly learn the model solution, based on an *a priori* (set of) underlying model(s). Each of these approaches corresponds to a different balance in the roles of physical models and collected data. In this text, we do not claim any exhaustivity, and mathematical descriptions will remain formal. The idea is to provide a fairly broad overview of the field while remaining accessible to the vast majority of mathematician readers.

3.1 Optimal control: From calibration to filtering

General setting

Let Λ be a vector space of parameters. For every $\lambda \in \Lambda$, the solution $u_\lambda \in \mathcal{U}$ of the "best-knowledge" model is defined to be the solution of the partial differential system

$$\mathcal{A}(\lambda, u_\lambda) = 0 \quad \text{in } \mathcal{V}^*,$$

in a weak formal form, where $\mathcal{V}$ is a Hilbert space and $\mathcal{V}^*$ its dual. The parameter space Λ encompasses modelling choices, the domain shape and the boundary conditions, as well as forcing terms.

Additionally, the physical quantity

$$Z\colon \lambda \in \Lambda \mapsto Z(\lambda) \in \mathcal{Z}$$

is known, potentially with some noise, at the sampling points $(\lambda_k) \in \Lambda$ as $(\mathbb{Z}_k)$. The purpose of our quest is to take benefit from this information to estimate $Z(\lambda)$ outside the sampling points, or to improve the solution u_λ itself given by the model.

Calibration and model correction. As an example, let us augment Λ into $\Lambda \times \mathcal{V}$ in order to account for a modelling error term in $\mathcal{V}$ as we consider the augmented system

$$\mathcal{A}(\lambda, u_{\lambda,\xi}) + \xi = 0 \quad \text{in } \mathcal{V}^*,$$

where $\xi \in \mathcal{V}$. For each sampling point λ_k, the idea is to find the most adequate parameter λ of the model close to λ_k and the model correction $\xi \in \mathcal{V}$ in order to best account for the measurement $\mathbb{Z}_k$. The calibration and model correction in the vicinity of λ_k can be formulated as finding

$$(\lambda_k^*, \xi_k^*) = \arg\inf_{\lambda,\xi} \Big\{ \frac{1}{2} |J(\lambda, u) - \mathbb{Z}_k|^2 + \frac{\alpha}{2} |\lambda - \lambda_k|^2 + \frac{\beta}{2} |\xi|^2 \Big\}, \tag{1}$$

where $\mathcal{A}(\lambda, u) + \xi = 0$ in $\mathcal{V}^*$, and $|\cdot|$ stands for the Hilbertian norms in $\mathcal{Z}$, Λ and $\mathcal{V}$. The scalar coefficients $\alpha, \beta \in \mathbb{R}_+^*$ are taken sufficiently large. As classically given by the techniques of optimal control [30], the solution is characterised by the following system:

$$\mathcal{A}(\lambda, u) - \frac{1}{\beta} p = 0, \tag{2}$$

$$\langle d_u \mathcal{A}, p \rangle_{\mathcal{V}^*,\mathcal{V}} + \Delta \cdot d_u J = 0, \tag{3}$$

$$\langle d_\lambda \mathcal{A}, p \rangle_{\mathcal{V}^*,\mathcal{V}} + \Delta \cdot d_\lambda J + \alpha(\lambda - \lambda_k) = 0, \tag{4}$$

in $\mathcal{V}^*$, $\mathcal{U}^*$ and Λ^* respectively, with $\Delta = J(\lambda, u) - \mathbb{Z}_k$. Equation (3) is called the adjoint problem and equation (4) defines the gradient of the cost function with respect to λ, that must vanish. The gradient expression enables an iterative calibration in order to avoid the resolution of the coupled system (2)–(3)–(4). One has $\xi_k^* = -\frac{1}{\beta} p$ and we use the notation $p_k^* \coloneqq p$ in $\mathcal{V}$.

Once the calibration and error corrections are performed at sampling points, let us introduce the functions $\hat{\lambda}^*(\lambda), \hat{\xi}^*(\lambda), \hat{p}^*(\lambda)$ obtained by kriging under the form

$$\hat{\square}^*(\lambda) = \sum_k f_k(\lambda) \square_k^*, \quad \square \in \{\lambda, \xi, p\},$$

that comply with values (λ_k^*), (ξ_k^*), (p_k^*) at sampling points (λ_k). For every $\lambda \in \Lambda$, an updated model can be formulated as finding the solution $\hat{u}_\lambda \in \mathcal{U}$ such that

$$\mathcal{A}(\hat{\lambda}^*(\lambda), \hat{u}_\lambda) + \hat{\xi}^*(\lambda) = 0 \quad \text{in } \mathcal{V}^*, \tag{5}$$

with the estimator $\hat{Z}(\lambda) = J(\hat{\lambda}^*(\lambda), \hat{u}_\lambda)$. For purposes of efficiency, the spaces $\mathcal{U}$ and $\mathcal{V}$ can be replaced by some reduced basis approximation in (5).

Bayesian inference. Parameters λ and measurements $Z(\lambda)$ can be considered as random variables. The relevance of this point of view is supported by their potential discrete natures and by the uncertainties and noise attached to them. Assume p (resp. q) is the probability density followed by λ (resp. Z). Bayesian inference provides the conditional density

$$\mathrm{p}(\lambda|Z) = \frac{\mathrm{q}(Z|\lambda)\mathrm{p}(\lambda)}{\int \mathrm{q}(Z|\lambda)\mathrm{p}(\lambda)\, d\lambda} \propto \mathrm{q}(Z|\lambda)\mathrm{p}(\lambda),$$

where $\mathrm{p}(\lambda)$ is called the prior, i.e., the *a priori* distribution expected on λ; $\mathrm{q}(Z|\lambda)$ is the output likelihood given λ, i.e., the uncertainty on the output measurement or simulation. It results in an assessment of the $\lambda|Z$ distribution, known as posterior, that can be used as a new prior and so on, until uncertainties are judged satisfactory [22, 26].

From a practical standpoint, a Markov Chain Monte Carlo approach can be used to simulate samples according to the distribution followed by λ, and a surrogate model – for instance relying on reduced bases – can be used to diminish the computational cost required to determine the output Z.

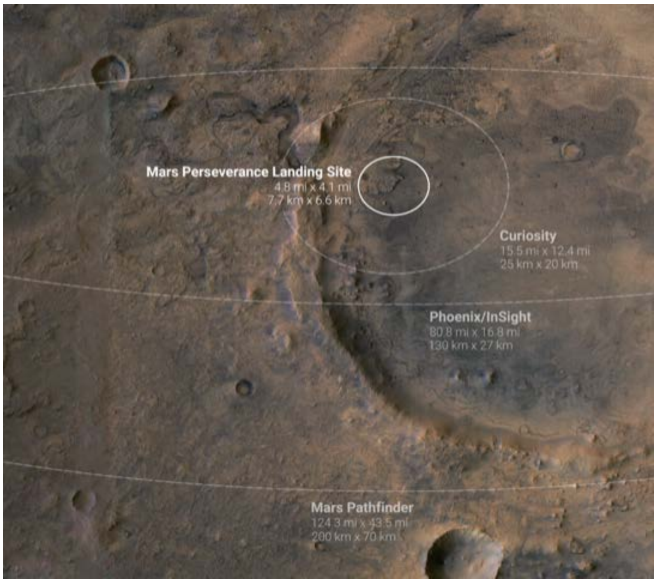

Figure 1. Perseverance Rover landing map on Mars (Credit: ESA/DLR/FU-Berlin/NASA/JPL-Caltech, visit mars.nasa.gov/mars2020)

Selection of parameters. In order to make the approaches as efficient as possible, the parametric space Λ must be reduced. A rather natural approach relies on variance decomposition, i.e. on the identification of subspaces within Λ for which the covariance matrix $\mathbb{E}[Z' \otimes Z']$ possesses the largest components; $Z' = Z - \mathbb{E}(Z)$. This can be done by Sobol decomposition or Principal Component Analysis for linear models [23]. Observe that in the vicinity of λ, the privileged most influential subspaces are given by the singular value decomposition of $d_\lambda Z \otimes d_\lambda Z$ in $T_\lambda\Lambda \times T_\lambda\Lambda$, the differential being computed by the adjoint method.

Sometimes, the parameters correspond to local characteristics of the model, such as material laws, and can be selected without involving the full resolution of the model. This is the case in Ortiz et al. [24, 28], which rely on local measurements or simulations with a closest-point projection approach.

Filtering: From space conquest to cardiovascular modelling

The complementarity between first principle modelling and data analytics was pioneered in a unique manner by space conquest. This clever combination makes it possible to benefit from the predictive power of simple dynamic models and the ability to cope with noise and uncertainties within the environment; as a result, it offers the possibility of automated decisions when long transmission times allow for neither full real-time feedback on the system state nor efficient human steering. We describe the main associated ideas within a linear framework; cf. [5, 6, 13, 30].

Filtering and optimal control. The above motivation was the main boost for the development of Kalman filtering, which is closely related to optimal control. Let $u(t) \in \mathbb{R}^n$ describe the state of the system at time $t \in [0, T]$ such that

$$\dot{u} = Au + B\lambda + F, \quad u(0) = u_0 + \xi, \tag{6}$$

where $\lambda \in \mathbb{R}^p$, $p < n$, stands for an unknown forcing contribution and ξ for the uncertainty on the initial condition. Measurements $Z = Hu + \varepsilon \in \mathbb{R}^m$ over $[0, T]$, with $m < n$, up to an error ε, are available in order to help estimate the real trajectory, through the determination of λ and ξ such that

$$\frac{1}{2}|\xi|_N^2 + \frac{1}{2}\int_0^T |Hu - Z|_M^2 + \frac{1}{2}\int_0^T |\lambda|_L^2$$

is infimised. Optimality is achieved for $\lambda(t) = L^{-1}B^\top p(t)$ and $\xi = N^{-1}p(0)$, where the adjoint state p is a solution of the backward system in time

$$\dot{p} = -A^\top p + H^\top M(Hu - Z), \quad p(T) = 0.$$

In order to avoid the difficulty of a two-end problem in u and p, the optimal solution u can be proven to decompose as $u = \hat{u} + Pp$, where P is the operator solution from Riccati's equation

$$\dot{P} - PA^\top - AP + PH^\top MHP - BL^{-1}B^\top = 0, \quad P(0) = N^{-1}.$$

The component $\hat{u}$ obeys the filtered dynamics

$$\dot{\hat{u}} = A\hat{u} + F + K(Z - H\hat{u}), \quad \hat{u}(0) = u(0),$$

where $K = PH^\top M$ is Kalman's gain; it reaches the optimal trajectory at time $t = T$, as $p(T) = 0$. Beyond this Linear Quadratic setting, filtering can incorporate robust control, and adapt in many ways to the case of nonlinear systems (Extended Kalman Filter, Unscented Kalman Filter).

Hamilton–Jacobi–Bellman. Dynamic programming is of particular importance in order to proceed to nonlinear extensions. For every $q \in \mathbb{R}^n$ and $t \in [0, T]$, let us define the cost-to-come function

$$V(q, t) = \min_{\lambda;\, u(t) = q} \left\{ \frac{1}{2}|\xi|_N^2 + \frac{1}{2}\int_0^t |Hu - Z|_M^2 + \frac{1}{2}\int_0^t |\lambda|_L^2 \right\},$$

where the minimum is taken over controls $\tau \in [0, t] \mapsto \lambda(\tau) \in \mathbb{R}^p$ and trajectories $\tau \in [0, t] \mapsto u(\tau) \in \mathbb{R}^n$ with end-points $u(0) = u_0 + \xi$ and $u(t) = q$. The cost function V is a solution of the Hamilton–Jacobi–Bellman equation

$$\dot{V} - \mathcal{H}^*(u, \nabla_u V(u, t), t) = 0,$$

in the sense of viscosity solutions, where

$$\mathcal{H}^*(u, p; t) = \mathcal{H}(u, p, L^{-1}B^\top p; t),$$
$$\mathcal{H}(u, p, \lambda; t) = \frac{1}{2}|Hu - Z|_M^2 + \frac{1}{2}|\lambda|_L^2 - \langle Au + B\lambda + F, p \rangle.$$

Observe that the optimality equations above read

$$\dot{u} = -\frac{\partial \mathcal{H}^*}{\partial p}, \quad \dot{p} = \frac{\partial \mathcal{H}^*}{\partial u}.$$

Taking $V(u,0) = \frac{1}{2}|u-u_0|_N^2$ as an initial condition, let us assume the HJB equation admits a solution $V \in C^1(0,T;\mathbb{R}^n)$. Then, for all $t \in [0,T]$ the optimal command is given by

$$\lambda^*(t) = \arg\min_{\lambda \in \mathbb{R}^p} \mathcal{H}(u(t), \nabla_u V(u(t),t), \lambda, t).$$

This provides the estimated dynamics

$$\dot{\hat{u}} = A\hat{u} + F - (\nabla_u^2 V)^{-1} H^\top M(Z - H\hat{u}), \quad \hat{u}(0) = u_0.$$

In the present linear setting, one has [37]

$$V(u,t) = \frac{1}{2}(u-\hat{u}(t))^\top P(t)^{-1}(u-\hat{u}(t)) + \frac{1}{2}\int_0^t |H\hat{u}(s) - Z(s)|_M^2 \, ds.$$

Stochastic perspective. Assume equation (6) is interpreted as a stochastic differential equation, where $\lambda(t)$ and $\varepsilon(t)$ are zero-mean independent Gaussian processes with covariance matrices Q and R respectively. The best mean square estimator $\hat{u}(t) = \mathbb{E}[u(t)|Z(t)]$ follows the same equation as in the previous paragraphs with $M = R^{-1}$ and $L = Q^{-1}$. The covariance matrix $P(t) = \mathbb{E}[(u-\hat{u}) \otimes (u-\hat{u})]$ obeys Riccati's equation.

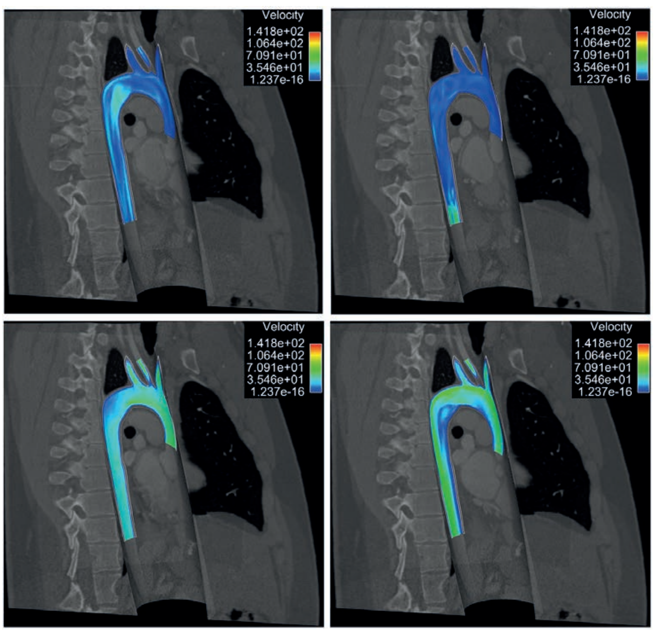

Figure 2. Aortic simulation for which viscoelastic boundary conditions are calibrated from medical imaging (courtesy of Moireau et al., see [38]).

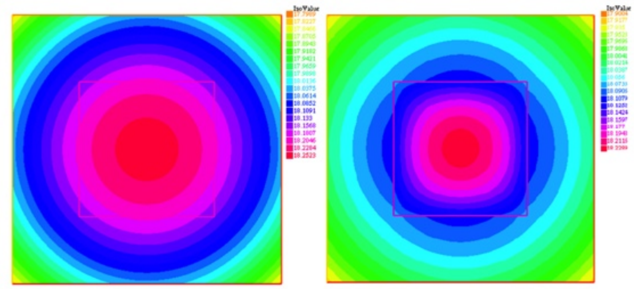

Figure 3. Heat equation resolved from an initial condition to a given end-time. The best-knowledge model uses single material, when the true solution corresponds to a bi-material. 121 measurement points are spread over the domain. Left: solution for the bk model (values from 17.80 °C to 18.25 °C). Right: Synthetic true solution using a bi-material plate (values from 17.90 °C to 18.23 °C) relying on seven basis vectors (courtesy of Benaceur, see [4]).

Stability and control. Kalman's gain achieves control optimality in the sense detailed above. Nevertheless, it can prove costly to determine, and difficult to access for distributed systems. As a matter of fact, some feedback terms acting as Lyapunov functions can suffice for practical purposes. Let us assume in the above that the gain K is chosen in the form $K = \alpha H^\top \mathcal{M}$ with a symmetric definite positive matrix $\mathcal{M}$ and a coefficient α to determine. It follows that the error $e = u - \hat{u}$ satisfies

$$\dot{e} = (A - KH)e + B\lambda - K\varepsilon, \quad e(0) = \varepsilon,$$

and can be made rapidly decreasing provided that λ and ε remain moderate and α is taken sufficiently large. This approach has been implemented with multiple refinements by Moireau et al. [39], comparing in-depth the displacement vs. velocity controls.

3.2 Solution space reduction

Reduced bases. The notion of reduced bases makes it possible to resolve the equations of the model in a low-dimensional subspace of solutions, rather than a large full finite element space for instance. It is reputed to go back to Rayleigh's intuition, and is key to benefit from surrogate models that comply with the first physical principles, with high computational efficiency. It is particularly useful for high-dimensional models, as in Quantum Mechanics for instance. The approach has been made particularly popular by the work of Maday, Patera et al. [32]. Some interesting challenges rise when handling highly nonlinear problems [33]; it is particularly striking when contact mechanics is involved [4], when structure preservation is concerned [20, 21], or when local accuracy is of particular importance [46]. The efficiency of the method is well established when the Kolmogorov n-width of the solution manifold rapidly decreases as $n \to +\infty$; cf. [2, 8].

Reduced order bases allow physical models to be interrogated (almost) as efficiently as data sets, making it possible to foster standard statistical learning methods [1, 19].

Parameterised-Background Data-Weak (PBDW) approach. Let

$$\mathcal{M}^{\mathrm{bk}} := \{u^{\mathrm{bk}}(\lambda);\, \lambda \in \Lambda\} \subset \mathcal{U}$$

be the solution manifold for the "best-knowledge" model. It can be gradually approximated with diminishing errors by solutions of the model in reduced spaces $\mathcal{R}_1 \subset \cdots \subset \mathcal{R}_N \subset \cdots \subset \mathcal{U}$.

The true solution $u^{\mathrm{true}} \in \mathcal{U}$ is unknown, but can be partially captured by experimental observations $\ell_m^{\mathrm{obs}} \in \mathcal{U}^*$ given as continuous linear forms of the solution: $1 \le m \le M$. They provide the scalar quantities $\mathbb{Z}_m = \ell_m^{\mathrm{obs}}(u^{\mathrm{true}}) \in \mathbb{R}$. Let us write (q_m) for the associated liftings satisfying $(q_m, v) = \ell_m^{\mathrm{obs}}(v)$ for all $v \in \mathcal{U}$, where $(\cdot,\cdot)$ denotes the inner product in the Hilbert space $\mathcal{U}$. It is imperative that the sensors be numerous enough ($N \le M$) to control the components of the reduced solution in the selected space $\mathcal{R}_M$; more specifically, one must have $\mathcal{R}_N \cap \mathcal{U}_M^{\perp} = \{0\}$ where $\mathcal{U}_M^{\perp}$ is orthogonal to $\mathcal{U}_M := \mathrm{span}\{q_m,\ 1 \le m \le M\}$ in $\mathcal{U}$.

The PBDW method [34–36] determines the approximation $u_{N,M}$ of the solution u^{true} in the form

$$u_{N,M} = r_{N,M} + \eta_{N,M} \in \mathcal{R}_M \oplus \mathcal{U},$$

where

$$(u_{N,M}, q) = (u^{\mathrm{true}}, q) \quad \text{for all } q \in \mathcal{U}_M, \tag{7}$$

and the norm $\|\eta_{N,M}\|_{\mathcal{U}}$ is infimised. It boils down to finding $r_{M,N} \in \mathcal{R}_M$ and $\eta_{M,N} \in \mathcal{U}_M$ such that

$$\begin{aligned} (r_{N,M}, q) + (\eta_{N,M}, q) &= (u^{\mathrm{true}}, q) \quad && \text{for all } q \in \mathcal{U}_M,\\ (\eta_{N,M}, r) &= 0 && \text{for all } r \in \mathcal{R}_N. \end{aligned}$$

This has been extended to a dynamic setting by Benaceur [4], in collaboration with Patera. The method allows for a real-time correction of the solution, based upon available measurements. In case of noisy measurements, a regularisation is required and the following functional:

$$\gamma \|\eta_{N,M}\|_{\mathcal{U}}^2 + \frac{1}{M} \sum_{m=1}^{M} |\ell_m^{\mathrm{obs}}(r_{N,M} + \eta_{N,M}) - \mathbb{Z}_m|^2, \quad \gamma > 0,$$

is infimised in order to compromise between the minimisation of the term $\|\eta_{N,M}\|_{\mathcal{U}}$ and the constraint (7), thus avoiding overfitting.

Data-driven reduced modelling. Each time the above reconstruction generates a prediction $u_{N,M}(\lambda_k)$ for a given state $\lambda_k \in \Lambda$ of the system, the vector $u^{\mathrm{bk}}(\lambda_k)$ can be replaced by $u_{N,M}(\lambda_k)$ in the reduced basis for the system. This can be done by the dynamic reduced basis low rank adaption proposed by Peherstorfer and Wilcox [42].

Another point of view consists in fitting the expression of the operators involved in the best-knowledge model, when projected onto an *a priori* reduced basis. This idea was proposed by Peherstorfer and Wilcox [43], who named it operator inference.

Finally, composite spaces can be constructed by the assembling of subdomains in which key driving parameters are retained and local reduced bases are adopted; coupling between subdomains can be performed via Lagrange multipliers, and the hidden parameters (for instance describing certain levels of damage within a structure) can be determined by statistical classification methods. Such approaches have been developed by Patera et al., and are called Port-Reduced Reduced-Basis Component (PR-RBC) methods; cf. [7, 16, 48].

Tensor approximation of solutions. Since functions in separable form are dense in spaces of sufficiently regular functions, one can decompose the solution of the parameterised model problem in the form

$$u_\lambda = \sum_\ell w_\ell(\lambda) u_\ell \quad \text{in } \mathcal{U}, \quad \text{for all } \lambda \in \Lambda,$$

by means of Proper Generalised Decompositions [10]. Of course, λ can easily incorporate a wide variety of correction terms, so as to account for the current state of the system. This general philosophy has been widely popularised by Chinesta and Nouy; see [9, 40] and the references therein.

3.3 *Multi-fidelity prediction and co-kriging*

In many practical cases, several sources of information (simulation and measurement campaigns) provide predictions of various accuracies for the observables. Let Λ be the parameter space under consideration. For every $1 \le j \le J$, the j-th measurement or simulation campaign is performed at sampling points $D_j = (\lambda_k^j)_{1 \le k \le K_j}$ with the model Z_j and provides data points $\mathbb{Z}_{j;k} = Z_j(\lambda_k^j)$. The

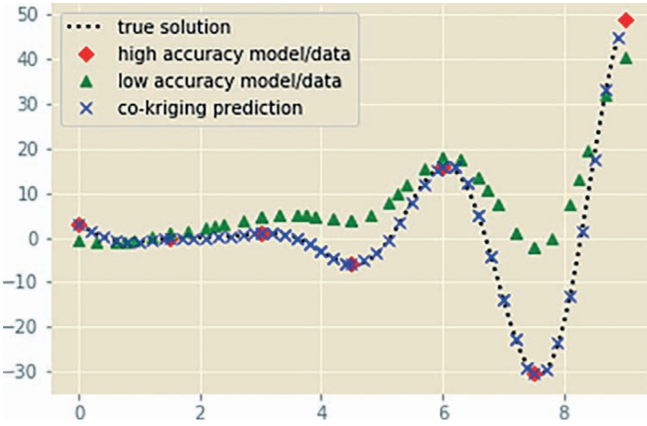

Figure 4. Co-kriging makes it possible to take joint advantage of (i) accurate but costly data (in red) and (ii) fast but inaccurate models (in green); computed using `OpenMDAO` and `Scikit Learn` Python packages.

accuracy increases with j and the most accurate forecast therefore corresponds to $j = J$.

Kriging. We set $j = 1$, and recall the construction of the universal kriging $\hat{Z}_1(\lambda)$ as a non-biased approximation of $Z_1(\lambda)$ with minimum variance in the form

$$\hat{Z}_1(\lambda) = \sum_{k=1}^{K_1} w_{1;k}(\lambda)\mathbb{Z}_{1;k} = w_1(\lambda)^\top \mathbb{Z}_1.$$

Universal kriging additionally assumes that $Z_1(\lambda)$ is a Gaussian process with unknown average of the form

$$\mathbb{E}(Z_1(\lambda)) = \sum_{i=1}^{I_1} f_{1;i}(\lambda)\beta_{1;i} = f_1(\lambda)^\top \beta_1,$$

and space correlation $\sigma_1^2 r_1(\lambda, \lambda'; \theta)$, where σ_1 is a variance scaling factor and θ a parameter of the space-correlation function r_1. The parameters β_1, σ_1 and θ can be determined through maximum likelihood. The non-biased minimum variance predictor is achieved for $w_1(\lambda) = C^{-1}(c(\lambda) + F^\top(FC^{-1}F^\top)^{-1}b(\lambda))$, where the matrices F, C are given by their components

$$F_{ik} = f_{1;i}(\lambda_k^1), \quad C_{k\ell} = \mathrm{cov}(\mathbb{Z}_{1;k}, \mathbb{Z}_{1;\ell}),$$

and the vectors c, b by

$$c_k(\lambda) = \mathrm{cov}(Z_1(\lambda), \mathbb{Z}_{1;k}), \quad b(\lambda) = f_1(\lambda) - FC^{-1}c(\lambda).$$

Recursive co-kriging. Co-kriging was pioneered by Kennedy and O'Hagan [25]. The recursive multi-fidelity adaptation introduced by Le Gratiet and Garnier [29] reads

$$\begin{cases} Z_{j+1}(\lambda) = \rho_j(\lambda)\tilde{Z}_j(\lambda) + \delta_j(\lambda), \\ \quad \tilde{Z}_j(\lambda) \text{ independent of } \delta_j(\lambda), \\ \quad \rho_j(\lambda) = g_j(\lambda)^\top \gamma_j, \end{cases}$$

where the following Gaussian processes are defined by their means and covariance matrices as

$$\delta_j(\lambda) \sim \mathrm{GP}\big(f_j(\lambda)^\top \beta_j; \sigma_j^2 r_j(\lambda, \lambda')\big),$$
$$Z_1(\lambda) \sim \mathrm{GP}\big(f_1(\lambda)^\top \beta_1; \sigma_1^2 r_1(\lambda, \lambda')\big);$$

$\tilde{Z}_j(\lambda)$ is a Gaussian process with distribution

$$[Z_j(\lambda) | Z^{(j)} = \mathbb{Z}^{(j)}, \beta_j, \gamma_{j-1}, \sigma_j^2],$$

where $Z^{(j)} = (Z_1(D_1), \dots, Z_j(D_j))$ and $\mathbb{Z}^{(j)} = (\mathbb{Z}_1, \dots, \mathbb{Z}_j)$.

The combined use of reduced models, full accuracy simulations and measurements clearly allow for very efficient and accurate surrogate models. The co-kriging above, further developed in [44], has the huge advantage of recursiveness and as a result, a very accessible computational cost.

3.4 *Physics-Informed Neural Network*

Neural networks have had great success with classification problems, together with Support Vector Machines for instance [1]. They generate functions that are dense among continuous functions (cf. Cybenko [12]), and conciliate smoothness with the ability to represent thresholds quite accurately. Fitting can be performed by a back descent gradient inspired by optimal control techniques. Furthermore, robust and powerful Python libraries, like `Tensor Flow`, are freely available.

Physics-Informed Neural Networks were introduced by Raissi, Perdikaris and Karniadakis [45]. They combine the statistical learning of the solution, say u on the space-time domain $[0, T] \times \Omega$ sampled on points (t^n, x_i) as u_i^n, under the penalised constraint that u is expected to resolve a partial differential equation of the form $\mathcal{A}(u) = 0$ in $[0, T] \times \Omega$. This reads as the infimisation

$$\inf_{\tilde{u}} \Big\{ \sum_{n,i} |\tilde{u}(t_n, x_i) - u_i^n|^2 + \sum_{n,i} |\mathcal{A}(\tilde{u})(t^n, x_i)|^2 \Big\},$$

which is close to (1) when $J(\lambda, u) = u$. A Bayesian approach can be used to identify the parameters from the model, as developed in [49]. Lucor et al. have developed the approach for the thermo-mechanical simulation of an incompressible viscous flow [31]; cf. Figures 5 and 6.

4 Conclusion

The combined use of first principle models and data analytics is an avenue for predictive sciences. It is a privileged way to synergise the modelling knowledge present in simulation software with the relevance of available data, while guaranteeing a high level of predictiveness in operation. Beyond the necessity of a growing mathematical toolbox to handle problems of optimal control with extreme efficiency, several challenges are implied: (i) the necessity of developing porosity at the interface between competences (numerical analysis, optimal control and automatism, high performance computing, statistics, computer sciences), (ii) the need for integrated development environments, with a role to play in the question of open software, and (iii) data protection, as data becomes an outstanding source of value.

Acknowledgement. I thank Krzysztof Burnecki for the kind invitation to write this article. I am indebted to Philippe Moireau, Amina Benaceur, Didier Lucor and Anne Sergent for providing the illustrations displayed in Section 3, and for our exchanges on the corresponding advances. The promotion of Digital Twins also owes thanks to my colleagues from the EU-MATHS-IN Industrial Core Team, with special thanks to Dirk Hartman, Zoltán Horváth, Wil Schilders, Cor van Kruijsdijk and Hubert Tardieu. In this regard, Section 1 presents several collective statements.

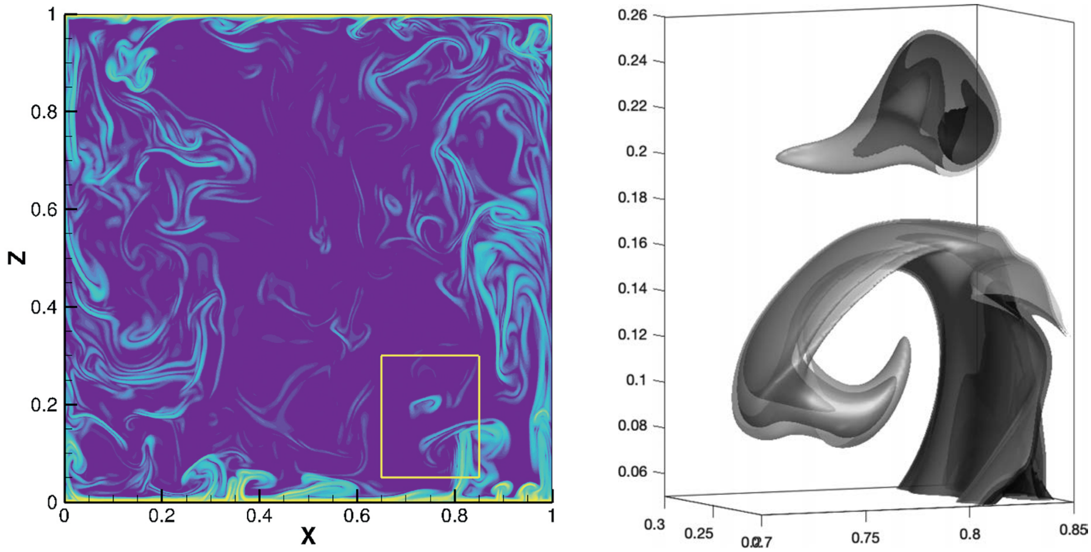

Figure 5. Left: computation domain containing an incompressible viscous flow heated at the bottom (the yellow box materialises the training subdomain); right: iso-temperature surfaces as reconstructed in the training subdomain by the Physics-Inspired Neural Network (courtesy of Didier Lucor, Atul Agrawal and Anne Sergent; see [31]).

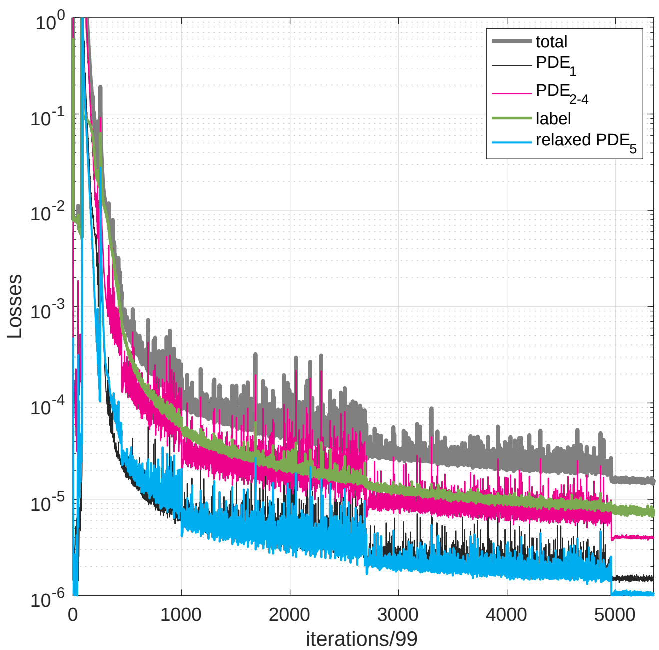

Figure 6. Learning curve displaying the cost function to infimise along with iterations (courtesy of Didier Lucor, Atul Agrawal, and Anne Sergent; see [31]).

References

[1] Y. S. Abu-Mostafa, M. Magdon-Ismail and H-T. Lin, *Learning from Data*. AMLbook.com (2012)

[2] M. Barrault, Y. Maday, N. C. Nguyen and A. T. Patera, An 'empirical interpolation' method: application to efficient reduced-basis discretization of partial differential equations. *C. R. Math. Acad. Sci. Paris* **339**, 667–672 (2004)

[3] P. Bauer, B. Stevens and W. Hazeleger, A digital twin of earth for the green transition. *Nature Climate Change* **11**, 80–83 (2021)

[4] A. Benaceur, *Réduction de modèles en thermo-mécanique*. PhD thesis, Paris Est University (2018)

[5] A. Bensoussan, *Filtrage optimal des systèmes linéaires*. Dunod (1971)

[6] A. Bensoussan, *Stochastic Control of Partially Observable Systems*. Cambridge University Press, Cambridge (1992)

[7] M. A. Bhouri and A. Patera, A two-level parameterized model-order reduction approach for time-domain elastodynamics. *Comput. Methods Appl. Mech. Engrg.* **385**, 114004 (2021)

[8] P. Binev, A. Cohen, W. Dahmen, R. DeVore, G. Petrova and P. Wojtaszczyk, Convergence rates for greedy algorithms in reduced basis methods. *SIAM J. Math. Anal.* **43**, 1457–1472 (2011)

[9] F. Chinesta, E. Cueto, E. Abisset-Chavanne, J. L. Duval and F. El Khaldi, Virtual, digital and hybrid twins: a new paradigm in data-based engineering and engineered data. *Arch. Comput. Methods Eng.* **27**, 105–134 (2020)

[10] F. Chinesta and P. Ladevèze (eds.), *Separated Representations and PGD-Based Model Reduction*. CISM International Centre for Mechanical Sciences. Courses and Lectures 554, Springer, Vienna (2014)

[11] CMI, Etude de l'impact socio-économique des mathématiques en france [CMI, A study of the soci-economical impact of mathematics in France] (2015)

[12] G. Cybenko, Approximation by superpositions of a sigmoidal function. *Math. Control Signals Systems* **2**, 303–314 (1989)

[13] B. d'Andréa Novel and M. Cohen de Lara, *Commande linéaire des systèmes dynamiques*. Modélisation. Analyse. Simulation. Commande. [Modeling. Analysis. Simulation. Control], Masson, Paris (1994)

[14] Deloitte, Measuring the economic benefits of mathematical sciences research in the UK. Deloitte (2012)

[15] Deloitte, Mathematical sciences and their value for the dutch economy. Deloitte (2014)

[16] J. L. Eftang and A. T. Patera, Port reduction in parametrized component static condensation: approximation and *a posteriori* error estimation. *Internat. J. Numer. Methods Engrg.* **96**, 269–302 (2013)

[17] EU-MATHS-IN Industrial Core Team, Modelling, simulation and optimization in a data rich environment: a window of opportunity to boost innovation in Europe. Foundation EU-MATHS-IN, European Service Network of Mathematics for Industry (2018)

[18] D. Hartmann and H. Van der Auweraer, Digital twins. Siemens. arXiv:2001.09747 (2019)

[19] T. Hastie, R. Tibshirani and J. Friedman, *The Elements of Statistical Learning*. 2nd ed., Springer Series in Statistics, Springer, New York (2009)

[20] P. Hauret, Multiplicative reduced basis for elasticity. submitted (2021)

[21] J. Hesthaven and C. Pagliantini, Structure-preserving reduced basis methods for Hamiltonian systems with a nonlinear poisson structure (2018)

[22] D. Higdon, J. Gattiker, B. Williams and M. Rightley, Computer model calibration using high-dimensional output. *J. Amer. Statist. Assoc.* **103**, 570–583 (2008)

[23] B. Iooss and P. Lemaître, A review on global sensitivity analysis methods. In *Uncertainty Management in Simulation-Optimization of Complex Systems*, Oper. Res./Comput. Sci. Interfaces Ser. 59, Springer (2015)

[24] K. Karapiperis, L. Stainier, M. Ortiz and J. E. Andrade, Data-driven multiscale modeling in mechanics. *J. Mech. Phys. Solids* **147**, 104239, 16 (2021)

[25] M. C. Kennedy and A. O'Hagan, Predicting the output from a complex computer code when fast approximations are available. *Biometrika* **87**, 1–13 (2000)

[26] M. C. Kennedy and A. O'Hagan, Bayesian calibration of computer models. *J. R. Stat. Soc. Ser. B Stat. Methodol.* **63**, 425–464 (2001)

[27] M. Kerremans, D. Cearley, A. Velosa and M. Walker, Top 10 strategic technology trends for 2019: digital twins. www.gartner.com/en/documents/3904569/top-10-strategic-technology-trends-for-2019-digital-twin, Gartner (2019)

[28] T. Kirchdoerfer and M. Ortiz, Data-driven computational mechanics. *Comput. Methods Appl. Mech. Engrg.* **304**, 81–101 (2016)

[29] L. Le Gratiet and J. Garnier, Recursive co-kriging model for design of computer experiments with multiple levels of fidelity. *Int. J. Uncertain. Quantif.* **4**, 365–386 (2014)

[30] J.-L. Lions, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*. Dunod, Paris; Gauthier-Villars, Paris (1968)

[31] D. Lucor, A. Agrawal and A. Sergent, Physics-aware deep neural networks for surrogate modeling of turbulent natural convection, preprint, arXiv:2103.03565 (2021)

[32] Y. Maday, Reduced basis method for the rapid and reliable solution of partial differential equations. In *International Congress of Mathematicians. Vol. III*, Eur. Math. Soc., Zürich, 1255–1270 (2006)

[33] Y. Maday, N. C. Nguyen, A. T. Patera and G. S. H. Pau, A general multipurpose interpolation procedure: the magic points. *Commun. Pure Appl. Anal.* **8**, 383–404 (2009)

[34] Y. Maday, A. T. Patera, J. D. Penn and M. Yano, A parameterized-background data-weak approach to variational data assimilation: formulation, analysis, and application to acoustics. *Internat. J. Numer. Methods Engrg.* **102**, 933–965 (2015)

[35] Y. Maday, A. T. Patera, J. D. Penn and M. Yano, PBDW state estimation: noisy observations; configuration-adaptive background spaces; physical interpretations. In *CANUM 2014 – 42e Congrès National d'Analyse Numérique*, ESAIM Proc. Surveys 50, EDP Sci., Les Ulis, 144–168 (2015)

[36] Y. Maday and T. Taddei, Adaptive PBDW approach to state estimation: noisy observations; user-defined update spaces. *SIAM J. Sci. Comput.* **41**, B669–B693 (2019)

[37] P. Moireau, A discrete-time optimal filtering approach for non-linear systems as a stable discretization of the Mortensen observer. *ESAIM Control Optim. Calc. Var.* **24**, 1815–1847 (2018)

[38] P. Moireau, C. Bertoglio, N. Xiao, C. A. Figueroa, C. A. Taylor, D. Chapelle and J.-F. Gerbeau, Sequential identification of boundary support parameters in a fluid-structure vascular model using patient image data. *Biomechanics and Modeling in Mechanobiology* **12**, 475–496 (2012)

[39] P. Moireau, D. Chapelle and P. Le Tallec, Filtering for distributed mechanical systems using position measurements: perspectives in medical imaging. *Inverse Problems* **25**, 035010, 25 (2009)

[40] A. Nouy, Proper generalized decompositions and separated representations for the numerical solution of high dimensional stochastic problems. *Arch. Comput. Methods Eng.* **17**, 403–434 (2010)

[41] NSF and WTEC, www.nsf.gov/pubs/reports/sbes_final_report.pdf; www.wtec.org/sbes/SBES-GlobalFinalReport.pdf.

[42] B. Peherstorfer and K. Willcox, Dynamic data-driven reduced-order models. *Comput. Methods Appl. Mech. Engrg.* **291**, 21–41 (2015)

[43] B. Peherstorfer and K. Willcox, Data-driven operator inference for nonintrusive projection-based model reduction. *Comput. Methods Appl. Mech. Engrg.* **306**, 196–215 (2016)

[44] P. Perdikaris, D. Venturi, J. O. Royset and G. E. Karniadakis, Multi-fidelity modelling via recursive co-kriging and Gaussian–Markov random fields. *Proc. R. Soc. A.* **471** (2015)

[45] M. Raissi, P. Perdikaris and G. E. Karniadakis, Physics-informed neural networks: a deep learning framework for solving forward and inverse problems involving nonlinear partial differential equations. *J. Comput. Phys.* **378**, 686–707 (2019)

[46] S. Riffaud, M. Bergmann, C. Farhat, S. Grimberg and A. Iollo, The DGDD method for reduced-order modeling of conservation laws. *J. Comput. Phys.* **437**, 110336, 19 (2021)

[47] Digital twins – explainer video. youtu.be/ObGhB9CCHP8, Siemens AG (2019)

[48] T. Taddei, J. D. Penn, M. Yano and A. T. Patera, Simulation-based classification: a model-order-reduction approach for structural health monitoring. *Arch. Comput. Methods Eng.* **25**, 23–45 (2018)

[49] L. Yang, X. Meng and G. E. Karniadakis, B-PINNs: Bayesian physics-informed neural networks for forward and inverse PDE problems with noisy data. *J. Comput. Phys.* **425**, 109913, 23 (2021)

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A conversation with Reuben Hersh

Ulf Persson

In the spring of 2009 the NCM (Nationellt Centrum för Matematik) a didactic institute located in Gotheburg and founded by Bengt Johansson, invited Marcus du Sautoy to give a popular lecture. I was invited along to the luncheon and dinner and was asked to interview the guest, but he was too busy, so instead I 'made up' an interview on the basis of his lecture, the conversations at meals, the discussion at the center, and snatches of interchanges during walks between venues. Thus I wrote down both questions and answers and submitted it to du Sautoy, who made some minor changes. The whole interview was published in the Newsletter of the Swedish Math Society, for which I was conveniently the chief editor. The director Johansson liked it so much that he had the whole procedure repeated with Keith Devlin later that summer and the next year he wanted to invite Ian Stewart, who could not come, so instead I was dispatched to Warwick University. The following winter I was sent to Boston, where I had a few sessions with my old advisor David Mumford, and then I went to New Mexico where I spent a few days with Reuben Hersh. Later on I was sent to Paris talking to Cédric Villani, Yves Meyer and Luc Illusie. For those opportunities I am indebted to the generosity of the NCM, which is somewhat ironic as I have been a vocal critic of mathematical didacticians and their various claims to be scientific.

The routine I initiated with Sautoy has not only served me on the missions launched by NCM but also on other occasions, as when interviewing Fields Medalists at the ICM, the results of which have been published in the EMS. My inspiration has been Eckermann's *Gespräche mit Goethe* (Conversations with Goethe) and the procedure is straightforward. I have a discussion with the subject, ideally over several days, but that is not an option at an ICM, and I keep no notes, I make no recordings of anything but trust my memory. After the interview I may jot down some cryptic notes to myself as a support and then I sit down and make up a conversation, or if you prefer an interview. This is great fun as it is in the spirit of writing a play, for one thing you get to formulate both questions and answers, and the purpose is to give the illusion of a conversation which, at least in a literal sense, has never taken place. Some people may be shocked at this confession and dismiss it all as counterfeit; on the other hand the subject has the last word and can make as many amendments as they want (some subjects, such as Mumford and Hersh, got into the spirit of the thing and made extensive elaborations, but most have been satisfied with minor revisions). The conversation as such may have never taken place, nor have the involved used the exact formulations presented, but so what? As long as the subjects give their consents and blessings everything is fine. The form of an interview or, as I prefer, conversation, is just a way of conveying information and hopefully also the character of the people involved, and as such presents a lively way of so doing.

Ulf Persson, July 2021

The earlier 'interviews' were published in The Newsletter of the Swedish Math Society and it was my intention to collect them all in a book but that ambition has not yet come to fruition. In the meantime nothing prevents me to let them appear to a larger audience and below I present the interview with Reuben Hersh which was conducted in his home in Santa Fe in February 2011. It was published in Bulletin of the Swedish Math Society in 2015 to which Hersh made some minor revisions in light of the time which had passed. Early in 2020 Hersh died at the age of 92 and the interview was also published on his memorial page through his son Daniel Hersh. My stay with Reuben and his partner Vera was very

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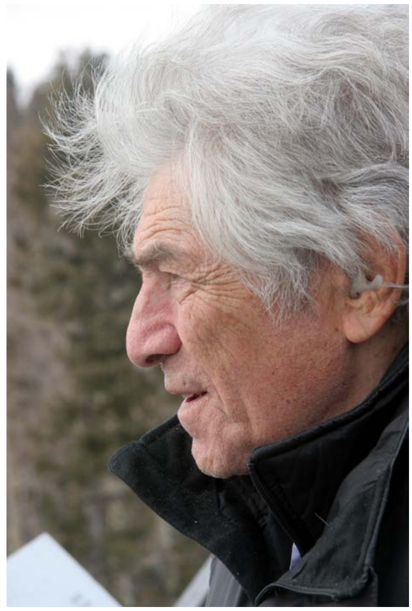
memorable and I regret that it turned out to be the only occasion I was going to have to meet him in the flesh; prior to this, we had kept up a rewarding e-mail exchange, which of course continued, and would do so until his death a decade later.

For an introduction to Reuben Hersh I refer to my recent obituary in the EMS Newsletter 116, June 2020. The original publication was in the Bulletin of the Swedish Math Society, October 2015, and we thank the society for permission to republish it.

Ulf Persson (UP): So you went to Harvard at fifteen, wasn't that very young?

Reuben Hersh (RH): There were a few sixteen-year olds as well. Smart Jewish kids from New York, who wouldn't have been let into Harvard in normal times. The war was on, this was 1943, and most of the regular Harvard boys were in the Navy, so we were let in to help fill the seats in the classrooms. Then after the war, things changed, it stopped being such a genteel snob school.



Reuben Hersh, February 2011

UP: Did you do any math at that time?

RH: I had enjoyed math a lot in high school. But my calculus course with David Widder was a disaster. Just plodding through motivationless technicalities. It killed all my interest. Instead I began to think of myself more as a writer. But Harvard didn't offer a degree in creative writing in those days. So instead I majored in English lit. I read a lot of fiction, poetry, drama. I wrote short lyric poems, and even won the Lloyd Mckim Garrison prize *for poetry by a Harvard undergraduate*. Two years running. (Not much competition, with most of the usual Harvard boys away in the Navy.) I still have the silver tombstone medal with my name on it, and I'm sure that the stuff that won the prize is still on file in some little room in Widener Library. My poetry professor Robert Hillyer nominated me to represent Harvard in an undergraduate poetry competition at Mt Holyoke, that's an Ivy League girl's college. I hitchhiked to Mt Holyoke, and showed up a bit late at the formal dinner. Wouldn't spend the money for a train ticket.

UP: So you gave up completely on math?

RH: I did have two friends,Johnny Wermer and Henry Helson, who later became successful mathematicians. They were a bit older than me. I just found out recently that Henry had died. That was sad. Have you heard of those guys? Wermer has a Swedish wife.

UP: Of course I have heard of them. So what did you do after you graduated?

RH: Well, I had assumed I was going to go into the Army to fight Hitler. As soon as I was 18, I could join the army without my parent's consent. But I was too late! The war ended in August 1945, and I wasn't 18 until December. I graduated in January 1946. But I had no other plans, no idea what else to do, and I expected to be drafted eventually any way, so I joined the army. They had 18-month enlistments available. I somehow survived basic training, and ended up classified as a clerk/typist. I spent seven months as a clerk typist in the U.S. Military Government headquarters in Seoul, Korea.

UP: What did you do next?

RH: I was discharged at Camp Stoneman in California and hitchhiked home to Mount Vernon, New York, my parent's home. I moved back in with them, and tried just sitting at home and writing, in order to become a writer …

UP: But it was not working, you were frustrated.

RH: Desperately. I just couldn't do it. None of it was any good. I was too young. I had no experiences worth writing about. Of course everyone has the experience of growing up – childhood and family life – but that has been written to death. I had nothing original to say.

UP: So you had to give up on that?

RH: Yes, I had to give it up. And I had to support myself, I had to get out of my parent's house. Somehow my mother knew someone who knew someone who was an editor on a magazine, and he knew Leon Svirsky, who was one of the editors of Scientific American. You know about Scientific American?

UP: Yes, of course. I remember they published retrospectives from 50 and 100 years earlier.

RH: That magazine had really deteriorated, it was almost dead, and three science writers from Time and Life magazine decided to buy it up, get the right to use the name, and do something different. This was 1947. Their idea was publish articles by scientists themselves, telling the public about their own work.

UP: It was a real discontinuity.

RH: Definitely. Of course most scientists can't write in a style accessible to the general public, but with a lot of help from editors, something actually readable may come out. They had just promoted the office boy, so there was an opening for me, working in the mail room. I spent a year packing up copies of Albert G. Ingalls' books on amateur telescope making. Then gradually I got more interesting things to do, minor writing and editing. I was assigned to read old issues and write those retrospectives you just referred to.

UP: But you did not want to spend the rest of your life there.

RH: I couldn't see myself doing this for the rest of my life. After four years there, I quit and tried to become a machinist. Being a veteran, I could take advantage of the GI Bill and get some paid schooling in the evenings, at the Machine and Metal Trades High School on York Avenue. Do you know what a lathe is?

UP: Of course I know.

RH: A lathe is a machine that cuts cylindrical parts. Machine parts that have to rotate are cylindrical. But manual work didn't come easy to me.

UP: You had trouble understanding the instructions.

RH: Right. I had a hard time understanding without clear instructions. They left so many obvious things out, things which were not obvious to me. But eventually I got reasonably good at it. Not great, but good enough to hold a job and make a living. You can learn a lot of things as long as you put your mind to it.

UP: But you did not spend your whole life there either.

RH: No, I did not. I had a very stupid accident using a band saw. I sawed off the upper half of my right thumb.

UP: And at the time there was no point to retrieve it and have it sewn on again. Micro-surgery had not yet been invented.

RH: It really scared me, and I decided I had to do something else. Everything I had done up to then – joining the army, trying to be a writer, trying to be a working class activist – it all was a way of trying to help change the world, fight Fascism and racism and oppression and so on. But it had all been a delusion, almost a waste of time. I had tried to change the world, and I just couldn't do it. Moreover, all this happened at almost the same time as Nikita Khrushchev's famous *secret speech* to the congress of the CPSU, where the head of the CPSU revealed that he and his associates had been servants of a paranoid sociopathic mass murderer. So finally I decided that if my efforts to change the world had been useless or worse, I might as well just do whatever I enjoyed. It took a while to figure out what I really enjoyed. Then I remembered that I used to enjoy mathematics and decided to apply to graduate school.

UP: And so you got into mathematics? But you had had very little mathematics, why should you be accepted as a graduate student?

RH: Fortunately, I applied at NYU, which had a somewhat open mind about off-beat applicants. I wanted to stay in New York, and between NYU and Columbia, NYU was a better bet. I was interviewed by a guy I had never heard of, a professor named Fritz John. Of course, later I knew very well who he was. He was skeptical, so I told him that I had gotten a perfect score on the math part of the Graduate Record Exam. He answered, *And what is the graduate record exam?* I explained that to him, and he thought about it, and said, *Probably means something*. So he told me to take advanced calculus in summer school, and then if I did OK I could apply for admission as a graduate student in the fall. Summer school was where I met Harold Shapiro. He had a summer job at NYU teaching advanced calculus. He is another mathematician who has a Swedish wife.

UP: You were quite old by then.

RH: Twenty-nine. A rather mature age to start in mathematics.

UP: But it had advantages?

RH: Unlike many other burgeoning mathematicians I was not burdened with unrealistic ambitions. I didn't expect to do great things, I would be happy if I could just get a job and support myself. I was married. I had married very young. At twenty. Far too young.

UP: I always thought that teenage marriages were very romantic.

RH: What a mistake! Anyway we lived in New Jersey in a town near Hackensack called Teaneck. Living in Manhattan was impossible unless you were willing to live in a slum. So I ended up commuting every day across the river.

UP: You were dedicated.

RH: I enjoyed it. And I had the great good luck that Peter Lax offered to be my advisor. He was just one year older than me, he had already been famous for over a decade. He was a Hungarian prodigy at fifteen. Paul Erdos introduced him to Einstein as *a Hungarian math prodigy*. *Why Hungarian?* Einstein asked. Peter showed me his list of problems, and I picked the one that looked the easiest, because it was just algebra. I got nowhere. As a matter of fact, it was a very hard problem that took decades before anyone was able to solve it. In the end I worked on the mixed initial-boundary value problem. At that time a major active field was developing the theory of linear partial differential equations from second to higher order. The classical theory was limited almost entirely to second order. You know about the Laplace, the wave and the heat equations? Of course you do, every mathematician does, even if you don't admit it. Louis Nirenberg did a lot of work generalizing the Laplace equation to higher order elliptic equations. Lax's specialty was hyperbolic equations, generalizing the wave equation, so I got involved in the hyperbolic case. You want to hear all the gory details?

UP: Sure.

RH: Well, my job was to find the most general correct boundary conditions for a general hyperbolic system in a half-space. I was stuck for a long time. Then I ran across a trick in a textbook on applied mathematics. You can combine a Laplace transform in time with Fourier transforms in the unbounded space variables, to reduce the mixed initial-boundary problem in a half-space to an ordinary differential equation in a single spatial variable, the variable orthogonal to the boundary. The resulting ODE problem is a kind of boundary value problem, with one boundary at infinity. This trick, which the textbook used in one particular limited context, could be used in a much more general setting. After that insight, I just had to write it up and show it to Lax. His reaction was *Laplace transform? I haven't used the Laplace transform in years*.

UP: And Laplace transform was Widder's specialty! Quite a coincidence. So it became your thesis?

RH: It was good enough to land me a two-year instructor-ship at Stanford, Way beyond my humble expectations.

UP: So whom did you meet in Stanford?

RH: Polya was there, but I had very little interaction with him at that time. Hörmander was there.

UP: He was visiting Stanford?

RH: He was a regular professor. I had worked on his first book on PDEs and found one silly little misprint, and when I told him so, he looked really disturbed. It took a few seconds for him to see the error – a misspelling of *see* as *se*, which of course is correct in Swedish! Then he looked relieved, and slightly amused. At the time he was teaching a course on several complex variables. That subject wasn't interesting to me, but I was very impressed by the way he handed out typed lecture notes before each lecture. When I expressed my admiration, he said, *There's nothing to it, just go like this*, and pretended typing with his fingers in the air, with a little smile.

UP: It must have come in very handy when he had his book published. I remember it very well. It was in our school-library. I understood nothing. That excited me a lot. I guess he is very efficient.

RH: I ran into him again at Stanford years later, at a week's celebration for Peter Lax turning sixty. He still remembered me. He had put the result of my thesis into his multi-volumed work on PDE. But he made a qualification. He said I had not solved it completely, only in the rough. Reiko Sakamoto had convinced him that she had the first complete proof. That's not true. I solved it, then she did it over again in a much more obscure manner. But who cares? It doesn't matter.

UP: Do you have any other stories about Hörmander?

RH: He gave a talk once while his famous predecessor Arne Beurling was in the audience. Hörmander was doing everything with Fourier transforms, and just to show that he too had a finger in the pie, Beurling asked about doing it with Fourier series. Hörmander quickly and briefly dismissed Beurling's remark as too trivial a case to even mention.

UP: Paul Cohen was there at Stanford too?

RH: Yes, I knew Cohen fairly well. He could be very aggressive. He always wanted to be on top. He would ask you a question and if you weren't prepared to battle with him, you had to either admit defeat or just ignore him. As if he was still a prodigy. He grew up in impoverished circumstances, and was introduced to calculus when he was nine or so. He had been asked by Scientific American magazine for an article about the continuum hypothesis, but they found his contribution impossible to edit for publication. He was looking for help. I did rewrite it successfully, it was published as a joint article. Later on he suggested that we work together on a popular book, but I declined. From a career viewpoint that was probably a mistake, but I just didn't feel comfortable working with him. You know about Courant–Robbins, don't you?

UP: Of course, it's a classic.

RH: Do you know that Robbins wasn't supposed to be listed as a co-author? Courant was the senior author, and he expected sole credit, of course with an acknowledgment to Robbins in the preface. After all, Courant–Hilbert was written by Courant, not Hilbert, with a lot of help from junior authors, and Hilbert was listed as the senior author only as a token of respect. But Robbins forced Courant to list him as a co-author. He got no information about royalties, though. Every once in a while a check would arrive in the mail, with a friendly greeting, but no explanation.

UP: So you ran into Courant when you were at the Courant institute, before it was named Courant?

RH: It was named after him after he retired. I even used a desk where he had once sat. People were really impressed by that. Later on Jerry Berkowitz assigned me as a graduate assistant to work with Courant on the English translation of volume 2 of Courant–Hilbert – Partial Differential Equations. I was just supposed to do copy editing of the galley proofs, but as an experienced editor I couldn't help making occasional suggestions for editorial improvement. This always amused him greatly. Sometimes he accepted my advice. Once I went to a course he was giving. He spoke in such a soft voice that only the people in the front row could hear him. He started out by saying he had sometimes been told that people couldn't always hear him, so would anyone who couldn't hear him please raise their hand, and then he would know he should talk louder. No one raised a hand. Courant was famous for playing up to the rich and powerful. He got a lot of funding that way, and used it very successfully, but some mathematicians turned up their noses at such vulgar behavior.

UP: Anyone else you recall from your Courant days.

RH: There was another Harold Shapiro in addition to the *Swedish* Harold S. Shapiro, I mean Harold N. Shapiro, the number theorist. A loud guy. They used to say, *S is for skinny, N is for noisy.* Do you know what Harold N. did to a promising student of his?

UP: Something unmentionable?

RH: He gave him as a thesis problem the twin prime conjecture. Can you believe it?

UP: I can believe anything.

RH: The poor student! That's the kind of problem you give if you hope to become famous through your student.

UP: The chances would be slim, though.

RH: And the student could be destroyed. Let's go back to Cohen.

UP: The Continuum hypothesis?

RH: You know the story of how it happened?

UP: Cohen had no high regard for logicians, and told them, give me your hardest problem and I will solve it.

RH: And they did! And he went ahead and solved it! Just imagine how those guys must have felt.

UP: End of story.

RH: It hasn't ended yet. Forcing is still keeping the logicians busy. Once he had it solved, Cohen had to go to Princeton to show it to Gödel. He knocked on Gödel's door. Gödel opened the door, peered out, snatched the manuscript and closed the door.

UP: Just like that?

RH: A couple of days later, after Gödel had read the manuscript, Paul was invited inside.

UP: Mathematicians are strange. What do you think of the current fashionable theories of Asperger being so prevalent among mathematicians? Masha Gessen in her recent book on Perelman makes a big deal out of it.

RH: I don't think labeling it as *Asperger's syndrome* helps very much. And in her book, which otherwise I found quite impressive, I thought the section on Asperger's was tasteless and unnecessary. I said so in my review in the Intelligencer.

UP: To me this is just a manifestation of the intolerance for eccentricity. The dictatorship of mediocrity. It is the flip side of genius adulation. The bottom line being that those geniuses may be very clever and all that and do things beyond our conceptions, but they are defective, not to say fatally flawed. It gives some consolation.

RH: Could be.

UP: When I was a beginning graduate student I heard a rumor from a visiting student of logic, that Gödel had already solved the CH. That he in fact knew it much deeper than Cohen, who was humbled.

RH: No, that's not true. After all, there has been a thorough investigation of Gödel's Nachlass.

UP: So Gödel gave Cohen his blessing?

RH: He did. In due time,that is. Cohen became a bit impatient for Gödel's public endorsement. Gödel reassured him and told him to relax.

UP: That sounds very human.

RH: Did you know that Cohen also married a Swedish woman?

UP: Had no idea. Wermer, Shapiro and now Cohen, where will this end?

RH: Cohen met Christina while he was in Sweden to visit the Mittag-Leffler Institute. With that name, there was no way he could pass her off as Jewish. He was very secretive about it. Finally he found a rabbi willing to marry them. Rabbis don't usually consent to marry someone to a non-Jew.

UP: And they lived happily ever after.

RH: Yes, as far as Cohen could be happy. He told me that they were a good couple because they were both childish. But as I said, he wasn't an easy character. At Stanford he usually argued against hiring or promoting anybody. No candidate was ever good enough. He had very few students.

UP: Sarnak was a student of his.

RH: Sarnak is a tough cookie, he could stand up to Cohen. He wrote a eulogy on Cohen in the Notices. Cohen developed some strange disease and died in his early seventies. He had spent decades trying as hard as he could to prove the Riemann conjecture. He actually said to someone I know, *I'll show those bastards I'm not dead yet*. There were four people he considered worth talking to about it. Selberg and Bombieri, I don't remember the other two.

UP: But I do not want to drop this issue of Asperger yet. Regardless or not whether you take the kind of diagnosis seriously, and I believe that it is anyway applied frivolously ignoring some more clinical criteria available; one may perhaps speak of certain character traits of mathematicians.

RH: To tell the truth, most mathematicians are boring. Most of them have no real intellectual interests, they just have a knack for doing mathematics well enough to make a living, teaching the same course over and over, and every now and then coming up with some theorem. It's the same way with artists. You tend to think of them romantically, but most of them are very mundane.

UP: So the vulgar conception of mathematicians as a kind of engineers may not be too far off?

RH: Not too far off. But numerical analysis is looked down on by most mathematicians just because mathematicians want to be above engineers. Peter Lax is an outstanding exception. He combines a great mathematical mind, deeply theoretical and abstract, along with original, effective down-to-earth calculations.

UP: This is supposed to be rather rare.

RH: His interest in computation isn't just to give examples of general principles. He's genuinely interested in it for its own sake, as well as for its practical utility.

UP: What about Polya?

RH: We invited him to speak at New Mexico. People in the department were impressed by my connections at Stanford. Phillips and Cohen came too, but you asked about Polya. He gave two talks. His lecture at the College of Education was called *Let Us Teach Guessing*. He used a problem which I later learned is a special case of *Steiner's problem*. Into how many regions is space divided by five planes chosen at random? You simplify from five planes to four, then three, then two, and from three dimensions to two, and then you guess. He was admirably patient. And pedagogical. For example, he used a ruler to represent a line, and divided it in two with a finger, then divided it once more using a finger on his other hand, then he ran out of hands and used his nose! The audience loved it. Later on I realized how good his books on problem solving are. It was easy to underestimate him. Hermann Weyl once had to comment on him. He said something to the effect that Polya likes to solve nice little problems one after another, but Weyl himself could never work like that. That was unfair. Polya chose important problems. He extracted the essence of some difficulty and presented it very concretely.

UP: I read that Polya considered himself too smart to be a philosopher, but not smart enough to be a physicist, so he chose mathematics.

RH: Replace the word 'smart' with 'good' and you would have the right quotation.

UP: Polya got to be very old, almost a hundred. But not as old as Cartan or Struik. Not to mention Vietoris, who got to be 111.

RH: When did Vietoris stop publishing mathematical papers?

UP: He published some when he was well over a hundred.

RH: Remarkable. Polya spent his last years in misery. He became blind. But he was a wonderful man. Old world civility. We took him to a Mexican restaurant once. He ordered a chile relleno. My wife cried out to him not to eat the seeds. He smiled and explained that he was Hungarian, he knew very well how to deal with hot stuff. Then his face turned red and his eyes were popping, but he kept smiling. He certainly didn't let on the pain. Wonderful guy.

UP: Going to New Mexico was not the end of your career as a mathematician?

RH: Why should it be? I started to collaborate with a young probabilist, Richard Griego. We got into applications of probability theory to partial differential equations. We wrote a popular article on how to solve the Dirichlet problem using Brownian motion. It was published in Scientific American in the late sixties.

UP: In fact I very much remember that article. It must have been in 1969. I had just finished high-school. I recall my father was intrigued by it.

RH: Happy to hear it. In fact, I have met other people who benefited from it. Griego and I started a new method of probabilistic solution of differential equations. It all started with an obscure paper by Mark Kac, which we realized could be vastly generalized by operator methods. We studied operator differential equations controlled by a stochastic process. Peter Lax gave it the right name, *random evolutions*. Do you want to hear more details? To make sense of what I just said?

UP: Sure.

RH: Well, the main point was to use the central limit theorem from probability as a tool to prove singular perturbation theorems about differential equations. A nice example is to start with a large number of Newtonian particles moving at constant speeds, and suffering collisions which make them switch direction and speed at random. If you put in a couple of properly scaled small parameters, you can make the mean free path between collisions go to zero. Our random evolution model permits us to use the central limit theorem to prove that this system of transport equations, a hyperbolic system, goes to a certain diffusion equation, a parabolic equation, in the limit. In physical language, this is a rigorous proof of the diffusion approximation for a high-density gas.

UP: But then it came to an end. Your career as a research scientist was short but glorious.

RH: You said it, I didn't. It was only about fifteen years. But my work exceeded my expectations. As I already explained. I never expected to become an above average researcher. At the end, there was a paper I was co-authoring. When it came back from being reviewed, I realized that what we were trying to do was essentially routine and uninteresting. Mistakes you can usually correct, often they indicate that you are on to something important and challenging. But not in this case. I apologized to my co-author. I simply couldn't go on with it. It was a tough time. My marriage was falling apart. We had married too young, we had eventually grown apart. I sought and found professional help, trying to sort out my life. And then I fell in love! How wonderful! I felt guilty, of course, but I couldn't help myself. I got divorced, and life started over. As I tell people, and especially you, life begins at sixty. Look at me!

UP: So it was your new love that provided your resurrection?

RH: And also having vibrant intellectual interests that I was passionate about. I had something to think about. And that is the aspect of my resurrection that will interest you.

UP: It was on the philosophy of mathematics and its practice, I understand. What made you into the Reuben Hersh that you are now, and for which people will remember you when all else is gone?

RH: Forget about being remembered, don't expect it. So many people are clamoring to be remembered, and who gets chosen is usually just a matter of chance. But I had an encompassing interest, and I had two talents which are seldom combined, a knack for mathematics and a knack for writing. You seem to have them too.

UP: That is very kind of you to say so. It was 'The Mathematical Experience' which launched you.

RH: I had gotten hooked on philosophy of math when I volunteered to teach a course that was listed in my department's catalogue as *Foundations of Mathematics.* No one had ever offered it, before or since. I expected to just do my usual thing when teaching a subject I know nothing about – pick the best textbook I can find, and stay a chapter ahead of the students. Not this time! All the textbooks I found simply presented three viewpoints – logicist, intuitionist, and formalist, and left it plain that all three were inadequate, unsatisfactory, failures. End of course!! As a teacher I found that situation deeply unacceptable. After all, I ought to at least know what was my own personal philosophy of math. But I found that I simply didn't know. So I had to find out where I stood, what was my understanding of the nature of the subject to which I devoted my life. On my part, *The Mathematical Experience* was a stage in my struggle to figure out my own answers. Then also, my career as a mathematician had given me a special kind of experience, which had not been much exploited in a literary way. I was very lucky to find Phil Davis as a collaborator. We never dreamed that the book would make such a splash. It was far short of our original intentions, but we were desperate and submitted what we had. It seemed only a rag-bag at the time, but nevertheless, it worked, after all!

UP: Yes, I remember very well reading the book in the early 80's. I was impressed by it. I felt that the authors, of whom I had no idea, were really up to something. I also recall Borel at a lunch at the Institute praising the book at the time as a serious work done by people who understood mathematics and what it meant to be a mathematician.

RH: Borel said that? By the way, is Borel related to Emile Borel, the famous French analyst?

UP: Not that I know. I doubt it though. Armand Borel was Swiss for one thing.

RH: The book was reviewed by Martin Gardner. You know him, of course.

UP: Yes. I got a book of his essays translated into Swedish when I was a child. Later I read his columns in the Scientific American. And just a year or so before he died I got into epistolary contact with him. He typed letters the old-fashioned way, even sent me a paper model of a Klein bottle he had made. You cannot do that on e-mail. The reason for getting in touch with him was to refute your anti-Platonist stand in the EMS Newsletter. You may recall the occasion.

RH: I certainly do. You know Gardner believed in God? Literally. He even wrote a chapter advocating the effectiveness of prayer. Gardner was assigned to review our book for the New York Review of Books. Of course that was wonderful. Attention to this kind of book in the NYR, what more can you ask for? You know the NYR, of course?

UP: I have subscribed to it since the mid-seventies.

RH: Good for you. So we have something more in common. Anyway, Gardner liked our book on the whole, but he attacked our anti-Platonist philosophy.

UP: You got a mixed review, in other words.

RH: Gardner was a Platonist. That makes sense for someone who believes in God. If you believe in God, you have an obvious place to put mathematics *out there*. I understand why my anti-Platonism upset or offended him. I never met him personally, but we did have a sort of connection by our common connection to the Scientific American. There were attacks from some other people that I can't so easily tolerate. The worst was from a certain computer list-serve called FOM, meaning Foundations of Mathematics. It belongs to a clique of logicians who not only work on axiomatic set theory, they worship it as The Foundation Of Mathematics. In what sense does mathematics have or need a foundation, let alone what might such a foundation be? I got lured into signing on to this activity. When I refused to convert to their ideology they made me an object of abuse and ridicule. Eventually I escaped by signing off from their computer list. And then, much before all that, there was Professor Hilary Putnam.

UP: A logician at Harvard. Nothing to do with the Putnam exam I take it, although I always made the naive connection when I first encountered his name.

RH: No connection. Jewish mother, WASP father. You know what a WASP is?

UP: I spent several years in the States. In a sense part of my formative ones.

RH: Sorry. You never know. I don't want to take anything for granted. Anyway I had sent a piece on philosophy of mathematics to the Monthly. Putnam was the referee. He referred to my piece as doggerel. I guess he thought of his own stuff as real poetry. As a consequence, The Monthly rejected it, which was a good thing, as Gian-Carlo Rota quickly published it in his journal – the Advances. A much better place for it.

UP: Rota was a dictator.

RH: Sure. And an excellent editor! You should know about that. I've never been an editor. Being one gives you a lot of power, and you need to use it wisely in order to do a good job.

UP: By the way I think of philosophy as the poetry of science. Philosophers do not take kindly to this notion. I mean it as a compliment though. What I mean is that philosophy proceeds by evocation rather than argument, and that it is very important that you present it in an elegant way. Among mathematicians expounding on philosophy I find that Yuri Manin stands out. He is a real pleasure to read.

RH: Manin is great. He has such wide and penetrating interests. Did you ever read his book on logic? It's written from the perspective of a mathematician. And as I understand, written from scratch. He taught himself logic.

UP: Yes. I was very much influenced by it. I came across it when I once tried to teach mathematical logic to undergraduates at Columbia.

RH: The problem with most academic philosophy of mathematics is that it's not about actual mathematics, it's about other philosophers of mathematics, their little clique. They aren't interested in mathematics or mathematicians, they aren't even interested in regular ordinary philosophers, they are just writing to answer each other and argue with each other. Look at Quine, for God's sake. Very well respected within the logic community. But such an arrogant pedant. He didn't know about the Riemann hypothesis – OK. I can understand that. But what was far worse, he wasn't even interested! The supposed greatest living philosopher of mathematics, and he neither knows nor cares about the most important open problem in mathematics. This man wrote that everything in mathematics can be *got down* to sets. In plain words, to do philosophy of mathematics, it's unnecessary to know anything about mathematics beyond set theory. How ignorant, presumptuous and arrogant! I am too blunt, I know. I have actually met a few philosophers who have a taste for mathematics. and I have finally met one here in New Mexico who is willing to talk and listen to me. Apart from this new acquaintance, there is really no one around here with whom I can discuss those matters, who really wants to listen. Some do it politely for a few minutes. My wife tries to do it, but she can only take so much. It's really exhilarating to have someone who listens as attentively as you. It's wonderful. It makes me blabber, and now I fear I am going beyond all bounds. Are you really going to write all this down?

UP: As much as I will be able to recall. Your powers of recollection are remarkable, once you start unwinding the threads of your memory. So much is retained. Not immediately accessible of course. You have to pull at it. But eventually one thing will lead to another.

RH: Still, it makes me a bit nervous. Where were we?

UP: We were speaking of the ignorance of philosophers when it comes to mathematics. I admit that the more ignorant you are the easier it is to hold firm opinions.

RH: Take Alonzo Church. An important, influential logician, certainly. No question about that. Church wrote down a long formula, involving an X; then he needed another formula identical to the first, except that X was replaced by Y. After mentioning that of course he could simply write something like *let X be replaced by Y*, he decided that the safest thing was to just write the whole thing all over again, but using Y instead of X. Super careful. Incredible. When Gian-Carlo Rota was an undergraduate at Princeton he attended Church's course. Solomon Lefschetz looked into the room, saw Rota sitting there, and shook his head in disapproval. And then, what about Ludwig Wittgenstein? *Mathematics is nothing but calculations. It has nothing to do with concepts or ideas*. How absurd! He is saying such a thing, even while mathematicians are trying hard to explain to him that we are interested in IDEAS ABOUT CALCULATION. With Alan Turing sitting right there in front of him, Wittgenstein is saying mathematics has nothing to do with concepts!

UP: I guess we are in a sense talking about Church's thesis. The point of mathematics is to make sense of calculations and to decide what calculations are to be done.

RH: You can put it that way if you want. Or their idea that mathematics essentially consists of deductive proofs. But in reality, nobody could follow all the way through a completely explicit detailed formal proof of any substantial interesting piece of mathematics. Unless it's a very simple one, like the examples that Hardy pulled out in order to convince people of the beauty and compelling power of mathematics.

UP: It is a commendable ambition.

RH: But misleading.

UP: Very much so. What makes for a convincing argument is not a long deductive chain but the way it fits into the web of mathematics.

RH: Well, in order to include him in my book *What is Mathematics, Really?*, I had to read Wittgenstein.

UP: He seemed very influenced by Russell, thinking of mathematics in a so to speak mechanical way, as a sequence of tautologies. Ultimately this view implies that mathematics contains no new knowledge, everything is in the axioms. It strikes me as somewhat peculiar that the richness of number theory is hidden in the simple axioms of Peano. There seem not to be enough information in them.

RH: That was the early Wittgenstein, the Wittgenstein of the Tractatus. The later Wittgenstein was completely different. He had some good points and some very bad ones. He emphasized that mathematics, like language, is a human activity. Excellent! But he went on to claim that a mathematician is free to do anything he pleases, anything at all. That is not true, it is ridiculous.

UP: So you agree that there are constraints. A mathematician is bound by rules beyond his control.

RH: Exactly. That is the essence of the mathematical experience, as eloquently described by Hardy.

UP: So you do not deny its validity?

RH: Not at all. Why should I?

UP: You have said that mathematics is objective as far as the individual is concerned, and subjective as far as the collective. Would you care to elaborate?

RH: Leslie White was the one who first said that plainly and clearly. Of course mathematics has a very high degree of objectivity. It doesn't matter what is your race, nationality, or religion, root 2 is irrational and pi is transcendental.

UP: So women do not think another mathematics.

RH: No. The Cauchy–Kovalevskaya theorem is neither male nor female.

UP: In what sense is mathematics subjective?

RH: Mathematics is a collective invention, like law or art or language. It's external from the viewpoint of the individual studying it, but it's internal with respect to human culture as a whole. It exists within the shared consciousness of human beings. Of course it's still very different from law, language or art. In particular, mathematics certainly is not just a language, although some people do thoughtlessly say so.

UP: I have a colleague who seriously claims that the difficulties students have with mathematics are linguistic. They have simply not understood 'mathematish' so to speak. They need to have the definitions of mathematics and the formulas translated into plain everyday language. According to this theory some of us instinctively acquire 'mathematish' but the rest need to be explicitly instructed as to its 'grammar' and vocabulary.

RH: That is dumb.

UP: I am glad that you agree. What is worse that this colleague seems to catch the ears of mathematical educators. What about law and art?

RH: Law is about more or less arbitrary regulations and their rational interpretation, and of course that doesn't have the same force as mathematical reasoning. And art, although many mathematicians claim that they are really artists, is likewise softer than down-to-earth mathematics and does not command the same kind of consensus, not even the same kind that law inspires.

UP: When you speak about mathematics are you not really speaking about the practice of mathematics? Mathematics is practiced by human beings, and we do not see it practiced anywhere than by humans, thus the argument that it is a human invention and would not make sense outside humanity is more or less tautological, in the sense of being circular and trivial. And of course what is considered important and beautiful in mathematics is subjective and vulnerable to the forces of fashion. Definitions and concepts are human inventions, but like all inventions, in the mental as well as the physical world, they have unintended consequences.

RH: But you exempt truth?

UP: Yes, I exempt truth. What is true in mathematics is not up to our discretion, certainly not as individuals.

RH: But in practice truth is agreed on by a process of social confirmation. I can give you a specific concrete example. As I told you before, I worked on linear partial differential equations with constant coefficients. My work was later extended by Heinz-Otto Kreiss to the case of variable coefficients. His theorem was quickly accepted as a *known* result that anyone else can freely quote and use. The proof is long and complicated. I could never really understand it all. But in the course of my mathematical education and research there have been many things that I accepted without completely understanding the proof. I would just assume it was my own fault, either I didn't know enough or I wasn't smart enough or I wasn't trying hard enough. Lax decided Kreiss's theorem was true. I don't know for certain how thoroughly he went into it. He knew Kreiss well and had a high opinion of his mathematical work. This particular result fitted well into what one might expect, based on general knowledge of the subject. It used the appropriate tools and methods, it encountered and overcame the expected difficulties. I would expect that he listened to Kreiss explaining it to him in his office until he was convinced. Once Lax decided it was true, no one doubted it. When Kreiss wrote it up for publication in NYU's Communications on Pure and Applied Mathematics, he didn't have to struggle to make every detail clear and explicit. He could publish it in an incomplete, cryptic form, because it had already been accepted by everyone. I suspect that you know of similar examples in your own field.

UP: Sure. One obvious example is Hironaka's resolution of singularities. I doubt that anyone has really gone through all the details. Most people like me, who have appealed to it in their work have not even made the attempt to read the paper, but trust it anyway, because that is socially acceptable. In a way it can be seen as an axiom, something you can rely on without understanding. And an even more generally known example, the proof that there are only 26 sporadic groups. The proof of that, scattered through tens of thousands of journal papers, is too long for any single mind to fathom in all its devilish details. And sure enough, as I understand it, small defects are continually being discovered and fixed, the general idea being that all the mistakes are fixable.

RH: So you agree, even when it comes to truth in mathematics it is a matter of social convention.

UP: But the remarkable thing is that this convention is so consensual. As I have already noted, deductive reasoning is not congenial to humans, when we as referees accept a paper we use other supplementary ways of being convinced. I agree with you that in practice mathematical truth is based on social consensus. In fact everything you say on the practice of mathematics we agree on. But I think that there is something beyond the practice of mathematics, beyond the human fallible way of doing mathematics. Outside of mathematics, socially accepted truths may be successfully challenged. And even in mathematics, if there is a counter-example to a previously authorized theorem, that will surely trump. Just as in science, our accepted truths are only provisional, although many of them have stood the test of time for a remarkably long time.

RH: Absolutely right. Nevertheless, I hold that the practice of mathematics is all there is to it. I would also emphasize that the most fundamental mathematical practices – counting on your fingers, and spatial intuition – are grounded, like all human activity, in our physical beings, in our bodies and in being in the world. Anything beyond that is mysticism. Wittgenstein's great insight, which was bitterly contested, is that the role of philosophy should be *to show the fly the way out of the fly bottle*. So many philosophical quandaries are illusory and artificial. The fact that language allows a certain question to be asked, by no means implies that a meaningful answer is possible or even conceivable. A famous fascinating question was first asked by Leibniz, and then repeated by Heidegger: 'Why is there something rather than nothing?' It is a useless question. It does not make sense, and no conceivable answer to it could make sense. The only reason that the question is asked is that it is possible to formulate it. Mathematical Platonism is a similar kind of fallacy. It arises from the unfounded idea that there must be something to mathematics beyond the practice of mathematics. You and I can agree on every basic issue of mathematics and disagree on this transcendental issue, which is not even an issue.

UP: What is your position on physical laws? Do they exist or are they just social constructs?

RH: Of course they exist, they are existing social constructs. To talk as if social constructs are things that don't really exist is untenable nonsense. Your electric bill exists, you'd better pay it or you'll be sorry. As to the laws of physics, they are not observed with our eyes or our instruments, they are formulated as part of our effort to make sense of the physical world. That means of course that we can't just make them up any way we please. There is a physical reality out there. The most devout anti-materialist doesn't doubt that his teeth are real, when he is having a really agonizing toothache.

UP: This is of course the standpoint of Karl Popper. Physical theories are just human constructs, but belonging to the objective world of thought – World 3 in fact, to use his somewhat unimaginative terminology, to be distinguished from the World 2 of individual thought and consciousness, all of them distinct from World 1 of the physical world. They are provisional. Theories are only 'true' as long as they are not contradicted. This is inductive reasoning according to Poppers interpretation of induction, which most of his critics do not seem to get. It concords beautifully with R. G. Collingwoods' distinction between deductive and inductive logic, the former is compelling the latter is permitting. Now, Popper failed really to consider mathematics seriously, probably because like most modern philosophers, and here I very much include Wittgenstein, he did not know much about mathematics and had certainly done no work in mathematics, which is a prerequisite for understanding mathematics. Thus he tended to exempt mathematics from science. He did not consider it empirical and thus not liable to the fallacies provided by inductive reasoning. He thought of it as an island of pure and incontestable truth, and hence as somewhat uninteresting. But when it comes to the practice of mathematics we know that it is not really deductive, mathematical truths are also products of social consensus. The difference is that traditional truths of mathematics can be challenged just as traditional beliefs are in science *as I have already mentioned*. And just as in science there are objective ways of coming to a verdict. By objective I mean ways that are agreed on prior to their conclusions. It is not like the case of fashion, when one fashion replaces another, the transformation is incontestable. The new fashion simply takes over as a social force trumping the old one who no longer has any say. This is not the way 'truths' are overthrown in science, although the in my opinion over-rated Thomas Kuhn and his theory of paradigm shifts, seem to imply something like that. Popper is clearer on the issue. The change is through a test. A test is not of universal validity, it is simply designed as to be accepted by two warring parties, by finding so to speak the 'biggest common divisor'. This is democratic. Not in the sense of voting, but always seeking and finding common ground. Popper's vision, and as such it is meta-physical and transcendental, is that there is a 'Truth out there' but we humans will only be able to approximate it. Intrinsic to his vision is that when one approximation replaces another this new approximation will be a 'better one'. Science, as a human enterprise is accumulative and progresses. Unlike the humanities and philosophy changes are not random and frivolous. As Kuhn remarks, and here I agree with him, progress is based on repudiation, by closing off certain lines of thought we are, in my words, able to penetrate deeper into the configuration space of ideas. This is how evolution works.

RH: That was quite a mouthful. I thought I was the one being interviewed, not the one who needs to be lectured to. I have also noticed that Popper seemed to ignore mathematics, putting it on a sort of pedestal. But his student Imre Lakatos applied Popperian thinking to mathematics, and profitably too. His writings on mathematics offended the cliques of academic philosophy of mathematics, and so they didn't get the attention they deserved until long after his death.

UP: Sometimes this is an advantage. Your disciples may propagate your ideas and then you do not have to worry about internal consistencies, on the contrary the more inconsistently they are presented, the wider the potential audience. Just think of the case of someone like Marx.

RH: Your jokes do enliven the conversation. Science is not the physical world, as I told you, it is our collective attempt to make sense of the physical world. Your notion seems to be that there has to be an actual *Mathematics* playing the role of the physical world, apart from us, residing in some Platonic heaven. And then apart from that transcendental *Mathematics*, there is also the practice of mathematics, which is the human effort to make sense of the inhuman transcendental *Mathematics*. You are the fly that needs to be led out of the bottle. You are seduced by false analogies. Let's make this discussion a bit more concrete, Does infinity exist?

UP: Existence has so many meanings. You can easily get confused.

RH: That's my point. But you know the meaning of the question, even if it's embarrassing to you.

UP: I agree with you that it is a key question, a kind of litmus test when it comes to the Platonic conception of mathematics. Truly it is very hard to manifest infinity in a physical way. Even if the universe would be infinite, which some cosmologists seem to believe, how would we ever verify it? All I can say is that Gauss did not believe in the actual infinity, only the potential.

RH: Long before Gauss, that goes back to Aristotle. It was Georg Cantor who by one sweeping gesture collected all the integers into one set.

UP: This was a very powerful thing to do.

RH: Infinitely powerful, it would seem. But by that very token, clearly illusory. It's one more example of language letting us reify an act which has only verbal meaning. Take the fact that every number can be doubled, so that there are as many even numbers as there are numbers. This was first noticed by Galileo. In the language of set theory, it gives the surprising fact that a subset of a set can be as numerous as the set itself. But all it really says is that every number can be doubled. And this is actually not so easy, if your number consists of a really very, very great many decimal digits. The notion of infinity is really a negative one, not a positive one. It means that we agree to ignore the boundary of the domain we are studying, it's very far away and we can just ignore it. For example, in theory (but not in practice) we can ignore the fact that when numbers get very large they become very difficult to factor. Or in geometry, what is the Euclidean plane but a very, very large sandbox? So big that we never need to draw a circle so big that it hits the boundary. So we can just pretend that there isn't a boundary at all. In fact, the word *infinity* just means *no boundary*. There is no such thing as an infinitude of riches. Imagining that you have collected all the integers, and calling that imagined collection N, does not enable you to take all the numbers under your control.

UP: It is in fact much harder and much more vertiginous to think of very large finite numbers, you know the number of digits of which takes so many digits to write down that it in itself must be expressed by a number with so many digits and so on a number of times the digits of which, you get the idea …

RH: … I get the idea …

UP: … than to think of infinity itself which is trivial.

RH: It is trivial because *infinity* simply says we wish to ignore the boundary. It simplifies, not to say trivializes. We simply ignore technical difficulties. We sweep them under the rug.

UP: The rug which is infinity and which allows everything to be swept under it. Are we not coming full circle?

RH: There is no need to go full circle. Infinity is just a stratagem to simplify our thinking. Mankind will never reach infinity. Why worry about large numbers we will never reach? Surely there is a number M large enough to delimit the ambitions of all humans. If we want to check something, anything, it would be enough to check it up to that number.

UP: Now you are getting carried away. That number M certainly becomes elusive. It is aptly named by the letter M for being metaphysical. It cannot be manipulated like an ordinary number, because it is a meta-physical number. It cannot be specified, at least not by humans, because if specified and pinned down, so would $M + 1$. You remind me of a boy who thinks that numbers are buttons. Through immense diligence and dedication he collects all the buttons in the world and then he says that adding one is impossible, because after all there are no buttons left to add with. What would you say to that boy? That he should start collecting grains of sand instead like Archimedes?

RH: I would have a long, serious conversation with him. Still you must admit that infinity is a pretty slippery concept. And if you don't think so, it's because you're so used to the concept that you no longer find it strange and contradictory as mathematical innocents find it. On the other hand, you think that those incredibly high cardinals, inaccessible, measurable or whatever they are called, that are thought up by logicians, have a transcendent reality? If so, God chooses strange vessels for his insights.

UP: I must admit that I find those things very fishy indeed.

RH: Yes.

UP: So if you deny infinity you deny that there is any meaning to the notion of an infinitude of primes?

RH: Euclid never said *infinity of primes*. He simply showed how, given any collection of primes, you can construct a new one.

UP: Yet even if you believe in the potential infinity as opposed to the actual, you have some faith in an inexhaustible supply. What you are saying is that there are two levels of existence, one potential and one actual. The former somehow weaker than the latter. You are denying the infinitude of the actual but not of the potential.

RH: There is nothing mysterious about that. Accepting infinity is simply agreeing to ignore complications at the horizon by pretending there is none. The same goes for primes. When it comes down to producing an inexhaustible supply of primes, Euclid's method becomes impractical. Humans can make long lists of primes, but I will never be surprised if every such attempt can be superseded.

UP: But that by itself is a testimony to infinity itself, no matter how many occasions, you will never be surprised. It reminds me that a single counter example to a theorem compels you to reject a potentially infinite number of purported proofs sight unseen, admittedly based on the transcendental faith in the consistency of mathematics.

RH: That is interesting.

UP: Now in analysis you are dealing all the time with infinite sets, especially countably infinite. And think nothing about it. Giving an infinite series, any finite sub sum gives no clue as to whether it is convergent or not, in a sense you need to 'see' all the terms to make sense of it. The same with constructions of Cantor sets and other fractal animals. To stop half-way would leave you with something silly, it is only when you go all the way to infinity those creatures become truly interesting. Now the negative result of the uncountability of the reals is the only thing you need to take into account when you are an analyst. Modern measure theory would be impossible without it. Thus in a sense the countable infinities are actualities for the analyst, while the uncountable of reals is merely potential and in a sense metaphysical. To go beyond this in human mathematical practice is simply pointless, no serious mathematics involves anything beyond the continuum. It might be different would we be able to do arguments involving an actual infinite number of steps, then every theorem in number theory could be verified using case by case study. It would be infinitely boring. In a very literal way to boot.

RH: With some care you could easily do away with those countable actual infinities, which are as chimerical as the set of all integers. But I agree that it would be painful. Infinity is just a shorthand designed for convenience. And as to fractals, their applications to nature are suggestive enough. It's really beside the point that on a physical level those structures can't go on indefinitely. The wonders of infinity can be well approximated.

UP: The idea of infinity is very much connected to the desire for immortality. No one wants to live forever, because eternity is an awful long time, yet everyone would like to postpone dying indefinitely.

RH: Speak for yourself.

UP: The idea of your own mortality is a scary concept, especially when you are young. It does not matter whether you live to a hundred, a thousand and even a million, the very idea that you yourself will at some time be at the brink of extinction is what is terrifying. The hidden assumption, which seems so natural when you are young, is the identity of your 'I' over time. This is no trivial assumption, in fact it begs a lot, as you realize when you start to get a more intimate acquaintance with aging. My point is that mathematical concepts such as infinity ties with some very fundamental existential issues.

RH: That only goes to show what I have been trying to say, namely, that mathematical concepts have no transcendental origin, but are perfectly explainable by the human psyche. As to actual infinity, have you ever come across the name of Tipler?

UP: Did he not co-author a book on the Anthropomorphic principle in Cosmos, to the effect that everything in the universe was fine-tuned to prepare the way for the developments of humans, or at least theoretical physicists. I guess this was just within the boundaries of reputable science.

RH: Whether within or not I don't know, certainly he has gone beyond them in later years. I came across a short article of his on the Internet recently. Using some simple physical principles, such as the indestructibility of information and the eventual evaporation of black holes, he predicted with unassailable logic that we humans would all be downloaded into infinite information traveling at the speed of light, all over the place.

UP: This seems like wishful thinking.

RH: Indeed it is. And he becomes really weird when he claims that this final state will be God, and the Christian God to boot.

UP: This shows a certain lack of imagination. It reminds me of an old idea of mine, namely that the past injects into the future, that no information is lost, that every event no matter how insignificant leaves a tiny trace no matter how elusive and diluted that can be in principle used to reconstruct the event. Otherwise what meaning would there be to say that a thing has occurred in the past, without we having no way of finding out. Psychologically it is easier to imagine that two different causes have the same effect than the same cause having two different effects. It was my way of turning this upside down.

RH: Once again, you are …

UP: … were …

RH: … OK, were the fly in the bottle needing to be led out.

UP: To return to more concrete issues. You recently published a book – Loving and hating mathematics. The very title seems to indicate that your feelings about mathematics are ambivalent.

RH: Aren't yours? Don't you hate it at times?

UP: I guess I have to admit that. I presume that *Loving and Hating Mathematics* is even more focused on the human interaction with mathematics than was *The Mathematical Experience*. Some might say it is gossipy.

RH: I like gossip. Within limits, of course.

UP: This time you co-authored the book with your wife, who is not a mathematician. How did that affect the writing of the book? Was that a major factor in emphasizing the human perspective?

RH: She said, *Let's do something together!* So we had to find a subject that we had in common. In fact, I think it was something that I always wanted to do.

UP: I like to say that you can be very emotional about mathematics, but mathematics offer you no way of expressing your emotions. Maybe this is a clue to the frustration it certainly provokes.

RH: Our book is very much about being emotional about mathematics. What else are loving and hating it? As to not being able to express emotion through mathematics, I am not exactly sure what you really mean by that. I guess to some extent you may be overly influenced by your professed Platonist view of mathematics.

UP: Is it not clear what I mean? Mathematics is completely unconcerned with humans and human emotions.

RH: The standard convention in mathematics is to strictly exclude humans and human emotions from what one writes down. On the other hand, when on occasion someone violates that convention, and their mathematical writing includes something human or even humanly emotional, it often turns out to be very popular and successful!

UP: *But that is exactly my point.* We may leave that topic. Your initial book with Davis was a great success as we have already confirmed. Do you think that this one will be as well?

RH: It's impossible to predict commercial success when it comes to books. If it wasn't, publishing would be so much easier. I could tell you a secret, provided you don't tell anyone, or include it in this interview …

UP: … but if I do not include any names? …

RH: … that might be fine. Anyway, a certain writer published popular columns in a well-known newspaper. When he collected them in a book it was expected to sell very well, but it didn't. As to our latest book, we have participated in a couple of book-signing events here in New Mexico. They were reasonably successful, but we both are known locally. I doubt that we would have such success on a national scale. However, I am trying to enhance the publicity of the book by enlisting U-Tube. If I could get a video on the book propagating on the Web, that would do wonders for its sales.

UP: So you are concerned about the sales of your book?

RH: Don't be so haughty. Just wait until you publish a book. I bet you will find the matter of its sales of utmost importance. Your books are like your children, you wish them every success …

UP: … and your only ticket to immortality?

RH: Speak for yourself. The key is to get a very good video. I had been thinking of using animation, but when it's done by professionals it gets very expensive.[1]

UP: In 'loving and hating' and also in many of your articles you bring up racism in general and anti-Semitism in particular. Is being Jewish very important to you?

RH: Yes and No. I'd like to say No, but there's no getting away from recent history. My memoir on Jews in U.S. mathematics has been chosen for Princeton's next anthology of the best recent articles on math. As I told you, my teen-age ambition was to

[1] In the original version there was a longer digression on this projected video, but as naught come of it, he asked me to delete it as being irrelevant, when I asked for his permission to publish the interview.

fight Hitler. I'm not a Zionist, but my father was. He sent me to Zionist summer camps when I was a child, in order to learn Yiddish, among other things. I resisted his pressure, and learned very little Yiddish. I am very strongly opposed to Israel's policies toward the Palestinians. I have been included on lists of so-called *self-hating Jews*. Concerning religious participation, I feel most at home with Quakers. They try to change the world in a modest and humble way. I very much sympathize with that, even if I often despair. You can't always despair. Sometimes you have to force yourself to be optimistic, to feel that you can make a difference in the world.

UP: As to the notion of races in general and Jewishness in particular, is that not mere social constructs. When it comes to races one can at least try and base it on some objective criteria such as DNA. But that does not work for deciding who is a Jew or not.

RH: Social constructs can have nasty consequences.

UP: I also think that concomitant with the kind of xenophobia we associate with racism there is also a sentimental fascination for exotic elements among your ancestors. Both tendencies can probably be found in many individuals, testifying to the intrinsic inconsistencies in our desires. I myself harbor some hopes that I may have Lappish blood (or more precisely mitochondria). Likewise to follow the historical flow of ancient populations, as reflected in present day DNA or cultural traits is a fascinating exercise, although it has by some been attacked as racism. Then humans are as a species remarkably genetically uniform, supposedly as a consequence of a fairly recent bottle-neck which almost wiped us out. It is noteworthy that any child can learn to speak the prevalent tongue without accent regardless of race …

RH: That was another mouthful. What are you really up to? I thought you were trying to bring up anti-Semitism.

UP: By all means. I am thinking of George Birkhoff. He was notorious was he not?

RH: Not to be unfair, but he was a real bastard.

UP: But he was not alone in the States at the time. Anti-Semitism, if in a relatively milder form was rampant, just as assumptions about the inferiority of the blacks.

RH: That's right, he wasn't alone. Some people were anti-Semitic, some were not. James Alexander, the topologist at Princeton, used his upper-class connections to force the Princeton administration to hire Solomon Lefschetz as a professor of mathematics. Unique in 1923, a Jewish professor in such an Ivy League college. At Columbia, Cassius Jackson Keyser was instrumental in hiring their first Jewish math professor, Edward Kasner. Their example shows that there was a choice whether to be anti-Semitic or not. Of course, Birkhoff had a theory. He explained that Jews mature earlier, and hence gentiles should be protected against them.

UP: It was a kind of affirmative action.

RH: Thank you for another amusing comment. James reports a conversation between Birkhoff and an officer of the Rockefeller Foundation, who noted afterwards, '*B. speaks long and earnestly concerning the Jewish question and the importation of Jewish scholars. He has no theoretical prejudice against the race and on the contrary every wish to be absolutely fair and sympathetic. He does however think that we must be more realistic than we are at present concerning the dangers in the situation and he is privately and entirely confidentially more or less sympathetic with the difficulties of Germany. He does not approve of their methods, but he is inclined to agree that the results were necessary.* No doubt he didn't know that within a few years the results would be the murder of millions of men, women and children, including nearly three dozen of my own cousins. Here's a funny story about Birkhoff that is certified by someone I know who knows someone who was there when it happened. Birkhoff actually was trying to get Rochester to hire a Jewish refugee mathematician. They refused. He replied in anger, *Who do you think you are, Harvard?* You get the joke?

UP: Not really.

RH: For a second-rate university like Rochester, it was pretentious to be anti-Semitic. For an elite institution like Harvard, it was only natural.

UP: It would have been different if he had been Jewish. Then he would have been classified as self-hating and been forgiven as an eccentric.

RH: What on earth are you talking about? Forgiven by who? Are you serious, or just baiting me? But I gladly admit that Birkhoff's anti-Semitism was nowhere near as bad as Hitler's.

UP: If he had been exposed to the Nazi variant he may have changed his views, as many moderate anti-Semitics did after the war. No one has done as much as Hitler as to discredit anti-Semitism. But at what a price!

RH: Yes, we must thank Dear Adolf for that. Birkhoff lived until 1944. So, to be fair, by then he may no longer have been *inclined to agree that the results were necessary*. For all we know, he might have voted against the Holocaust. When Ralph Phillips wrote about Birkhoff's active malignant influence, Saunders Mac Lane, who collaborated with Birkhoff's son Garrett, was sufficiently irritated to write an article in defense of Birkhoff. No surprise – his defense was, *It's not fair to single him out, everybody was like that in those days*. But then, to be fair, maybe MacLane never heard of Keyser or Alexander.

MacLane studied in Göttingen during the 30's. He later reported that he had experienced nothing untoward. It would be honest to write and report that at the time you didn't notice anything wrong, now you know that you were badly mistaken, blind to what was going on. I could respect that. But to still pretend after all those years that nothing was really bad, because you didn't notice it, that's bizarre, to put it politely.

UP: In retrospect, for obvious reason, we tend to emphasize the anti-Semitic elements in early Nazi propaganda. I do not believe, pace Goldhagen, that this was what attracted people to Nazism at the time. Anti-Communism I think was a far more serious factor. I guess that the anti-Semitic rhetoric was more of an embarrassment.

RH: So let's not be angry at those early Nazi-supporting voters, they may really just have been premature anti-communists. Well, to be fair, they got what they wanted, and a little bit more. War against Russia, yes! And the battle of Stalingrad! Destruction of the Reichswehr! Suicide of Adolf Hitler! A communist dictatorship over half of die Heimat! And the murder of my grandparents, whom I never met. Murdered, wantonly and openly, in Vinnitsa, Ukraine, in 1945. If it's not unfair or off-subject to say so.

UP: People cast their votes for all kinds of silly reasons. I would not be surprised that Hitler got votes because he was a vegetarian. Yes supporters, whatever their motivations, obviously have a moral responsibility and there is all the reason to be angry at them. (Angry by the way is a mild word, it holds out the possibility of forgiveness, you may want a stronger.) Yet if you are searching for psychological explanations, it is fully legitimate to look beyond the obvious ones such as anti-Semitism.

RH: Did you know that Nevanlinna was a Nazi?

UP: Osmo Pekonen at Math Intelligencer told me.

RH: Yes, Pekonen wrote about it. That was instructive, and somewhat courageous of him. Nevanlinna was not only a Nazi, he was a Nazi who made up a story claiming he had saved a Jew! You know the story about André Weil visiting Finland, being accused of being a spy, about to be executed, when Nevanlinna saved him?

UP: Yes, I do. I recall being told about it by Ahlfors wife, long before it appeared in print. What about it?

RH: Not to be unfair, it was a lie. Nevanlinna just made it up after the war, to make himself look a little better.

UP: He fooled Weil!

RH: He fooled everybody.

UP: Not Pekonen.

RH: He wasn't even born then.

UP: What is your next project about?

RH: I'm starting to write a biography of my old advisor Peter Lax. What I really want to do is to write his autobiography. To make his life and work really come alive. I don't know whether I'm up to the challenge. I've never done anything like this before.

UP: But you have dreamt of doing it. Come on, you were a budding writer once. Now rise to the occasion. It must be very exciting. You can do much more than you think.

RH: Thank you.

UP: And I would advise you to title the book 'The autobiography of Peter Lax' and have you as the sole author.

RH: That was already done by Gertrude Stein. She wrote *The Autobiography of Alice B. Toklas*.

UP: Maybe we should stop now. You must be exhausted.

RH: I am not. I could keep on talking for ever.

UP: Potentially or actually?

RH: Actually of course.

Building virtual bridges everywhere: A report on the 2020 and 2021 Bridges online conferences

Eve Torrence

The international Bridges Conference is the world's largest conference on mathematical connections in art, music, architecture, and culture. Since 1998, Bridges has traveled to North America, Europe, and Asia, and has attracted participants from over thirty countries.

In a typical year the Bridges conference is an eclectic mix of events ranging from traditional academic talks to a mathematical art exhibition and a fashion show. There are hands-on workshops, a mathematical poetry reading, a short film festival, theater and dance presentations, musical events, and a public Family Day of informal workshops for the local community. Participants include artists, mathematicians, computer scientists, and curious and creative people from many other fields. Academics and non-academics alike are welcome and comfortable in this rich environment of collaboration.

The Bridges conferences were born from the Art and Mathematics conferences held annually by mathematician and sculptor Nat Friedman at the State University in Albany, New York from 1992–1997. Carlo Sequin was one of many attendees who found the 1997 conference "particularly stimulating and contagious". Carlo was inspired to host the 1998 Art and Mathematics conference at the University of California, Berkeley. Many people were clearly excited about the possibilities for such meetings, as a total of six math and art conferences were held in 1998, including the first Bridges conference organized by Reza Sarhangi. Nat formed the International Society of the Arts, Mathematics, and Architecture (ISAMA) in 1998 and held the first ISAMA conference in San Sebastian, Spain in 1999.

With Reza's endless energy and warm personality, the Bridges conferences quickly became an annual gathering of people interested in exploring the rich interdisciplinary links between mathematics and the arts. The annual Bridges Proceedings, which contains a paper for every talk presented at the conference, became an important record of mathematics and arts research. In the Preface to the first Bridges Conference Proceedings Reza wrote, "A major reason for developing the Bridges Conference and this collection of papers is our desire to come together from a diverse set of apparently separated disciplines, to share and recognize abstract similarities, common patterns, and underlying characteristics".

While the Bridges Proceedings are an important collection of work, there are many restrictions that must be imposed on the papers given that the Proceedings are produced and published before the conference each year. A desire to allow more time to develop and refine articles and to publish art and mathematics research year-round led to the establishment of the Journal of Mathematics and the Arts (JMA) in 2007. JMA is not a Bridges publication, but many of the same people who love Bridges have been instrumental in founding, editing, and writing for JMA.

Reza organized the first five Bridges conferences at his home universities, Southwestern College in Winfield, Kansas (1998–2001) and Towson University in Maryland (2002). In 2003 Bridges became an international gathering when the meeting was held jointly with ISAMA at the University of Granada in Spain. After one more meeting in Winfield in 2004 the growing popularity of Bridges led to invitations to hold the conference at institutions across the globe. The 2005 meeting in Banff, Canada was followed by meetings in England, Spain, the Netherlands, Hungary, Portugal, Korea, Finland, Sweden, and Austria, with periodic meetings in North America scattered in between.

The 2020 conference was scheduled to be held at Aalto University in Helsinki and Espoo, Finland. The pandemic forced a cancellation of the in-person conference and a virtual version of the conference was quickly developed. Authors were encouraged, but not required to submit short videos or links to websites on their papers. The poetry reading was also assembled from contributed videos. Discussions were possible in the comments for each author's page, but this exchange was a poor substitute for in-person interactions. The 2020 conference website, with links to all these resources is permanently available at https://2020.bridgesmathart.org.

The Bridges conference Proceedings from every year are available for free at http://archive.bridgesmathart.org. The 2020 Proceedings contains papers relating mathematical topics such as topology, geometry, knot theory, combinatorics, and algebra to a huge variety of art forms, from fiber arts to poetry. Scrolling through the Proceeding one can find a wealth of exciting ideas to explore. From papers on experiencing geometric spaces such

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as *Non-Euclidean Billiards in VR*, by Jeff Weeks, and *Dancing the Quaternions*, by Karl Schaffer, to fiber arts like *Folding Fabric: Fashion from Origami*, by Uyen Nguyen, and *Knotty Knits are Tangles in Tori*, by Shashank G Markande and Elisabetta Matsumoto. Topology is explored in papers such as *Topological Classification of Vittorio Giorgini's Sculptures*, by Daniela Giorgi, Marco Del Francia, Massimo Ferri, and Paolo Cignoni, and *Maximizing the Symmetry of Knots,* by Peter Alexander Generao and Carlo H. Séquin. It is easy to spend hours roaming these enticing papers. You may find a paper like *Wallpaper Patterns from Nonplanar Chain Mail*, by Frank Farris or *Hilbert's Portrait via his Space-Filling Curve*, by Judy Holdener, is the perfect way to introduce your students to these topics.

As in other years, the pieces from the Exhibition of Mathematical Art are available at gallery.bridgesmathart.org/exhibitions. A beautiful color glossy catalog of the art exhibition is also published annually. But nothing can substitute for the experience of wandering the gallery and experiencing these eclectic displays in person. We also greatly missed the rich exchange of ideas possible when talking to each artist about their work.

A wide variety of artistic media were represented in the 2020 exhibition, including drawing, painting, beadwork, weaving, ceramics, woodwork, stained glass, metalwork, quilting, paper cutting and folding, and 2D and 3D digital prints. Artists drew inspiration from the mathematics of fractals, polyhedra, non-Euclidean and four-dimensional geometry, tiling, knot theory, number theory, and more. The incredible range of materials, techniques, and concepts is a visible display of the breadth and depth of mathematics and its applications. Professional artists exhibit their work alongside mathematicians and others with less formal artistic training. There is also some outstanding work by students. The result is a fascinating display that can be enjoyed by people with a huge variety of knowledge of art and mathematics.

A small example of the variety of media, techniques, and mathematics from the 2020 exhibition can be seen in the images in Figure 2. Ulrich Mikloweit is an artist who makes gorgeous polyhedra based paper sculptures cut and assembled entirely by hand. His 2020 contribution was a portrayal of the three stellations of the dodecahedron. Kerry Mitchell is a digital artist who uses his sophisticated knowledge of mathematics to produce beautiful designs. Judy Holdner is a mathematician who works in many media. Her 2020 contribution was a portrait of Hilbert using a 3D printed Hilbert curve. Of course, it wouldn't be 2020 without at least one abstract sculpture of a virus. Kacper Dobras and Briony Thomas created this colorful rendition of a polio virus.

Videos from the annual Bridges Short Film Festival are also posted at gallery.bridgesmathart.org/exhibitions. The creative diversity of the eight 2020 contributions is impressive. For example, *Spatial Variants of a Propeller*, by the Kocaeli Team, shows the filmmakers' inspiration for and development of a kinetic sculpture based on an iconic propeller displayed in Kocaeli, Turkey. George Hart's *Warped-Grid Jigsaw Puzzles* shows how to design complex puzzles using algorithms and transformations. The results are stunning works of art that can be assembled over and over. *The Arts of the Finite Topology Conjecture*, by Katrin Leschke, Chloe Aligianni, Lee Boyd Allatson, Jenny Hibberd, and Andrew Johnston, explores a collaboration between a mathematician, two musicians and a dancer and their interpretations of this conjecture. It is just a taste of what was clearly an exciting and rich experiment and is a great example for mathematicians and artists interested in such partnerships.

Figure 1. Bridges participants on the polyhedral climbing structure in the Mathematical Garden of the Tekniska Museet in Stockholm, Sweden (Equirectangular projection of spherical photograph by Henry Segerman)

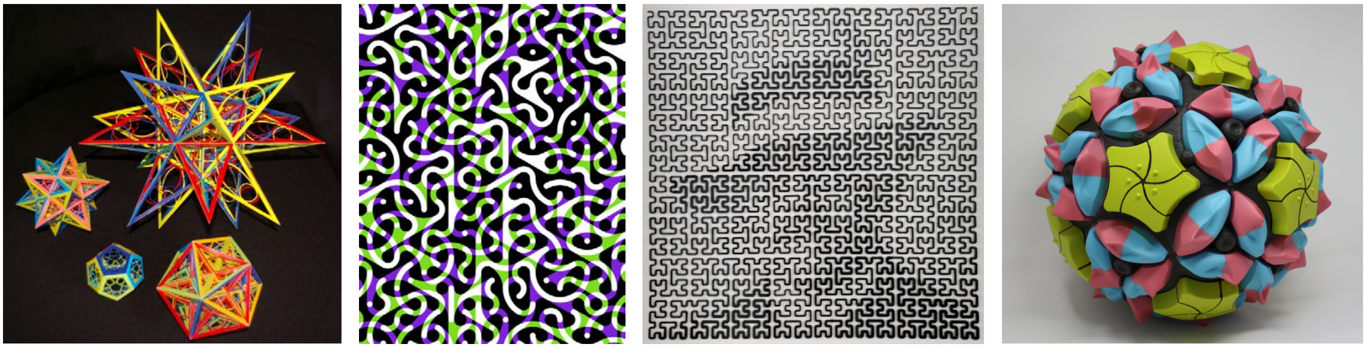

Figure 2. (a) *Four Dodecahedra*, by Ulrich Mikloweit, Freelance Artist, Germany. (b) *Truchet Bugaloo*, by Kerry Mitchell, Artist, USA. (c) *Hilbert*, Judy Holdener, Professor of Mathematics, Kenyon College, USA. (d) *PolioMechanics 1.0*, by Kacper Dobras, Research Assistant, and Briony Thomas, Lecturer in Design Science, School of Mechanical Engineering, University of Leeds, UK

The annual Bridges Mathematical Poetry reading is organized by Sarah Glaz. In 2020 this popular event was enacted via a collection of videos of poets reading their work and can be accessed through the 2020 virtual conference page https://2020.bridgesmathart.org under the Poetry Reading heading. The written versions of the poems can be found in the printed *Bridges 2020 Poetry Anthology*. Links to how to purchase this anthology and the three previous volumes are listed on the Poetry Reading page. The poems have multi-dimensional connections to mathematics and a wide range of styles, from traditional to lyrical and visual interpretations of this medium. 2020 titles include *I Forgot the Turnkey to the Void* by Carol Dorf, *The Mathematician's December*, by Sarah Glaz, *A Mother's Math is Never Done*, by Gizem Karaali, *How Taylor Series can Resonate on a First Date* by Lisa Lajeunesse, and *Singularity* by Mike Naylor. Mathematical Poetry is the perfect way to soothe and inspire a world in the midst of a pandemic.

Among the few live events held over Zoom for Bridges 2020 were four 90-minute workshops. Free registration was required to obtain the Zoom code and over 200 people from around the world signed up. Many first-time attendees were excited to be able to participate in a Bridges workshop without the expense of travel. Stephen Erfle and Katherine Erfle showed how to use Excel to explore a rich collection of symmetric patterns in *Exploring Symmetry using Aestheometry in Classrooms and Beyond*. This workshop would be a wonderful way to inspire students to explore the relationships between geometry, algebra, and simple number theory. In *A Two-Dimensional Introduction to Sashiko*, by Carol Hayes and Katherine Seaton, participants learned the history of this traditional Japanese needlework and the underlying mathematics as well has how to design and construct these pieces. Participants could experiment with needle and thread or pencil and paper. In António Araújo's workshop, *Dürer Machines Running Back and Forth*, he introduced his method for drawing anamorphic images. Attendees cheerfully shared photos of their drawings taken from the exact viewpoint that displays the perspective illusion. All the workshops generated wonderful discussions and were much needed opportunities for social interaction among Bridges friends new and old.

A few, less formal live events were also organized. These were listed under Social Events and Informal Gatherings and included a short meeting to exchange ideas about math and dance and a social hour. The last Social Event was an online version of Informal Music night, a campy mix of singing, dancing, and other performances by Bridges participants. In recent years the final act has been a conference song written and performed by Doug Norton. A video of Doug singing this year's rendition can be seen at http://2020.bridgesmathart.org/ under the heading "Wrap Up". The lyrics, sung to the tune "Rudolph the Red Nosed Reindeer," give a sense of the fun and creative environment of a Bridges conference:

> We've had Bridges Linz, Waterloo, Banff, and Alhambra,
> Towson, Jyväskylä, London, and Coimbra.
> But what will Fate decree for bizarro year twenty-twenty?
> Welcome to Bridges Nowhere! There's no need to rent a room.
> No need to book that airfare: sign up for your space in Zoom.
>
> They've wrapped up the Proceedings
> (though we didn't quite proceed):
> Lots of great math/art reading; entertainment guaranteed!
> Knotty knits and trefoil knots, steganography,
> Virtual reality, fractal cohomology,
> Fashion-fold origami, labyrinths and spiroplots,
> Lampshade Miura-ori, Morton's tritangentless knots.
>
> Orbifolds and gyrations, tiles dendritic and Truchet,
> Girih and pied-de-poule ones, hyperbolic plane crochet.
>
> Coptic bananas, heptagons,
> perhaps the plaintive numbers flow;

Hallå STEAM, platonicons, derivision, sgraffito.
Aalto, Espoo, Helsinki, Otaniemi.
Plans wrecked by COVID-19;
Unforeseen, quarantine, more hygiene, please, vaccine!
Very keen to reconvene, back on routine,
Back to the live math-art scene:

Beauty beyond perfection, augmented reality,
Math and art intersection, unveiling infinity!

The paradox of this pandemic is that we've learned how important face-to-face interaction is for learning and the exchange of ideas, yet we have been able to achieve surprisingly successful events online. The creative growth that this pandemic forced has led to many innovations.

For Bridges 2021 we are planning a greatly expanded virtual conference. There will be many more live events, including paper presentations, social events, a Mathematical Art Exhibition opening, workshops, and of course, Informal Music night. We will be using virtual spaces to host interactive events and to allow for more personal exchanges. For example, we will try to approximate the experience of the art exhibition with a virtual exhibition that participants can "walk through" and talk to the artists. Time will be scheduled for participants to discuss papers and ask questions of authors.

We anticipate a large international gathering. This is a wonderful chance for people to experience the Bridges conference from home and to increase awareness of the exciting current research in the growing field of mathematics and the arts. Information about how to register and attend is available at bridgesmathart.org.

It is with great hope that we plan to finally host Bridges Aalto in person in 2022. But for 2021, we hope to see you virtually!

Eve Torrence is a professor of mathematics at Randolph-Macon College in Virginia, USA and a member of the Bridges Organization Board of Directors. She is the author of *Cut and Assemble Icosahedra: Twelve Models in White and Color* (Dover). Eve co-edited the 2016 and 2018 Bridges Conference Proceedings with her partner, Bruce Torrence. Other Torrence and Torrence collaborations include *The Student's Introduction to Mathematica and the Wolfram Language* (Cambridge), Mathematics Awareness Month 2014, and the raising of two wonderful children. Eve enjoys designing mathematical sculpture and incorporating the arts into teaching mathematics.

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Figure 3. An audience enjoys the outdoor debut of "Witches of Agnesi" at the 2018 Bridges conference (Photo by Bruce Torrence)

Institute of Mathematics of the Czech Academy of Sciences

Jiří Rákosník and Miroslav Rozložník

The Institute of Mathematics of the Czech Academy of Sciences, located in the very centre of Prague with a small group of researchers working in Brno, is the leading research institution in mathematics in the Czech Republic. The mission of the Institute is to foster fundamental research in mathematics and its applications and to provide the necessary infrastructure. In cooperation with universities, the Institute carries out doctoral study programmes and provides training for young scientists.

The Institute was established in 1947 as the Institute for Mathematics of the Czech Academy of Sciences and Arts (Česká akademie věd a umění). The initiator and the first director was Eduard Čech. In 1953, the Institute was reorganized and incorporated into the newly established Czechoslovak Academy of Sciences. In 1993, when Czechoslovakia split, the Institute became a part of the newly established Czech Academy of Sciences. In 2007, the Institute, together with 53 other institutes of the Czech Academy of Sciences, was transformed into a public research institution. This status provides much broader autonomy, especially in research and personnel policy, but still under public control as the vast majority of funds come from public sources. Institutional funding forms about 60 % of the resources and is provided by the Czech Academy of Sciences based on regular evaluation of research quality. About 35 % of the revenue is earned through competitions for grant projects, and 5 % results from economic activities related mainly to the publication of research journals.

The Institute currently employs around 90 researchers. Almost half of them are foreigners, with more than 20 nationalities. Postdoctoral fellows and PhD students represent more than 25 % of the research staff. All researchers are hired in open competitions for 2–5 year contracts with the possibility of further extension based on a successful personal evaluation. Unfortunately, getting good PhD students is complicated by the legislation that allows the Institute to be involved in their education only in conjunction with a university. This means that a number of students are coming from abroad. The infrastructure is supported by 20 staff members who provide the services of the library, IT, project support, editorial office, administration and management.

Research

The research strategy of the Institute is based on bottom-up activities that are supported, encouraged and guided by the management in close cooperation with the Board of the Institute. The International Advisory Board is asked for advice on important decisions.

The fields of research include those connected with the best tradition of Czech mathematics as well as newly developed areas. The traditional fields are inherently connected with the founding members and strong personalities of the Institute such as Eduard Čech (Stone–Čech compactification in topology, Čech cohomology, Čech closure operator), Jaroslav Kurzweil (Henstock–Kurzweil integral, stability theory for ordinary differential equations), Ivo Babuška (theory of finite element method, Ladyzhenskaya–Babuška–Brezzi condition), Jindřich Nečas (regularity of generalized solutions of elliptic equation, theory of elasto-plastic bodies in continuum mechanics), Miroslav Fiedler (Fiedler algebraic connectivity in graph theory, Fiedler vector in linear algebra and matrix theory), and Vlastimil Pták (Pták topological vector spaces, Pták subtraction theorem and the notion of the critical exponent of iterative processes).

Following the high standards set by these distinguished personalities, research teams have been cultivating the traditional and strong mathematical disciplines while also opening new research directions. Current research focuses on mathematical analysis (differential equations, numerical analysis, functional analysis, theory of function spaces), mathematical logic and logical foundations of computer science, complexity theory, combinatorics, set theory, numerical linear algebra, general and algebraic topology, category theory, optimization and control theory, algebraic and differential geometry and mathematical physics.

The research at the Institute is organized in five departments that are described in the following paragraphs.

Abstract Analysis

Originally called Department of Topology and Functional Analysis, this department represents a continuation of one of the traditionally strong research directions in the Institute. Under the leadership of Wieslaw Kubiś, the team recently reassessed their research focus.

The emphasis has shifted from the traditional topics of the theory of Banach spaces, operator theory, classical topology and functional analysis to those areas where mathematical logic plays a significant role, even though it is not the main object of study, namely descriptive set theory, algebraic topology, category theory, and the theory of C*-algebras. For this reason, the department has been renamed Abstract Analysis.

Several team members are currently involved in the prestigious EXPRO project of excellence funded in 2020–2024 by the Czech Science Foundation and lead by Wieslaw Kubiś. The project aims to explore and classify generic mathematical objects appearing in the above-mentioned areas of abstract analysis.

Algebra, Geometry and Mathematical Physics

Formed in 2014 on a bottom-up initiative of several members of other teams, this department steadily grows and continuously proves to be one of the most successful within the Institute. They investigate algebraic and differential geometry and closely related areas of mathematical physics. Their research focuses on mathematical aspects of modern theoretical physics, mathematical models aiming at understanding the nature of matter, fields, and spacetime. Research topics include representation theory and its applications to algebraic geometry, homological algebra, algebraic topology, applied category theory, tensor classification, mathematical aspects of string field theory, generalized theory of gravitation, and study of Einstein equations.

The team achieved excellent results in the theory of gravity, analytical solutions of Einstein equations, and modified theories of gravity. Using their conformal-to-Kundt method, Vojtěch Pravda and Alena Pravdová with their colleagues from the Charles University identified and studied several classes of new static spherically symmetric vacuum solutions of the field equations of modified gravity, including a new non-Schwarzschild black hole. This discovery attracted widespread attention and was even reported in the media. Martin Markl and his collaborators achieved the ultimate result on loop homotopy algebras in closed string field theory and constructed the disconnected rational homotopy theory. In 2018, he received the Praemium Academiae award of the Czech Academy of Sciences, connected with generous funding that allowed him to hire several talented postdocs and establish his own ambitious research group.

Evolution Differential Equations

The research of this department focuses on theoretical analysis of complex multi-field evolution processes in physics, in particular continuum mechanics and thermodynamics. Special attention is paid to the description of interacting phenomena of different physical natures, such as biological systems, stratified or viscoelastic fluids, contact mechanics between fluids and solids or between rigid, elastic, or elastoplastic solids, fluid diffusion in deformable porous media, electric and magnetic effects in moving solids and fluids, magnetohydrodynamics, liquid crystals, hysteresis, thermal effects and radiation, or temperature-induced phase transitions in a large parameter range. The systems under consideration are based on physical laws of conservation of mass, momentum, energy, balance of entropy, including also energy exchange principles between mechanical, thermal, and electromagnetic energy in multifunctional materials.

Eduard Feireisl, the principal investigator of the ERC Advanced Grant MATHEF (Mathematical Thermodynamics of Fluids), 2013–2018.

An outstanding achievement was the ERC Advanced Grant MATHEF (Mathematical Thermodynamics of Fluids) awarded to Eduard Feireisl in 2013–2018. He and his collaborators built a complete mathematical theory describing the motion of compressible viscous heat-conducting fluids, including aspects of stochastic forcing and construction of convergent numerical schemes. The novel and original approach to the interpretation of the principles of continuum thermodynamics in modelling heat-conducting fluid flow turned out to be a rich source of results for the general theory, as for example, the concept of dissipative measure-valued solutions. Further essential results concerned well-posedness, regularity and stability of the Euler system and similar partial differential equations, including the construction of a stable finite volume scheme and proof of its convergence via dissipative measure-valued solutions.

The team members are involved in the Nečas Center for Mathematical Modeling, a research platform established by the Institute, the Charles University and the Institute of Computer Science of the Czech Academy of Sciences with the ambition of coordinating and supporting research and education activities in the theoretical and applied mathematics, particularly in the field of continuum mechanics. They are also active in the network for industrial mathematics EU-MATHS-IN.CZ (part of the European network EU-MATHS-IN).

Mathematical Logic and Theoretical Computer Science

The research programme of this department concerns mathematical problems arising from theoretical computer science, logic, set theory, finite combinatorics, and control theory. The main topics

Tomáš Vejchodský, Director of the Institute, in the promotional video presenting the cooperation with the company Doosan-Bobcat EMEA, youtube.com/watch?v=_I2KN-z_fo4.

studied by its members include proof and computational complexity, logical foundations of arithmetic, quantum information theory, graph theory, and set theory. The problems studied have foundational importance in themselves, and potentially also practical applications, for example in data security.

In the area of the logical foundations of mathematics, the team is one of the world's leading centres of research in bounded arithmetic and proof complexity. Computational complexity is a discipline with a short history that has only recently been recognized as an important field not only in computer science but also in mathematics. It is also due to the fact that fundamental questions in this domain (e.g. the famous "P versus NP" problem) belong to the set of mathematical problems which resist being solved for decades. Pavel Pudlák's group attacks these problems using methods of mathematical logic. He believes that the reason why we cannot answer these questions is fundamental in nature, and therefore their logical aspects should be studied. The research domain in which he and his colleagues work and have already reached important results is called proof complexity. While computational complexity deals with how difficult it is to compute something, proof complexity asks how difficult it is to prove it.

Pavel Pudlák, the principal investigator of the ERC Advanced Grant FEALORA (Feasibility, Logic and Randomness in Computational Complexity) in 2014–2018.

Numerical Analysis

Following a decades-long tradition, this department investigates both theoretical and practical aspects of computational science, mainly numerical methods for partial differential equations and numerical linear algebra, whereas classical and strong areas have been complemented with new research topics. Its members focus on questions of convergence, efficiency, and reliability of numerical methods for partial differential equations, including matrix computations and high-performance implementations on parallel computer architectures. Members of the team led by Michal Křížek are experts in the finite element method, saddle-point systems, preconditioning, domain decomposition methods, rounding error analysis, high-performance computing and computational fluid dynamics.

The team is involved in the Nečas Center for Mathematical Modeling and in the network for industrial mathematics EU-MATHS-IN.CZ. It has succeeded in competitions for the CPU time at large European computers and cooperates with the IT4Innovations National Supercomputing Center of the Technical University in Ostrava.

Members of the five above-mentioned departments organize a dozen regular seminars and about the same number of international workshops and conferences. Around 150 foreign researchers visit the Institute every year. In 2016, the Institute established Eduard Čech Distinguished Visitor Programme with the aim of significantly enhancing its creative environment by attracting highly distinguished mathematicians for a longer period of time. One visitor is selected every year to deliver a series of lectures and to essentially develop scientific collaboration with researchers in the Institute. The visitor is also expected to deliver the prestigious Eduard Čech Lecture for the general mathematical community.

Other activities and service to the community

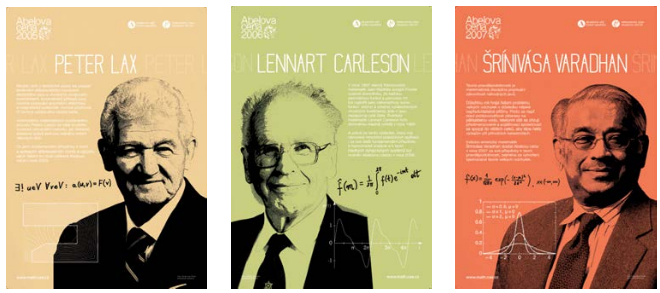
Although the emphasis is on fundamental research, attention is also paid to connections with applications. The Institute is involved in the Strategy AV21 programme "Hopes and Risks of the Digital Era" run by the Czech Academy of Sciences. The role of the Institute is to develop mathematical models for engineering applications. The Institute cooperates on a long-term basis with the Innovation Centre of the company Doosan Bobcat EMEA, the renowned producer of compact loaders and excavators.

Students during the Open House Days and the exhibition of Imaginary posters demonstrating the beauty of mathematical surfaces.

Close attention is paid to the popularization of mathematics. Public lectures in the annual Open House Days used to be attended by more than a thousand visitors, mostly high-school students. The restrictions connected with the Covid-19 pandemic inspired us to create a webpage for students and the general public presenting various mathematical problems, popular lectures and other interesting materials like posters celebrating the laureates of the Abel Prize.

The Institute publishes three mathematical journals. The Czechoslovak Mathematical Journal and Mathematica Bohemica are continuations of Časopis pro pěstování mathematiky a fysiky (Journal for Cultivation of Mathematics and Physics) established in 1872. The aim of these two journals is to publish original research papers of high scientific quality in all fields of mathematics. The third journal, Applications of Mathematics, specializes in mathematical papers directed at applications in various branches of science.

The Institute also provides several services for the wide mathematical community and public. Its library, with almost 100,000 volumes including 35,000 monographs and 1,300 journal titles, is the largest public mathematical library in the country. Since 1996, the Prague editorial Group cooperates with zbMATH to produce metadata and reviews of mathematical publications. Since 2009, the Institute has been developing the Czech Digital Mathematics Library (DML-CZ, https://dml.cz) with the aim of digitizing, organizing and archiving the relevant mathematical literature published throughout history in the Czech lands, and providing free access to metadata and full texts. DML-CZ currently includes 17 journal titles, proceedings of 8 conference series, and about 300 books. The Institute is a member of the international consortium that has developed the European Digital Mathematics Library (EuDML, https://eudml.org).



Posters presenting the winners of the Abel Prize.

The well-being of the Institute employees and their work-life balance is supported in various ways. There is a tradition of cultivating a friendly atmosphere and an effort to approach and comply with individual needs of employees. The main objective of the currently running project "Institute of Mathematics CAS goes for HR Award – implementation of the professional HR management" is to improve the stimulating and attractive work environment in the Institute and to apply for the HR Excellence in Research Award (known as the HR Award) granted by the European Commission.

To learn more about the Institute, please visit the webpage www.math.cas.cz.

A group photo of members of the Institute at the annual bike trip, July 30, 2020.

Jiří Rákosník obtained his CSc (PhD equivalent) in 1980 at the Charles University in Prague. Since then, he has been working in the Institute of Mathematics of the Czech Academy of Sciences, in the position of the director in 2014–2019. His research interests focus on the theory of function spaces. For 25 years, he has been active in the digitization of mathematical literature, in close cooperation with zbMATH, and in building the Czech Digital Mathematics Library and the European Digital Mathematics Library. He serves as the current Secretary of the EMS.

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Miroslav Rozložník graduated in Mathematics in 1992 and obtained his PhD in Applied Mathematics in 1997 at the Faculty of Nuclear Sciences and Physical Engineering of the Czech Technical University in Prague. In 1998–2000, he worked on a postdoctoral fellow position at the Swiss Federal Institute of Technology (ETH) in Zürich. He is the author or co-author of one book and more than 40 journal publications. His research interests include numerical linear algebra, saddle point problems, parallel computing and rounding error analysis. From 2001 he was a research fellow at the Institute of Computer Science of the Czech Academy of Sciences until 2017, when he moved to the Institute of Mathematics, where he currently serves as the deputy director.

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The International Association of Applied Mathematics and Mechanics

Jörg Schröder

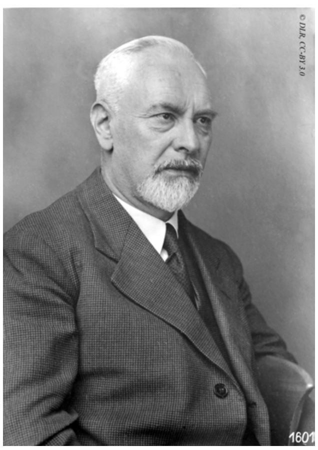
The Society of Applied Mathematics and Mechanics (GAMM, "Gesellschaft für Angewandte Mathematik und Mechanik") has its roots in the joint meetings of the German Mathematical Society, the German Physical Society, and the German Society for Technical Physics. It was founded in 1922 by Ludwig Prandtl and Richard von Mises. Following the motivation of the founding fathers, our scientific organization encourages the international cooperation of applied mathematics with all subfields of mechanics and physics, which are among the foundations of engineering sciences. Thus, GAMM promotes the scientific development of applied mathematics and mechanics and has been able to contribute significantly to progress in hydro- and aerodynamics, solid mechanics, numerical mathematics and mathematics for industrial applications. The society has an international orientation and today comprises about 1400 members.

The foundation of GAMM is closely related to the foundation of the Journal of Applied Mathematics and Mechanics (ZAMM, "Zeitschrift für Angewandte Mathematik und Mechanik") by Richard von Mises in 1921. Motivated by the economic situation after the First World War, engineers, among others, displayed a particular responsibility for the reconstruction of Germany, which is reflected in the versatile activities of the VDI, the Association of German Engineers. Committees for technical mechanics and physics were formed in some of the district associations, which devoted themselves to topics such as the calculus of differences and vector calculus as well as their applications in engineering, elastic and inelastic deformations for special constructions, and a logarithmic integration device. A finding in the meeting of the board of directors of the VDI on September 19, 1920 stated: "It is astonishing how simple methods [vector calculus, difference calculus, conformal mappings] can be used to solve many technical problems (Es ist erstaunlich, auf wie einfachem Wege […] man viele technische Probleme hierdurch lösen kann.)" (Z.-VDI 65, 1921, page 54). Although the VDI had already been publishing research papers in the form of individual booklets for 19 years, it was of great importance to the Committee for Mathematics and Mechanics to continuously publish short critical reports on current topics in addition to sporadically appearing extensive original papers. The foundation of ZAMM, which began to appear in 1921, was based on this foundation. The guiding principle of Richard von Mises is particularly noteworthy here:

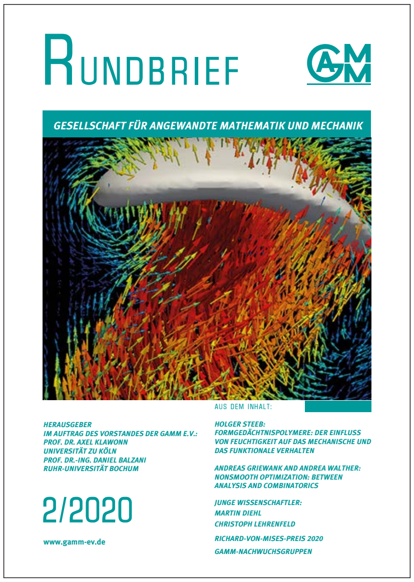
> To overcome the boundaries between pure mathematics and the application of mathematical theories, especially in the engineering sciences.

This policy was and is of great importance for our interdisciplinary acting scientific community. The 100th anniversary of ZAMM was taken as an opportunity to produce a series of selected publications highlighting the developments since the beginning of the journal. The first article [2] in this series recounts the beginnings of ZAMM.

GAMM regularly initiates annual meetings at different locations in Germany and nearby European countries. A first meeting, or rather, an informal gathering with representatives of hydro- and aeromechanics, took place in Innsbruck in 1922. In 1925, GAMM held its first scientific meeting in Dresden. There were annual meetings until 1938, which in the following years took place only under difficult and limited conditions until 1943. After the Second World



Left: Richard von Mises (April 19, 1883 – July 14, 1953).
Right: Ludwig Prandtl (February 4, 1875 – August 15, 1953).



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War, GAMM resumed its activities in 1950 with the Darmstadt meeting. This meeting is of particular importance because a new foundation of GAMM was not necessary due to the merger of the societies of the English and American zones. A broad spectrum of keynote lectures on the disciplines represented in GAMM and the Ludwig Prandtl Memorial Lecture organized jointly with the German Aerospace Society (DGLR, "Deutsche Gesellschaft für Luft- und Raumfahrt") since 1957 are complemented at the annual meetings by mini-symposia on current developments in applied mathematics and mechanics and further sections with short lectures. The sections are dedicated to particular fields and offer the opportunity, especially for younger participants, to introduce themselves and their work to the scientific community. An outline of the history of GAMM up to the 1970s can be found in [1].

GAMM establishes GAMM activity groups ("GAMM Fachausschüsse") on request of its members for current research topics, aimed at further differentiation and specialization of scientific issues. They were established in the early 1950s and today contribute significantly to developing important future areas of applied mathematics and mechanics. There are currently 17 activity groups, which largely shape the scientific activities of the society outside the annual meetings. For this purpose, they organize seminars and workshops, participate in the organization of large national and international conferences, and prepare statements on particular problems from the point of view of the respective committee. The activity groups are established for a term of 11 years and are evaluated twice during this period.

Since 1989, GAMM has annually awarded the Richard von Mises Prize for outstanding scientific achievements in applied mathematics and mechanics. The prize promotes young scientists and includes a certificate, a free two-year membership, and prize money. Traditionally, this award is presented during the opening session of the GAMM annual meeting, at which the awardees prominently present their research results in a keynote lecture.

Thanks to the legacy of Dr. Klaus Körper, the Dr. Klaus Körper Foundation was established in 2011. It awards four prizes annually for the best dissertations of the past year in applied mathematics and mechanics.

An important task of GAMM is to recruit new members and to support these young scientists in building their careers. The GAMM Juniors deserve special mention in this context. They have established themselves as an essential part of our organization. For example, since 2017, a YAMM (Young Academics Meet Mentors) Lunch has been organized at our annual meeting. Here, young academics have the opportunity to ask experienced scientists questions about career paths and opportunities. Furthermore, in 2021, the GAMM Juniors organized a pre-GAMM event for the first time as part of our annual meeting. This event was a great success in preparing young scientists for the keynote lectures of our 91st annual meeting.

The GAMM Student Chapters, introduced in 2018, have taken their first steps and form another important building block for new impulses to make GAMM attractive and sustainable. Currently, eight Young Investigator Groups strengthen interdisciplinary collaboration and exchange in applied mathematics and mechanics between master students, Ph.D. students, and scientists. They organize a variety of activities ranging from barbecue parties to workshops and excursions.

In addition to the already mentioned ZAMM, the GAMM-Mitteilungen with scientific contributions are published quarterly, the

GAMM-Rundbrief with generally understandable editorials and socio-political information is published semi-annually, the PAMM (Proceedings in Applied Mathematics and Mechanics) with contributions of the GAMM annual meetings is published once a year, and recently also a GAMM student journal has been established.

In my view, today more than ever, the sciences have a responsibility to society as a whole to provide adequate advice to other institutions through knowledge-based findings. Interdisciplinary work and a lively exchange with other scientists are a necessity for these purposes. GAMM meets this challenge through its continuous development and the commitment of its status groups.

For further information on GAMM and our activities, please visit the website www.gamm-ev.de.

References

[1] H. Gericke, "50 Jahre GAMM". Beiheft zum *Ingenieur-Archiv* **41** (1972)

[2] R. Siegmund-Schultze, "The joy that engineers and mathematicians have come together." Richard von Mises' foundation of ZAMM, and its "tasks and goals" (1920/21). *ZAMM Z. Angew. Math. Mech.* **100**, e202002017, 26 (2020)

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Pop Math: find math everywhere!

Andreas Matt and Roberto Natalini
Raising Public Awareness Committee of the EMS

Communication of Mathematics is growing. Some years ago, communication was done by writing some books, giving a few public talks and preparing the occasional exhibit, or in rare cases, opening a mathematics museum. Most of these efforts were aimed at young people, and very few serious academic mathematicians were interested in taking part in these activities.

Nowadays, outreach activities are essential in every math department or scientific institution, and even outside the world of academics one can find dedicated mathematics outreach organizations or professional math communicators. Promoting math, and more generally STEM disciplines, is considered not only useful but crucial for the future development of the field and for the educational and economic growth of different countries. It has become something of a fundamental duty for all mathematicians.

Today, apart from an increasing production of books and public talks, there is a flourishing of new ideas: hands-on museums of mathematics, interactive exhibitions, theater shows and movies, web sites and YouTube channels, podcasts, documentaries, graphic novels, and even activities related to dance and other performing arts.

As the Raising Public Awareness Committee of the European Mathematical Society, we have been observing, registering, and even taking part in this trend for many years, and we have long been aware of the difficulty in creating a true European or international network connecting these initiatives. All too often, similar initiatives are started in different countries from scratch, there are few opportunities to share the different experiences, and it is very difficult to have an overview of what is happening in other regions. Even a simple database recording all math awareness events across Europe was missing.

For all these reasons, as a committee, we decided to start a sort of map/calendar on which all the outreach events in mathematics in Europe could take their place. It was based on similar ideas of interactive maps with events used by the international non-profit organization IMAGINARY and the International Day of Mathematics. The big difference with our idea is that it is intended to be a joint effort and an open calendar, so that everybody – without even the need for a user account – can easily publish and promote their own mathematics outreach events. And of course, everybody should be able to use the calendar to locate ongoing and upcoming events around their own city!

But just as this project was under construction, the world was hit as we all know by the pandemic, and we had to stop and rethink how to restructure this map/calendar to take into account the big changes that were taking place everywhere. Suddenly, all the face-to-face events that we would have liked to collect, advertise and archive had disappeared, replaced, maybe temporarily, by dozens of online events on various platforms, often with participants dispersed in various corners of the world.

It was certainly a dramatic turning point, and even activities to promote mathematics had to quickly adapt to this new situation, which in spite of all the negative consequences we faced also brought surprising new opportunities. In just a few months, everybody's habits have changed; now, connecting from Europe to follow an event in Australia or the United States has become completely normal. Naturally, our project had to take all this into account.

Finally, in January 2021 we came out with our new website dedicated to the collection of outreach events in Europe and the rest of the world: Pop Math www.popmath.eu

The website displays, on a world map, the announcements of all mathematics outreach events for a general audience, as well as academic or professional events concerning mathematics outreach. The events can be commercial or non-profit, online or in-person, and have a limited duration. To feed the database, we decided to

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rely on the math community. Everybody who wishes can submit an event from this page: www.popmath.eu/submit-event

While the main goal is still the creation of a virtual place to find and share all the events of mathematical outreach in a simple way, Pop Math is also an archive of everything happening in the field, since it has the ability to preserve links to all past events; this is particularly useful for the online events. You will be surprised by the amount and type of events on Pop Math: from an international mathematics engagement conference organized in collaboration between Paris, New York and Berlin to talks in local languages on numbers, mathematics and climate or artificial intelligence, from large national math festivals to new traveling exhibitions. And some events are also physical, or even hybrid, combining the real and the virtual worlds.

In the future, and with the collaboration of the mathematical community, we aim to improve our initiative in various directions. One of the possible new features we hope to implement is the creation of an applet to embed our map in every (third party) site, by giving some customization choices to the systems managers to take into account some regional realities. Another direction would be to create the first directory of all individuals and groups active in math outreach, so as to aid and simplify the organization of local events and to assist people in academic institutions to improve their communication of math to every type of audience.

We hope everyone will be able to take advantage of, appreciate and contribute to Pop Math!

—

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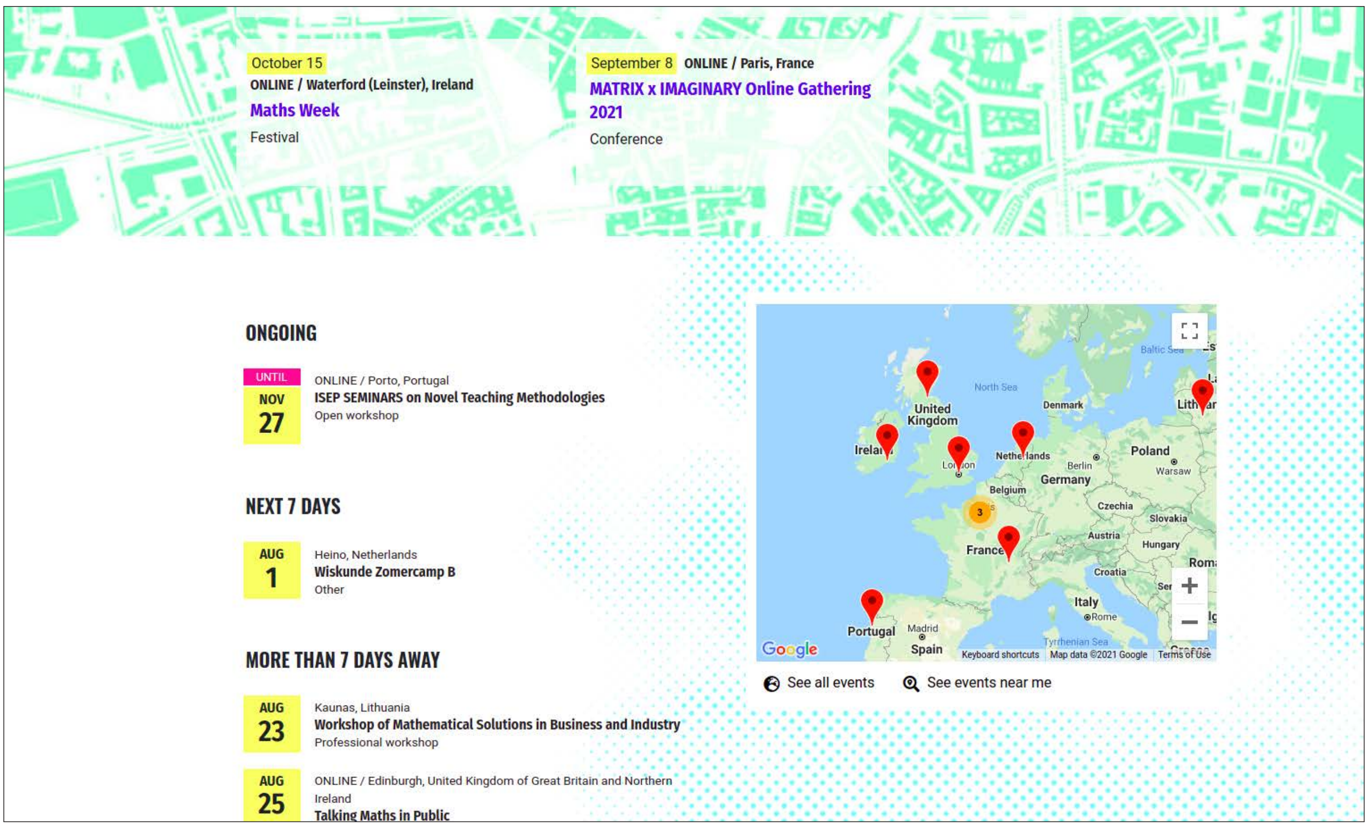

ICMI column

Susanne Prediger

Connecting from far and near – a successful hybrid ICME conference

ICME, the International Congress on Mathematical Education, is like the Olympics of worldwide mathematics education conferences, taking place only every four years and gathering more than 3500 participants from all over the world.

Due to the pandemic, ICME 14 in 2020 had to be postponed, and even in July 2021 it could only take place in a hybrid mode. Mainly the Chinese colleagues were on site, while the rest of the world was online. The ICMI EC was grateful that at least our brave ICMI president, Frederick Leung, took on himself the burden of a two week quarantine to be able to attend the congress in Shanghai in person.

The hybrid mode allowed 3988 participants from 129 countries to participate in the congress, among them about 1100 participants from China. A program with seven plenary activities, 65 invited lectures, four survey teams and five lectures of ICME awardees offered substantial input on mathematics education research and mathematics education practices. Within the 62 Topic Study Groups, 1817 accepted presentations were provided, which shows the wide and active participation that shapes this conference.

Although the hybrid mode made informal conversations very difficult and forced many people to work during strange night hours, we are grateful for the wonderful organization by the Chinese local organizing committee and the international program committee, which led to a very successful conference.

Additionally, the hybrid mode even had some interesting advantages. More than 1700 participants from underrepresented countries were able to participate with waived fees, and this was a fantastic opportunity to include many more researchers from underrepresented countries than the usual solidarity fund is able to finance!

My personal highlights of the conference were the following five, but this is of course completely subjective:

- The fact that country representatives from banned countries could take part in the assembly of ICMI country representations for the first time – even if only creative technical VPN solutions made this important participation possible.
- Cédric Villani's plenary lecture in which he attracted the listeners' attention and fascination to the nature of mathematical research practices for a full hour without relying on any visualization aids like powerpoint. Cédric Villani is an ideal person for initiating deep and insightful communication between the fields of mathematics research and mathematics education!
- Early career researchers from all over the world who worked so hard in our ECRD workshop and asked such substantial questions.
- Lingyuan Gu's plenary lecture in which he presented a 45 year-long research and development project for Mathematics Teaching Reform in Shanghai. It massively helped to understand how China was able to develop so quickly from a third world country into one of the most dynamic STEM-led economies. The ambitious mathematics education reform program played a major role in this, with the sincerity and intensity of collaboration between teachers, teacher educators and mathematics education researchers who connected experimenting and developing curricula and teacher education in an impressive way.
- The multiple occasions in plenary panels, TSGs and workshops in which we discussed equity challenges and potentials to strengthen equity via mathematics education. The diverse voices from all over the world made clear that issues of equity have a great many different faces, but they are a highly relevant concern in nearly every country and will continue to challenge us as a scientific community.

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ERME column

regularly presented by Jason Cooper and Frode Rønning

In this issue, with a contribution by
Laura Black, Anette Bagger, Anna Chronaki, Nina Bohlmann and Sabrina Bobsin Salazar

ERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME), holds a biennial conference (CERME), in which research is presented and discussed in Thematic Working Groups (TWG). We continue the initiative of introducing the working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. Our aim is to enrich the ERME community with new participants, who may benefit from hearing about research methods and findings and contribute to future CERMEs.

Introducing CERME's Thematic Working Group 10 – Social, Cultural and Political Aspects of Mathematics Education

Group Leaders: Laura Black, Anette Bagger, Anna Chronaki, Nina Bohlmann and Sabrina Bobsin Salazar

This working group discusses mathematics and mathematics education within the realms of the cultural, the social and the political. TWG10 builds on the premise that engaging with mathematics is always more than an encounter between an individual and a mathematical object, whether it be in a classroom, a workplace or a university setting. Instead, it views such encounters as shaped and produced by wider cultural and societal contexts that are inherently social and political. As such, the group addresses questions such as:

- How is mathematics valued in society?
- How does mathematics act in society?
- Who is mathematics for?
- What are legitimate sources of reference?
- Who decides what is legitimate?
- What mathematics should be taught and learned in schools and universities?
- Who decides what mathematics is taught in these contexts?

As an example, in a recent paper [5], Maheux, Proulx, L'Italien-Bruneau and Lavallée-Lamarche question the traditional view of seeing "professional mathematics" as the reference by which all other forms of mathematics are judged. Instead, the authors suggest considering school mathematics as an alternative reference for what mathematics is.

TWG10 began in 2004 at CERME 3 with a focus on teaching and learning mathematics in multicultural classrooms. De Abreu, Grogorió and Boistrup [1] reported that this stemmed from an interest that was gathering momentum in the mathematics education community at the time, due to the increased levels of migration in European countries, contributing to increased diversity in classrooms. Consequently, the group has a long-standing interest in diversity in relation to mathematics from a number of angles:

(1) diversity as expressed in terms of the attributes of people who engage with mathematics (either professionally or in classrooms), such as gender, ethnicity, language, socio-economic status, social class, (dis)abilities, and so on;
(2) diversity in terms of ways of perceiving the world and giving structure to it, such as aspirations, worldviews, ideologies, school systems, and governance structures;
(3) diversity in relation to the variety of sites where doing mathematics takes place, such as schools and universities but also homes, workplaces, after-school organisations, communities; and finally,
(4) diversity in relation to who is doing the research and who is being researched, posing methodological issues of an ethical nature. Clearly this generates a degree of openness in terms of what counts as mathematics and mathematical thinking, but it also recognises the need to adopt an inclusive approach in considering who has the right to do, think, and learn with mathematics.

Whilst culture is viewed as central to thinking, doing, learning and teaching mathematics, more recently "the socio-political turn" in mathematics education has come to the fore [2]. TWG10 adopts a critical perspective in recognising that mathematics may be used in ways that reproduce existing power relations in the world as weapons of capitalism, but crucially, and perhaps more significantly, the group is also focused on how mathematics can be used to challenge or disrupt activities, events, and practices that produce social inequalities. This draws on work in the field of critical mathematics education which has been central to developing teaching approaches that utilise mathematics in ways that help learners understand their own struggles against oppression and injustice. See for instance the work of Gutstein [3, 4] and colleagues on teaching mathematics for social justice in the US.

The work of TWG10 is characterised by a strong openness to perspectives and methods that are not yet established within the field of mathematics education, and is interdisciplinary in that it draws on broader fields in social and political theory, anthropology and cultural studies. This adds to the agenda outlined above by identifying innovative and creative ways to understand, critique and address issues of social inequality in relation to mathematics, and proposes creative openings and new imageries. The work of the group also emphasises an ethos of reflexivity, as noted in TWG10's group introductory report for CERME 11: "Research in this group is characterised by an effort to reflect on its own double role in not only analysing but also shaping the possibilities of seeing and inventing mathematics education practices" [6]. This all adds up to a body of work that is both critical and architectural – it both questions the elite gatekeeping function of mathematics (particularly in education, but also in wider society) and also highlights alternative practices and ways of being that position mathematics as progressive rather than restrictive.

References

[1] G. de Abreu, N. Gorgorió and L. B. Boistrup, Diversity in mathematics education. In *Developing research in mathematics education*, edited by T. Dreyfus, M. Artigue, D. Potari, S. Prediger and K. Ruthven, Routledge, London, 211–222 (2018)

[2] R. Gutiérrez, The sociopolitical turn in mathematics education. *J. Res. Math. Educ.* **44**, 37–68 (2013)

[3] E. Gutstein, Teaching and learning mathematics for social justice in an urban, Latino school. *J. Res. Math. Educ.* **34**, 37–73 (2003)

[4] E. R. Gutstein, "Our issues, our people—Math as our weapon": Critical mathematics in a Chicago neighborhood high school. *J. Res. Math. Educ.* **47**, 454–504 (2016)

[5] J.-F. Maheux, J. Proulx, R.-A. L'Italien-Bruneau and M. L. Lavallée-Lamarche, Referring and proffering: An unusual take on what school mathematics is about. In *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11, February 6–10, 2019)*, edited by U. T. Jankvist, M. van den Heuvel-Panhuizen and M. Veldhuis, Utrecht University and ERME, Utrecht, 1920–1927 (2019)

[6] H. Straehler-Pohl, A. Bagger, L. Black, A. Chronaki and D. Kollosche, Introduction to the work of TWG10: Diversity and Mathematics Education: Social, Cultural and Political Challenges. In *Proceedings of the Eleventh Congress of the European Society for Research in Mathematics Education (CERME11, February 6–10, 2019)*, edited by U. T. Jankvist, M. van den Heuvel-Panhuizen and M. Veldhuis, Utrecht University and ERME, Utrecht, 1819–1822 (2019)

Laura Black is professor of education at the University of Manchester in the UK. She has been involved in TWG10 since CERME 9, and has been a member of the leading team since CERME 11. Her research interests focus on learner identities in mathematics and their relationship with social class, gender, and ethnicity and family aspirational narratives. She is particularly interested in how mathematical identities are mediated by pedagogic practice(s), the production of capital exchange (using Bourdieu) and the use and exchange value attached to mathematics.

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Sabrina Salazar is a tenured faculty member at the Mathematics and Statistics Department at Universidade Federal de Pelotas in Pelotas, Brazil. She is interested in studying teaching practices and their relationship with reproduction and rupture of social institutions in different levels of schooling. She also studies how such practices can be learned by novice teachers in their initial preparation and practicing teachers in continuing professional development programs.

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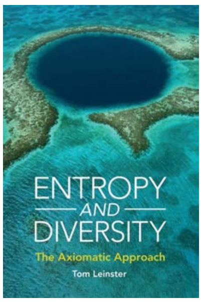
Entropy and Diversity: The Axiomatic Approach by Tom Leinster

Reviewed by Stefan Forcey



At least since the recording of the Noahic directive "two of every kind", humans have instinctively felt that extinction is loss. Because the viable forms of DNA-based life are developed by such arduous processes, it seems like a profound waste of hard-won information to let species become extinct. Furthermore, we know that there are many dependencies between relatively unrelated taxa: for instance birds may depend on berries, or insects, or fish, and humans depend on all the above. Due to habitat loss, pollution and climate change, the potential for a catastrophic domino effect appears to be increasingly imminent. It could be wished that there was not such a clear need for a book like Tom Leinster's new book *Entropy and Diversity: The Axiomatic Approach*.

However, in times of natural or artificial disaster, we may be forced into triage situations. With finite resources at our disposal, we may be compelled to decide which biome or which species to save, and measurements of biological diversity help make that decision. Greater diversity means less chance of cascading loss, at least as a first approximation. Further justifications are found in the study of drug resistant bacteria, as well as the study of human gut health, as Dr. Leinster points out in the introduction. Recent experience demonstrates that sometimes we may even desire to minimize diversity, as in the example of virus mutations that have the potential to be worse than their common ancestor.

The thesis of *Entropy and Diversity* is that by beginning with the list of properties we wish our diversity measurement to obey, we can often completely describe the function or family of functions that will fit those requirements. This is what is called the axiomatic approach. Showing a tight relationship between structure and properties allows the author to review a long list of measurements of diversity (and inversely, entropy) to demonstrate how they are specializations of general principles. At another level, we see that both the contribution of a single individual and the diversity of a community are special cases of a concept of *value* that is axiomatically determined. Furthermore, many mathematical invariants measuring size (such as cardinality, volume, surface area, fractional dimension, and Euler characteristic) arise from a single concept, of a general invariant called the *magnitude of an enriched category*. It is shown that this magnitude is closely related to maximum diversity: indeed in some cases they are precisely the same.

Perhaps the most distinctive new contribution here is Leinster's work (with C. A. Cobbold) on defining a family of diversity measures that depends on both the relative abundances of the species in a population and the pairwise differences between them. The similarity (or dissimilarity) between two species can be measured in many different ways: genetic, phylogenetic, or functional. By defining the value of a species to be its expected similarity (on average) to a randomly chosen individual from the population, it is shown that Leinster and Cobbold's diversity measures are special cases of an aggregate value function, which also captures the Hill numbers and the phylogenetic diversity of Chao, Chiu and Jost. Not only does the new family of diversity measures respect the similarity matrix (finite metric), and obey the desired properties, it also has the surprising feature of being simultaneously maximizable. Given a similarity matrix, it is shown that the entire family of diversity measures is maximized for a single probability distribution of the species in that ecosystem. The common maximum is yet another invariant measurement, but one which measures (the magnitude of) the metric itself.

In the early chapters, Dr. Leinster motivates and explains the basic problem of deciding how to measure biological diversity, and covers the steps of solving an equation to find a missing function. Then he begins answering those questions with an exposition of Shannon entropy from information theory. Deformations and relative versions of entropy are also covered, each with its corresponding inverse concept of diversity. The central chapters introduce the concepts of mathematical size, value, means and magnitude, and relate them back to the special cases of diversity measures. Along the way, there is a chapter on using probabilistic methods to solve functional equations. Finally there is a nice axiomatic characterization of information loss, discussion of entropy modulo a prime number, and the promised deep dive into the category-theoretical foundations.

Entropy and Diversity is a thorough presentation of the mathematics of measuring diversity, including many new results of uniqueness, unification and utility. Beyond the practical value, it is also a display of mathematical art. Beautiful patterns, at deeper and deeper levels of abstraction, are exhibited to clarify the simplicity of what at first might appear to be abstruse formulas. Leinster approaches the subject like a craftsman, paying attention to every detail. The book is over 450 pages long, but it is so nicely organized and readable that I felt immediately drawn in rather than intimidated. The book is directly accessible to a general audience comfortable with mathematical reasoning. It will be a valuable reference for both mathematicians and mathematical ecologists. The new material has already engendered a lot of discussion on future directions, as can be seen in some recent online conversations:

- johncarlosbaez.wordpress.com/2011/11/07/measuring-biodiversity
- golem.ph.utexas.edu/category/2020/12/entropy_and_diversity_the_axio.html

Tom Leinster, *Entropy and Diversity: The Axiomatic Approach.* Cambridge University Press, 2021, 458 pages, Paperback ISBN 978-1-108-96557-6, eBook ISBN 978-1-108-96217-9.

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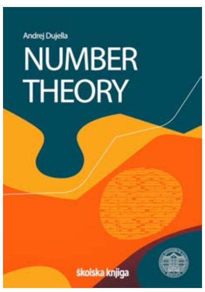
Number Theory by Andrej Dujella

Reviewed by Jean-Paul Allouche



As a student of Number Theory, I really appreciated the famous book of G. H. Hardy and E. M. Wright, while some of my friends frequently mentioned the book of Z. I. Borevich and I. R. Shafarevich. Of course, even these two books did not cover the whole (huge) field of Number Theory, and several other excellent books could be cited as well. More recently, many books have been devoted to (parts of) this vast field whose characteristic is to be both primary and not primary (pun intended). The meaning of the expression "Number Theory" itself has changed over time – partly because the domain has exploded – to the point that some contemporary authors now refer to "Modern Number Theory" …

A very recent book, entitled *Number Theory* and based on teaching materials, has been written by A. Dujella. Devoted to several subfields of this domain, this book is both extremely nice to read and to work from. It starts from primary results given in the first three chapters, ranging from the Peano axioms to the principle of induction, from the Fibonacci numbers to Euclid's algorithm, from prime numbers to congruences, and so on. Chapter 3 ends with primitive roots, decimal representation of rationals, and pseudoprimes. Chapters 4 and 5 then deal with quadratic residues (including the computation of square roots modulo a prime number) and quadratic forms (including the representation of integers as sums of 2, 4, or 3 squares). Chapters 6 and 7 are devoted to arithmetic functions (in particular, multiplicative functions, asymptotic behaviour of the summatory function of classical arithmetic functions, and the Dirichlet product), and to the distribution of primes (elementary estimates for the number of primes less than a given number, the Riemann function, Dirichlet characters, and a proof that an infinite number of primes are congruent to ℓ modulo k when $\gcd(\ell, k) = 1$). Chapter 8 deals with first results on Diophantine approximation, from continued fractions to Newton approximations and the LLL algorithm, while Chapter 9 studies applications of Diophantine approximation to cryptography (RSA, attacks on RSA, etc.). Actually two more chapters are devoted to Diophantine approximation, Chapter 10 (linear Diophantine approximation, Pythagorean triangles, Pellian equations, the Local-global principle, …) and Chapter 14 (Thue equations, the method of Tzanakis, linear forms in logarithms, Baker–Davenport reduction, …). Chapters 11, 12, and 13 deal with polynomials, algebraic numbers, and approximation of algebraic numbers. The book ends with Chapters 15 and 16 which cover elliptic curves and Diophantine problems.

This quick and largely incomplete description clearly shows that this book addresses many jewels of number theory. This is done in a particularly appealing way, mostly elementary when possible, with many well-chosen examples and attractive exercises. I arbitrarily choose two delightful examples, the kind of "elementary" statements that a beginner could attack, but whose proofs require some ingenuity, namely the unexpected statements 4.6 and 4.7:

Example 4.6. Let $p > 5$ be a prime number. Prove that there are two consecutive positive integers that are both quadratic residues and two consecutive positive integers that are both quadratic nonresidues modulo p.

Example 4.7. Let n be an integer of the form $16k + 12$ and let $\{b_1, b_2, b_3, b_4\}$ be a set of integers such that $b_i \cdot b_j + n$ is a perfect square for all i, j such that $i \neq j$. Prove that all numbers b_i are even.

The book also comprises short historical indications and 426 references. It really made me think of my first reading of Hardy and Wright, and I almost felt regret that I cannot start studying Number Theory again from scratch, but using this book! I highly recommend it not only to neophytes, but also to more "established" scientists who would like to start learning Number Theory, or to refresh and increase their knowledge of the field in an entertaining and subtle way.

Andrej Dujella, *Number Theory.* Textbook of the University of Zagreb, Školska knjiga, Zagreb, 2021, 621 pages, translated by Petra Švob, ISBN 978-953-0-30897-8.

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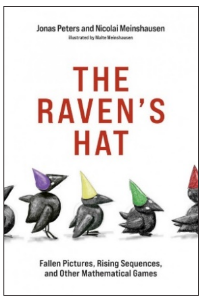
The Raven's Hat by Jonas Peters and Nicolai Meinshausen

Reviewed by Adhemar Bultheel



This book introduces, with some variations, eight mathematically flavoured games or puzzles. As the authors accurately explain in their preface, the type of problems they present look at first sight almost impossible to solve. It is only after a careful analysis, reducing it to a formal (say mathematical) reformulation, that it becomes clear that a solution strategy can be designed that is in some sense even optimal. Each time, the discussion of the solution to a problem is taken as an excellent pretext to explain some piece of mathematics. A reader with a minimal mathematical background will learn what a Hamming code is, what a cyclic group is, and some elements of linear algebra, probability, and even broader topics such as information theory, projective geometry, and algebraic topology. What starts as a playful game with a seemingly impossible solution becomes, after placing it in an appropriate mathematical context, relatively easy to solve. Moreover, by isolation and abstraction of the essentials, it becomes simple to consider more general situations. A better advertisement for the power of mathematics and a stronger motivation to study mathematical formalism and mathematical structures can hardly be found.

The book opens with a classic game, which is also for the title of the book. So let's take this as an illustration of the concept used by the authors throughout. Consider three players (in this book the players are adorable ravens illustrated in graphics by Malte Meinhausen). Each player has a blue or red hat on his head. They see the other players but they cannot see the colour of their own hat. All players have to guess their own hat's colour (red, blue, or don't know) without communicating with each other. The players, as a team, win the game if at least one player is correct and none is wrong (where the don't know answer is not considered wrong). There exist innumerable variations of this type of game, generally called "hat-problems". The present problem, which asks for a successful collective strategy, was originally formulated by Todd Ebert in 1998. It was Elwyn Berlekamp who later connected the solution to coding theory and solved it for n players when n is of the form $2^m - 1$. The solution for general n is still an open problem today. This kind of (brief) historical background of the problem is also given for the problems presented in the other chapters, which is a nice feature of the book. To work towards solving the problem, the authors first propose some guessing strategies or naive trials, possibly introducing a first formalism such as coding the red-blue colours as 0-1 and the configurations of three hats as binary numbers from 000 to 111, with the first bit for player 1, the second for player 2, and the last for player 3. This shows that there are a total of 8 possible states, and each player knows 2 of the 3 bits in the true number and must guess the bit of her own position. The probability of winning for the 3 players is maximized if each player chooses her own colour (0-1 bit) so that the "distance" to one of the $8 = 2^3$ possibilities is minimal. The crux is to define this distance as the Hamming distance, which is the mathematical contribution to the solution in the form of coding theory. Once the principle is clear, it is easy to generalize the solution to $n = 2^m - 1$ players.

This problem involves some elementary probability, and probability is also an ingredient for several of the other problems discussed in this book (both of its authors are professors of statistics). Several variants of the game correspond to the following description: a set of players needs to guess something on the basis of partial information that is available to them, and the goal is to agree on a strategy that will maximize their chance of winning the game as a team. For example, in the second game of this book, the n players have their name hidden in n boxes, and they have to find the box with their own name in a minimal number of trials for the whole team. Here the mathematical tool is the factorization of a permutation into cycles. In the existing literature, the players are often presented as prisoners that are collectively freed if they win. In this book, the stories vary, but all the illustrations portray ravens with hats.

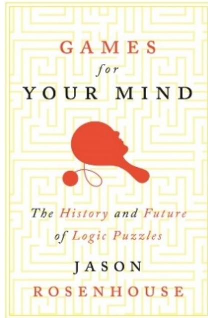
Let me skim more quickly over the other chapters. Somewhat related to the two hat-problems mentioned above is a problem where there are more colours for the hats and where the players are lined up in such a way that each player can only see the other players (and their hats) who are positioned in front of them. This seems to contain strictly less information than when they can all see each other, but when the players start guessing the colour of their own hat one after another, starting with the one at the back, the extra information is provided. Here it is cyclic groups that come to the rescue to solve the problem. Another game is a magical card trick which is based on understanding correctly how much randomness is introduced in an ordered card deck by riffle shuffling or cutting the deck. While the previous problems involved simple counting, a bit of probability, and basic algebraic concepts, there is no way around introducing the logarithm when information theory is involved. Not that the logarithm is an advanced topic, but still it requires some understanding of mathematics beyond counting. A further game introduces some projective geometry with a pinch of graph theory. Somewhat of a different nature is the problem involving two globes placed randomly on the table, in which one is asked to find the place or country that are in the same (or opposite) position on the two globes. This means that the axis defined by the position of that place on globe 1 and the globe centre should be parallel to the corresponding axis for globe 2. For this problem, one needs to realize that there is a mathematical one-to-one match of the two globes by a translation (to match up the centres) and a single rotation. The axis of the latter gives the solution. This requires the introduction of some linear algebra to see that the problem is a simple eigenvalue problem for a 3D rotation matrix. The final game is, once again, a classic, involving hanging a picture on a wall. If there is one nail on which the frame string can be hooked, then the picture will fall when the nail is removed. The game is to wind the string of the frame around two (or more) nails in such a way that if one nail is removed, the frame will not drop to the floor. Here the naive reader is confronted with group theory and algebraic topology to find a general solution.

For an inexperienced puzzler, the problems look challenging at first sight, so she is gradually guided by the authors towards a solution with several variants of naive trials, pointing at the shortcomings, and just enough mathematics is used to deal with the problem at hand. If, in a top-down approach, the mathematics were to be introduced first with the problem solution as an a posteriori application, this approach would not have the same motivating effect. A simple problem that has a hard solution leads to mathematics that not only solves the original problem, but that can immediately be applied to generalizations and other variants of the problem. For readers who are attracted to solving puzzles and games, but who have a weak (or no) mathematical background, the authors provide several appendices with additional explanation of notation, binary and complex numbers, converging sequences, probability, etc. as well as some extra details on specific problems discussed in the chapters. This is an engaging book that all puzzlers, and certainly novices to the field, will love.

Jonas Peters and Nicolai Meinshausen, *The Raven's Hat.* MIT Press, 2021, 192 pages, Paperback ISBN 978-0-262-04451-6.

Games for Your Mind by Jason Rosenhouse

Reviewed by Adhemar Bultheel



If you ask a connoisseur where to look for logic puzzles, then she will almost certainly mention Raymond Smullyan (1919–2017), and perhaps also Lewis Carroll (1832–1898) and some of the columns of Martin Gardner (1914–2010). Obviously, this kind of puzzle has existed since antiquity, but these three names are certainly among those that popularized the topic as we know it today. Jason Rosenhouse is a mathematics professor at James Madison University. He has written a book on the *The Monty Hall Problem* (Oxford, 2009), he is also the co-author of one entitled *Taking Sudoku Seriously* (Oxford, 2011), and he is a co-editor of three volumes on *Mathematics of Various Entertaining Subjects* (Princeton, 2015–2019); with all this, he has gained quite a reputation in the gamers-and-puzzlers community. This book will considerably add to his authority.

In his preface, Rosenhouse mentions that his original idea was to write a book with some entertaining logic puzzles à la Carroll and Smullyan. However, the actual result is a book that goes well beyond a mere collection of entertaining brain teasers. It explains the mechanisms and principles needed to solve the puzzles, but it also instructs the reader about the history of logic and its interpretation by different philosophical disciplines. Some example puzzles are solved and discussed, and just as in a mathematics textbook, some solved exercises are inserted to illustrate the theory. After each principle is explained, a list of puzzles is given for the reader to solve. They feel like exercises after the lesson; solutions are given at the end of each chapter.

It is clear that logic influences and interacts with mathematics. Deciding what is true and what is not, means deciding what to accept as being proved or not. This is done by checking the rules that were followed to arrive at the result. Therefore, one has to agree on what rules should be followed and what axioms one should start from. This quickly results in a discussion about the foundations of mathematics and philosophical considerations. Thus, while the puzzles as such are challenging but entertaining, it is also necessary to assimilate some background for which a leisurely scanning of the text is insufficient; it really requires staying focused while reading.

Rosenhouse divides the book into five parts: (1) A general introduction to logic and puzzles, (2) Lewis Carroll and Aristotelian logic, (3) Raymond Smullyan and mathematical logic, (4) Puzzles based on nonclassical logics, (5) Miscellaneous topics. The titles of the first four are self-explanatory. First, in part one, some general considerations about logic are given as well as some sample puzzles to whet the appetite. Logic is boringly simple in everyday life, but when philosophy is involved, it needs more precise definitions of its atoms, called propositions, and one must understand the mechanism of categorical syllogisms if one wants to explore puzzles based on Aristotelian logic.

Parts 2 and 3 are the main parts of the book. In the part about Lewis Carroll, a short introduction of Aristotle's syllogism serves as an introduction to a discussion of Carroll's book *The game of logic* (1886), in which he used certain diagrams to visualize the syllogism that solves the puzzles. In his book *Symbolic logic* (1896), Carroll solves sorites puzzles, meaning that one must deduce a conclusion from more than two categorical propositions. Rosenhouse explains how Carroll did this with more analytical techniques and tree graphs. Finally, the book discusses two journal papers that Carroll published in *Mind*. The first one involves if-then propositions and *modus ponens* and *modus tollens* arguments. The other paper is a regression problem. If p and $p \to q$ are true, then one would normally conclude that q is true, using $(p \,\&\, (p \to q)) \to q$. But what if we do not accept the latter form of the *modus ponens*? A naive solution is to add it to the system, but that sets into action a recursive addition of propositions ad infinitum like in Zeno's paradox of Achilles and the tortoise. Rosenhouse gives an extensive discussion about how this problem has been tackled by different authors.

In part 3, we leave syllogisms and move on to newer forms of formal logic, with Smullyan as the main puzzler. Smullyan's puzzles often involve knights (who always speak the truth) and knaves (who always lie). Some examples of this type of puzzle are given with a bit of formal logic in an appetizing chapter. This is however then followed by several chapters on the history of logic, ranging from Aristotle through John Locke, George Boole and John Venn, to the formal system of Bertrand Russell and Kurt Gödel's incompleteness theorems. There is less room for puzzles in these chapters; however, some elements can still be illustrated in puzzle form, typically the Smullyan type puzzles in which we meet people who may be either knights or knaves. In this type of problem, one usually is tasked with asking a question which reveals either who is what or what is true. In several puzzles where the problem is to find out whether p is true, the appropriate question to ask has the form: Is p true if and only if you are a knight? Smullyan attributes this principle to Nelson Goodman.

Part 4 gives an introduction to several more recent forms of logic. For example, three-valued logic is involved if people can be knights, knaves or neutral. Or probability can be involved in fuzzy logic if people are not either knights or knaves, but can be picked from a continuum between the two extremes, so that they answer questions truthfully or not according to a certain probability distribution.

Finally in the last part, several further topics are discussed. One is the *Hardest Logic Puzzle Ever* published by George Boolos in 1996, which has attracted a lot of academic interest, becoming a bit of a legend with a life of its own. This puzzle involves three gods: one answers with a lie, one with the truth, and one answers randomly. You do not know who is who and their answers are "da" and "ja", but you do not know which word means "yes" and which one means "no". The problem is to find out how many questions you need to ask, and which ones, in order to discover who is who. Other topics in this part are paradoxes and so called metapuzzles (in which some extra hidden information can be deduced). The concluding chapter gives some examples from fiction in the form of film and literature where some logic is involved. It ranges over a broad spectrum from Mr. Spock in Star Trek to the famous super-intelligent detectives Auguste Dupin, Sherlock Holmes, Hercule Poirot, and many others who solve crimes using logic. An appendix contains a very useful glossary of definitions of logic-related terms, along with an extensive index.

Even though this book is entertaining and addressed to a lay public, it goes well beyond a mere popularizing puzzle book, situated somewhere between entertainment and an introductory course in logic. The part on Gödel's theorems is, for example, not just entertaining but quite a good explanation of the problem for an interested lay reader, including a Turing-like machine. The puzzles presented range from simple to extremely difficult. In most cases, the origin of the puzzle is mentioned. They are usually formulated as stories, in imitation of the initiator of the genre, Lewis Carroll, who always formulated his puzzles for children. Lovers of the Smullyan puzzles or Lewis Carroll will be happy to read the background material compiled in this book. Conversely, it may also introduce readers to Smullyan's many puzzle books. Moreover, extending the type of underlying logic, Rosenhouse opens a door to more general types of entertaining puzzles. To raise interest in logic is a good thing not only for everyday life or for mathematics, but also for computer science, where it plays in important role in topics such as logic programming and its use in machine learning.

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Solved and unsolved problems

Michael Th. Rassias

The present column is devoted to Game Theory.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

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We consider a setting where there is a set of m candidates

$$C = \{c_1, \ldots, c_m\}, \quad m \geq 2,$$

and a set of n voters $[n] = \{1, \ldots, n\}$. Each voter ranks all candidates from the most preferred one to the least preferred one; we write $a \succ_i b$ if voter i prefers candidate a to candidate b. A collection of all voters' rankings is called a *preference profile*. We say that a preference profile is *single-peaked* if there is a total order $\lhd$ on the candidates (called the *axis*) such that for each voter i the following holds: if i's most preferred candidate is c and $a \lhd b \lhd c$ or $c \lhd b \lhd a$, then $b \succ_i a$. That is, each ranking has a single 'peak', and then 'declines' in either direction from that peak.

(i) In general, if we aggregate voters' preferences over candidates, the resulting majority relation may have cycles: e.g., if $a \succ_1 b \succ_1 c$, $b \succ_2 c \succ_2 a$ and $c \succ_3 a \succ_3 b$, then a strict majority (2 out of 3) voters prefer a to b, a strict majority prefer b to c, yet a strict majority prefer c to a. Argue that this cannot happen if the preference profile is single-peaked. That is, prove that if a profile is single-peaked, a strict majority of voters prefer a to b, and a strict majority of voters prefer b to c, then a strict majority of voters prefer a to c.

(ii) Suppose that n is odd and voters' preferences are known to be single-peaked with respect to an axis $\lhd$. Consider the following voting rule: we ask each voter i to report their top candidate $t(i)$, find a median voter i^*, i.e.

$$|\{i : t(i) \lhd t(i^*)\}| < \frac{n}{2} \quad \text{and} \quad |\{i : t(i^*) \lhd t(i)\}| < \frac{n}{2},$$

and output $t(i^*)$. Argue that under this voting rule no voter can benefit from voting dishonestly, if a voter i reports some candidate $a \neq t(i)$ instead of $t(i)$, this either does not change the outcome or results in an outcome that i likes less than the outcome of the truthful voting.

(iii) We say that a preference profile is *1D-Euclidean* if each candidate c_j and each voter i can be associated with a point in $\mathbb{R}$ so that the preferences are determined by distances, i.e. there is an embedding $x\colon C \cup [n] \to \mathbb{R}$ such that for all $a, b \in C$ and $i \in [n]$ we have $a \succ_i b$ if and only if $|x(i) - x(a)| < |x(i) - x(b)|$. Argue that a 1D-Euclidean profile is necessarily single-peaked. Show that the converse is not true, i.e. there exists a single-peaked profile that is not 1D-Euclidean.

(iv) Let P be a single-peaked profile, and let L be the set of candidates ranked last by at least one voter. Prove that $|L| \leq 2$.

(v) Consider an axis $c_1 \lhd \cdots \lhd c_m$. Prove that there are exactly 2^{m-1} distinct votes that are single-peaked with respect to this axis. Explain how to sample from the uniform distribution over these votes.

These problems are based on references [4] (parts (i) and (ii)), [2] (part (iii)) and [1, 5] (part (v)); part (iv) is folklore. See also the survey [3].

References

[1] V. Conitzer, Eliciting single-peaked preferences using comparison queries. *J. Artificial Intelligence Res.* **35**, 161–191 (2009)

[2] C. H. Coombs. Psychological scaling without a unit of measurement. *Psychological Review* **57**, 145 (1950)

[3] E. Elkind, M. Lackner and D. Peters. Structured preferences. In *Trends in Computational Social Choice*, edited by U. Endriss, Chapter 10, AI Access, 187–207 (2017)

[4] H. Moulin, *Axioms of Cooperative Decision Making*. Cambridge University Press, Cambridge (1991)

[5] T. Walsh, Generating single peaked votes. arXiv:1503.02766 (2015)

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Consider a standard prisoners' dilemma game described by the following strategic form, with $\delta > \beta > 0 > \gamma$:

	C	D
C	β β	δ γ
D	γ δ	0 0

Assume that any given agent either plays C or D and that agents reproduce at a rate determined by their payoff from the strategic form of the game plus a constant f. Suppose that members of an infinite population are assorted into finite groups of size n. Let q denote the proportion of agents playing strategy C ("altruists") in the population as a whole and q_i denote the proportion of agents playing C in group i. We assume that currently $q \in (0, 1)$.

The process of assortment is abstract, but we assume that it has finite expectation $E[q_i] = q$ and variance $\mathrm{Var}[q_i] = \sigma^2$. Members within each group are then randomly paired off to play one iteration of the prisoners' dilemma against another member of their group. All agents then return to the overall population.

(a) Find a condition relating q, σ^2, β, γ, δ and n under which the proportion of altruists in the overall population rises after a round of play.

(b) Now interpret this game as one where each player can confer a benefit b upon the other player by individually incurring a cost c, with $b > c > 0$, so that $\beta = b - c$, $\delta = b$ and $\gamma = -c$. Prove that, as long as (i) there is some positive assortment in group formation and (ii) the ratio $\frac{c}{b}$ is low enough, then the proportion of altruists in the overall population will rise after a round of play.

References

[1] B. Cooper and C. Wallace, Group selection and the evolution of altruism. *Oxford Economic Papers* **56**, 307–330 (2004)

[2] R. Povey, Punishment and the potency of group selection, *J. Evolutionary Economics* **24** (2014)

[3] G. R. Price, Selection and covariance, *Nature* **227**, 520–521 (1970)

[4] G. R. Price, Extension of covariance selection mathematics, *Ann. Human Genetics* **35**, 485–490 (1972)

[5] E. Sober and D. S. Wilson, *Unto Others*, Harvard University Press, Cambridge, Massachusetts (1999)

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Consider a village consisting of n farmers who live along a circle of length n. The farmers live at positions $1, 2, \ldots, n$. Each of them is friends with the person to the left and right of them, and each friendship has capacity m where m is a non-negative integer. At the end of the year, each farmer does either well (her wealth is $+1$ dollars) or not well (her wealth is -1 dollars) with equal probability. Farmers' wealth realizations are independent of each other. Hence, for a large circle the share of farmers in each state is on average 1.

The farmers share risk by transferring money to their direct neighbors. The goal of risk-sharing is to create as many farmers with OK wealth (0 dollars) as possible. Transfers have to be in integer dollars and cannot exceed the capacity of each link (which is m).

A few examples with a village of size $n = 4$ serve to illustrate risk-sharing.

- Consider the case where farmers 1 to 4 have wealth

  $$(+1, -1, +1, -1).$$

  In that case, we can share risk completely with farmer 1 sending a dollar to agent 2 and farmer 3 sending a dollar to farmer 4. This works for any $m \geq 1$.

- Consider the case where farmers 1 to 4 have wealth

  $$(+1, +1, -1, -1).$$

  In that case, we can share risk completely with farmer 1 sending a dollar to farmer 2, farmer 2 sending two dollars to farmer 3 and farmer 3 sending one dollar to farmer 4. In this case, we need $m \geq 2$. If $m = 1$, we can only share risk among half the people in the village.

Show that for any wealth realization an optimal risk-sharing arrangement can be found as the solution to a maximum flow problem.

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This exercise is a continuation of Problem 247 where we studied risk-sharing among farmers who live on a circle village and are friends with their direct neighbors to the left and right with friendships of a certain capacity. Assume that for any realization of wealth levels the best possible risk-sharing arrangement is implemented and denote the expected share of unmatched farmers with $U(n, m)$. Show that $U(n, m) \to \frac{1}{2m+1}$ as $n \to \infty$.

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In a *combinatorial auction* there are m items for sale to n buyers. Each buyer i has some valuation function $v_i(\cdot)$ which takes as input a set S of items and outputs that bidder's value for that set. These functions will always be monotone ($v_i(S \cup T) \geq v_i(S)$ for all S, T), and satisfy $v_i(\emptyset) = 0$.

Definition 1 (Walrasian equilibrium). A price vector $\vec{p} \in \mathbb{R}^m_{\geq 0}$ and a list $B_1, \ldots, B_n$ of subsets of $[m]$ form a *Walrasian equilbrium* for $v_1, \ldots, v_n$ if the following two properties hold:

- Each $B_i \in \arg\max_S \{v_i(S) - \sum_{j \in S} p_j\}$.
- The sets B_i are disjoint, and $\bigcup_i B_i = [m]$.

Prove that a Walrasian equilibrium exists for $v_1, \ldots, v_n$ if and only if there exists an integral[1] optimum to the following linear program:

$$\begin{aligned} \text{maximize } & \sum_i \sum_S v_i(S) \cdot x_{i,S} \\ \text{such that, for all } i, \quad & \sum_S x_{i,S} = 1, \\ \text{for all } j, \quad & \sum_{S \ni j} \sum_i x_{i,S} \leq 1, \\ \text{for all } i, S, \quad & x_{i,S} \geq 0. \end{aligned}$$

Hint. Take the dual, and start from there.

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Consider a game played on a network and a finite set of players $\mathcal{N} = \{1, 2, \ldots, n\}$. Each node in the network represents a player and edges capture their relationships. We use $\mathbf{G} = (g_{ij})_{1 \leq i,j \leq n}$ to represent the adjacency matrix of a undirected graph/network, i.e. $g_{ij} = g_{ji} \in \{0, 1\}$. We assume $g_{ii} = 0$. Thus, $\mathbf{G}$ is a zero-diagonal, squared and symmetric matrix. Each player, indexed by i, chooses an action $x_i \in \mathbb{R}$ and obtains the following payoff:

$$\pi_i(x_1, x_2, \ldots, x_n) = x_i - \frac{1}{2} x_i^2 + \delta \sum_{j \in \mathcal{N}} g_{ij} x_i x_j.$$

The parameter $\delta > 0$ captures the strength of the direct links between different players. For simplicity, we assume $0 < \delta < \frac{1}{n-1}$.

A Nash equilibrium is a profile $\mathbf{x}^* = (x_1^*, \ldots, x_n^*)$ such that, for any $i = 1, \ldots, n$,

$$\pi_i(x_1^*, \ldots, x_n^*) \geq \pi_i(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*) \quad \text{for any } x_i \in \mathbb{R}.$$

In other words, at a Nash equilibrium, there is no profitable deviation for any player i choosing x_i^*.

Let $\mathbf{w} = (w_1, w_2, \ldots, w_n)'$, $w_i > 0$ for all i (the transpose of a vector $\mathbf{w}$ is denoted by $\mathbf{w}'$), and $\mathbf{I}_n$ the $n \times n$ identity matrix. Define the *weighted* Katz–Bonacich centrality vector as

$$\mathbf{b}(\mathbf{G}, \mathbf{w}) = [\mathbf{I}_n - \delta \mathbf{G}]^{-1} \mathbf{w}.$$

Here $\mathbf{M} := [\mathbf{I} - \delta \mathbf{G}]^{-1}$ denote the inverse Leontief matrix associated with network $\mathbf{G}$, while m_{ij} denote its ij entry, which is equal to the discounted number of walks from i to j with decay factor δ. Let $\mathbf{1}_n = (1, 1, \ldots, 1)'$ be a vector of 1s. Then the *unweighted* Katz–Bonacich centrality vector can be defined as

$$\mathbf{b}(\mathbf{G}, \mathbf{1}) = [\mathbf{I} - \delta \mathbf{G}]^{-1} \mathbf{1}_n.$$

1. Show that this network game has a unique Nash equilibrium $\mathbf{x}^*(\mathbf{G})$. Can you link this equilibrium to the Katz–Bonacich centrality vector defined above?
2. Let $x^*(\mathbf{G}) = \sum_{i=1}^n x_i^*(\mathbf{G})$ denote the sum of actions (total activity) at the unique Nash equilibrium in part 1. Now suppose that you can remove a single node, say i, from the network. Which node do you want to remove such that the sum of effort at the new Nash equilibrium is reduced the most? (Note that, after the deletion of node i, we remove all the links of node i, and the remaining network, denoted by $\mathbf{G}_{-i}$, can be obtained by deleting the i-th row and i-th column of $\mathbf{G}$.)
   Mathematically, you need to solve the *key player problem*
   $$\max_{i \in \mathcal{N}} \big(x^*(\mathbf{G}) - x^*(\mathbf{G}_{-i}) \big).$$
   In other words, you want to find a player who, once removed, leads to the highest reduction in total action in the remaining network.
   Hint. You may come up with an index c_i for each i such that the key player is the one with the highest c_i. This c_i should be expressed using the Katz–Bonacich centrality vector defined above.
3. Now instead of deleting a single node, we can delete any pair of nodes from the network. Can you identify the key pair, that is, the pair of nodes that, once removed, reduces total activity the most?

References

[1] C. Ballester, A. Calvó-Armengol and Y. Zenou, Who's who in networks. Wanted: the key player. *Econometrica* **74**, 1403–1417 (2006)

[2] M. O. Jackson and Y. Zenou, Games on networks. In *Handbook of Game Theory. Vol. 4*, edited by H. P. Young and S. Zamir, Handbooks in Economics, 91–157, Elsevier/North-Holland, Amsterdam (2015)

[3] Y. Sun, W. Zhao and J. Zhou, Structural interventions in networks. arXiv:2101.12420 (2021)

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[1] That is, a point such that each $x_{i,S} \in \{0, 1\}$.

II Open problem

Equilibrium in Quitting Games

by Eilon Solan (School of Mathematical Sciences, Tel Aviv University, Israel)[2]

Alaya, Black, and Catherine are involved in an endurance match, where each player has to decide if and when to quit, and the outcome depends on the set of players whose choice is larger than the minimum of the three choices. Formally, each of the three has to select an element of $\mathbb{N} \cup \{\infty\}$: the choice ∞ corresponds to the decision to never quit, and the choice $n \in \mathbb{N}$ corresponds to the decision to quit the match in round n. Denote by n_A (resp. n_B, n_C) Alaya's (resp. Black's, Catherine's) choice, and by $n_* := \min\{n_A, n_B, n_C\}$. As a result of their choices, the players receive payoffs, which are determined by the set $\{i \in \{A, B, C\} : n_i > n_*\}$ and on whether $n_* < \infty$. As a concrete example, suppose that if $n_* = \infty$, the payoff of each player is 0, and if $n_* < \infty$, the payoffs are given by the table in Figure 1.

Each entry in the figure represents one possible outcome. For example, when $n_* = n_A = n_B < n_C$, the payoffs of the three players are $(1, 0, 1)$: the left-most number in each entry is the payoff to Alaya, the middle number is the payoff to Black, and the right-most number is the payoff to Catherine. This game is an instance of a class of games that are known as *quitting games*.

How should the players act in this game? To provide an answer, we formalize the concepts of *strategy* and *equilibrium*. As the choice of each participant may be random, a *strategy* for a player is a probability distribution over $\mathbb{N} \cup \{\infty\}$. Denote a strategy of Alaya (resp. Black, Catherine) by σ_A (resp. σ_B, σ_C), and by $\gamma_i(\sigma_A, \sigma_B, \sigma_C)$ the expected payoff to player i under the vector of strategies $(\sigma_A, \sigma_B, \sigma_C)$. A vector of strategies $(\sigma_A^*, \sigma_B^*, \sigma_C^*)$ is an *equilibrium* if no player can increase her or his expected payoff by adopting another strategy while the other two stick to their strategies:

$$\gamma_A(\sigma_A^*, \sigma_B^*, \sigma_C^*) \geq \gamma_A(\sigma_A, \sigma_B^*, \sigma_C^*)$$

for every strategy σ_A of Alaya, and analogous inequalities hold for Black and Catherine.

The three-player quitting game with payoffs as described above was studied by Flesch, Thuijsman, and Vrieze [2] who proved that the following vector of strategies $(\sigma_A^*, \sigma_B^*, \sigma_C^*)$ is an equilibrium:

	1	2	3	4	5	6	7	8	9	…	∞
σ_A^* :	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	$\frac{1}{8}$	0	0	…	0
σ_B^* :	0	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	$\frac{1}{8}$	0	…	0
σ_C^* :	0	0	$\frac{1}{2}$	0	0	$\frac{1}{4}$	0	0	$\frac{1}{8}$	…	0

Under $(\sigma_A^*, \sigma_B^*, \sigma_C^*)$, with probability 1 the minimum n_* is the choice of exactly one player: $n_* = n_A$ with probability $\frac{4}{7}$, $n_* = n_B$ with probability $\frac{2}{7}$, and $n_* = n_C$ with probability $\frac{1}{7}$. It follows that the vector of expected payoffs under $(\sigma_A^*, \sigma_B^*, \sigma_C^*)$ is

$$\gamma(\sigma_A^*, \sigma_B^*, \sigma_C^*) = \frac{4}{7} \cdot (1, 3, 0) + \frac{2}{7} \cdot (0, 1, 3) + \frac{1}{7} \cdot (3, 0, 1) \\ = (1, 2, 1).$$

Can a player profit by adopting a strategy different than σ_A^*, σ_B^*, or σ_C^*, assuming the other two stick to their prescribed strategies? It is a bit tedious, but not too difficult, to verify that this is not the case, hence $(\sigma_A^*, \sigma_B^*, \sigma_C^*)$ is indeed an equilibrium.

In fact, Flesch, Thuijsman, and Vrieze [2] proved that under *all* equilibria of the game, with probability 1 the minimum n_* coincides with the choice of exactly one player. Moreover, a vector of strategies is an equilibrium if and only if the set $\mathbb{N}$ can be partitioned into blocks of consecutive numbers, and up to circular permutations of the players, the support of the strategy of Alaya (which is a probability distribution over $\mathbb{N} \cup \{\infty\}$) is contained in blocks number $1, 4, 7, \dots$, and the total probability that n_A is in block $3k - 2$ is $\frac{1}{2^k}$ (for each $k \in \mathbb{N}$), the support of the strategy of Black (resp. Catherine) is contained in blocks number $2, 5, 8, \dots$ (resp. $3, 6, 9, \dots$), and the total probability that n_B (resp. n_C) is in block $3k - 1$ (resp. $3k$) is $\frac{1}{2^k}$ (for each $k \in \mathbb{N}$).

Does an equilibrium exist if the payoffs are not given by the table in Figure 1, but rather by other numbers? Solan [6] showed that this is not the case. He studied a three-player quitting game that differs from the game of [2] in three payoffs:

- the payoffs in the entry $n_* = n_A = n_B < n_C$ are $(1 + \eta, 0, 1)$,
- the payoffs in the entry $n_* = n_A = n_C < n_B$ are $(0, 1, 1 + \eta)$,
- the payoffs in the entry $n_* = n_B = n_C < n_A$ are $(1, 1 + \eta, 0)$;

and showed that provided η is sufficiently small, the game has no equilibrium. For example, the strategy vector $(\sigma_A^*, \sigma_B^*, \sigma_C^*)$ described above is no longer an equilibrium, because Catherine is better off selecting $n_C = 1$ with probability 1, thereby obtaining expected payoff $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (1 + \eta) = 1 + \frac{\eta}{2}$, which is higher than her expected payoff under $(\sigma_A^*, \sigma_B^*, \sigma_C^*)$ (that is still 1).

Yet in Solan's variation [6], for every $\varepsilon > 0$ there is an *ε-equilibrium*: a vector of strategies such that no player can profit more than ε by deviating to another strategy, in other words,

$$\gamma_A(\sigma_A^*, \sigma_B^*, \sigma_C^*) \geq \gamma_A(\sigma_A, \sigma_B^*, \sigma_C^*) - \varepsilon,$$

for every strategy σ_A of Alaya, and analogous inequalities hold for Black and Catherine. Indeed, given a positive integer m, consider the following variation of $(\sigma_A^*, \sigma_B^*, \sigma_C^*)$, denoted $(\hat{\sigma}_A, \hat{\sigma}_B, \hat{\sigma}_C)$, where the set $\mathbb{N}$ is partitioned into blocks of size m: block k contains the integers $\{(k-1)m + 1, (k-1)m + 2, \dots, km\}$, for each $k \in \mathbb{N}$. $\hat{\sigma}_A$ is the probability distribution that assigns to each integer in block $3k - 2$ the probability $\frac{1}{m \cdot 2^k}$, for every $k \in \mathbb{N}$. Similarly, $\hat{\sigma}_B$ (resp. $\hat{\sigma}_C$) is the probability distribution that assigns to each integer in block $3k - 1$ (resp. $3k$) the probability $\frac{1}{m \cdot 2^k}$, for every $k \in \mathbb{N}$. As

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Black's choice: columns; Alaya's choice: rows; Catherine's choice: matrix.

$n_C > n_*$

	$n_B > n_*$	$n_B = n_*$
$n_A > n_*$		0, 1, 3
$n_A = n_*$	1, 3, 0	1, 0, 1

$n_C = n_*$

	$n_B > n_*$	$n_B = n_*$
$n_A > n_*$	3, 0, 1	1, 1, 0
$n_A = n_*$	0, 1, 1	0, 0, 0

Figure 1. The payoffs to the players in the game when $n_* < \infty$. In red, purple, and green the choices and payoffs of respectively Alaya, Black, and Catherine. Alaya chooses a row, Black a column, and Catherine a matrix.

mentioned above, the strategy vector $(\hat{\sigma}_A, \hat{\sigma}_B, \hat{\sigma}_C)$ is an equilibrium of the game whose payoff function is given in Figure 1, and one can verify that provided $m \geq \frac{1}{\varepsilon}$, it is an ε-equilibrium of Solan's variation [6].

It follows from [5] that an ε-equilibrium exists in *every* three-player quitting game, for every $\varepsilon > 0$, regardless of the payoffs. One of the most challenging problems in game theory to date is the following.

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Does an ε-equilibrium exist in quitting games that include more than three players, for every $\varepsilon > 0$?

For partial results, see [1, 3, 4, 7–9], which use different tools to study the problem: dynamical systems, algebraic topology, and linear complementarity problems. The open problem is a step in solving several other well-known open problems in game theory: the existence of ε-equilibria in stopping games, the existence of uniform equilibria in stochastic games, and the existence of ε-equilibria in repeated games with Borel-measurable payoffs.

It is interesting to note that if we defined

$$n_* := \max\{\mathbf{1}_{\{n_A < \infty\}} \cdot n_A, \mathbf{1}_{\{n_B < \infty\}} \cdot n_B, \mathbf{1}_{\{n_C < \infty\}} \cdot n_C\},$$

then an ε-equilibrium need not exist for small $\varepsilon > 0$. Indeed, with this definition, the three-player game in which the payoff of player i is 1 if $\infty > n_i = n_* > n_j$ for each $j \neq i$, and 0 otherwise, has no ε-equilibrium for $\varepsilon \in (0, \frac{2}{3})$.

References

[1] G. Ashkenazi-Golan, I. Krasikov, C. Rainer and E. Solan, Absorption paths and equilibria in quitting games. arXiv:2012.04369 (2021)

[2] J. Flesch, F. Thuijsman and K. Vrieze, Cyclic Markov equilibria in stochastic games. *Internat. J. Game Theory* **26**, 303–314 (1997)

[3] R. S. Simon, The structure of non-zero-sum stochastic games. *Adv. in Appl. Math.* **38**, 1–26 (2007)

[4] R. S. Simon, A topological approach to quitting games. *Math. Oper. Res.* **37**, 180–195 (2012)

[5] E. Solan, Three-player absorbing games. *Math. Oper. Res.* **24**, 669–698 (1999)

[6] E. Solan, The dynamics of the Nash correspondence and n-player stochastic games. *Int. Game Theory Rev.* **3**, 291–299 (2001)

[7] E. Solan and O. N. Solan, Quitting games and linear complementarity problems. *Math. Oper. Res.* **45**, 434–454 (2020)

[8] E. Solan and N. Vieille, Quitting games. *Math. Oper. Res.* **26**, 265–285 (2001)

[9] E. Solan and N. Vieille, Quitting games – an example. *Internat. J. Game Theory* **31**, 365–381 (2002)

III Solutions

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We take for our probability space (X, m): the unit interval $X = [0, 1]$ equipped with Lebesgue measure m defined on $\mathcal{B}(X)$, the Borel subsets of X and let (X, m, T) be an invertible measure preserving transformation, that is $T\colon X_0 \to X_0$ is a bimeasurable bijection of some Borel set $X_0 \in \mathcal{B}(X)$ of full measure so that and $m(TA) = m(T^{-1}A) = m(A)$ for every $A \in \mathcal{B}(X)$.

Suppose also that T is ergodic in the sense that the only T-invariant Borel sets have either zero- or full measure ($A \in \mathcal{B}(X)$, $TA = A \Rightarrow m(A) = 0, 1$).

Birkhoff's ergodic theorem says that for every integrable function $f\colon X \to \mathbb{R}$,

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{} \mathbb{E}(f) := \int_X f\,dm \text{ a.s.}$$

The present exercise is concerned with the possibility of generalizing this. Throughout, (X, m, T) is an arbitrary ergodic, measure preserving transformation as above.

Warm-up 1. Show that if $f\colon X \to \mathbb{R}$ is measurable, and

$$m\left(\left[\varliminf_{n\to\infty}\left|\sum_{k=0}^{n-1} f \circ T^k\right| < \infty\right]\right) > 0,$$

then $\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$ converges in $\mathbb{R}$ a.s.

Warm-up 1 is [1, Lemma 1]. For a multidimensional version, see [1, Conjecture 3].

Warm-up 2. Show that if $f\colon X \to \mathbb{R}$ is as in Warm-up 1, there exist $g, h\colon X \to \mathbb{R}$ measurable with h bounded so that $f = h + g - g \circ T^n$.

Warm-up 2 is established by adapting the proof of [3, Theorem A].

Problem. Show that there is a measurable function $f\colon X \to \mathbb{R}$ satisfying $\mathbb{E}(|f|) = \infty$ so that

$$\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k$$

converges in $\mathbb{R}$ a.s.

The existence of such f for a specially constructed ergodic measure preserving transformation is shown in [2, Example b]. The point here is to prove it for an arbitrary ergodic measure preserving transformation of (X, m).

Jon Aaronson (Tel Aviv University, Israel)

Solution by the proposer

We'll fix sequences $\varepsilon_k, M_k > 0$, $N_k \in \mathbb{N}$ $(k \geq 1)$. For each $\varepsilon, M > 0$, $N \geq 1$, we'll construct a small coboundary $f^{(\varepsilon,M,N)}$. The desired function will be of the form $F := \sum_{k\geq 1} f^{(\varepsilon_k,M_k,N_k)}$ for a suitable choice of $\varepsilon_k, M_k > 0$, $N_k \in \mathbb{N}$ $(k \geq 1)$.

To construct $f^{(\varepsilon,M,N)}$, choose, using Rokhlin's lemma, a set $B \in \mathcal{B}$ such that $\{T^kB : |k| \leq 2N\}$ are disjoint and $m(A) = \varepsilon$ where $A := \bigcup_{|k|\leq 2N} T^kB$. Let

$$f = f^{(\varepsilon,M,N)} := M\sum_{k=1}^{2N}(-1)^k 1_{T_kB}.$$

It follows that

$$S_nf(x) \in \{0, M, -M\} \quad \text{for all } n \geq 1,\ x \in X;$$

$$S_nf(x) = 0 \quad \text{for all } 1 \leq n \leq N,\ x \notin A;$$

$$E(|f|) = Mm\left(\bigcup_{j=1}^{2N} T^jB\right) = \frac{M\varepsilon 2N}{4N+1} > \frac{M\varepsilon}{3}.$$

Set $\varepsilon_k := \frac{1}{5^k}$, $M_k = 6^k$, $N_k = 7^k$, and define $F^{(k)} := f^{(\varepsilon_k,M_k,N_k)}$ as above.

Since

$$\sum_{k\geq 1} m([F^{(k)} \neq 0]) \leq \sum_{k\geq 1}\varepsilon_k < \infty,$$

this is a finite sum and so

$$F := \sum_{k\geq 1} F^{(k)} : X \to \mathbb{R}.$$

Proof that $E(|F|) = \infty$. For each $K \geq 1$,

$$\begin{aligned}
|F| &\geq \left|F^{(K)} + \sum_{1\leq j\leq K-1} F^{(j)}\right| 1_{[F^{(k)}=0\ \forall k>K]} \\
&\geq \left(|F^{(K)}| - \sum_{1\leq j\leq K-1}|F^{(j)}|\right) 1_{[F^{(k)}=0\ \forall k>K]} \\
&\geq \left(M_K - \sum_{1\leq j\leq K-1} M_j\right) 1_{[F^{(K)}\neq 0\ \&\ F^{(k)}=0\ \forall k>K]} \\
&\geq \frac{4}{5}M_K 1_{[F^{(K)}\neq 0\ \&\ F^{(k)}=0\ \forall k>K]}
\end{aligned}$$

and

$$E(|F|) \geq \frac{4}{5}M_K m([F^{(K)} \neq 0\ \&\ F^{(k)} = 0\ \forall k > K]).$$

Next,

$$\mathcal{E}_K := [F^{(k)} = 0\ \forall k > K]^c = \bigcup_{k\geq K+1}\ \bigcup_{1\leq j\leq 2N_k} T^kB_k$$

whence

$$m(\mathcal{E}_K) \leq \sum_{k\geq K+1}\frac{\varepsilon_k}{2} = \frac{1}{2}\sum_{k\geq K+1}\frac{1}{5^k} = \frac{\varepsilon_K}{40}.$$

It follows that

$$m([F^{(K)} \neq 0] \setminus \mathcal{E}_K) = m\left(\bigcup_{j=1}^{2N_K} T^jB_K \setminus \mathcal{E}_K\right) > \frac{\varepsilon_K}{3} - \frac{\varepsilon_K}{40} = \frac{37\varepsilon_K}{120},$$

whence

$$E(|F|) \geq \frac{4}{5}M_K m([F^{(K)} \neq 0] \setminus \mathcal{E}_K) > \frac{37\varepsilon_K M_K}{150} \xrightarrow[K\to\infty]{} \infty.$$

Proof that $S_nF = o(n)$ a.s. There is a function $\kappa\colon X \to \mathbb{N}$ so that for a.s. $x \in X$, $x \in A_k^c$ for all $k \geq \kappa(x)$. Suppose that $k \geq \kappa(x)$ and $2N_k \leq n < 2N_{k+1}$, then

$$|S_nF(x)| = \left|\sum_{j=1}^{k} S_nF^{(j)}(x)\right| \leq \sum_{j=1}^{k} M_j < \frac{6}{5}\cdot\left(\frac{6}{7}\right)^k \cdot N_k$$

and

$$\frac{|S_nF(x)|}{n} \xrightarrow[n\to\infty]{} 0 \text{ a.s.}$$

References

[1] P. Hagelstein, D. Herden and A. Stokolos, A theorem of Besicovitch and a generalization of the Birkhoff ergodic theorem. *Proc. Amer. Math. Soc. Ser. B* **8**, 52–59 (2021)

[2] D. Tanny, A zero-one law for stationary sequences. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **30**, 139–148 (1974)

[3] D. Volný and B. Weiss, Coboundaries in L_0^∞. *Ann. Inst. H. Poincaré Probab. Statist.* **40**, 771–778 (2004)

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Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\{X_n : n \geq 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables on Ω. Assume that there exists a sequence of positive numbers $\{b_n : n \geq 1\}$ such that $\frac{b_n}{n} \leq \frac{b_{n+1}}{n+1}$ for every $n \geq 1$, $\lim_{n\to\infty}\frac{b_n}{n} = \infty$, and $\sum_{n=1}^{\infty}\mathbb{P}(|X_n| \geq b_n) < \infty$. Prove that, if $S_n := \sum_{j=1}^{n} X_j$ for each $n \geq 1$, then

$$\lim_{n\to\infty}\frac{S_n}{b_n} = 0 \text{ almost surely.}$$

Comment. The desired statement says that, if such a sequence $\{b_n : n \geq 1\}$ exists, then $\{X_n : n \geq 1\}$ satisfies the (generalized) Strong Law of Large Numbers (SLLN) when averaged by $\{b_n : n \geq 1\}$.

If $X_n \in L^1(\mathbb{P})$ for every $n \geq 1$, then the desired statement follows trivially from Kolmogorov's SLLN, since in which case, with probability one,

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}[X_1],$$

and hence

$$\frac{S_n}{b_n} = \frac{S_n}{n} \cdot \frac{n}{b_n}$$

must converge to 0 under the assumptions on $\{b_n : n \geq 1\}$. Therefore, the desired statement can be viewed as an alternative to Kolmogorov's SLLN for i.i.d. random variables that are not integrable.

Linan Chen (McGill University, Montreal, Quebec, Canada)

Solution by the proposer

As explained above, we will need to prove the desired statement without assuming integrability of X_n's. For every $n \geq 1$, we truncate X_n at the level b_n by defining $Y_n = X_n$ if $|X_n| < b_n$, and $Y_n = 0$ if $|X_n| \geq b_n$. Then, $\{Y_n : n \geq 1\}$ is again a sequence of independent random variables. It follows from the assumption on $\{b_n : n \geq 1\}$ that

$$\sum_{n=1}^{\infty} \mathbb{P}(X_n \neq Y_n) = \sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq b_n) < \infty,$$

which, by the Borel–Cantelli lemma, implies that the sequence of the truncated random variables $\{Y_n : n \geq 1\}$ is *equivalent* to the original sequence $\{X_n : n \geq 1\}$ in the sense that

$$\begin{aligned} &\mathbb{P}(X_n \neq Y_n \text{ infinitely often}) = 0, \text{ or equivalently,} \\ &\mathbb{P}(X_n = Y_n \text{ eventually always}) = 1. \end{aligned} \tag{1}$$

Next, by setting $b_0 = 0$, we have that

$$\begin{aligned} \sum_{n=1}^{\infty} \mathbb{P}(|X_n| \geq b_n) &= \sum_{n=1}^{\infty} \sum_{k=n+1}^{\infty} \mathbb{P}(b_{k-1} \leq |X_1| < b_k) \\ &= \sum_{k=2}^{\infty} \sum_{n=1}^{k-1} \mathbb{P}(b_{k-1} \leq |X_1| < b_k) \\ &= \sum_{k=2}^{\infty} (k-1)\mathbb{P}(b_{k-1} \leq |X_1| < b_k) \\ &= \sum_{k=1}^{\infty} k\mathbb{P}(b_{k-1} \leq |X_1| < b_k) - 1, \end{aligned}$$

and hence the assumption on $\{b_n : n \geq 1\}$ implies that

$$\sum_{k=1}^{\infty} k\mathbb{P}(b_{k-1} \leq |X_1| < b_k) < \infty. \tag{2}$$

Our next goal is to establish the desired SLLN statement for $\{Y_n : n \geq 1\}$. To be specific, we want to show that if $T_n \coloneqq \sum_{j=1}^{n} Y_j$ for each $n \geq 1$, then $\lim_{n\to\infty} \frac{T_n}{b_n} = 0$ almost surely. We will achieve this goal in two steps.

Step 1 is to treat the convergence of $\frac{\mathbb{E}[T_n]}{b_n}$. To this end, we derive an upper bound for this term as

$$\begin{aligned} \frac{\mathbb{E}[|T_n|]}{b_n} &\leq \frac{1}{b_n} \sum_{j=1}^{n} \mathbb{E}[|Y_j|] = \frac{1}{b_n} \sum_{j=1}^{n} \int_{\{|X_1| < b_j\}} |X_1|\, d\mathbb{P} \\ &= \frac{1}{b_n} \sum_{j=1}^{n} \sum_{k=1}^{j} \int_{\{b_{k-1} \leq |X_1| < b_k\}} |X_1|\, d\mathbb{P} \\ &\leq \frac{1}{b_n} \sum_{k=1}^{n} (n-k+1) b_k \mathbb{P}(b_{k-1} \leq |X_1| < b_k) \\ &\leq \frac{2n}{b_n} \sum_{k=1}^{n} b_k \mathbb{P}(b_{k-1} \leq |X_1| < b_k). \end{aligned}$$

Then (2) implies that

$$\sum_{k=1}^{\infty} \frac{b_k \mathbb{P}(b_{k-1} \leq |X_1| < b_k)}{(b_k/k)} = \sum_{k=1}^{\infty} k\mathbb{P}(b_{k-1} \leq |X_1| < b_k) < \infty,$$

which, by Kronecker's lemma, leads to

$$\lim_{n\to\infty} \frac{n}{b_n} \sum_{k=1}^{n} b_k \mathbb{P}(b_{k-1} \leq |X_1| < b_k) = 0.$$

Hence, we conclude that $\lim_{n\to\infty} \frac{\mathbb{E}[|T_n|]}{b_n} = 0$.

Step 2 is to establish the convergence of $\frac{T_n - \mathbb{E}[T_n]}{b_n}$, for which we will use a martingale convergence argument. We note that if

$$M_n \coloneqq \sum_{j=1}^{n} \frac{Y_j - \mathbb{E}[Y_j]}{b_j}$$

for each $n \geq 1$, then $\{M_n : n \geq 1\}$ is a martingale (with respect to the natural filtration) and for each $n \geq 1$,

$$\begin{aligned} \mathbb{E}[M_n^2] &\leq \sum_{j=1}^{n} \frac{\mathbb{E}[Y_j^2]}{b_j^2} = \sum_{j=1}^{n} \frac{1}{b_j^2} \int_{\{|X_1| < b_j\}} X_1^2\, d\mathbb{P} \\ &\leq \sum_{j=1}^{n} \sum_{k=1}^{j} \frac{b_k^2}{b_j^2} \mathbb{P}(b_{k-1} \leq |X_1| < b_k) \\ &\leq \sum_{k=1}^{n} \left(\sum_{j=k}^{n} \frac{1}{j^2} \right) k^2 \mathbb{P}(b_{k-1} \leq |X_1| < b_k) \\ &\leq C \sum_{k=1}^{n} k\mathbb{P}(b_{k-1} \leq |X_1| < b_k), \end{aligned}$$

where the second last inequality follows from the assumption that $\frac{b_n}{n}$ is increasing in n, and the last inequality is due to the fact that there exists constant $C > 0$ such that $\sum_{j=k}^{\infty} \frac{1}{j^2} \leq \frac{C}{k}$ for every $k \geq 1$. Hence, (2) implies that $\{M_n : n \geq 1\}$ is bounded in $L^2(\mathbb{P})$. A standard martingale convergence result implies that $\lim_{n\to\infty} M_n$ exists in $\mathbb{R}$ almost surely[3], which, by Kronecker's lemma again, leads to

$$\lim_{n\to\infty} \frac{T_n - \mathbb{E}[T_n]}{b_n} = 0 \text{ almost surely.}$$

[3] One can also use Kolmogorov's maximal inequality to prove the almost sure existence of the limit of M_n.

Finally, we write $\frac{S_n}{b_n}$ as

$$\frac{S_n}{b_n} = \frac{S_n - T_n}{b_n} + \frac{T_n - \mathbb{E}[T_n]}{b_n} + \frac{\mathbb{E}[T_n]}{b_n},$$

where the last two terms have been proven to converge to 0 almost surely, and (1) implies that, with probability one, the limit of the first term is also 0. We have completed the proof.

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In Beetown, the bees have a strict rule: all clubs must have exactly k members. Clubs are not necessarily disjoint. Let $b(k)$ be the smallest number of clubs that the $n \geq k^2$ bees can form, such that no matter how they divide themselves into two teams to play beeball, there will always be a club all of whose membees are on the same team. Prove that

$$2^{k-1} \leq b(k) \leq Ck^2 \cdot 2^k$$

for some constant $C > 0$.

Rob Morris (IMPA, Rio de Janeiro, Brasil)

Solution by the proposer

This is an old result of Erdős, and a classic application of the probabilistic method. Let us think of the two teams as being red and blue, so that a club is 'monochromatic' if all of its membees are on the same team.

First, for the lower bound, we need to show that if $m < 2^{k-1}$, then for any collection of m clubs there exists a colouring with no monochromatic club. To do so, we choose the teams randomly, and observe that the expected number of monochromatic clubs is less than 1. To be precise, let $\Pr(b \text{ is red}) = \frac{1}{2}$, independently for each bee b, and let S count the number of monochromatic clubs. Then, by linearity of expectation, $\mathbb{E}[S] = m \cdot 2^{-k+1} < 1$, since each club is monochromatic with probability exactly 2^{-k+1}. But this implies that $\Pr(S = 0) > 0$, so there exists a colouring with no monochromatic club, as required.

For the upper bound, we choose the clubs randomly. To be precise, choose $N = k^2$ bees, and choose each club uniformly and independently from the k-subsets of these N bees. The idea is that, for any colouring of the bees, the expected number of monochromatic clubs is at least k^2, so the probability of having no monochromatic club should be at most e^{-k^2}. Since there are 2^{k^2} colourings of these bees, the expected number of colourings with no monochromatic clubs is less than 1, so there must exist a choice for which it is zero.

To spell out the details, fix a colouring, and suppose that x of the N chosen bees are red. The probability that a random club is monochromatic is

$$\left(\binom{x}{k} + \binom{N-x}{k}\right)\binom{N}{k}^{-1} \geq 2 \cdot \binom{N/2}{k}\binom{N}{k}^{-1} \geq 2^{-k-c}$$

for some constant $c > 0$, where in the final inequality we used the fact that $N \geq k^2$.

Now, let T count the number of colourings of the N bees with no monochromatic club, and observe that if there are $m = k^2 2^{k+c}$ clubs, then

$$\mathbb{E}[T] \leq \sum_{\substack{\text{colourings} \\ \text{of the } N \text{ bees}}} (1 - 2^{-k-c})^m \leq 2^{k^2} e^{-k^2} < 1.$$

It follows that there exists a choice of m clubs such that $T = 0$, as required.

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N agents are in a room with a server, and each agent is looking to get served, at which point the agent leaves the room. At any discrete time step, each agent may choose to either shout or stay quiet, and an agent gets served in that round if (and only if) that agent is the only one to have shouted. The agents are indistinguishable to each other at the start, but at each subsequent step, every agent gets to see who has shouted and who has not. If all the agents are required to use the same randomised strategy, show that the minimum time to clear the room in expectation is $N + (2 + o(1)) \log_2 N$.

Bhargav Narayanan (Rutgers University, Piscataway, USA)

Solution by the proposer

Here is a simple strategy that works in expected time $N + (2 + o(1)) \log_2 N$. The agents all toss independent fair coins to decide whether to shout or not in each of the first $k = (2 + o(1)) \log_2 N$ rounds. It is easy to see that with high probability, after these k rounds, every agent (still in the room) has a unique 'history', i.e. no two agents have the exact same sequence of turns (shouting/staying quiet). Now the agents are all distinguishable, and we are done in N more steps; for example, the agents can interpret each others histories as numbers in binary, and can get served in increasing order. Below, we show that no strategy can do significantly better.

At any time, we can partition all the agents into clusters based on their history so far: two agents go into the same cluster if they have chosen to do the same thing in all previous rounds. By the requirement that the agents all have the same randomised strategy, we know that at any time, all the agents in the same cluster must have the same strategy. Let X be the number of times an agent from a cluster of size at least 2 gets served and leaves the room, and let Y be the number of times either

1. exactly two agents from the same cluster, and nobody else, ask to be served, or
2. nobody asks to be served at all.

An easy computation shows that

$$\mathbb{P}(\text{Bin}(m, p) = 1) \leq \mathbb{P}(\text{Bin}(m, p) = 0) + \mathbb{P}(\text{Bin}(m, p) = 2)$$

for all $m > 1$ and any $0 \leq p \leq 1$; consequently, it is easy to see that Y stochastically dominates X. So, if for some strategy,

$$\mathbb{P}(X > 2(\log_2 N)^2) > \frac{1}{\log_2 N},$$

then the expected time to clear the room, which is at least $N + Y$, is at least $N + 2\log_2 N$ in expectation. So we may assume that $X < 2(\log_2 N)^2$ with high probability for any strategy under consideration.

Let S be the set of agents who leave the room only when they belong to their own singleton cluster. As we just observed, the number of such agents $|S| = M = N - X$ may be assumed to be at least $N - 2(\log_2 N)^2$. The key observation is this: if someone leaves the room in a particular step, the cluster structure of S does not change in that step. To see this, note that when an agent not from S leaves the room, that agent shouts and everyone in S does not, so there is no change to the cluster structure of S. On the other hand, when an agent from S leaves the room, that agent is, by definition, already in their own singleton cluster, and every other agent in S does not shout in this step; again, there is no change in the cluster structure of S.

But we know that at the end of the process, which let us say takes $N + \Delta$ rounds, S has been split from a single cluster into M singleton clusters. Nothing changes in the cluster structure of S in the N rounds when someone leaves the room, so S gets broken down into singleton clusters in the remaining Δ steps.

Consider these Δ steps where nobody leaves the room. Deterministically, in the first $\log_2 M - 1$ of these steps, we can produce at most $\frac{M}{2}$ singletons in S. The remaining $\frac{M}{2}$ agents in S are all in clusters of size at least 2. Divide all these cluster into sub-clusters each of size 2 (by ignoring agents if necessary). The result is at least $\frac{M}{6}$ 2-clusters that we still need to break down into singletons (the worst case being when the $\frac{M}{2}$ agents are each in a cluster of size 3). The probability that a 2-cluster breaks down into two singletons at any given time step, with any strategy, is at most $\frac{1}{2}$. So in any strategy, we need at least another $\log_2 M - \log_2 \log_2 M$ time steps, say, for all these 2-clusters to separate into singletons. Thus, $\Delta \geq 2\log_2 M - \log_2 \log_2 M$ with high probability, which with our previous bound on M, tells us that any strategy takes at least $N + (2 - o(1))\log_2 N$ steps to clear the room.

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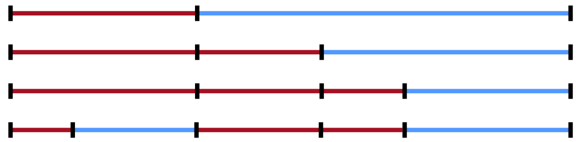
Consider the following sequence of partitions of the unit interval I: First, define π_1 to be the partition of I into two intervals, a red interval of length $\frac{1}{3}$ and a blue one of length $\frac{2}{3}$. Next, for any $m > 1$, define π_{m+1} to be the partition derived from π_m by splitting all intervals of maximal length in π_m, each into two intervals, a red one of ratio $\frac{1}{3}$ and a blue one of ratio $\frac{2}{3}$, just as in the first step. For example π_2 consists of three intervals of lengths $\frac{1}{3}$ (red), $\frac{2}{9}$ (red) and $\frac{4}{9}$ (blue), the last two are the result of splitting the blue interval in π_1. The figure above illustrates $\pi_1, \ldots, \pi_4$, from top to bottom.



Let $m \in \mathbb{N}$ and consider the m-th partition π_m.

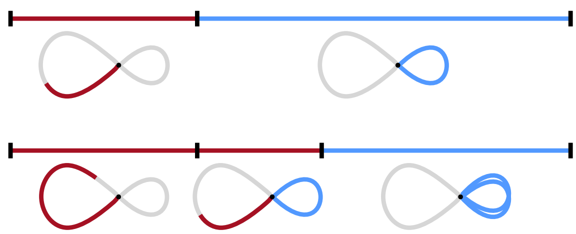
1. Choose an interval in π_m uniformly at random. Let R_m be the probability you chose a red interval. Does the sequence $(R_m)_{m \in \mathbb{N}}$ converge? If so, what is the limit?
2. Choose a point in I uniformly at random. Let A_m be the probability that the point is colored red. Does the sequence $(A_m)_{m \in \mathbb{N}}$ converge? If so, what is the limit?

Yotam Smilansky (Rutgers University, NJ, USA)

Solution by the proposer

The proposed solution is based on path counting results [1] on an appropriately defined graph, and can be generalized to higher dimensions and to more complicated sequences of partitions [2].

Let G be a weighted graph with a single vertex and two directed loops: a red one of length $-\log(\frac{1}{3})$ and a blue one of length $-\log(\frac{2}{3})$, and consider directed walks along the edges of G that originate at the vertex and terminate on a point of a colored loop. The first important observation is that there is a 1-1 correspondence between colored intervals in π_m and walks of length ℓ_m on G, where $(\ell_m)_{m \in \mathbb{N}}$ is the increasing sequence of lengths of closed orbits on G. In the following illustration, the top is partition π_1 and the corresponding two walks of length $\ell_1 = -\log(\frac{2}{3})$, and the bottom is partition π_2 and the corresponding three walks of length $\ell_2 = -2\log(\frac{2}{3})$.



In general, a splitting of an interval corresponds to an extension of a walk that terminates at the vertex to two new walks, one that extends onto the red loop and the other onto the blue. Therefore, R_m is the relative part of walks of length ℓ_m that terminate on the red loop. For A_m, consider random walks on G and prescribe probabilities to the two outgoing loops: a walk along G is extended onto the red loop when reaching the vertex with probability $\frac{1}{3}$, and onto the blue loop with probability $\frac{2}{3}$. These are chosen because $\frac{1}{3}$ of a split interval is colored red and $\frac{2}{3}$ colored blue. It follows that A_m is the probability that a walker is located on the red loop after walking a walk of length ℓ_m.

The second observation is that in order to compute the asymptotic behavior of R_m and A_m, one can apply the well-known Wiener–Ikehara theorem, originally devised to approach the prime number theorem. The theorem states that if there exists $\lambda \in \mathbb{R}$ for which the Laplace transform of a counting function is analytic for $\Re(s) > \lambda$, has a simple pole at $s = \lambda$ and no other singular points on the vertical line $\Re(s) = \lambda$, then the main term of the growth rate is $ce^{\lambda x}$, with c the residue of the Laplace transform at $s = \lambda$.

A direct computation shows that the Laplace transform for the number of walks that terminate on the red loop is

$$\frac{1}{s} \cdot \frac{1 - (\frac{1}{3})^s}{1 - (\frac{1}{3})^s - (\frac{2}{3})^s}.$$

Inspecting the term $1 - (\frac{1}{3})^s - (\frac{2}{3})^s$ one sees that $s = 1$ is a simple root of maximal real part, and so to apply the Wiener–Ikehara theorem it suffices to establish that there are no other roots of the form $1 + it$. Indeed, a careful but elementary inspection shows that otherwise, the loops of G must have commensurable lengths, or equivalently $\log_2 3 \in \mathbb{Q}$, which is of course false. The Laplace transform of the total number of walks is similar but has numerator $2 - (\frac{1}{3})^s - (\frac{2}{3})^s$, and so R_m tends to the ratio of the residues of these two transforms at $s = 1$, that is, $\lim_{m\to\infty} R_m = \frac{2}{3}$. Similarly, the Laplace transform for A_m is

$$\frac{\frac{1}{3}}{s} \cdot \frac{1 - (\frac{1}{3})^s}{1 - (\frac{1}{3})^{s+1} - (\frac{2}{3})^{s+1}},$$

with the same poles but shifted by -1. It follows that A_m converges to the residue at $s = 0$, namely

$$\lim_{m\to\infty} A_m = \frac{-\frac{1}{3}\log\frac{1}{3}}{-\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3}}.$$

Note that the limit of R_m is simply the length of the blue interval in π_1, and the limit of A_m can be viewed as the relative contribution of the red interval to the partition entropy of π_1. This interpretation leads me to suspect that there may exist a more direct and illuminating approach to these problems, possibly based on tools from probability and dynamics, and I would be very happy to discuss any ideas or suggestions.

References

[1] A. Kiro, Y. Smilansky and U. Smilansky, The distribution of path lengths on directed weighted graphs. In *Analysis as a Tool in Mathematical Physics*, Oper. Theory Adv. Appl. 276, Birkhäuser/Springer, Cham, 351–372 (2020)

[2] Y. Smilansky, Uniform distribution of Kakutani partitions generated by substitution schemes. *Israel J. Math.* **240**, 667–710 (2020)

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Prove that there exist $c < 1$ and $\varepsilon > 0$ such that if $A_1, \dots, A_k$ are increasing events of independent boolean random variables with $\Pr(A_i) < \varepsilon$ for all i, then

$$\Pr(\text{exactly one of } A_1, \dots, A_k \text{ occurs}) \le c.$$

(What is the smallest c that you can prove?)

Here $A \subset \{0,1\}^n$ is an "increasing event" if whenever $x \in A$, then the vector obtained by changing any coordinates of x from 0 to 1 still lies in A.

A useful fact is the Harris inequality, which states that for increasing events A and B of boolean random variables, $\Pr(A \cap B) \ge \Pr(A)\Pr(B)$.

I learned of this problem from Jeff Kahn.

Yufei Zhao (MIT, Cambridge, USA)

Solution by the proposer

We will show that the claim is true for every $\varepsilon > 0$ and $c = \frac{1+\varepsilon}{2}$.

If $\Pr(A_1 \cup \dots \cup A_k) \le c$, then the conclusion is automatic. So let us assume that $\Pr(A_1 \cup \dots \cup A_k) > c$. Since $\Pr(A_i) < \varepsilon$ for each i, there exists some j such that $\Pr(A_1 \cup \dots \cup A_j)$ lies within $\frac{\varepsilon}{2}$ of $\frac{1}{2}$. Let $B = A_1 \cup \dots \cup A_j$ and $C = A_{j+1} \cup \dots \cup A_k$. We write $\overline{B}$ and $\overline{C}$ for the complementary events.

If exactly one of $A_1, \dots, A_k$ occurs, then exactly one of B and C can occur. So

$$\begin{aligned}\Pr(\text{exactly one of } A_1, \dots, A_k \text{ occurs}) &\le \Pr(B \cap \overline{C}) + \Pr(\overline{B} \cap C)\\ &\le \Pr(B)\Pr(\overline{C}) + \Pr(\overline{B})\Pr(C)\\ &\le \max\{\Pr(B), \Pr(\overline{B})\}\\ &\le \frac{1+\varepsilon}{2}\end{aligned}$$

where the second inequality is due to Harris' inequality.

Remark. It is conjectured that for any $c > \frac{1}{e}$ there exists some $\varepsilon > 0$ for which the statement is true. Here $\frac{1}{e}$ is optimal, since if A_i are independent Bernoulli random variables with mean $\frac{1}{k}$, then the number of occurrences is asymptotically Poisson with mean 1, with so that the probability of single occurrence is $\frac{1}{e} + o(1)$.

We are eager to receive your solutions to the proposed problems, and any ideas that you may have on open problems. Send your solutions to Michael Th. Rassias (Institute of Mathematics, University of Zürich, Winterthurerstrasse 190, 8057 Zürich, Switzerland; michail.rassias@math.uzh.ch).

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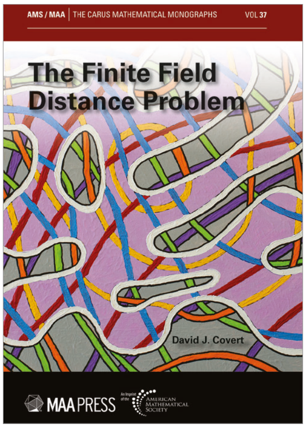

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David J. Covert, University of Missouri

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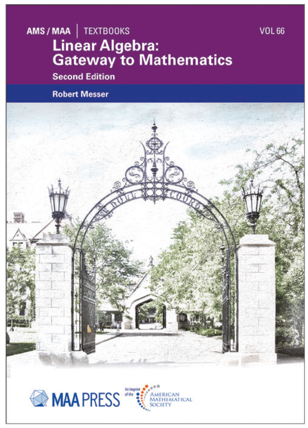

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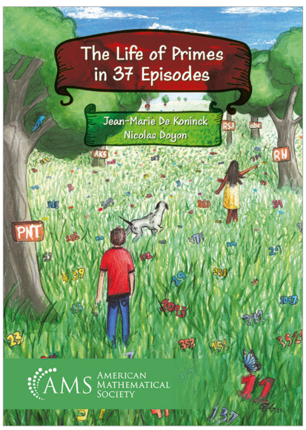

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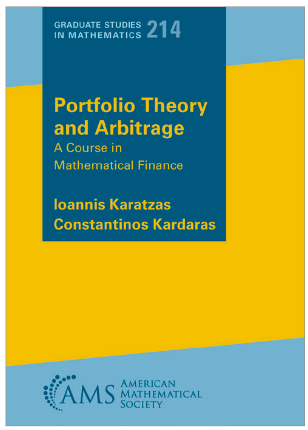

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