European Mathematical Society Magazine
ISSN 2747-7894 / Issue 125 / September 2022

EMS Magazine

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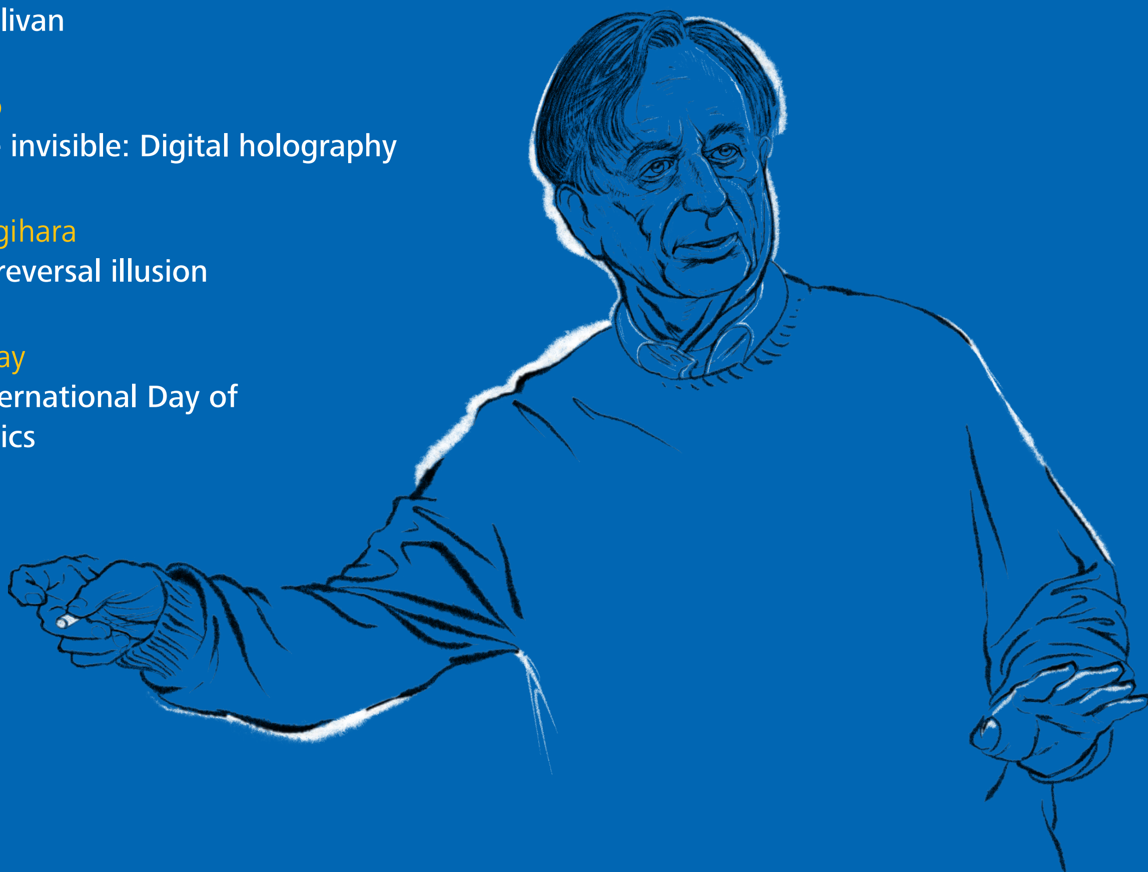

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EMS Magazine 125

European Mathematical Society Magazine

Published by EMS Press, an imprint of the
European Mathematical Society – EMS – Publishing House GmbH
Technische Universität Berlin, Institut für Mathematik,
Straße des 17. Juni 136, 10623 Berlin, Germany

https://ems.press

Typesetting: WisSat Publishing + Consulting GmbH, Fürstenwalde, Germany
Printing: Beltz Bad Langensalza GmbH, Bad Langensalza, Germany

ISSN (print) 2747-7894; ISSN (online) 2747-7908

The cover is a sketch of the 2022 Abel Prize winner Dennis Sullivan, drawn by A. B. Araújo. It is based on a photograph by John Griffin for Stony Brook University and the Abel Prize.

Dear EMS members,

After more than two years of online meetings, the EMS council finally met in person on June 25–26 to elect the new president and members of the executive committee and also to decide on several important issues that will change the operation of the EMS. First of all, I congratulate the new president Jan Philip Solovej, the new vice president Beatrice Pelloni, the new treasurer Samuli Siltanen, and the new member-at-large Victoria Gould, who will start office January 1, 2023. I also thank the departing EC members, vice president Betül Tanbay and treasurer Mats Gyllenberg for their service to the EMS. It was great to work with you over such a long time and steer the EMS through these difficult times. I also want to thank the Slovenian colleagues for organizing the council in the wonderful town of Bled.

Besides several changes of the statutes which will allow considerably more procedures with virtual meetings, the following important decisions were made.

(For more details on these three new activities, see the EMS website.)

EMS Young Academy

To strengthen the support and integration of the young generation of mathematicians in Europe, the EMS Council in Bled has approved the formation of an EMS Young Academy (EMYA), for which each EMS member society/member institute can every year nominate two early career mathematicians (3rd year PhD students up to 5 years after PhD, allowing for career breaks). The first nomination period ends September 30, 2022. Every year a committee formed by the EMS EC will select up to 30 members of the EMYA, for a four-year membership duration. The EMYA will establish itself bylaws and an organizational structure and it is supposed to suggest procedures for the future development of the EMS. The EMYA will be supported by a separate budget and it will also nominate a representative for the EMS EC who will be elected for a two-year period by the council.

EMS Topical Activity Groups

To address the large diversity of the European mathematical community and to integrate the scientific cooperation across Europe and all mathematical fields, starting 2023, a group of (at least 7) members of the EMS can apply to form an EMS topical activity group (EMS-TAG). The activity groups will be selected and evaluated on a four-year basis by an evaluation committee. EMS TAGs can organize meetings and workshops. They can apply for support from the EMS, they can nominate EMS committee members and speakers for conferences, and they are expected to plan and organize applications to European funding programs.

Support of large events

To promote mathematics across the board, across geographic boundaries, and across discipline boundaries, the EMS also calls for large, inclusive, and cross-institutional events that complement or extend existing infrastructure for such meetings. Supported events can range from special semesters to interdisciplinary study groups and large showcase events; calls will be made four times a year, with a first call on December 1, 2022. Proposals are required to utilize a significant amount of funding supporting regions and communities that do not have the infrastructure or the finances to organize such large-scale events. Applicants from low- and middle-income countries are particularly encouraged to apply.

Volker Mehrmann
President of the EMS

Seeing the invisible: Digital holography

Ana Carpio

For the past years there has been an increasing interest in developing mathematical and computational methods for digital holography. Holographic techniques furnish noninvasive tools for high-speed 3D live cell imaging. Holograms can be recorded in the millisecond or microsecond range without damaging samples. A hologram encodes the wave field scattered by an object as an interference pattern. Digital holography aims to create numerical images from digitally recorded holograms. We show here that partial differential equation constrained optimization, topological derivatives of shape functionals, iteratively regularized Gauss–Newton methods, Bayesian inference, and Markov chain Monte Carlo techniques provide effective mathematical tools to invert holographic data with quantified uncertainty. Holography set-ups are particularly challenging because a single incident wave is employed. Similar tools could be useful in inverse scattering problems involving other types of waves and different emitter/receiver configurations, such as microwave imaging or elastography, for instance.

1 Introduction

Experimental sciences have traditionally been a source of challenging mathematical problems with a double edge: while mathematical theories are created, technology moves fast and industry develops. Imaging sciences provide a remarkable example. Typical imaging systems, such as radar [28], magnetic resonance tomography, ultrasound, echography [25], and seismic imaging [34], pose inverse scattering problems with a similar mathematical structure. In all of them, waves generated by a set of emitters interact with a medium under study and the wave field resulting from the interaction is recorded at a set of receivers [10]. Different imaging systems resort to different types of waves and arrange emitters and receivers according to varied geometries. The nature of the employed waves depends on factors such as the size of the specimens under study, the contrast between components, and the damage caused to the sample during the imaging procedure. Knowing the emitted and recorded waves, we aim to infer the structure of the medium.

Approximating the solutions of inverse scattering problems is a challenging task because such problems are severely ill posed [10]. Given arbitrary data, the problem under study may not admit a solution, the solution may not be unique, or it may not depend continuously on the given data. This means that small errors may lead to a solution different from the searched one. In view of the relevant technological applications in a host of fields, such as medicine, security, geophysics, or materials testing, to mention a few, there is a need of even better mathematical techniques for classical imaging problems, as well as a need of new ideas to tackle new imaging set-ups.

We focus here on recent developments in digital holography, summarizing work done during the past 10 years in collaboration with experimentalists designing holographic microscopes. This collaboration started in 2012 thanks to the interdisciplinary communication environment created at the Harvard University's Kavli Institute seminars. Since then, we have developed analytical and computational tools to handle inverse problems arising in digital holography, in collaboration with researchers from Harvard University and Tesla, Universidad Complutense de Madrid, Universidad Politécnica de Madrid, Universidad de Oviedo, Université de Technologie de Compiègne, and New York University.

Digital in-line holography is a noninvasive tool for accelerated three-dimensional imaging of soft matter and live cells [16, 23, 26, 37] that achieves high spatial (nanometers) and temporal (microseconds) resolution without the need of toxic fluorescent markers or stains. In this context, a hologram is a two-dimensional light interference pattern encoding information about the optical and geometrical properties of a set of objects [35]. Shining a properly chosen light beam back through the hologram we can recreate the original three-dimensional image. Instead, digital holography is designed to produce numerical reconstructions of the objects in an automatic way, which amounts to solving computationally an inverse scattering problem. We will show next that optimization schemes with partial differential equation constraints, analysis of the topological derivative of objective functions, regularized Gauss–Newton iterations, and Bayesian inference are effective tools to invert holographic data in the presence of noise while quantifying uncertainty.

 EMS MAGAZINE 125 (2022) — DOI 10.4171/MAG/99

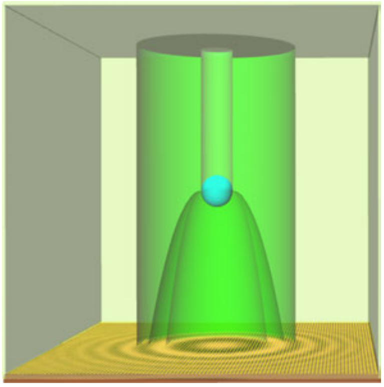

Figure 1. Formation of an in-line hologram. A laser beam hits an object. The scattered and undiffracted beams form an interference pattern on a screen, which is recorded at a mesh of detectors. Laser lights have wavelengths varying from about 405 nm (violet light) to about 660 nm (red light). Typical object sizes are in the micron range (1 μm = 10^{-6} m, 1 nm = 10^{-9} m).

2 The forward problem

The forward problem is a mathematical model of how a hologram is generated. Figure 1 illustrates how an in-line hologram is formed, though more complicated set-ups are possible. First, a laser light beam interacts with a sample. Then, interference of the scattered light field with the undiffracted beam generates the hologram on a detector screen past the object [23]. The light wave field obeys the Maxwell equations. Typically, the emitted laser beams are time harmonic, that is, $\mathcal{E}_{\rm inc}(\mathbf{x}, t) = \mathrm{Re}[e^{-\imath\omega t}\mathbf{E}_{\rm inc}(\mathbf{x})]$. The resulting wave field is also time harmonic, namely, $\mathcal{E}_{\Omega,\kappa}(\mathbf{x}, t) = \mathrm{Re}[e^{-\imath\omega t}\mathbf{E}_{\Omega,\kappa}(\mathbf{x})]$, with complex amplitude $\mathbf{E}_{\Omega,\kappa}(\mathbf{x})$ governed by the stationary Maxwell equations. The resulting forward problem is

$$
\begin{aligned}
\mathbf{curl}\Big(\frac{1}{\mu_e}\,\mathbf{curl}\,\mathbf{E}\Big) - \frac{\kappa_e^2}{\mu_e}\mathbf{E} &= 0 && \text{in } \mathbb{R}^3 \setminus \overline{\Omega},\\
\mathbf{curl}\Big(\frac{1}{\mu_i}\,\mathbf{curl}\,\mathbf{E}\Big) - \frac{\kappa_i^2}{\mu_i}\mathbf{E} &= 0 && \text{in } \Omega,\\
\hat{\mathbf{n}} \times \mathbf{E}^- &= \hat{\mathbf{n}} \times \mathbf{E}^+ && \text{on } \partial\Omega,\\
\frac{1}{\mu_i}\hat{\mathbf{n}} \times \mathbf{curl}\,\mathbf{E}^- &= \frac{1}{\mu_e}\hat{\mathbf{n}} \times \mathbf{curl}\,\mathbf{E}^+ && \text{on } \partial\Omega,\\
\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}| \Big| \mathbf{curl}(\mathbf{E} - \mathbf{E}_{\rm inc}) \times \frac{\mathbf{x}}{|\mathbf{x}|} - \imath\kappa_e(\mathbf{E} - \mathbf{E}_{\rm inc})\Big| &= 0,
\end{aligned}
\tag{1}
$$

where μ_i, ε_i and $\kappa_i = \omega_i^2\varepsilon_i\mu_i$ are the permeabilities, permittivities and wavenumbers of the imaged objects Ω, while μ_e, ε_e and κ_e correspond to the ambient medium [3] and are known. In biomedical applications, $\mu_i \sim \mu_e \sim \mu_0$, μ_0 being the vacuum permeability. The upper signs $-$ and $+$ represent limit values from inside and outside Ω, respectively, and $\hat{\mathbf{n}}$ denotes the outer unit normal vector. Incident waves are polarized in a direction $\hat{\mathbf{p}}$ orthogonal to the direction of propagation $\hat{\mathbf{d}}$, that is, $\mathbf{E}_{\rm inc}(\mathbf{x}) = E_0\hat{\mathbf{p}}\,e^{\imath\kappa_e\hat{\mathbf{d}}\cdot\mathbf{x}}$, where E_0 stands for the magnitude of the incident field.

For any smooth region $\Omega' \subset \mathbb{R}^3 \setminus \overline{\Omega}$ and any real $\kappa_e > 0$, system (1) has a unique solution [31] in the Sobolev space $H^{2,0}(\Omega') = \{\mathbf{E} \in H^2(\Omega'), \operatorname{div}\mathbf{E} = 0\}$ that is continuous in Ω' (see [19]). For collections of spheres and piecewise-constant κ_i, one can calculate Mie series solutions [3]. Starshaped object parametrizations with piecewise-constant μ_i allow for fast spectral solvers [20, 24]. Coupled BEM/FEM formulations [29, 31] are convenient for more general parametrizations, while discrete dipole approximations [36, 38] solve the problem avoiding the use of parametrizations.

In principle, the hologram is obtained evaluating the solution of the forward problem (1) at detectors placed on the screen: $I_{\Omega,\kappa_i} = |\mathbf{E}_{\rm inc} + \mathbf{E}_{{\rm sc},\Omega,\kappa_i}|^2 = |\mathbf{E}_{\Omega,\kappa_i}|^2$. In practice, the measured holograms $\mathbf{I}_{\rm meas}$ are corrupted by noise.

3 Deterministic inverse problem

Given a hologram $\mathbf{I}_{\rm meas}$ measured at screen points $\mathbf{x}_j$, $j = 1, \ldots, N$, the inverse holography problem seeks objects $\Omega = \bigcup_{\ell=1}^{L}\Omega_\ell$ and functions $\kappa_i\colon \Omega \to \mathbb{R}^+$ such that

$$I_{\rm meas}(\mathbf{x}_j) = |\mathbf{E}_{\Omega,\kappa_i}(\mathbf{x}_j)|^2, \quad j = 1, \ldots, N,$$

where $\mathbf{E}_{\Omega,\kappa_i} = \mathbf{E}_{\rm inc} + \mathbf{E}_{{\rm sc},\Omega,\kappa_i}$ is the solution of the forward problem (1) with an object Ω and the wavenumber κ_i (see [5]). Since the measured data are not exact, in practice one seeks shapes Ω and functions κ_i for which the error between the recorded hologram and the synthetic hologram that would be generated solving (1) for the proposed objects and wavenumbers is as small as possible.

3.1 Constrained optimization

We recast the inverse problem as an optimization problem with a partial differential equation constraint: find Ω and κ_i minimizing the cost functional

$$J(\Omega, \kappa_i) = \frac{1}{2}\sum_{j=1}^{N} |I_{\Omega,\kappa_i}(\mathbf{x}_j) - I_{\rm meas}(\mathbf{x}_j)|^2, \tag{2}$$

where $I_{\Omega,\kappa_i} = |\mathbf{E}_{\Omega,\kappa_i}|^2$ and $\mathbf{E}_{\Omega,\kappa_i}$ is the solution of (1). Here Ω and κ_i are the design variables and the stationary Maxwell system (1) is the constraint. For exact data, the true objects would be a global minimum at which the functional (2) vanishes. In general, spurious local minima may arise.

3.2 Topological derivative based approximations

A topological study of the shape functional (2) for κ_i fixed provides first guesses of the imaged objects without a priori information on them. The topological derivative of a shape functional [32] quantifies its sensitivity to removing and including points in an object. Given a point $\mathbf{x}$ in a region $\mathcal{R}$, we have the expansion

$$J(\mathcal{R} \setminus \overline{B_\varepsilon(\mathbf{x})}) = J(\mathcal{R}) + \frac{4}{3}\pi\varepsilon^3 D_T(\mathbf{x}, \mathcal{R}) + o(\varepsilon^3), \quad \varepsilon \to 0, \qquad (3)$$

for any ball $B_\varepsilon(\mathbf{x}) = B(\mathbf{x}, \varepsilon)$ centered at $\mathbf{x}$ with radius ε. The factor $D_T(\mathbf{x}, \mathcal{R})$ is the topological derivative of the functional at $\mathbf{x}$ (see [32]). If $D_T(\mathbf{x}, \mathcal{R})$ is negative, $J(\mathcal{R} \setminus \overline{B_\varepsilon(\mathbf{x})}) < J(\mathcal{R})$ for $\varepsilon > 0$ small. We expect the cost functional to decrease by forming objects Ω_{ap} with points below a large enough negative threshold [9, 14, 27]:

$$\Omega_{ap} \coloneqq \{\mathbf{x} \in \mathcal{R} \mid D_T(\mathbf{x}, \mathcal{R}) < -C_0\}, \quad C_0 > 0. \qquad (4)$$

When $\mu_e = \mu_i$, $\mathcal{R} = \mathbb{R}^3$ and $\mathbf{E}_{inc}(\mathbf{x}) = \hat{\mathbf{p}} e^{\imath\kappa_e z}$, asymptotic expansions yield the formula [27]

$$D_T(\mathbf{x}, \mathbb{R}^3) = 3\,\mathrm{Re}\left[\frac{\kappa_e^2(\kappa_e^2 - \kappa_i^2)}{(\kappa_i^2 + 2\kappa_e^2)} \mathbf{E}(\mathbf{x}) \cdot \overline{\mathbf{P}}(\mathbf{x})\right], \quad \mathbf{x} \in \mathbb{R}^3, \qquad (5)$$

where $\mathbf{E} = \mathbf{E}_{inc}$ and

$$\overline{\mathbf{P}}(\mathbf{x}) = \sum_{j=1}^{N} \mathbf{curl}\,\mathbf{curl}\left(\frac{2}{\kappa_e^2} G_{\kappa_e}(\mathbf{x} - \mathbf{x}_j)\,(I_{meas}(\mathbf{x}_j) - |\mathbf{E}_{inc}(\mathbf{x}_j)|^2)\,\overline{\mathbf{E}_{inc}(\mathbf{x}_j)}\right)$$

with $G_{\kappa_e}(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|} e^{\imath\kappa_e|\mathbf{x}|}$ denoting the outgoing Green function of the Helmholtz equation [31]. Once Ω_{ap} is constructed, we fit a parametrized contour $\mathbf{q}_{ap}$ to its boundary. Starshaped parametrizations are typical choices. Figure 2 exemplifies the procedure. The method is robust to noise, in the sense that perturbations of the data with random 10 % or 20 % noise, for instance, produce similar results. Notice that the value of κ_i enters through a factor that we may scale out in (5) and it is not really needed to localize the object. Similar results are obtained using the topological energy [6]

$$E_T(\mathbf{x}, \mathbb{R}^3) = |\mathbf{E}(\mathbf{x})|^2 |\mathbf{P}(\mathbf{x})|^2,$$

which does not involve κ_i at all. No knowledge of κ_i is needed to construct a first guess of the objects.

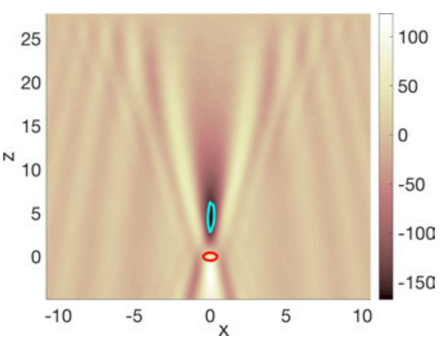

Figure 2. Slice $y = 0$ of the topological derivative computed using expression (5) for holographic data $\mathbf{I}_{meas}$ corresponding to a sphere of radius 0.45 µm illuminated by polarized light of wavelength 520 nm and placed at a distance 28 µm of a CMOS screen. Axis units are microns. The red contour marks the location of the true object, while the cyan contour represents the approximation. Redrawn from [5].

3.3 Regularized Gauss–Newton iterations

Fast methods to improve our knowledge of the objects starting from an initial guess are based on the following result. Let us consider two Hilbert spaces X, Y and a Fréchet differentiable operator $\mathcal{F} : D(\mathcal{F}) \subset X \to Y$. Assuming that the exact data $y \in Y$ are attainable (that is, there is $x \in X$ such that $\mathcal{F}(x) = y$), but only noisy data y^δ verifying $\|y^\delta - y\|_Y \le \delta$ are accessible, the iteratively regularized Gauss–Newton (IRGN) method [1] constructs a sequence x_{k+1}^δ as follows. We linearize the equation at x_k^δ at each step, approximate the solution of $\mathcal{F}(x_k^\delta) + \mathcal{F}'(x_k^\delta)\xi = y^\delta$ through the minimization problem

$$\xi_{k+1} = \underset{\xi \in X}{\mathrm{Argmin}} \|\mathcal{F}(x_k^\delta) + \mathcal{F}'(x_k^\delta)\xi - y^\delta\|_Y^2 + \alpha_k \|x_k^\delta + \xi - x_0\|_X^2$$

and set $x_{k+1}^\delta = x_k^\delta + \xi_{k+1}$. The Tikhonov term $\alpha_k \|x_k^\delta + \xi - x_0\|_X^2$ has regularizing properties and promotes convergence for specific choices x_0 and α_k (see [21]). The theory of linear Tikhonov regularization guarantees that

$$\xi_{k+1} = -(\mathcal{F}'(x_k^\delta)^* \mathcal{F}'(x_k^\delta) + \alpha_k I)^{-1} [\mathcal{F}'(x_k^\delta)^* (\mathcal{F}(x_k^\delta) - y^\delta) + \alpha_k (x_k^\delta - x_0)],$$

where $\mathcal{F}'(x_k^\delta)^*$ denotes the adjoint of the Fréchet derivative $\mathcal{F}'(x_k^\delta)$. The noise level δ affects the stopping criterion, the so-called discrepancy principle.

In a holography set-up, the map $\mathcal{F}$ is the operator that to each parametrization of objects $\mathbf{q}$ assigns the synthetic hologram $\mathbf{I}(\mathbf{q})$ generated by solving the forward problem for those objects. Starshaped parametrizations are a standard choice for simple objects. They describe each object by a few parameters: its center and a radius function represented by a finite combination of spherical harmonics [5, 20]. Given a starshaped parametrization $\mathbf{q}_k$ and a recorded hologram $\mathbf{I}_{meas}$ with a level of noise δ, the IRGN method first solves the linearized equation

$$\mathbf{I}(\mathbf{q}_k) + \mathbf{I}'(\mathbf{q}_k)\xi = \mathbf{I}_{meas}$$

by addressing the nonlinear least squares problem

$$\xi_{k+1} = \mathrm{Argmin}_\xi \{\|\mathbf{I}_{meas} - \mathbf{I}(\mathbf{q}_k) - \mathbf{I}'(\mathbf{q}_k)\xi\|_2^2 + \alpha_k \|\mathbf{q}_k + \xi - \mathbf{q}_{ap}\|_{H^s(\mathbb{S}^2)}^2\},$$

where $H^s(\mathbb{S}^2)$, $s > 0$, is an adequate Sobolev space [5], and then sets $\mathbf{q}_{k+1} = \mathbf{q}_k + \xi_{k+1}$. The initial parametrization $\mathbf{q}_0 = \mathbf{q}_{ap}$ represents the first guess of the objects constructed by topological methods. The updated objects Ω_k correspond to the parametrizations $\mathbf{q}_k$. The stopping criterion for the noise level δ is as follows. If the synthetic hologram calculated numerically for the current approximation of the objects $\mathbf{I}(\mathbf{q}_k)$ satisfies

$$\|\mathbf{I}(\mathbf{q}_k) - \mathbf{I}_{meas}\|_2 \le \tau\delta,$$

we stop the algorithm, $\tau > 0$ being a parameter adjusted to guarantee a reasonable approximation while preventing early stops.

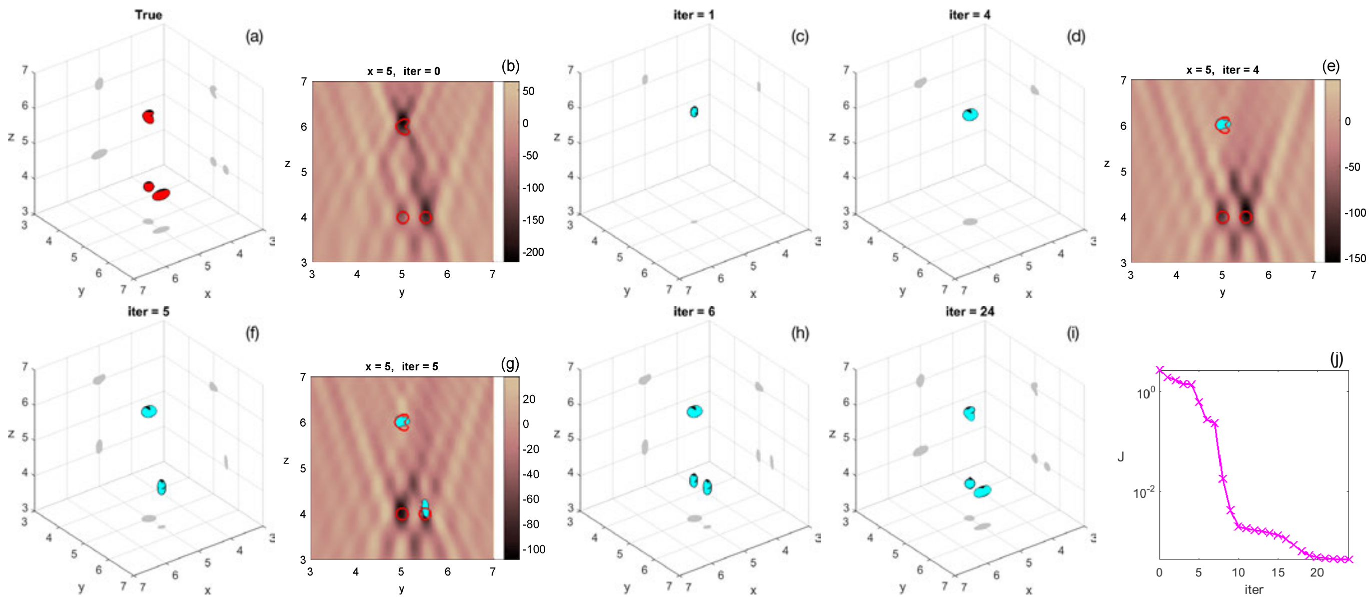

Figure 3. For the hologram in Figure 4 (redrawn from [5]): (a) True geometry. (b) Slice $x = 5$ of the topological derivative (5). Red contours are true objects. (c) Initial guess defined by (4). (d) Approximation after 4 steps of the IRGN method. (e) Slice of the topological derivative (6) at step 4. Cyan regions are approximate objects. (f) Approximation after creating an object at step 4 and applying once the IRGNM. (g) Slice of the topological derivative (6) at step 5. (h) Approximation after creating an object at step 5 and applying once the IRGNM. (i) Final approximation. Axis units are μm. (j) Decrease in the cost.

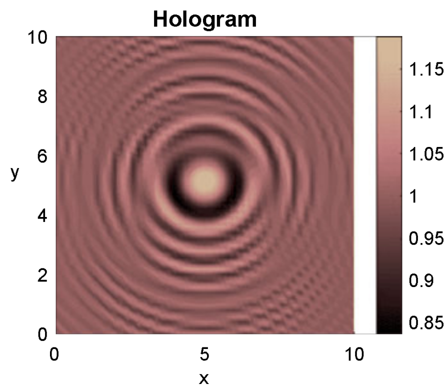

Figure 4. Hologram generated by the three objects represented in Figure 3(a), obtained with violet light having a wavelength of 405 nm emitted at $z = 0$ and recorded at $z = 10$. Axis units are microns. Redrawn from [5].

Figures 3 and 4 illustrate the process. Figure 4 depicts the hologram generated by the configuration with three objects shown in Figure 3 (a). We use the topological derivative (5) to spot a first dominant object at the top and locate an object there, see panel (b). Then we apply the IRGN method, see panels (c) and (d). At step 4 the cost functional, depicted in panel (j), stagnates without fulfilling the stopping criteria. This suggests that more objects should be created. This can be done by hybrid methods, as we explain next.

3.4 Topologically informed IRGN methods

Approaches that use initial object parametrization as reference have a drawback: the initial guess of the number of objects may be wrong. To overcome it, we have developed hybrid algorithms [5] combining topological derivatives and regularized Gauss–Newton iterations [5]. We fit an initial parametrization $\mathbf{q}_{\mathrm{ap}}$ to the first guess of the objects constructed by topological methods. Then, we apply the IRGN method and check that the cost (2) decreases. When the cost stagnates without fulfilling the stopping criteria, we reset Ω_{ap} equal to the current guess of the objects Ω_k for the last parametrization obtained $\mathbf{q}_k$ and calculate the topological derivative of the cost for $\mathcal{R} = \mathbb{R}^3 \setminus \overline{\Omega_{\mathrm{ap}}}$. This is given by (3) if $\mathbf{x} \in \mathcal{R} = \mathbb{R}^3 \setminus \overline{\Omega}$ and its equivalent

$$
\begin{aligned}
&J(\mathbb{R}^3 \setminus \overline{(\Omega \setminus \overline{B_\varepsilon(\mathbf{x})})}) \\
&\quad = J((\mathbb{R}^3 \cup B_\varepsilon(\mathbf{x})) \setminus \overline{\Omega}) \\
&\quad = J(\mathbb{R}^3 \setminus \overline{\Omega}) - \frac{4}{3}\pi\varepsilon^3 D_{\mathrm{T}}(\mathbf{x}, \mathbb{R}^3 \setminus \overline{\Omega}) + o(\varepsilon^3)
\end{aligned}
$$

if $\mathbf{x} \in \Omega$. Asymptotic calculations yield the formula [5, 9]

$$
D_{\mathrm{T}}(\mathbf{x}, \mathbb{R}^3 \setminus \overline{\Omega}) = \begin{cases} 3\,\mathrm{Re}\left[\dfrac{\kappa_e^2(\kappa_e^2 - \kappa_i^2)}{(\kappa_i^2 + 2\kappa_e^2)}\mathbf{E}(\mathbf{x}) \cdot \overline{\mathbf{P}}(\mathbf{x})\right], & \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}, \\ 3\,\mathrm{Re}\left[\dfrac{\kappa_i^2(\kappa_e^2 - \kappa_i^2)}{(\kappa_e^2 + 2\kappa_i^2)}\mathbf{E}(\mathbf{x}) \cdot \overline{\mathbf{P}}(\mathbf{x})\right], & \mathbf{x} \in \Omega, \end{cases} \qquad (6)
$$

when $\mu_e = \mu_i$, with forward and conjugate adjoint fields satisfying transmission Maxwell problems with object $\Omega = \Omega_{\rm ap}$:

$$\begin{aligned}
\mathbf{curl}(\mathbf{curl}\,\mathbf{E}) - \kappa_e^2 \mathbf{E} = 0 \qquad & \text{in } \mathbb{R}^3 \setminus \overline{\Omega},\\
\mathbf{curl}(\mathbf{curl}\,\mathbf{E}) - \kappa_i^2 \mathbf{E} = 0 \qquad & \text{in } \Omega,\\
\hat{\mathbf{n}} \times \mathbf{E}^- = \hat{\mathbf{n}} \times \mathbf{E}^+ \qquad & \text{on } \partial\Omega,\\
\hat{\mathbf{n}} \times \mathbf{curl}\,\mathbf{E}^- = \hat{\mathbf{n}} \times \mathbf{curl}\,\mathbf{E}^+ \qquad & \text{on } \partial\Omega,\\
\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}| |\mathbf{curl}(\mathbf{E} - \mathbf{E}_{\rm inc}) \times \hat{\mathbf{x}} - \imath\kappa_e(\mathbf{E} - \mathbf{E}_{\rm inc})| = 0, &\\
\mathbf{curl}(\mathbf{curl}\,\overline{\mathbf{P}}) - \kappa_e^2 \overline{\mathbf{P}} = 2\sum_{j=1}^{N} (\mathbf{I}_{\rm meas} - |\mathbf{E}|^2)\overline{\mathbf{E}}\delta_{\mathbf{x}_j} \qquad & \text{in } \mathbb{R}^3 \setminus \overline{\Omega},\\
\mathbf{curl}(\mathbf{curl}\,\overline{\mathbf{P}}) - \kappa_i^2 \overline{\mathbf{P}} = 0 \qquad & \text{in } \Omega,\\
\hat{\mathbf{n}} \times \overline{\mathbf{P}}^- = \hat{\mathbf{n}} \times \overline{\mathbf{P}}^+ \qquad & \text{on } \partial\Omega,\\
\hat{\mathbf{n}} \times \mathbf{curl}\,\overline{\mathbf{P}}^- = \hat{\mathbf{n}} \times \mathbf{curl}\,\overline{\mathbf{P}}^+ \qquad & \text{on } \partial\Omega,\\
\lim_{|\mathbf{x}|\to\infty} |\mathbf{x}| |\mathbf{curl}\,\overline{\mathbf{P}} \times \hat{\mathbf{x}} - \imath\kappa_e \overline{\mathbf{P}}| = 0, &
\end{aligned}$$

where $\hat{\mathbf{n}}$ is the unit outer normal, $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$ and $\delta_{\mathbf{x}_j}$ are Dirac masses concentrated at the detectors $\mathbf{x}_j$, $j = 1, \dots, N$.

We create a new approximation $\Omega_{\rm new}$ from $\Omega_{\rm ap}$ by removing the points in $\Omega_{\rm ap}$ at which the topological derivate surpasses a positive threshold $c_{\rm new}$ and adding the points outside $\Omega_{\rm ap}$ at which the topological derivate falls below a negative threshold $-C_{\rm new}$, see [6, 9]:

$$\begin{aligned}
\Omega_{\rm new} := \{\mathbf{x} \in \Omega_{\rm ap} \mid D_{\rm T}(\mathbf{x}, \mathbb{R}^3 \setminus \overline{\Omega}_{\rm ap}) < c_{\rm new}\} &\\
\cup \{\mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}_{\rm ap} \mid D_{\rm T}(\mathbf{x}, \mathbb{R}^3 \setminus \overline{\Omega}_{\rm ap}) < -C_{\rm new}\}.&
\end{aligned}$$

The constants $C_{\rm new}$, $c_{\rm new}$ are selected to ensure a decrease in the cost functional (2) keeping κ_i fixed. Once $\Omega_{\rm new}$ is constructed, we fit a parametrization $\mathbf{q}_{\rm new}$ to its contour and restart the IRGN procedure for $\mathbf{q}_{\rm ap} = \mathbf{q}_{\rm new}$. The procedure stops when the changes in the cost and the parametrizations fall below selected thresholds.

Let us revisit the example studied in Figures 3 and 4. At step 4 of the IRGN method the cost stagnates without fulfilling the stopping criteria. We calculate the topological derivative (6) of the cost for the current approximation of the objects, illustrated in Figure 3 (e). A new region where the topological derivative attains large negative values appears. We create a new object there and update the parametrization, see panel (f). Then we apply the IRGN method again. Since the cost functional still stagnates without fulfilling the stopping criteria, we recalculate the topological derivative (6) for the available object approximation. Panel (g) suggests the creation of a third object. We update the IRGN method using this new configuration, and evolve the resulting object configuration, represented in panel (h), until the stopping criterion is met at panel (i) after 24 steps. Panel (j) illustrates stagnation and decrease of the cost as new objects are added to the parametrization using topological information and the updated IRGN method evolves, in a logarithmic scale. These simulations assume κ_i known and fixed. Once first guesses for κ_i are available, we can implement this procedure considering constant values for κ_i at each component of the parametrization. Obtaining first guesses for κ_i that are reliable enough is a hard task [7] and the optimization procedure can encounter difficulties. Bayesian approaches provide alternative procedures that can handle these difficulties while quantifying uncertainty associated to noise and missing information.

4 Bayesian inverse problem

Bayesian formulations consider all unknowns in the inverse problem as random variables. Given a recorded hologram $\mathbf{I}_{\rm meas}$, we seek a finite-dimensional vector of parameters $\boldsymbol{\nu}$ characterizing the imaged objects. When we assume the presence of L objects, $\boldsymbol{\nu}$ is formed by L blocks, one per object. Using Bayes' formula [22, 33]

$$p_{\rm pt}(\boldsymbol{\nu}) := p(\boldsymbol{\nu}|\mathbf{I}_{\rm meas}) = \frac{p(\mathbf{I}_{\rm meas}|\boldsymbol{\nu})}{p(\mathbf{I}_{\rm meas})} p_{\rm pr}(\boldsymbol{\nu}), \tag{7}$$

where $p_{\rm pr}(\boldsymbol{\nu})$ represents the prior probability of the variables, which incorporates our previous knowledge on them, while $p(\mathbf{I}_{\rm meas}|\boldsymbol{\nu})$ is the conditional probability or likelihood of observing $\mathbf{I}_{\rm meas}$ given $\boldsymbol{\nu}$. The solution of the Bayesian inverse problem is the posterior probability $p_{\rm pt}(\boldsymbol{\nu}|\mathbf{I}_{\rm meas})$ of the parameters given the data. Sampling the posterior distribution, we obtain statistical information on the most likely values of the object parameters with quantified uncertainty.

4.1 *Likelihood choice*

Assuming additive Gaussian measurement noise, the measured hologram and the synthetic hologram obtained for the true object parameters are related by $\mathbf{I}_{\rm meas} = \mathbf{I}(\boldsymbol{\nu}_{\rm true}) + \boldsymbol{\varepsilon}$, where the measurement noise $\boldsymbol{\varepsilon}$ is distributed as a multivariate Gaussian $\mathcal{N}(0, \boldsymbol{\Gamma}_{\rm n})$ with zero mean and covariance matrix $\boldsymbol{\Gamma}_{\rm n}$. A possible choice for the likelihood $p(\mathbf{I}_{\rm meas}|\boldsymbol{\nu})$ is [8]

$$p(\mathbf{I}_{\rm meas}|\boldsymbol{\nu}) = \frac{1}{(2\pi)^{N/2}\sqrt{|\boldsymbol{\Gamma}_{\rm n}|}} \exp\Big(-\frac{1}{2}\|\mathbf{I}(\boldsymbol{\nu}) - \mathbf{I}_{\rm meas}\|^2_{\boldsymbol{\Gamma}_{\rm n}^{-1}}\Big) \tag{8}$$

with $\|\mathbf{v}\|^2_{\boldsymbol{\Gamma}_{\rm n}^{-1}} = \overline{\mathbf{v}}^t \boldsymbol{\Gamma}_{\rm n}^{-1}\mathbf{v}$. Here, $\mathbf{I}(\boldsymbol{\nu})$ represents the synthetic hologram obtained solving the forward problem (1) for objects characterized by parameters $\boldsymbol{\nu}$, see Section 2.

4.2 *Topological priors*

A typical choice for the prior distribution is a multivariate Gaussian

$$p_{\rm pr}(\boldsymbol{\nu}) = \frac{1}{(2\pi)^{n/2}} \frac{1}{\sqrt{|\boldsymbol{\Gamma}_{\rm pr}|}} \exp\Big(-\frac{1}{2}(\boldsymbol{\nu} - \boldsymbol{\nu}_0)^t \boldsymbol{\Gamma}_{\rm pr}^{-1}(\boldsymbol{\nu} - \boldsymbol{\nu}_0)\Big) \tag{9}$$

if $\boldsymbol{\nu}$ is "admissible", and $p_{\rm pr}(\boldsymbol{\nu}) = 0$ when $\boldsymbol{\nu}$ is "not admissible", that is, it does not satisfy known constraints on the parameter set, see [8] for details. Here, $\boldsymbol{\Gamma}_{\rm pr}$ is the covariance matrix and n is the total number of parameters characterizing the objects. The mean $\boldsymbol{\nu}_0$

is typically a set of parameter values characterizing an initial guess of the objects. Sharp priors are obtained fitting parametrizations to first guesses of the objects obtained from the study of topological fields associated to deterministic shape costs, as explained in Section 3.2.

4.3 Markov chain Monte Carlo sampling

Combining (7), (8) and (9), the posterior probability becomes (neglecting normalization constants)

$$p_{\mathrm{pt}}(\boldsymbol{\nu}) \propto \exp\Big(-\frac{1}{2}\|\mathbf{I}(\boldsymbol{\nu}) - \mathbf{I}_{\mathrm{meas}}\|^2_{\Gamma_{\mathrm{n}}^{-1}} - \frac{1}{2}\|\boldsymbol{\nu} - \boldsymbol{\nu}_0\|^2_{\Gamma_{\mathrm{pr}}^{-1}}\Big)$$

when $\boldsymbol{\nu}$ is admissible, and $p_{\mathrm{pt}}(\boldsymbol{\nu}) = 0$ otherwise. Markov chain Monte Carlo (MCMC) methods provide tools to sample unnormalized posteriors. Classical MCMC methods, such as Hamiltonian Monte Carlo or Metropolis–Hastings [30] construct a chain of n-dimensional states $\boldsymbol{\nu}^{(0)} \to \boldsymbol{\nu}^{(1)} \to \cdots \to \boldsymbol{\nu}^{(k)} \to \cdots$ which evolve to be distributed in accordance with the target distribution $p_{\mathrm{pt}}(\boldsymbol{\nu})$. After sampling an initial state $\boldsymbol{\nu}^{(0)}$ from the prior distribution (9), the chain advances from one state $\boldsymbol{\nu}^{(k)}$ to the next $\boldsymbol{\nu}^{(k+1)}$ by means of a transition operator that varies with the method employed [30]. More recent ensemble MCMC samplers [13, 18] draw W initial states from the prior distribution (the "walkers" or "particles") and transition to new states while mixing the previous ones to generate several chains. This approach allows for parallelization and can handle multimodal posteriors [8].

Figure 5 illustrates the results in a two-dimensional geometry, to reduce the computational cost in the tests. A few million samples were generated, which requires solving an identical number of forward problems. In two-dimensional set-ups we replace the stationary transmission problem for the Maxwell equations by a transmission problem posed for the Helmholtz equation [8]. Assuming κ_i is piecewise constant, we resort to fast boundary elements to solve the Hemholtz transmission problems in two dimensions [12]. Once a large enough collection of samples is generated [17], we extract statistical information describing the imaged object: the most likely shapes, sizes, locations, as well as uncertainty in the predictions. While starshaped two-dimensional objects can be reasonably characterized with 10–20 parameters, three-dimensional objects require 80–90. Full characterization of the posterior probability by MCMC sampling becomes more expensive as the number of parameters and the time required to solve forward problems increase.

4.4 Laplace approximation

The full characterization of the posterior probability is a challenging and costly probability problem for moderate- and high-dimensional parameters $\boldsymbol{\nu}$. Low-cost approximations of the posterior distribution often rely on finding the maximum a posteriori (MAP) point, that is, the set of parameters that maximize the posterior probability.

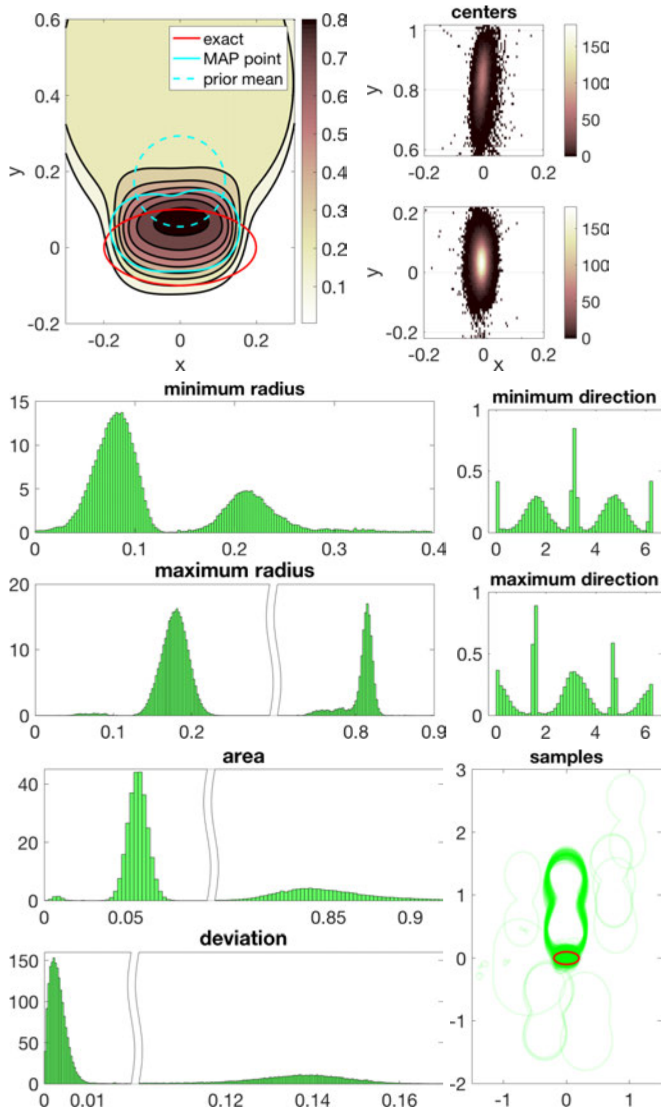

Figure 5. For the two-dimensional object depicted in red, with violet light having a wavelength of 405 nm emitted at $y = -5$ and recorded at $y = 5$, we present statistical information obtained from the samples generated by MCMC sampling. A contour projection of a two-dimensional histogram represents the probability of belonging to the object, compared to the contours of the true object, the prior mean and the MAP (maximum a posteriori) approximation. The probability is multimodal, as evidenced by additional two-dimensional histograms representing the probability of being the center of mass of the object, the most likely values for the largest/smallest object radii and their orientation, the most likely areas, and the deviation from a spherical shape. Two main peaks are identified. The main mode corresponds to a majority of samples wrapping around the object, while the second mode represents the contribution of additional large samples elongated in the direction of incidence of the incoming wave and represents an aberration of this imaging system, which uses only one incident wave. This effect may not be observed for other shapes, sizes, or light wavelengths – it depends on the geometry. Axis units are microns. Redrawn from [8].

Upon taking logarithms, maximizing the posterior probability of the parameter set $\boldsymbol{\nu}$ given the data $\mathbf{I}_{\text{meas}}$ is equivalent to minimizing the regularized cost functional [2]

$$J(\boldsymbol{\nu}) := \frac{1}{2}\|\mathbf{I}(\boldsymbol{\nu}) - \mathbf{I}_{\text{meas}}\|^2_{\boldsymbol{\Gamma}_{\text{n}}^{-1}} + \frac{1}{2}\|\boldsymbol{\nu} - \boldsymbol{\nu}_0\|^2_{\boldsymbol{\Gamma}_{\text{pr}}^{-1}}. \tag{10}$$

This is a nonlinear least-squares problem of the form previously considered in deterministic inversion, including regularization terms provided by the prior knowledge. We can solve it efficiently by using an adapted Levenberg–Marquardt–Fletcher iterative scheme [15]. Starting from $\boldsymbol{\nu}^0 = \boldsymbol{\nu}_0$, we set $\boldsymbol{\nu}^{k+1} = \boldsymbol{\nu}^k + \boldsymbol{\xi}^{k+1}$, where $\boldsymbol{\xi}^{k+1}$ is the solution of

$$(\mathbf{H}^{\text{GN}}_{\lambda_k}(\boldsymbol{\nu}^k) + \omega_k \operatorname{diag}(\mathbf{H}^{\text{GN}}_{\lambda_k}(\boldsymbol{\nu}^k)))\boldsymbol{\xi}^{k+1} = -\mathbf{g}_{\lambda_k}(\boldsymbol{\nu}^k). \tag{11}$$

Here, $\mathbf{H}^{\text{GN}}$ is the Gauss–Newton approximation to the Hessian of the functional (10) and $\mathbf{g}$ is its gradient, while λ_k is a scaling factor for $\boldsymbol{\Gamma}_{\text{pr}}^{-1}$ that balances the different orders of magnitude of the two terms defining the cost in the first iterations, and becomes equal to 1 at a certain point. At each step, the adjustable parameter $\omega_k > 0$ increases until the cost $J(\boldsymbol{\nu}^k)$ decreases, and decreases otherwise, making the iteration closer to Gauss–Newton or gradient schemes as required.

Linearization about the resulting MAP point $\boldsymbol{\nu}_{\text{MAP}}$ (the so-called Laplace approximation) provides an approximation of the posterior distribution by a Gaussian with mean $\boldsymbol{\nu}_{\text{MAP}}$ and posterior covariance $\boldsymbol{\Gamma}_{\text{po}} = \mathbf{H}^{\text{GN}}(\boldsymbol{\nu}_{\text{MAP}})^{-1}$. Sampling this Gaussian, we extract statistical information representing the dominant mode at a much lower computational cost, see Figure 6. Reaching $\boldsymbol{\nu}_{\text{MAP}}$ takes about 20 steps of scheme (11). The whole process, sampling included, is finished in a few minutes, instead of a few days.

We have considered κ_i fixed and known in these tests. In case it is constant and unknown, it becomes an additional parameter included in $\boldsymbol{\nu}$. In the end, we obtain additional histograms reflecting uncertainty about the value with highest probability [8].

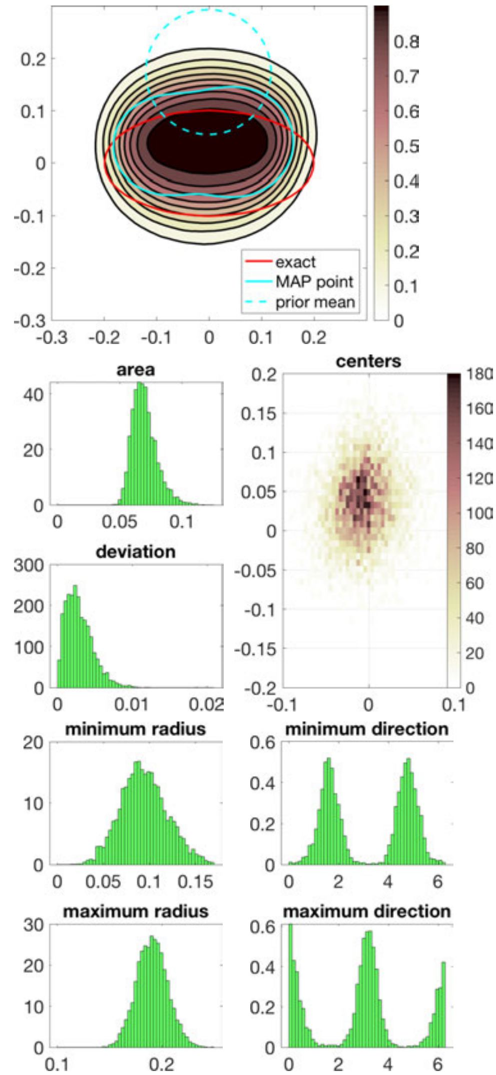

Figure 6. Counterpart of Figure 5 using the Laplace approximation of the posterior density. The main mode corresponding to the true object is captured. Axis units are microns. Redrawn from [8].

5 Perspectives

Digital holography poses challenging inverse problems which provide an opportunity to develop and test a variety of analytical and computational tools. First guesses of imaged objects are obtained by calculating the topological derivative of misfit functionals comparing the true hologram and the synthetic holograms that would be generated for different object configurations according to the selected forward model. Such guesses are robust to noise in the data. To reduce dimensionality, one can characterize the imaged objects by means of starshaped parametrizations. In a deterministic framework, we have shown that hybrid schemes combining iteratively regularized Gauss–Newton methods with topological derivative initializations and updates lead to good reconstructions of simple object configurations in a few steps, using stopping criteria that take into account the expected level of noise in the data. We are able to quantify uncertainty in such predictions by resorting to Bayesian formulations with topological priors. In two dimensions, Markov chain Monte Carlo methods provide a complete characterization of the posterior probability of the observed hologram being that is generated by a few starshaped objects. Three-dimensional tests are affordable for very simple shapes, such as a sphere or a cylinder [11]. Handling high-dimensional parametrizations, in three dimensions or just irregular shapes, requires the introduction of strategies to reduce the computational cost. Laplace approximations based on optimizing to find the highest probability parameter set and then linearizing the posterior probability about it to obtain a multivariate Gaussian distribution are useful tools for uncertainty quantification when there is a single dominant mode. Developing fast sampling methods which are robust as dimension grows would be an important step forward to handle more general situations.

Holography set-ups are particularly challenging due to the fact that a single incident wave is used. We have focused here on light imaging, though acoustic waves can also be used to resolve at different scales. We expect similar techniques to be useful in inverse scattering problems involving other types of waves and different emitter/receiver configurations, such as microwave imaging or elastography, for instance.

Acknowledgements. Thanks to Krzysztof Burnecki for the kind invitation to write this article. The author is indebted to the Kavli Institute Seminars at Harvard for the interdisciplinary communication environment that initiated this work, M. P. Brenner for hospitality while visiting Harvard University, R. E. Caflisch for hospitality while visiting the Courant Institute at New York University, V. N. Manoharan for discussions of light holography, and D. G. Grier for an introduction to acoustic holography. The work summarized here has been done in collaboration with Thomas G. Dimiduk (Harvard University and Tesla), María Luisa Rapún (Universidad Politécnica de Madrid), Virginia Selgas (Universidad de Oviedo), Frédérique Le Louër (Université de Technologie de Compiègne), Georg Stadler (New York University), and Sergei Iakunin (Universidad Complutense de Madrid, now at the Basque Center for Applied Mathematics). This research was partially supported by the FEDER/MICINN–AEI grants PID2020-112796RB-C21, MTM2017-84446-C2-1-R, Fundación Caja Madrid Mobility grants, and MICINN "Salvador de Madariaga" Mobility grant PRX18/00112.

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Left-right reversal illusion

Kokichi Sugihara

This article presents a class of 3D optical illusions, in which the apparent orientation of an object changes to the opposite in a mirror. We first show the mathematics behind such illusions, and then present a method for designing objects with desired appearances. Next, we show two simple subclasses, which can be realized using paper, and hence can be useful even for children to create their own illusion objects.

1 Introduction

The real world and its mirror image are plane-symmetric to each other with respect to the surface of the mirror. However, an object and its mirror image do not necessarily appear plane-symmetric because human visual perception can be distorted due to optical illusions [1, 3]. One typical case is a left-right reversal illusion [9]. An example is shown in Figure 1, where an object is placed on a desk and a mirror is placed vertically behind it. The object is an arrow pointing to the right, but it points to the left in the mirror. We call this class of objects the "left-right reversing objects." Their behaviors look impossible, and hence they belong to the class of "impossible objects" [11].

In this article, we focus on left-right reversing objects. We first show the mathematics behind them, and then present a method for designing such objects with desired appearances. Next, we exhibit two simple subclasses, which can be constructed using paper and hence can be useful even for children to create their own illusion objects.

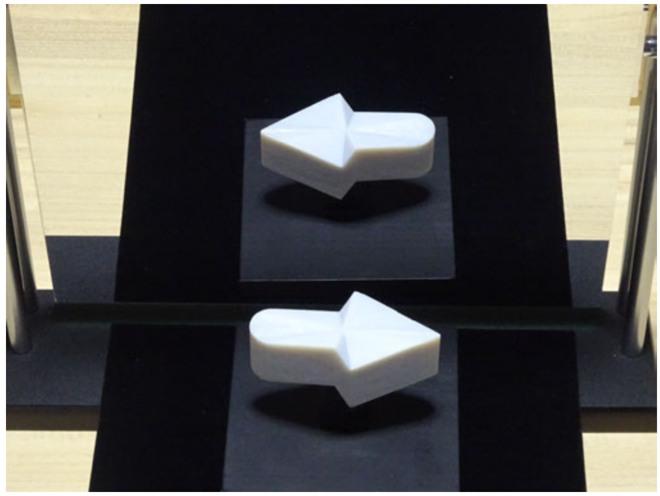


Figure 1. Arrow that changes direction when seen in a mirror.

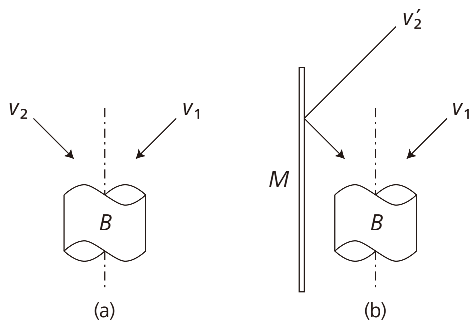


Figure 2. Line-symmetric object and two mutually opposite viewing directions.

2 Left-right reversal created by line-symmetric objects

Let B be a set of points in the 3D space, l be a straight line, and $\mathrm{rot}(B; l)$ be the set of points obtained by rotating B around l by 180°. If $\mathrm{rot}(B; l) = B$, B is said to be *line-symmetric* with respect to l, and l is called a *line of symmetry*.

Suppose that we fix a line-symmetric object B in space in such a way that the line of symmetry is vertical, as shown in Figure 2 (a), where the dot-dashed line represents the line of symmetry. Let v_1 and v_2 be two viewing directions that are parallel to a common plane containing the line of symmetry, with these views directed towards the object from opposite sides, with the same downward angle. Then B looks the same when we see it from the directions v_1 and v_2. This is the basic nature of a line-symmetric object. As shown in Figure 2 (b), if we use a vertical mirror M and see the reflected image along the direction v_2' instead of seeing it directly along v_2,

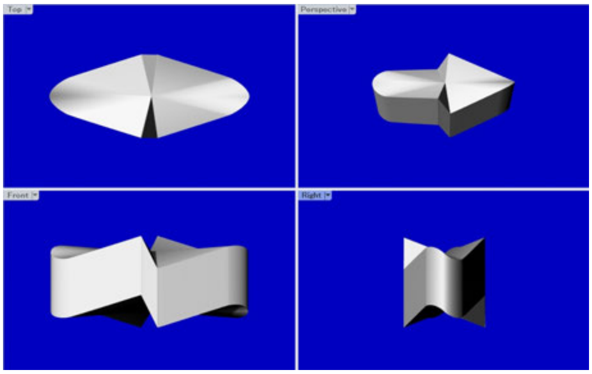


Figure 3. Computer graphics images of the object in Figure 1.

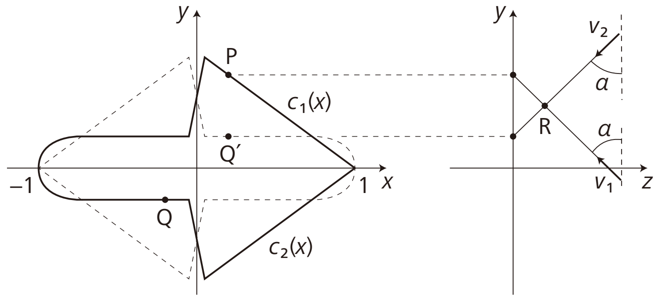


Figure 4. Construction of a left-right reversing object having a desired shape.

the left and right will be exchanged. Thus, if the original appearance along v_1 is a right-facing arrow, the appearance along v_2' will be a left-facing arrow.

Indeed, this optical process is seen in Figure 1. Figure 3 shows computer graphics images of the object. The left top is the plan view, from which we can see that the boundary of the object is point-symmetric with respect to the center. The line of symmetry of this object passes through the center and is perpendicular to the image plane. The left bottom image is the front view, and the right bottom image is the side view. From those images we may deduct that the object is line-symmetric with respect to a vertical line. The top right image shows the appearance along the special viewing direction, which makes the object look like a right-facing arrow.

As this example demonstrates, once we have a line-symmetric object, we can produce a left-right reversal illusion. The next question is how to create a line-symmetric object having a desired appearance.

3 How to produce a desired appearance

Let (x, y, z) be a Cartesian coordinate system in 3D space. Figure 4 shows this coordinate system in such a way that the left half shows the (x, y)-plane and the right half shows the (z, y)-plane, with the common vertical y-direction. Suppose that, as shown in the left part of Figure 4, we fix two x-monotone curves, $y = c_1(x)$ and $y = c_2(x)$, $-1 \le x \le 1$, on the (x, y)-plane that satisfy $c_1(-1) = c_2(-1)$, $c_1(1) = c_2(1)$, and $c_1(x) > c_2(x)$ for $-1 < x < 1$. Let S denote the closed curve composed of these two curves, and S′ denote the curve obtained when we rotate S by 180° around the z-axis. In Figure 4, S′ is represented by broken lines.

Our goal is to find the space curve, say T, that coincides with S when seen along the viewing direction $v_1 = (0, 1, -\tan\alpha)$ and with S′ when seen along the viewing direction $v_2 = (0, -1, -\tan\alpha)$. In Figure 4, we take the right-facing arrow as the curve S, and then $c_1(x) = -c_2(x)$, $-1 \le x \le 1$. However, this condition is not necessary in general; that is, the curve S is not necessarily symmetric with respect the x-axis.

For an arbitrary x, we consider two points $\mathrm{P} = (x, c_1(x), 0)$ and $\mathrm{Q} = (-x, c_2(-x), 0)$. When we rotate S by 180° around the origin, Q is transformed into $\mathrm{Q}' = (x, -c_2(-x), 0)$, and hence P and Q′ align along the same line parallel to the y-axis.

Let R denote the point that matches P when seen along v_1 and matches Q′ when seen along v_2. As shown on the right side of Figure 4, the point that matches P when seen along the direction v_1 is on a line passing through P and parallel to v_1. This point is represented by

$$z_1 = -(y - c_1(x)) \tan\alpha.$$

In its turn, the point that matches Q′ when seen in the direction v_2 is on the line passing through Q′ and parallel to v_2. This point is represented by

$$z_2 = (y + c_2(-x)) \tan\alpha.$$

The point R is obtained by setting $z_1 = z_2$ $(= z)$. This yields the formula

$$y = \frac{c_1(x) - c_2(x)}{2},$$

and substituting this expression in the formula for z_1, we obtain

$$z = \frac{c_1(x) + c_2(-x)}{2} \tan\alpha.$$

Finally, we obtain

$$R = \Big(x, \frac{c_1(x) - c_2(-x)}{2}, \frac{c_1(x) + c_2(-x)}{2} \tan\alpha\Big).$$

As x moves from -1 to 1, the point R traces a space curve that we denote by T_1. Let T_2 be the curve that is line-symmetric to T_1 with respect to the z-axis. Then

$$T_2 = \Big(-x, -\frac{c_1(x) - c_2(-x)}{2}, \frac{c_1(x) + c_2(-x)}{2} \tan\alpha\Big),$$

where $-1 \le x \le 1$.

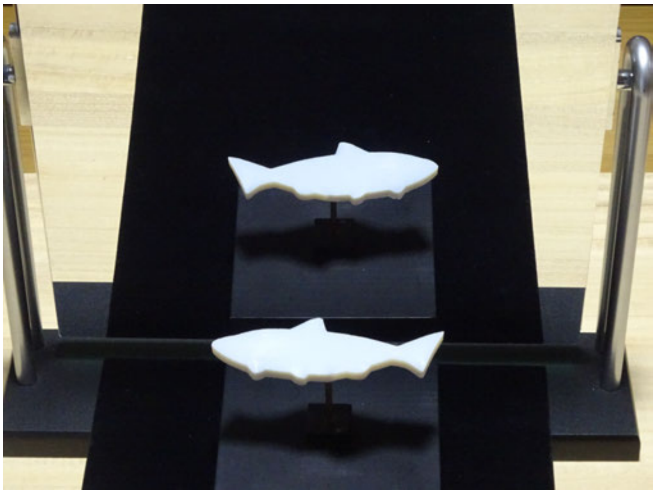


Figure 5. Left-right reversing fish.

The curves T_1 and T_2 together form a closed space curve denoted by T, which is our objective. That is, T coincides with S when seen along v_1 and coincides with S′ when seen along v_2.

We constructed the 3D object in Figure 1 by first computing the space curve T from the boundary of an arrow shape using the above method, then translating T in the vertical direction (the direction parallel to the z-axis) and obtaining the swept cylindrical surface, and finally by wrapping the top and the bottom with continuous surfaces.

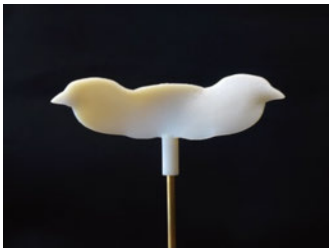
Figures 5, 6 and 7 show three more examples of left-right reversing objects computed by the method described above. The object in Figure 5 is a fish facing towards the left, which however faces towards the right in the mirror. Note that the upper and lower boundary curves of the fish shape are not symmetric with respect to the x-axis.

In Figure 6 (a), a bird faces towards the right, but its mirror image faces towards the left. In this case, we gave the initial boundary curve of the bird shape on the (x, z)-plane instead of on the (x, y)-plane, so that the resulting bird is almost vertical instead of being almost horizontal. Figure 6 (b) and (c) present the front view and the side view, respectively, of this object.

Figure 7 shows a jet airplane facing towards the left, but facing in the opposite direction in the mirror. In this case, the upper and lower boundaries are not x-monotone, but we can nevertheless construct the object. Indeed, the monotonicity condition is too strong; what we need is a one-to-one correspondence between the points on the given curve and those with the same x-coordinate on the 180° rotated curve.

In all examples, we first computed the space curve T, and then added the thickness by translating T in the direction perpendicular to the plane containing the initial curve S. Thus, the translation is vertical in Figures 5 and 7, and horizontal in Figure 6.

(a) Object and its mirror image



(b) Front view

(c) Side view

Figure 6. Left-right reversing bird.

Figure 7. Left-right reversing jet airplane.

4 Human factors of the illusion

Mathematically, the left-right reversal illusion is created by a line-symmetric 3D object. However, we also must consider human factors to strengthen the illusion.

In the objects shown in last section, we adopted a swept surface when the curve T moves along a straight line. The reason is as follows: T is a space curve and physically matches S when it is seen in the viewing direction v_1. However, there is no guarantee that S is perceived. One may perceive S, perceive T, or perceive any other curve that matches T in the viewing direction v_1. Thus, we need some additional trick for the viewer to perceive S instead of any other possible interpretations.

For this purpose, we used a remarkable characteristic of the human vision system, that is, the preference for rectangularity. The human brain prefers right angles to other angles when interpreting 2D pictures as 3D objects [4, 5, 12]. When we see a parallelogram, we are apt to interpret it as a rectangle seen in the slanted direction. This tendency is very strong and can be used to design various types of depth illusions, such as impossible motions [7], ambiguous cylinders [8], and topology-disturbing objects [10].

When T is translated vertically, the swept surface forms a cylinder whose height is the same wherever we measure it. Therefore, we may expect that the viewer interprets the top curve as the section obtained when we cut the cylinder by the plane perpendicular to the axis. This section is identical to the original curve S.

Another factor to note is the difference between a 3D object and its projected image. When we look at Figures 1, 5, 6 (a), and 7, most of us can enjoy the illusion without any special effort. However, we must note that these figures are 2D images taken by a camera. When we see an actual 3D object, in contrast, the illusion is not as strong because we have stereoscopic vision.

When we see a real object with two eyes, we can perceive the depth to the surface of the object by the triangulation principle [1, 2]. This function is called binocular stereoscopic vision. Hence, we can figure out the actual shape of the object relatively easily.

When we see an image taken by a camera, however, binocular stereoscopic vision is not a factor in perceiving the image. A camera has only one set of lenses, and hence taking a picture with a camera is equivalent to seeing an object with one eye while closing the other. As a result, our brain needs to choose some 3D structure among many possibilities and usually chooses one that has many right angles. For this reason, the left-right reversal illusion can be perceived more strongly when we see projected images than when we see actual objects.

5 Construction by rectangular cylinders

The left-right reversal illusion can be created when we construct a line-symmetric object. One simple way to accomplish this is to use a rectangular cylinder. The black lines in Figure 8 show a diagram of the unfolded surface of a rectangular cylinder. When we print it on a sheet of paper, fold it along the vertical lines, and glue it so that the left and right edges meet, we obtain a rectangular cylinder.

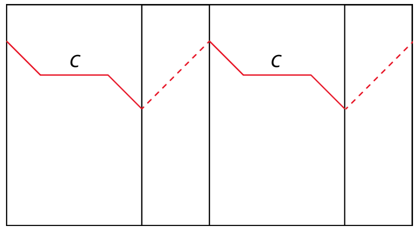


Figure 8. Unfolded surface of a rectangular cylinder.

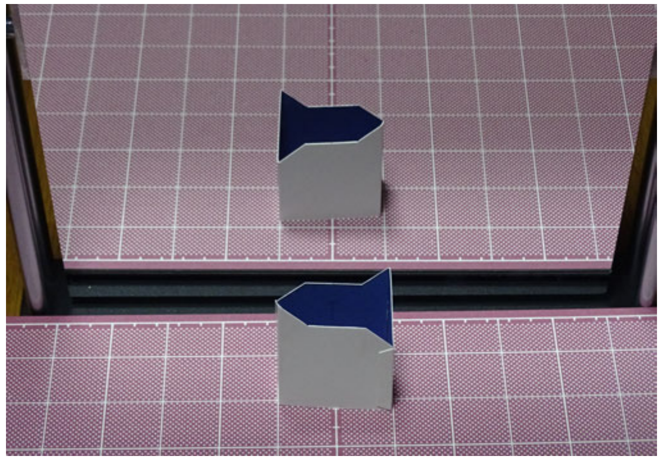
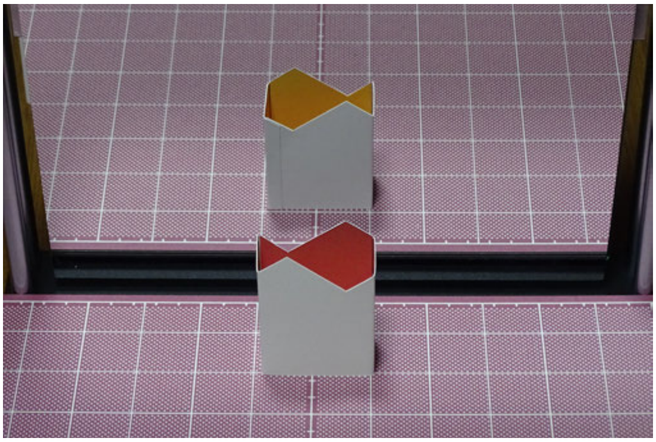


Figure 9. Left-right reversing rocket made from the diagram in Figure 8.

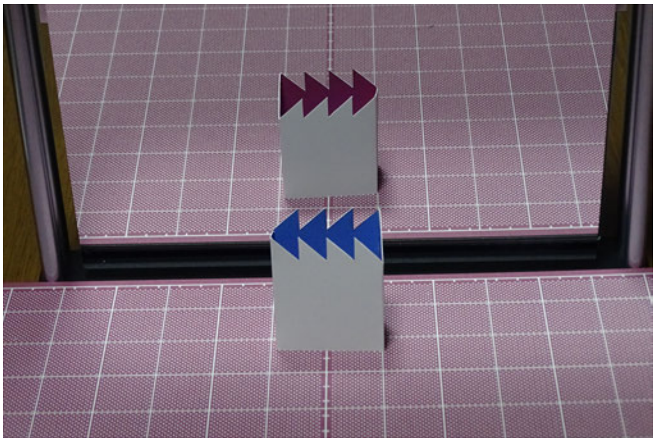
Next, as shown by the red lines in Figure 8, let us cut off the upper part in such a way that the leftmost side and the third side from the left are cut along the same curve, and the second and the fourth sides are cut along the straight lines connecting the end point of c and the starting point of the other c (the red broken lines in the figure). Then the resulting cylinder is a line-symmetric object whose line of symmetry is parallel to the axis of the cylinder and passes through the center of the rectangle section. Therefore, by placing the resulting cylinder vertically in front of a mirror and viewing it from a high angle, we can see the left-right reversal illusion.

Figure 9 shows the object constructed from the diagram in Figure 8. We painted the inner side of the cylinder in blue. The top of the cylinder appears to be a rocket facing towards the left, while it faces towards the right in the mirror. The thickness of the apparent shape as well as the lengths of the left- and right-side edges depend on the slant angle α along which we look down at the object. In the case of Figure 9, we adjusted the viewing angle so that one of the side edges degenerates to a point and consequently the head of the rocket forms a sharp corner.

Figure 10 displays two more examples: (a) shows a fish and (b) shows a cascade of arrows. The colors in the mirror change simply because the inner side of each cylinder was painted in two colors.



(a) Fish



(b) Cascade of arrows

Figure 10. Left-right reversal illusion made by rectangular cylinders.

The method is very simple. We need to use the same curve twice, as shown by the points labeled c in Figure 8. Therefore, it might be fun even for children to search for the curves that can create their own original shapes.

6 Construction by pictures

Another simple way to construct a line-symmetric object is to draw a picture. Figure 11 shows an example. The direct view of the drawing looks like a staircase going up from left to right, while in the mirror it goes up from right to left. Not only the staircase but also the upper and lower floors and walls are all left-right reversed in the mirror.

The drawing used in Figure 11 is shown in Figure 12. Note that this drawing is point-symmetric: if we rotate it by 180° around the center, we get the same picture as the initial image. This in turn means that the picture is line-symmetric with respect to the line

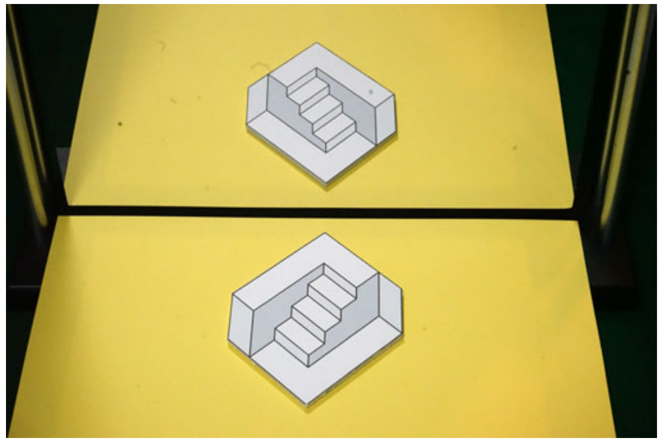


Figure 11. Left-right reversing drawing of a staircase.

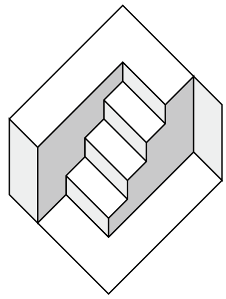


Figure 12. Drawing used in Figure 11.

that passes through the center of the drawing and is perpendicular to the picture plane. Therefore, the left and the right are reversed in the mirror because of the same reason as described above.

However, the perceptual process is a little more complicated because our brain automatically interprets 2D pictures as 3D objects when we look at the scene in Figure 11. If one does not interpret the drawing as a 3D structure, one could easily understand Figure 11, because the drawing in Figure 12 is just reflected by the mirror. In fact, the nearest point of the drawing is mapped to the farthest point in the mirror. However, the human brain has a stronger preference for rectangles than for general parallelograms [4, 5, 12]. Thus, when we look at Figure 11, our brain perceives a 3D object instead of a 2D drawing, and realizes that the mirror image is inconsistent.

Another example is shown in Figure 13. This object is also a horizontally placed drawing, and it is point-symmetric with respect to the center, and hence line-symmetric with respect to a vertical line. However, it is a perspective projection instead of the orthographic projection of a 3D object, and consequently, the impression of 3D structure is strong. Physically, the nearest part is mapped to the farthest part in the mirror. However, because we interpret the nearest part as the lowest part of the 3D structure, we try to find the corresponding counterpart around the nearest area in the mirror and fall into an inconsistent perception.

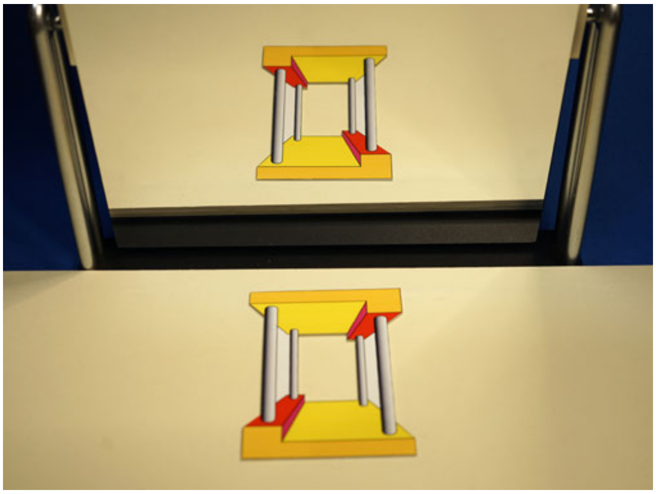


Figure 13. Another left-right reversing illusion made with a drawing.

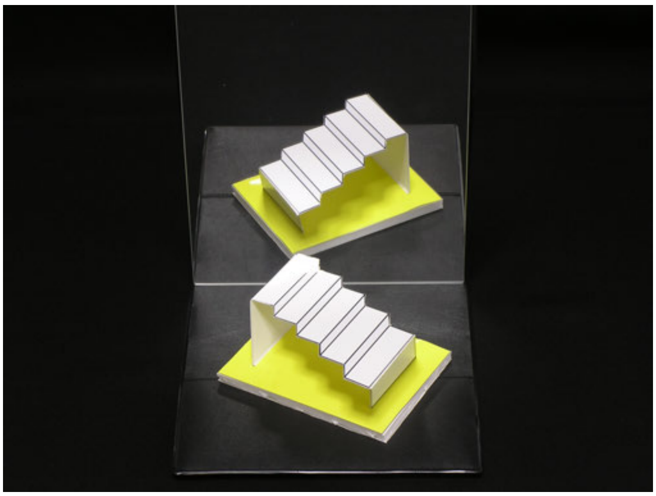


Figure 14. Mixture of a horizontal drawing (staircase) and non-horizontal structure (supporting walls).

The final example, shown in Figure 14, is a mixture of a horizontal drawing and an actual 3D structure. The staircase is a drawing fixed horizontally, with only the side walls not horizontal. The whole structure is line-symmetric with respect to a vertical line, and as a result, we can perceive the left-right reversal illusion.

7 Concluding remarks

We have demonstrated how a line-symmetric 3D structure can create a left-right reversal illusion and presented a method for designing illusion objects with desired appearances. We also presented two simple classes of line-symmetric structures: rectangular cylinders and point-symmetric 2D pictures. These simple classes can offer material for anyone, even for children, to create their own illusion objects, and to experience the illusion.

From an educational point of view, these two simple classes of illusion objects enjoy several advantages. First, children can create their own objects instead of just being handed existing objects. This may stimulate their active involvement. Second, children can experience the illusion using real 3D objects instead of just viewing images taken by a camera. This should provide an opportunity to understand the difference between seeing objects and seeing their images, and thus help children understand the importance of having two eyes. Third, illusion objects can give children an opportunity to understand the power of mathematics, which provides a framework to create illusions in a systematic manner rather than by heuristics.

Optical illusions in general cover a wide range of visual phenomena, including misperception of size, orientation, shape, color, brightness, and motion. Among them, the depth illusion, which includes the left-right reversal illusion, is remarkable in that the mechanism can be understood from a mathematical point of view more clearly than other classes of optical illusions. Indeed, the interpretation of 2D retinal images as 3D objects is an ill-defined problem, in the sense that the answer is not unique [2, 6], and by observing the behavior of human visual perception, we can guess what kind of possibilities are chosen more frequently. This understanding also helps us create new optical illusions. The left-right reversal illusion is one of the common illusions that can be discovered using mathematics.

Acknowledgments. This work was supported by the Japan Society for the Promotion of Science, KAKENHI Grant Nos. JP21H03530 and JP21K19801.

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Kokichi Sugihara is a Meiji University distinguished professor emeritus. His research area is mathematical engineering. In his work on computer vision, he found a method for constructing 3D objects from "impossible figures," and extended his research interests to human vision and optical illusions. He is acting also as an illusion artist by creating various impossible objects. He won the first prize four times in the Best Illusion of the Year Contest.

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Interview with Abel laureate 2022 Dennis Sullivan

Bjørn Ian Dundas and Christian F. Skau

Professor Sullivan, first we want to congratulate you on being awarded the Abel Prize for 2022 for your groundbreaking contributions to topology in its broadest sense and, in particular, its algebraic, geometric and dynamical aspects. You will receive the Abel Prize from His Majesty the King of Norway tomorrow.

Thank you!

You have worked in very many different fields, and, actually, your supervisor, William Browder, described you as sort of an intellectual vacuum cleaner. But it seems that you always had a guiding principle for what you are doing. If mathematics rests upon two pillars: space *and* number*, you have been partial to space to the extent that you want to replace number by space.*

A part of this quest of yours is the question: "What is a manifold?" And that is perhaps a good place to start; before we continue on your journey, as you say, from the outside to the inside, intuitively: what is a manifold?

It is *space*, expressed logically in terms of a set of points.

It's space, but it's sort of a special space, isn't it?

No. The idea of space is that you can move things around. There isn't an invisible wall that makes you stop here, but you can move around. Any object which is locally like that is called a manifold. Space itself is an intuitive word, that we all know about. But there is an actual concept called manifold, which is the logical version of that intuitive concept. It's an attractive notion when you first learn about it as a math student. And the first math theory about these manifolds that I learned about was sort of strange.

Tell us!

You attached to such an object, which you didn't really describe in terms of its logical definition, some other objects which were very abstract and part of algebraic topology. And when you had enough of those with the right conditions, you could build the manifold.

So you could actually reconstruct the manifold from these abstract objects?

You could build it up to equivalence. But you didn't really construct the points of the manifold in a canonical way. So, it has no points. It was like a black box. The information is stored there. And that is where numbers come in; all these concepts are based on numbers, the algebra, whereas the actual *texture of space* is not there.

Is it like the recipe for the cake versus the cake?

Yeah, I'd say it's exactly like that; it's a good idea. You must have prepared that?

No, we did not!

It's a very good interpretation. It's like a cake with no edges or layers. It's just this delicious cake going on for ever, right?

And you really want to get at the cake?

Well, that is what you are attracted to, the idea of space and its texture. And then, it turned out, that every time I would ask a professor a question, he gave me an answer that was in terms of number, which is algebraic topology and homotopy theory. So I had to learn that, as it were. I adjusted the geometrical problem so that it fitted with the numbers, so to speak. You know, some goals are not achievable and some are within reach, so I adjusted to get the ones within reach during that period.

Is what you describe here more or less what is called surgery, where you actually build the space according to the prescription?

Right, you have a prescription of the information: how many holes it has, how many handles etc., and you build an actual manifold with that description. And surgery allows you to build it. That was a powerful technique. Actually, it was a secondary technique following Thom's cobordism theory, which was very influential.

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Abel Prize laureate Dennis Sullivan receives the Abel Prize from His Majesty, King Harald of Norway.

But the important distinction here is between what can be deformed and pushed, well, in homotopy theory, in the homotopy type – to use technical jargon – as opposed to the actual manifold?

Right. First, it's interesting that the classification of closed manifolds is an interesting subject. It's not, a priori, clear that it will be so, but it's extremely interesting classifying manifolds that are closed. You know, no boundary, not going off to infinity.

Classically, one knows the classification for surfaces. That goes back to Abel and Riemann. They figured that out. The sphere, the genus number, abelian functions, abelian differentials, and so on. But already Poincaré discovered that in dimension three it's much more complex. And then it gets more and more complex as the dimension goes up.

It was kind of interesting that there is enough "number machinery", so to speak, to understand spaces of dimension five and higher. That was an amazing development, basically due to Thom, I would say, who started this, and surgery was completing this story. And I got in on the last big boat heading to … wherever.

With the surgery exact sequence?

Kind of. Browder – you mentioned Browder – he was presenting this theory. And it was in a complicated form. You could sort of change it around a little bit and get it simpler. And then you could see from the changed picture which areas could be developed completely. The smooth structure is still open, in some sense, up to finiteness. I mean, we know all the infinite part for the smooth structures.

And that is an area where the previous Abel Prize winner Milnor had a huge impact.

Yeah, on that one, certainly. His 1963 paper with Kervaire was my math bible.

This actually leads us to your thesis in Princeton. Princeton must have been a fascinating place to be at that time?

Absolutely! All these famous people around with their expertise.

So you could just ask them?

Yeah, you could just ask them every day at tea, you didn't have to make an appointment, because they all came to tea. You could ask them anything you wanted to.

There is a cute story about when you are closing in on your thesis, and you had a discussion with Milnor. Could you tell it?

Well, I had this sequence of steps, and if I could do them all, I could solve what I wanted. But each step had a clear surgery part and then it had a Milnor exotic sphere part. I didn't know how they where linked together, so I went into his office, because I had a serious question. At tea you could ask any question, but this was serious. He looked at it and said: "Why don't you just forget the Milnor-part". He didn't say it that way, but something like: "Why don't you just forget the exotic sphere part, and just do the first part." And this worked for piecewise differentiable manifolds.

So this is the combinatorial manifold case?

Yes, I call them combinatorial manifolds, or PL manifolds, or piecewise differentiable manifolds. You allow the differential structure to break, but you keep the combinatorial structure. And I said: "Thank you Professor Milnor", but I thought: "Oh, these piecewise linear manifolds, I like *smooth* manifolds, but these are just piecewise linear." And I thought about it: "Wait a minute, if I do that, I know the structure completely! That is, I know the local structure completely." And I just had to figure out the global structure, which took another year, but then it solved the whole problem.

And you asked your thesis advisor Browder: "Can this go into my thesis?"

Well, yeah, that's right. I asked him: "I have this sequence of steps, which have these coefficients, and if you can do all the steps you get this result. Can that be a part of my thesis?" And he said: "Well, I guess that *is* your thesis."

And that answered a long standing question that people had been wondering about for quite a while? We are thinking about the so-called Hauptvermutung.

That was actually the driving engine. There was this more famous question about whether the combinatorial structure was uniquely determined by the topological structure. And that was called the Hauptvermutung. And it turned out that whenever I could understand the theory of what I was discussing completely, I could use the technique of Novikov to prove my list of numbers were zero.

The next eight months was like a race, it was really a race against reality. Every time I could understand this global theory better, I could prove the Hauptvermutung. It turned out that I could prove everything was zero except one little thing in dimension four that wasn't zero, but had order two, and that was it. A few years later they actually found counterexamples in that little place there. So I proved as much as one could.

What you call "that little place" is an obstruction group in dimension four, right?

Yes, that was my obstruction group in dimension four. In a sense, that isn't the way I work. Well, I would love it if I could solve a well-known question, but I really like *understanding* things better. So, I actually like the theory that says that these are all the piecewise linear manifolds in a given homotopy type, and you can compute these numbers and then you know which one you have, and that is a complete discussion. It turns out that 99 out of a 100 of those numbers are also topological invariants. So you get this corollary. People today only know the corollary. And now they even have a simpler proof, so everything I have done is forgotten! So I'm glad I get this Prize so I can talk about it again.

Immediately from there you move on and do other amazing stuff. You discover that the Galois group has important consequences for the study of manifolds. Indeed, you solve a famous conjecture that way. Could you elaborate on that, focusing on the manifold aspect of it? Specifically, how come you have a Galois action on manifolds, it doesn't seem reasonable at all.

I would say that it's still not understood. In other words, there was this list of invariants – I'm simplifying it a little bit – but a big part of that list could be collected into one element in K-theory. And K-theory has this symmetry, the Adams operations. One knows that when you look at the roots of unity in the complex numbers, that is if you add the roots of unity and form that field, that gives you the abelian part of the Galois group. And the symmetry of those fields, more precisely, you have to complete the manifold theory – it's technically a little strange to topologists and geometers – you complete the number aspect of manifolds so to speak, and that has symmetry exactly the abelian part of the big Galois group. So we have Abel and Galois together.

And that symmetry exists in K-theory, so it acts on the invariants of manifolds. So, the manifolds were just given the information, the homotopy type and these other numerical invariants, and the Galois group acted on these invariants, and therefore it acted on the manifolds. That is how it came about. It doesn't come about in a natural explicit geometric way, and that gave rise to this Jugendtraum, or dream of youth, a term coined by Kronecker in a different context. This Jugendtraum, explaining this in elementary terms, is still open.

How can we view manifolds? As we would view algebraic varieties?

It's a little strange, you see. If you think of usual algebraic varieties with real numbers and complex numbers, they are normal topological spaces. And this topology comes from the topology of complex numbers or the real numbers, right? The Galois group doesn't preserve that topology. A lesson from algebraic geometry is that to understand things that are defined in terms of integers it is best understood by looking at each prime and looking at the real completion, and view the information that way. The "intersection" of all this information gives the integral information. It's kind of sophisticated. This was actually too much for my topological colleagues. They didn't want to hear about it. The geometric topologists, not the homotopy theorists. The homotopy theorists – they loved it!

So, you are assembling all this information, one prime at a time, plus the rational information?

Yeah, for a manifold the finite prime part splits into the prime two and all the odd primes. Individual odd primes behave the same way. Because of the Poincaré duality, it's like a quadratic form. It's well known that quadratic forms behave differently at the prime two than at the odd primes.

Could we for a moment segue into a different topic, though still associated with the name Poincaré. We are thinking of the term Poincaré moment, which refers to the experience Poincaré himself described where he in a flash saw the solution to a problem he had worked on for months. Have you had such Poincaré moments?

I search for them all the time, but they come very seldom.

Could you tell us about the fascinating experience you had when you were about to take the oral exam as part of your PhD?

Oh, yeah, yes, right. There is a little book by Milnor called *Topology from a differentiable viewpoint*. About how you could do all of the usual things, you know, the Königsberg bridge problem, continuing to Betti numbers, etc., etc. You could do all that more geometrically using smooth functions and regular values, preimages of the nice points, submanifolds and stuff like that. That was Milnor's beautiful description of the Thom theory from 1953, okay? So, we were studying that for the orals, and I knew it forwards and backwards, I could answer any question.

I was walking in to take the exam, and thought: "Let me look at it one more time before the exam". I went to the library, opened the book, looked at it. It's a small book, it's got ten theorems in it. But still, there are a lot of steps, and I was looking at it one more time, and then this basic picture appeared to me: You have a map to something like a sphere, and you take the preimage of a point – which is what is called a nice value – you get a nice submanifold by the Implicit Function Theorem. You get local coordinates, and then the neighborhood sort of funnels down, like you would push a slinky down and flatten it out completely. But this was saying something about the global map: There is the preimage of one point, and then I noticed: "Oh, wait a minute, the preimage of one point has all the information." The complement may be very complicated in the domain, but the complement of a point or a disk in the image sphere is contractible. It's like taking a point out of a balloon, it contracts, it's contractible! So you can extend the mapping to the contractible part uniquely. Any choice you make will be related by deformation to any other choice.

Dennis Parnell Sullivan – 2022 Abel Prize laureate.

Suddenly the whole book, or the whole theory, became clear. It just follows from this picture, from this slinky picture, with the logical remark that the complement here is contractible, so there is no more information. That is just pure logic, plus this simple picture. The whole book fell away, the entire theory fell away. If I got amnesia but was left with that picture in my mind, I could reproduce the whole book and the whole theory. And then I thought: "This is what it means to understand mathematics". I was a graduate student! So, I want to feel this again!

And have you?

Yes! However, it takes longer and longer.

Of your other main results in this area, is there any one that has such a picture in your mind, where you actually see the entire theory?

Well, I mean, basically this sequence of steps things I was talking about, where you take preimages and use this picture, I kept using it. For example you know how a screwdriver works, it goes into the slot and you turn it. You can take apart this house, you know. I mean, you can do anything. You have to have a simple tool, you have to understand it, and then use it. Well, that wasn't exactly a Poincaré moment.

The Poincaré moment I was thinking of, when you said that, was when he put his foot on the bus and he realized that the holomorphic bijections of the unit disk were the same as the symmetries or the congruences of non-euclidean geometry. And that was a fantastic connection. He knew both things. But, in a sense, the connection is the moment. This largely dictated the next century, and all the work of Thurston and so on.

But you must have had a similar experience, when you proved the Adams conjecture. You've commented that it wasn't really important that the Galois action corresponded to the Adams operations. Still, it must have been very important to you at the time when you were trying to solve the Adams conjecture, that they were the same. That must have been a revelation, that that actually could be true?

Well, it's not my creation, it was Quillen's observation that somehow these Adams operations, whatever they are, let's just say they are some symmetries of something that relates to manifolds and space.

The symmetry is related to the fact that, when you are working in the field of algebra, you may assume that p times anything is 0, where p is a prime, like 3 times anything is zero, 3 being the prime. There is an amazing fact that if you work, for example, with 3 times something is 0, and you take a number x and you cube it, and you take another number y and cube it, then if you add the two and then cube the sum of the two numbers, you get the same thing: $x^3 + y^3 = (x + y)^3$. This is because of what the binomial coefficient theorem says, that you get these 1, 3, 3, 1-terms, but 3 is zero, so you get 1 and 1. That shows that you have this symmetry in each of these prime worlds. So, you have this additional symmetry given by what is called the Frobenius automorphism. That is fantastic!

Quillen had already suggested that there is a relation between the Adams conjecture and Frobenius, but then that was a little too exotic for me. I wanted to *use* the answer to the Adams conjecture, I didn't want to prove it. And then I heard – I hadn't met him yet – that he wasn't going to work on it, because he first had to learn 200 pages of Grothendieck and transfer it into his setting. Okay, he only wrote perfect papers, it had to be perfect, or else he didn't write it.

It's Quillen you are talking about, right?

Yes, it's Quillen. Now I'm adding what I found out later, as I read more of his work: every paper is perfect. Perfect isn't the right word, it's optimal. You can't do better. So, I heard about this, and I said: "Okay, I'm going to pretend that this is true, because Quillen made this connection, and he could have written the proof out." And then I said: "But wait a minute, I can't just pretend that this is true, I've got to prove it myself." But if it's true, it's easier to prove. Because you know it's true. It's a topological theorem, so I just kept working on it.

I worked on it for six months, which in those times was a really long time because things were happening faster. I reduced it to something – it was equivalent to something – and then I tried for a long time to prove this something, but I couldn't do it. And then: I remember sitting on the lawn, I remember exactly that moment, August 19, 1967. I had just driven up from Mexico with my family to Berkeley. I was going to spend two years there. I was sitting on the lawn of the house where we were staying for a few days until we got our own place, and I thought: "What has Quillen said about this?" He said: "Frobenius! algebraic symmetry! at the primes!" It

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turned out that it gave my condition immediately, and so I had a proof of the Adams conjecture.

In some sense that was a Poincaré moment. It took me a year to write out the details. There were different details, less foreboding than what Quillen had envisaged, so I was able to do it.

And that spawned the so-called MIT notes, which became widely circulated and famous?

That spawned the MIT notes, yeah. You have to first localize, then complete and then do all the related homotopy theory.

And then you moved on to the quasiconformal manifolds and Lipschitz conditions. How did that transition happen?

You sort of skipped about ten years … but we don't have so many hours!

Yeah, we agreed to skip the rational homotopy theory, which really hurts, but …

Okay, but let me make one point about that. Algebraic geometry and stuff like that just does the finite primes. It turns out that all the information in this algebraic topology which is determined at the primes, has this extra symmetry in it, which is related to algebraic geometry. But then I thought: "Wait a minute, what about the infinite prime, the archimedean place?" I didn't know any analysis, or anything like that. "But, maybe it has to do with differential forms?" And it turned out that it did. It's sort of like algebra does *this* part, analysis and geometry do *this* part.

Which does open analysis to all of the rational theory.

Right!

And you then prove that cohomology in many situations determines the entire rational type; Kähler manifolds.

Yeah, it had nice corollaries. The idea was to express the information in terms that are natural. It's natural to express the information of the infinite part, rational numbers, real numbers, in terms of differential forms, which is natural for analysis and geometry.

So you have this information for the primes with the Galois action and you have analysis on the differential forms for the infinite piece?

All of which is related to topology, right. But then, to go on, all of this was frustrating, because it was outside the manifold. They were sort of invariants. I liked facts about things inside the manifold. Foliations or dynamical systems and fractal sets, these things are inside the manifold and they are constructed by infinite processes inside the manifold. So I started to learn about these infinite processes.

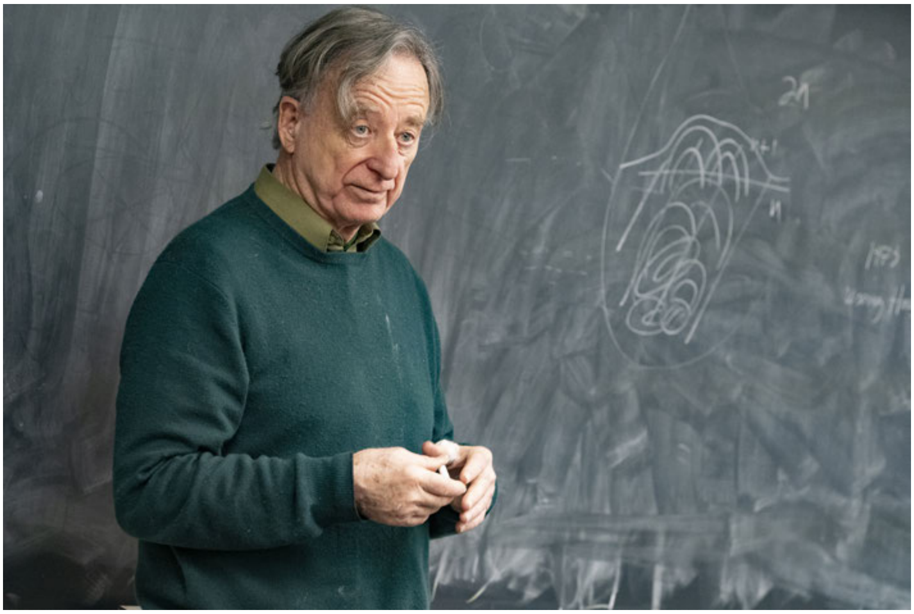
That began the dynamics part. It was sort of like just following this interest inside, there was no logical reason. I was starting over as a graduate student again, I'd say. It turns out that the best way to understand the holomorphic part of manifold theory in dimension two is not through the smooth structure, but in terms of the quasiconformal structure. That is the best way to understand dimension two. And it's amenable to certain infinite fractal processes. Anyway, it was natural to leave this highly sophisticated algebraic viewpoint and go back to the original interest in manifolds, like dynamics – and processes like dynamics – inside the manifold. I mean, physical processes take place in space, so this is all about everything else in science. You know, even medicine; your body has tubes with fluids and so on.

Let's talk a little more about these dynamical systems and their importance in studying manifolds. Perhaps we could start with something very concrete, namely Denjoy's answer in the 1930s to a question posed by Poincaré about circle diffeomorphisms without periodic points. This was taken up and extended enormously in the '70s by Michel Herman and his student Yoccoz, answering, among other things, a question posed by Arnold. With this as background, could we ask you how this theory impacted your desire, so to speak, to understand things inside the manifold? This in contrast to the picture you give of manifolds locally being like a puddle of milk looked at from the outside – there isn't much personality.

Let me answer this by first posing the question: "Why is it interesting to know about manifolds?" It's all about space. Okay, we have done the number aspect, but why is it really interesting? Well, all the processes that we see go on in space. All that stuff that is described by various other fields, ODEs, partial differential equations, functional analysis, that's all part of describing the processes. It's also combinatorics, computer algorithms. All that is about processes in time, but all these processes in time go on in space.

I didn't know all that then, but I wanted to know more about things going on inside manifolds. A little dynamical system could create an interesting fractal set inside the manifold. And if you perturb that dynamical system, that fractal set was still there. It was structurally stable. So I had to learn about things such as Cantor sets, fractals and stuff. So I started and I'd say it was almost a ten year period of time before I got to quasiconformal mappings.

This was at the end of the '70s. I was thinking about dynamics and foliations, like this idea of an onion that is foliated. That is a very attractive picture, and these were interesting objects. Thurston had arrived on the scene, and he blew everybody's mind away, including mine. Immodestly, I have to say that I was smart enough to appreciate that I was watching Mozart playing the piano. I mean, not everyone did, because Thurston wasn't so communicative.



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But he was one of your heroes along with Thom, wasn't he?

Yes, but he was younger than I was, he was my younger brother hero. All this fitted with this desire of mine to go inside the manifolds, and understand more geometric things. So I started studying dynamics, and I learned about the Smale school. And then, in France, I started going to Michel Herman's lectures, and I met Yoccoz, his student. Michel Herman was working on the problem you alluded to in your question. It happened like this: Denjoy died in 1974, and Michel Herman was working on his papers for the French Mathematical Society. Herman started to talk about the Denjoy argument. So, I learned that argument. And then Herman started answering these questions, refining what Denjoy had done. You have to remember that Poincaré was doing celestial mechanics, in particular, the three body problem. He came up with this question that was answered by Denjoy, who did this a couple of decades after Poincaré died, actually.

This is all about one dimensional manifolds. It turns out that they are actually among the hardest from this interior point of view. They are very difficult. Herman analyzed the very fine structure of diffeomorphisms of the circle, and we were learning as he was producing results. I was just intrigued about it. For example, there is a beautiful example involving the golden ratio number and Fibonacci numbers, and that intrigued me.

And this is while you are at IHÉS?

At IHÉS, yes. He was at Orsay, which was just a walk across the valley of the Yvette. The interesting thing about the real line is that there are three kinds of distortions that behave algebraically very nicely. There's the metric distance distortion, the ratio distance distortion and the cross ratio distortion; corresponding respectively to metric geometry, affine geometry and projective geometry. And there is the usual chain rule. You take the logarithm of that, it's a nice formula under composition, and now you can do two other compositions with these higher distortions. Those were the key things that I used to explain Herman's work to myself.

Michel Herman's theorem took a whole volume of Publications Mathématiques de l'IHÉS, and I wanted to get it down to something like just a few key moments of understanding. And you could – after a couple of years thinking about it – get it down to something you could tell on the phone to somebody. That was my challenge: Find a proof that I could tell to somebody on the phone. You have to understand it, you can't write down a lot of formulas and calculate and stuff like that, you have to understand it. It was just like that, the desire to understand, and it was just like fun, you know.

But then, in '82, I heard that physicists had discovered something startling related to phase change. You know, the water gets colder and colder, and suddenly it forms this crystal, right? It's when all this rigidity happens. That is called phase change. There are a lot of situations where that happens in physics. It turned out that physicists had calculated one such in a dynamics example, where you adjust a certain parameter to the freezing point, I'd say, and then you get this incredible thing: It could have depended on infinitely many parameters, and it doesn't depend on anything at all, it's universal!

That was what Feigenbaum first discovered, right?

Feigenbaum discovered that there was this *rate* (from the other direction). Then other physicists discovered – and Feigenbaum too, actually; he hadn't communicated it as well as the other ones – that it was this intrinsic geometry, like a crystal, I'd say.

What was interesting about this for me was that there were not enough techniques available to *prove* this at the time. It was numerically calculated. You can take this formula and that formula, and do this infinite process, calculate and – bingo! The Hausdorff dimension is 0.5308…, or something like that. So, here is a theorem that is true and it is precisely formulated. True with quotes, because it was numerically true. The available techniques weren't enough to prove it. It turned out that you just had to add three more things to the Michel Herman and Yoccoz stuff, and then you could prove it. But it took eight years.

The idea was, I could stop whatever I was doing, and just work on this, there wouldn't be any counterexamples, you know. And a proof would need new math.

And you were the one that came up with a proof?

Yeah, I found it and it took eight years.

And that was in '82?

It was in '90. It was '82 when I heard about it.

And in the meantime …

… in the meantime? I was just working on this. There might be other things that appeared in print, but I wasn't working on anything else.

For instance the non-wandering-domain theorem?

No, that is '81.

It was published in '85?

No no no, that was already over. I *was* in quasiconformal mappings; Ahlfors and Bers' theory goes into dynamics. That was already *fait accompli* by 1980.

That must have been very inspirational that you got this result about non-wandering sets.

It was sort of obvious. It was obvious from the understanding.

But it wasn't obvious to Fatou.

No, but he didn't have this theory of quasiconformal mappings, this deformation theory.

It must have been very satisfactory for you to prove that?

Well, no it's not, no no, you have misread me. These prizes and stuff are nice, but that's not the point. It's not the point to solve a problem, the point is to understand. And by this point, by the time, you understood what Ahlfors and Bers were doing, it was like a Poincaré moment, where you say: "This theory here could be very useful in this other theory". These are disjoint universes, and do this Fatou–Julia thing, and just transfer the technique over.

Are you now talking about your dictionary?

That is the first entry of my dictionary, right.

In the paper where you prove the non-wandering domain you state the dictionary in the introduction. But do you use your dictionary in order to prove, say, the non-wandering result?

I do. There is something called the Ahlfors finiteness theorem, and you take what makes that work, and you restructure it over in this other domain. It was really using the comparison, the correspondence.

The non-wandering result, the Fatou theorem, corresponds to a known theorem in this Kleinian group category. It's about the idea of understanding, not the names, not what field it is, but what is the math idea. The math idea is the same *here* and *here*.

Is this like you were telling us a moment ago, that once you know something is true, it's way easier to prove it? Was the dictionary some sort of guidance in that respect – you knew what would be true?

No, it's like when you arrange a party: you have to have enough drinks, enough food. I mean, you have to have enough stuff. You have to accommodate the correspondence. In retrospect you can say that the Fatou problem corresponds to something known over here, in Ahlfors and Bers, okay?

The underlying math is the same, and that is satisfying. But it was so obvious, it wasn't exciting. The idea is, if you think in terms of structures, the structure here and the structure there were the same, two examples, the same structure.

So we were talking about your dictionary between the Kleinian groups and quadratic or complex dynamics, if you like, right?

That is one item in the dictionary. The dictionary says: "For every item *here*, there should be a corresponding item *here*, because the basic elements of the two universes are the same. In fact, I once introduced Bers at a conference to Mostow. Bers asked: "Why are you introducing us? We've know each other for years, we're close friends, but we never talk math". Like he said it proudly. I said:

"Well, I have this one theorem. If you do *this* it is Mostow's theorem, if you do *this* it is your theorem."

How did they react to that?

You know, people are in their comfortable world, it's already rich and beautiful, they are happy there. I'm not like that, when I start to understand something, I start wanting to move sideways, somehow.

So, you have the dictionary and what you're telling us is that the underlying mathematics of the two things are the same. But not for any particular reason; it's just the same? It occasionally happens that you have two different mathematical problems, and the way you handle them, or the way their combinatorics work is just the same, for no apparent reason.

No. The question is: "what are the basic elements that are involved in the mathematics, in each situation?" In this case there is dynamics which has a certain form actually, a technical form called hyperfiniteness, related to von Neumann algebras, and also it has to do with Riemann's ideas of deforming the complex structure. Okay, so those are the two ideas.

There is an underlying complex structure, that is preserved by the dynamics. These are called holomorphic dynamical systems. This technique can be used in the entire field. But before this happened there was a field called Fatou–Julia theory and one separate field involving Poincaré limit sets and domains of discontinuity and so on. These were two different fields. *This* was occupied by complex analysts, and *this* was occupied, in modern time, by dynamical systems people. The basic elements of the underlying discussion were the same. Every advance here should correspond to something over there.

It's just to look at things in simple terms, without the words. I don't let my graduate students use names, they can't use any proper name. They have to say, in an English sentence, in terms of basic concepts, like linear algebra or integers what the hell they are talking about. And I slap them around if they don't, verbally.

You are known to be very broad in your interests in mathematics, and you see connections that other people do not see. But could we ask you a provocative question: is there some type of mathematics that you don't like?

No, because there is this one tapestry, it's all connected. It's like the tapestry behind you, it goes all around. Everything is interesting to me.

And now the fluid dynamics enters. Can you tell us about that and why? Okay, you have a punchline in the end here, we won't spoil it for you.

I forgot …

Oh, you promised to replace Newton's calculus by Poincaré's combinatorial topology.

Oh, right, of course yes, but that isn't a punchline, that's the theme. The idea is, yeah, so, quick history of math, right: We had the Greeks, they had their problems, more than two thousand years ago. Newton came along and he invented the calculus along with Leibniz. Suddenly, a bunch of problems the Greeks had could be solved. You can compute volumes of new things. Because with calculus you sort of ignore higher order error terms. Error of 0.1 decimal place, and errors of 0.001, you ignore all those, and you just try to get the first part. And then the formula is simple, and you get this beautiful theory.

But, you know, if you look a physicist in the eyes and ask, they'll say: "The continuum doesn't exist." The continuum doesn't exist, because, what do we know about it? The atomic models, elementary particles, there is no physics below 33 decimal places. There is no physical theory, you can't even talk about distance below that.

On the other hand, the calculus ideal works beautifully, we have gravity, Einstein's theory. By the way, Einstein's theory hasn't been connected to the standard model, which is the way the elementary particles interact, with these small distances getting down to Planck scale. In fact, Planck scale is sort of the scale in which gravity and the strong forces of nature are comparable.

Even the physicists use the continuum … like in a religious way! As if it exists! And they know it's not true, because Newton's calculus leads to classical physics, which is negated by quantum theory. But it's so beautiful! Representation theory, Lie groups, it's so beautiful, and they can make models, and the models work! But there is no basis somehow, there is something missing, right? In the physics theory.

So, fluid mechanics has been in between the classical and the quantum discussion, you might say, the statistical discussion. It has been in between, and in three dimensions … Well, in two dimensions it has theoretically been worked out, not computationally, but theoretically worked out. For the same mathematical reasons, this Ahlfors–Bers theory and this deformation theory works, analysis, it's related to that, and I understood that. That was one reason I got in, I understand that, and half of that theory works in dimension three, but not the other half.

I was astonished to hear, in '91 or '92, that these basic hydrodynamics equations in three dimensions weren't theoretically understood – whether they have solutions or not – because in dimension two it was all clearcut, and I understood why. They're used all over the world by engineers to produce oil and by doctors to fix aneurysms. The latter use a little turbulence inside the aneurysms and do a little support thing here, doctors can do several a day, and they can fix people up that might die at any point.

How could it be true that 3D hydrodynamics was so mysterious? Also about that same time one was able to put things on a computer quite well, but still there is now a limitation of a thousand of grid points or so in each direction. Thousand by thousand by thousand, that is a billion. You calculate, but then there is a matrix problem, billion by billion, so that is beyond reach. So there is this definite limitation to what one can compute. This mathematical problem, which actually became one of these millennium problems later, I was already working on it for about a decade before, is beautiful, precise and so on. But it's not practical. What is really important is: what can you understand at the scale where you can compute? And then maybe prove theorems too.

The idea I had was, this is all about space – processes happening in space. And you've got the Newton continuum, which gives you a beautiful algebra picture of space, you have differential forms, calculus, the Leibniz' rule for a product, you know. Great! It turns out that if you discretize the problem and put it on a computer, you've got to do difference quotients instead of derivatives, and they don't satisfy the product rule which has an h^2-error, divided by h, still an h-error. But then h goes to 0. That is in every computation, that error term. So the idea is, and they know this, the numerical analysts know this, of course, they know this much better than I do, but they don't seem to have a theoretical way to approach it. So, the idea is, or Poincaré told us, for all this topology, all the numbers games we were talking about before, which is quite deep, has to be done by breaking space into little chunks, and do some combinatorics with that. So, that is combinatorial topology, that allows you to understand the non-linear aspect, which has a product structure. That has been my theme of understanding, and now I have been working on it for three decades, and I think I have made some progress recently.

To take the discretization that we have to do in order to calculate anything in fluid mechanics and anything like that. Are you saying that we should make that *as a main object of study itself?*

Yes! We should study the full algebraic topology – this is going back to the beginning now: Poincaré duality, intersections, how things intersect, that's the ring structure. You know, these objects in a manifold can be intersected, and that gives a ring structure.

Do you think the Navier–Stokes problem, which we've been talking about, is one of the hardest Millennium Prize Problems?

No idea. I'm even not concerned with it as a Millennium Problem. I'd love to prove it, but I'd rather understand some variant of it. I mean, what made this dictionary stuff so interesting in a way, there were several Fields Medals there and stuff like that, was because they had these pictures of the Mandelbrot set. Once a waiter came along while we were working on it, and he said: "Oh, that's the Mandelbrot set". Everybody knows the Mandelbrot set, right? There are good computations of the Mandelbrot set, you can zoom in to any scale, it gets more and more complicated, it's beautiful, like a fern or something. And you go deeper, and then there is a new thing, you know, it's precise. And that has led to many statements and conjectures, half of which have become theorems, and half of which are still open. So, it has been a very active field. We don't have such good computations for fluids in general. We don't have enough understanding. We can just try, if it works: good. If it doesn't work, you know: bad. So, the idea is to put more kind of conceptual work on the problem.

To use Poincaré's ideas, to break space up to combinatorial pieces, see how they interact, put other pieces to cover the breaks which reveals the Poincaré duality, and put all that into the computer programs that is treating the Navier–Stokes equation.

You've said several times, that simplicity is the thing. When Atle Selberg was interviewed two years before he died, one of the things he stressed very much, was, and we quote him with a direct translation from Norwegian: "I believe that it is the simple things that will survive in mathematics." Would you agree with that?

Oh yeah, of course. *C'est evident!* You know, like a screwdriver. It's going to last forever, if it's simple, and it's useful. I'll go even further, the goal of mathematics is to simplify everything. I think that the complicated things can be simplified.

Actually, Selberg mentioned Hermann Weyl as a prime example of a person that could attack a problem, simplify it and solve it.

I think that is a good method, because there are these fundamental points, like the moments I was describing with the graduate students, organize everything. They aren't easy to find, you know. What are the central points? You don't know a priori. And you start by getting a sense of it, it has to do with the structure: what is the structure of the situation. A little "Grothendieck-like".

The time is …

I'm not tired! I know this phenomenon; if hours are late and the mathematician one is talking to is tired, then one just asks him a question about what he is doing, right? And he starts talking, and suddenly he's full of energy again!

This is going to be the last question, we promise! During our preparatory Zoom-meeting we mentioned a 1828 quote of Abel's we'd like you to comment on.

> One should give a problem such a form that it is possible to solve it, something one can always do with any problem. In presenting a problem in this manner, the actual wording

> of it contains the germ to its solution, and shows the route one should take. I have treated several topics in analysis and algebra in this manner, and although I have often posed myself problems that surpass my powers, I have nevertheless attained a great number of general results that have shed a broad light on the nature of these quantities, the knowledge of which is the object of mathematics.

Do you have any comment on this?

The formulation of the problem is very important. Even more, a given problem may not be the correct formulation of *the* problem. Every problem stands, if it's well-defined, but it could be that there is a slightly different version of the problem which is more natural and will be successfully solved, you know.

I'm willing to change the problem, while it sounds like Abel is trying to take the problem as given and put it in its best perspective. I'm also willing to change a problem slightly, to one that can be solved, right? But I certainly agree with that.

Another thing that I've noticed, as I've been around doing this for a long time, is that when a subject is sort of complete, you can look back, you know, it's very easy to close the barn door after the horse has escaped. You know that you should have done it before. When you look at the final story, you would say, "Jeez, if we had started over *here*, then it would be natural to do *this*, and then you would have gotten there very quickly." Using just a simple picture of what has happened.

So, if you are in a situation where you don't have that, look for it. That is kind of what Abel said.

On behalf of the Norwegian Mathematical Society and the European Mathematical Society and the two of us we would like to thank you very much for this most interesting interview.

It was my pleasure, I assure you!

Thank you!

The Rényi Institute

András Stipsicz

The Rényi Institute is located in the historic center of Budapest, in a beautiful palace from the end of the nineteenth century. The institute plays a central and dominant role in Hungarian mathematics. Its main mission is to advance mathematical research at the highest level, and (in cooperation with universities in Budapest) it also plays an important role in training young mathematicians.

The Rényi Institute was founded in 1950, as part of the chain of research institutes of the Hungarian Academy of Sciences (HAS). Its founding director, Alfréd Rényi, a well-versed and outstanding mathematician with important contributions to probability theory, graph theory and number theory, envisioned an institute producing significant mathematical results to serve the applied sciences. Therefore, initially the research activities of the institute concentrated on applied mathematics, and subsequently fields of more theoretical nature were slowly introduced. Today the institute hosts departments in all major fields of theoretical mathematics, including algebra, geometry, analysis, probability theory, number theory, topology, set theory, and many branches of discrete mathematics. In the past few years, the institute extended its research divisions towards more applied directions, such as network science, financial mathematics, and the mathematical foundations of artificial intelligence. The interplay between the theoretical and applied sides of research at the Rényi Institute already produced exciting results, and the institute is devoted to further foster this interchange of ideas and methods. In 2019 the institute became part of the chain of research institutes of the Eötvös Loránd Research Network.

In the recent past researchers of the Rényi Institute won an impressively large number of prestigious grants, both on the European and on the national scale, including 11 ERC grants of various levels (Consolidator and Advanced Grants, and a Synergy Grant), several Marie Skłodowska-Curie grants, and many Momentum (Lendület) and Frontier (Élvonal) grants from the Hungarian Academy of Sciences and from the Hungarian grant agency NKFIH.

Endre Szemerédi 2012
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László Lovász 2021
© Abel Prize

Important international mathematical prizes have been awarded to institute scientists, including the most coveted mathematical prize, the Abel prize, to Endre Szemerédi in 2012 and László Lovász in 2021.

The Rényi Institute was represented (by invited or plenary speakers) at all International Congresses of Mathematicians (ICM) in the past 30 years – in 2018 four mathematicians from the institute lectured in Rio at the ICM2018, and the ICM in 2022 (originally planned in St. Petersburg and moved online) will also have a mathematician from the institute as invited speaker. Institute researchers are frequent participants in the European Congresses of Mathematics (ECM) – at ECM8 in 2020 (held in 2021) a plenary speaker represented the institute.

1.	János Pintz	2008	Advanced
2.	Imre Bárány	2010	Advanced
3.	András Stipsicz	2011	Advanced
4.	Endre Szemerédi	2012	Advanced
5.	Balázs Szegedy	2013	Consolidator
6.	Miklós Abért	2014	Consolidator
7.	László Pyber	2016	Advanced
8.	Gábor Pete	2017	Consolidator
9.	László Lovász	2018	Synergy*
10.	János Pach	2019	Advanced
11.	Gábor Tardos	2021	Advanced

* joint with László Barabási Albert and Jaroslav Nešetřil

A Rényi ERC csapata. Standing row: Abért, Szemerédi, Pintz, Bárány, Lovász, Pyber, Pach. Squat row: Stipsicz, Pete, Tardos, Szegedy.
© Gergely Szilágyi / Rényi Institute

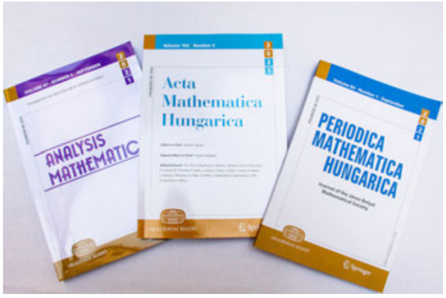
The Rényi Institute currently employs roughly 120 researchers, among them 60 tenured mathematicians and 60 visitors. Guests come on all levels, as PhD students, recent PhDs, postdoctoral researchers and senior mathematicians, and from all over the world, from Europe to the Americas and Asia. The infrastructure is supported by roughly 30 staff members, managing a library, an IT center, the administration, and keeping the building up and running.

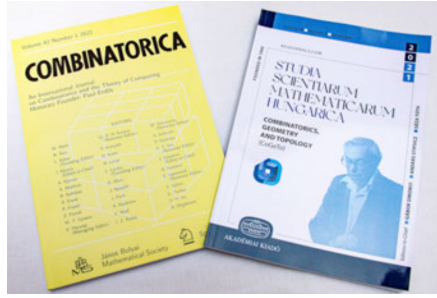
Following the footsteps of Paul Erdős (one of the most influential mathematicians of the twentieth century), Pál Turán, Alfréd Rényi and many other outstanding researchers, discrete mathematics plays a central role in the scientific activities of the institute. With the generation of László Lovász and Endre Szemerédi, together with an impressively large number of successful fellow researchers, the subject flourished further. Another dominant school (initiated by Rényi) is working in probability theory, and developed into a significant center through the work of Imre Csiszár, Gábor Tusnády, Domokos Szász, Bálint Tóth and others. Today the Rényi Institute is among the top research institutions worldwide in graph theory, combinatorics, limits of graphs, discrete probability, and many more subfields surrounding these areas.

The institute has a longstanding tradition in organizing international-level conferences, workshops, and summer and winter schools. The success of these activities led to the launching of the Erdős Center, which allows researchers to organize thematic semesters at the institute, strengthening its bonds to the international mathematical community throughout the world. These semesters then provide opportunities for mathematically oriented students in Budapest to get a first-hand experience of cutting-edge research in a wide variety of mathematical topics. Topics for the first two years are fixed (ranging from network science, including large networks, to automorphic forms, optimal transport, algebraic geometry and low-dimensional topology), while for the years to come a call for proposals has been launched.

The education of the next generation of scientists is among the institute's top priorities. Being a research institute rather than an educational center, the main emphasis has been put on helping young scientists on the doctoral and postdoctoral level. The institute receives an impressive number of visitors (for visits spanning from a few weeks to two–three years) in a wide spectrum of mathematical disciplines. It also offers a program for professors at Hungarian universities to spend sabbatical years within the walls of the institute, to submerge in the research activities there. The institute maintains a so-called hyphenated Fulbright-Rényi-BSM (Budapest Semesters in Mathematics) scholarship program both for scholars and talented recent graduates from US-based universities. Indeed, many professors at the institute serve as members of the faculty of Budapest Semesters in Mathematics, a widely known and acknowledged, excellence-based study abroad program in mathematics for advanced undergraduate students.

Realizing current trends in mathematics and other sciences, the Rényi Institute has started new research directions in the mathe-





matical foundations of artificial intelligence (AI), mainly concerning neural networks and deep learning. The AI team shows a careful balance of theoretical scientists and deep learning/machine learning practitioners. The practitioners in the team are well-versed and up to date in the extremely fast-paced world of deep learning, but nevertheless can contribute to foundational research. The theorists of the team are world-class experts in the highly abstract theoretical machinery, but they also do not shy away from running simulations. This balance creates an optimal environment for a free flow of ideas between theory and practice, and thus supports the general goal of bridging the gap between mathematical theory and machine learning practice. The AI team serves as a leading partner of the consortium of the Hungarian Artificial Intelligence National Laboratory, responsible for the theoretical foundations of AI. The participation in the National Lab provided a newly deployed high-performance computing center for AI within the institute's walls. Furthermore, the Rényi Institute offers help to many industrial partners in mathematics as well as in applying deep learning modeling and algorithms to their work.

The institute also participates in other aspects of service for the mathematical community. Researchers of the institute provide the backbone of scientific journals, by editing Studia Scientiarium Mathematicarum Hungarica (a mathematical journal founded by Alfréd Rényi) and Acta Mathematica Hungarica (a journal of the Hungarian Academy of Sciences) and working as editors in other significant international journals. The Rényi Institute also participates in disseminating mathematical research through the electronic journal Érintő (Tangent), aiming to present mathematical ideas and results on various levels for a widely varied audience, ranging from interested high school students to researchers of other disciplines. Researchers of the institute also participate in mentoring programs for talented high school students.

András Stipsicz, member of the Hungarian Academy of Sciences, and the director of the Rényi Institute since 2019. His research is devoted to low-dimensional topology, three- and four-dimensional manifold theory, and contact three-manifold topology.

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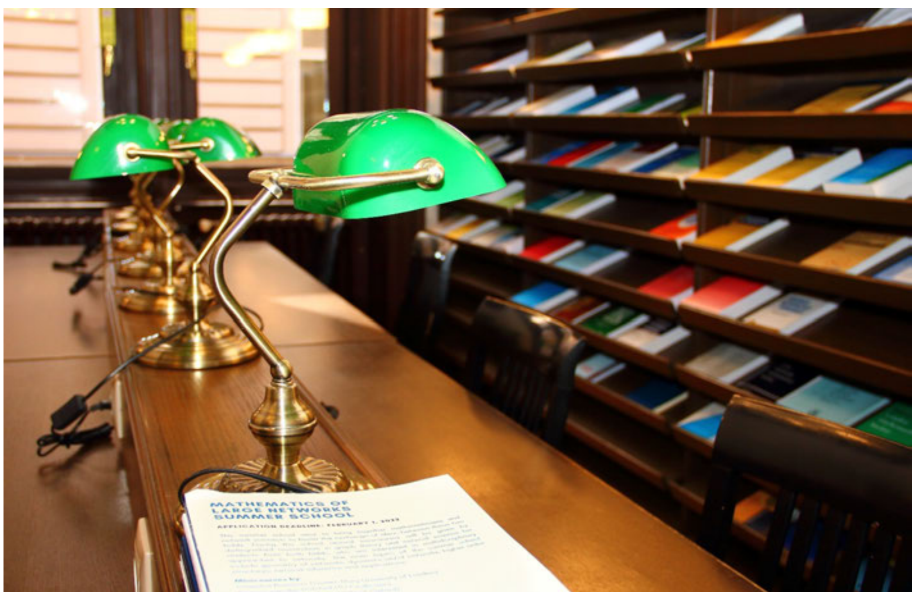

IMAG, the Institute of Mathematics of the University of Granada

Jose A. Cañizo, Ginés López and Joaquín Pérez

Instituto de Matemáticas
Universidad de Granada

Logo of IMAG

The institute building is located at the Campus Center, Rector López-Argüeta S/N.

The Institute of Mathematics of the University of Granada is an international center for research and advanced training in mathematics, where mathematicians from several countries collaborate on common strategic areas in theoretical and applied mathematics, with emphasis on differential geometry, functional analysis, differential equations and modeling, statistics, and operational research. The institute stands for excellence in research, has been quite successful in fundraising, and has become one of the main centers of mathematical research in Spain.

Structure, objectives, and facilities

IMAG is the acronym for the Institute of Mathematics of the University of Granada (UGR). The institute was officially created in May 2015, although its activities started in 2013 as one of the four venues of the Spanish Institute of Mathematics (IEMath), a competitive national project today discontinued.

The fundamental mission of IMAG is to carry out scientific research and specialized training in all areas of mathematics. This mission is developed by accomplishing actions of different nature:

1. Supporting external researchers that collaborate with members of the institute.
2. Holding research seminars and conferences of international importance.
3. Carrying out Master and PhD programs in mathematics.
4. Organizing strategic events aimed at bringing mathematics and society together (outreach, gender activities, etc.).

IMAG facilities are located in a building donated by the Spanish National Research Council (CSIC), at approximately 500 meters from the Science Campus of the UGR. In its three floors, the building includes a conference room, two class and seminar rooms, two meeting rooms, 13 individual offices, and 37 posts distributed in 8 multiple offices, reading area, library, printer and computer rooms, and a warehouse. Equipment includes blackboards, electronic interactive whiteboards, video projectors, and a videoconferencing system. All offices have desktop computers connected to the UGR network, and several laptops are available for use. Printing needs

EMS MAGAZINE 125 (2022) — DOI 10.4171/MAG/97

IMAG provides common areas for discussion and reading in a relaxed atmosphere, besides offices and seminar/conference rooms.

Conference room at IMAG

are covered by three copier-printers installed on network. Besides Ethernet connections in the offices, a wireless internet connection is available everywhere in the building. As part of the UGR research infrastructure, electronic access to databases and journals is also available.

Research lines and activities

The institute is currently populated by more than 100 researchers from different countries around the world, particularly interested in four main areas of research, listed below.

A. Modeling and differential equations
   - A.1. Modeling of physical and biological processes
   - A.2. Geometric and nonlocal partial differential equations

B. Geometric and physical shapes
   - B.1. Geometric analysis
   - B.2. Mathematical physics

C. Linear and nonlinear infinite-dimensional analysis
   - C.1. Geometry and structure of Banach spaces
   - C.2. Operators on Banach spaces and Banach algebras
   - C.3. Nonlinear geometry of Banach and metric spaces

D. Statistics and operations research
   - D.1. Stochastic modeling and forecasting
   - D.2. Data science
   - D.3. Applications/data-driven research

The institute participates in four Master and PhD math programs, all with Excellence Awards from the Spanish National Research Agency (AEI). Special mention is owed to the program "Mathematics", coordinated by UGR and participated by four other Andalusian universities, whose component at UGR is entirely developed at IMAG.

The above research and training activities at IMAG are complemented by Transfer of Knowledge and Culture/Outreach Services.

Some numbers and people at IMAG

The intense research activity of our institute is particularly shown by indicators pertaining to the pre-pandemic period. From 2016 to 2020, IMAG hosted 21 workshops and meetings, two three-month international Doc-Courses on specific topics, at least 270 seminars and colloquia, and more than 370 external visits of researchers, of which approximately one half were foreigners.

IMAG has a solid and visible position in the international arena. It is headed since its opening by Joaquín Pérez, an internationally reputed leader in the differential geometry of minimal surfaces.

Among international projects led by members of the institute, we highlight the recent ERC Advanced Grant in *Holomorphic Partial Differential Relations* (2022–2027) by F. Forstnerič (external member of IMAG and long-term collaborator, University of Ljubljana, Slovenia), the ERC Advanced Grant in *Nonlocal PDEs for Complex Particle Dynamics* (2020–2025) by J. A. Carrillo (external member of IMAG, currently professor at Oxford University, former graduate and PhD student at UGR, and one of the leaders of the PDE group at IMAG), and the ERC Starting Grant in *Analysis of Moving Incompressible Fluid Interfaces* (2015–2022) by F. Gancedo (external member of IMAG and now assistant professor at University of Seville).

Besides its role as a research institute of the UGR, IMAG is one of the two current venues (the other being IMUS, the Institute of Mathematics of the University of Seville) of the recently approved Andalusian Institute of Mathematics (IAMAT), a new structure that aims to strengthen research in mathematics at the regional level, together with other Andalusian universities.

EXCELENCIA
MARÍA
DE MAEZTU

Logo of the Program of Excellence "Maria de Maeztu" by the Spanish Research Agency (AEI)

Excellence Seal "María de Maeztu"

The excellent level of mathematical research at IMAG is supported by more than 25 research projects at international, national, and regional levels, awarded to members of the institute and currently in execution. As a consequence of its successful trajectory, starting in January 2022 IMAG enjoys a "María de Maeztu Excellence Unit" accreditation by the AEI, funded with almost two million euros for a period of four years. This is an extremely competitive program by the AEI which seeks institutional strengthening and is aimed at centers and units of excellence in the public sector and private non-profit research institutions, whose scientific leadership is proven at an international level, with capacities to decisively contribute to advancing on the frontier of knowledge and generate high-impact results, as well as to exert a driving effect on the Spanish science, technology, and innovation system.

Present and future projects

As part of the IMAG's international strengthening policy, the institute has recently associated with the Banff International Research Station (BIRS, Canada) in hosting a series of research workshops that are part of the scientific program of BIRS. This is one of the key

Interaction of researchers in different domains has proven essential for strengthtening the scientific life at IMAG.

actions undertaken by IMAG within the framework of the María de Maeztu Seal. The pilot program will begin in 2023, and it is planned that starting in 2024, IMAG will function as a center associated with BIRS. In this way, IMAG is the first European center associated with BIRS, offering its excellent scientific programs with easy access to the European mathematical community, with the rich natural and cultural backdrop of Andalusia.

Another direction in which the institute projects substantial growth in the coming years is in the area of artificial intelligence. The environment of Granada, with the IA Lab co-partnered by UGR and the international technology companies Indra and Google, will undoubtedly be a fertile ground for the development of applied research in mathematics for data analysis and the transfer of knowledge to society and business.

Despite having a relatively short trajectory, the institute is the continuation of an intense international research activity that has been carried out in the mathematics section of the UGR for more than 30 years. Several generations of UGR mathematicians laid the foundations, and now the IMAG is reaping the rewards while projecting its image with increasing weight at the international level.
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Jose A. Cañizo is vice-director of IMAG and assistant professor at UGR. He works in applied mathematics and is one of the research group leaders in the field of kinetic theory and aggregation processes in physics and biology. Author of a number of highly cited works, he also serves in the editorial board of the journal *Communications in Pure and Applied Analysis* since 2014. Former coordinator of a trilateral project including researchers in Spain, France, and Austria, he maintains fruitful contacts with Oxford and Cambridge Universities, among others.

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Ginés López is secretary of IMAG and professor at UGR, and one of the leaders of the functional analysis team at IMAG. He works on geometric properties in Banach spaces in connection to the Radon–Nikodym, point-of-continuity, Krein–Milman, Daugavet, and diameter-two properties, norm attaining functionals, etc. He enjoys strong international connections, collaborating with J. Langemets, G. Godefroy, and V. Kadets, among others.

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Joaquín Pérez is director of IMAG, professor at UGR, and one of the leaders of the geometric analysis group at IMAG, which after more than 30 years and several generations of geometers, has a long international recognition as one of the top teams in minimal surface theory. He has served in several strategic and organizational positions, both at the national and international levels (former coordinator of the French-Spanish network of geometric analysis, editor-in-chief of the Royal Spanish Mathematical Society, individual member delegate of the EMS council, to name a few).

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On the International Day of Mathematics

Betül Tanbay

To celebrate the beauty and importance of mathematics and its essential role in everyone's life, the International Mathematical Union (IMU) has led the initiative to have UNESCO proclaim March 14 as the International Day of Mathematics (IDM). On November 26, 2019, the 40th General Conference of UNESCO approved the proclamation.

March 14 was already known as *Pi Day* and was being celebrated in many countries around the world. This certainly helped the approval, but it all would not have happened without the efforts of Christiane Rousseau from the University of Montreal, the initiator of the project at the IMU.

The second name to mention right away is Andreas Daniel Matt, director of IMAGINARY, a non-profit organization dedicated to communicating modern mathematics. The IMAGINARY team won the call for hosting the IDM website: www.idm314.org.

The IMU executive board gathered a group of mathematicians to constitute the first IDM-Governing Board, and the first decision taken was to have a theme for each year. We started with an ambitious one for the first IDM, to be celebrated on March 14, 2020: *Mathematics is everywhere.*

A wonderful webpage was prepared in seven languages to show the use of mathematics in different subjects and issues: https://everywhere.idm314.org/, and a map was presented to access the activities all around the planet, on which more than a thousand activities were announced: www.idm314.org/2020-idm.html

Two parallel international launch events were planned, the first one in Paris at the UNESCO Headquarters, and the second one as a plenary event at the *Next Einstein Forum* 2020 in Nairobi, Kenya.

The whole world knows what happened just before: *Pandemic was everywhere*. Despite huge lockdowns, hundreds of activities still took place, one of the biggest ones being realized by the Istanbul municipality, an IMAGINARY exhibition in the main underground station hallways of the 16-million megalopolis.

Despite the general panic, the pandemic has also been an occasion to see that *Mathematics was indeed everywhere*, as the whole world started talking about rates of change, geometric or exponential growth, the R_0 reproduction index, analyzing graphs and understanding probabilities. Mathematics and statistics have been essential tools for decision makers in predicting the evolution of the disease and optimizing mitigation strategies with limited resources.

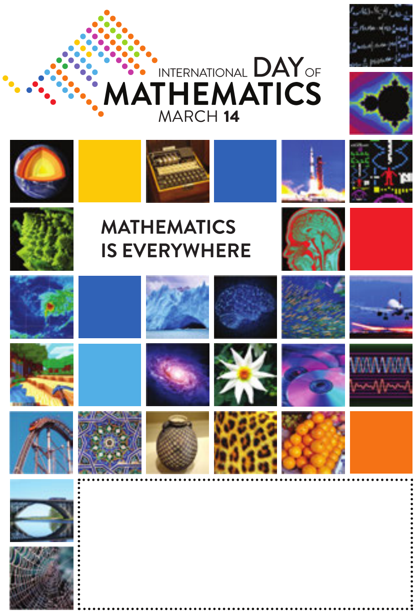
In view of the pandemic, the 2021 theme was chosen to be *"Mathematics for a Better World"*. As the role played by mathematics in building a better world goes well beyond the response to the pandemic, schools were invited to explore examples such as the mathematics of *fair division*, which has so many applications in designing economic and social policies.

This time, with the experience of the previous year, almost all activities have been prepared online. The result was still quite a success: more than seven hundred events throughout the world[1], a poster challenge to which more than 2000 posters were submitted[2], and again a webpage in five languages on the different uses of mathematics[3].

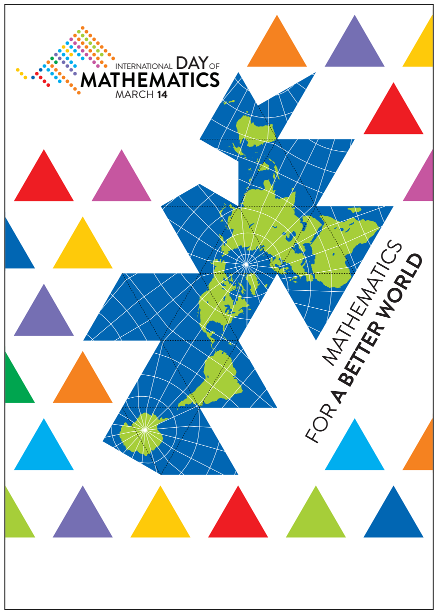
More and more theme proposals arrive to IDM theme calls, and the selected theme of IDM22 was proposed by Yuliya Nesterova, a graduate student from the University of Ottawa in Canada. She

A photo from the Mathematics Unites photo challenge: a Fibonacci spiral formation by the students of Sri. H. D. Devegowda Government First Grade College, Paduvalahippe, India.

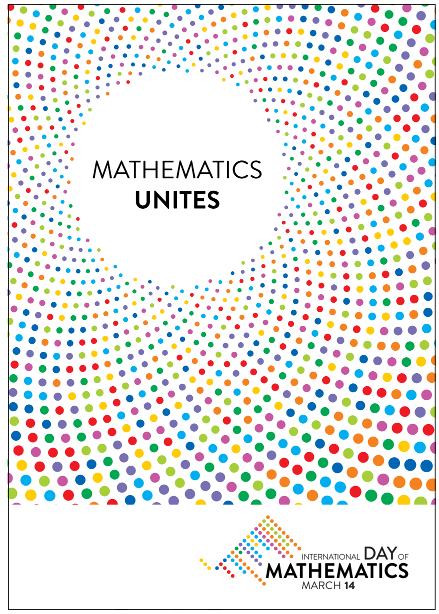
[1] https://www.idm314.org/2021-idm.html
[2] https://www.idm314.org/2021-poster-challenge-gallery.html
[3] https://betterworld.idm314.org/



Poster of IDM 2020



Poster of IDM 2021



Poster of IDM 2022

explains: "Mathematics unites, to signal that it is a common language we all have and a common subject with which to find one another."

At a time when we, as humans on this planet, urgently need a common language other than our mother tongue, a common value other than our credit cards, a common ground other than politics, to solve our common problems, the choice of the theme seemed more than adequate. A few months after this beautiful theme was chosen, it was even more disappointing than the pandemic to face a war in Europe just a few days before March 14, 2022.

Despite the terrible polarization the war hinted to, mathematicians tried to unite, and IDM 2022 is still being celebrated on all continents: from Uzbekistan to the Philippines, from Guinea to Rwanda, from Dominican Republic to Peru, from Moldova to Montenegro. An international live celebration[4] in five languages (Arabic, English, French, Portuguese, and Spanish) took place on March 14. Also, a 48-hour live coverage[5] on the IDM website started at 00:00 New Zealand time and ended at 24:00 Pacific time. The international celebration was complemented by national and local competitions, conferences, exhibitions, and talks, organized by mathematical societies, research institutes, museums, schools, universities. The *Mathematics Unites* photo challenge generated more than three thousand entries; some the most beautiful and inspiring photos are displayed in galleries: www.idm314.org/2022-photo-challenge-gallery-intro.html.

People and organizations all over the world announced almost two thousand events in their cities: www.idm314.org/#theme2022.

A special series of online teacher training sessions with participants from Africa and Latin America accompanying IDM will take place during the next five years. It will start in the fall of 2022 with a Portuguese workshop for primary and secondary mathematics teachers from Mozambique, Angola, Portugal, Cap Verde and São Tomé and Príncipe. The series is part of the Global-South IDM project and is supported by the Simons Foundation with the goal to further engage with Africa and Latin America and to expand the network for local IDM celebrations.

For IDM22, UNESCO has published a tool kit called *Mathematics for Action: Supporting Science-Based Decision Making*, launched[6] on March 14, 2022. The open access tool kit[7] consists of a collection of lively two-page briefs highlighting the role of mathematics in addressing the Sustainable Development Goals of the UN 2030 Agenda, for instance, how to monitor an epidemic, to model climate change, or to measure biodiversity.

So, here we are, heading for a new theme in the middle of the year 2022, in the midst of pandemics and wars. If the number 0 is an absorbing element for mathematical multiplication, war is an absorbing element for human multiplication and well-being. It is loss no matter what the results may be.

Were we too ambitious when we declared that "mathematics is everywhere"? Could mathematics help us live in "a better world"?

[4] https://www.idm314.org/2022-global-event-program.html

[5] https://www.idm314.org/launch-2022.html

[6] https://en.unesco.org/commemorations/mathematics

[7] https://unesdoc.unesco.org/ark:/48223/pf0000380883.locale=en

How can mathematics unite, in times when neighbors become enemies?

But still, I believe there is a reason why mathematicians choose and keep ambitious themes. Once we are convinced of a statement, we cannot abandon the goal of proving it right. We cannot afford discouragement. In a world of post-truth, I believe mathematicians are among the best placed to make affirmative statements. Maybe because mathematics is about interrogations, because it teaches us to ask questions! A true statement is reached by raising the right questions. If mathematicians feel that something important is to be proven, they know they must work a lot, consistently and beyond their own lifetime, using past experience and trusting future developments. Einstein's words are well known: "the important thing is not to stop questioning".

I wish all of us peace, health, and the courage to ask the right questions.

Betül Tanbay is a professor in functional analysis at the Boğaziçi University in Istanbul. She was founder and first co-director of the Istanbul Center for Mathematical Sciences. She was the first female president of the Turkish Mathematical Society, and she has also served and serves in many committees of the IMU or EMS. Tanbay received her undergraduate degree from Université Louis Pasteur, Strasbourg in 1982, and graduate degrees from UC Berkeley in 1989.

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ERME column

regularly presented by Jason Cooper and Frode Rønning

In this issue, with a contribution by
Michela Maschietto and Alik Palatnik

CERME12

The 12th Congress of the European Society for Research in Mathematics Education, CERME12, was held as an online conference from 2 to 6 February 2022, organized by the Free University of Bolzano, Italy. The congress was due to be held in 2021, but was postponed because of the COVID-19 pandemic. with the hope that it would be possible to run it as a physical conference. Unfortunately, this did not work out. However, the organizers in Bolzano did a great job turning the conference into an online event, including also social events in addition to the sessions in the Thematic Working Groups (TWGs).

A total of more than 900 participants attended CERME12, distributed over 27 TWGs. Around 540 papers and 130 posters were presented. Two plenary lectures were delivered: one titled *Enhancing language for developing conceptual understanding – a research journey connecting different research approaches*, by Susanne Prediger from TU Dortmund, Germany, and another, titled *Conceptualizing individual-context relationships in teaching: Developments in research on teachers' knowledge, beliefs and identity*, by Jeppe Skott from Linneaus University, Sweden, and University of Agder, Norway. A plenary panel around the topic of *Big Questions in Mathematics Education* was held on the final day of the conference.

Two new Thematic Working Groups began their activity at the 12th conference: TWG11 – Algorithmics (taking the number 11 from the discontinued *Comparative studies in mathematics education*), and TWG27 – *The Professional Practices, Preparation and Support of Mathematics Teacher Educators*.

Planning for CERME13 is well underway. CERME13 will be held at the Eötvös Loránd University, Budapest, Hungary from 10 to 14 July 2023. See http://erme.site/cerme-conferences/. Scheduling the conference in the summer is hoped to improve the chances of holding it face-to-face, and to bring us back to schedule with CERME14, to be held in February 2025.

CERME Thematic Working Groups

We continue the initiative of introducing the CERME working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for people working in pure and applied mathematics. Our aim is to enrich the ERME community with new participants, who may benefit from hearing about research methods and findings and contribute to future CERMEs.

Introducing CERME Thematic Working Group 4 – Geometry Teaching and Learning

Michela Maschietto and Alik Palatnik

CERME's Thematic Working Group on geometry was created at CERME3 in 2003 [5]. Throughout the history of CERME, this TWG has had different names, so as to take into account different aspects of teaching and learning geometry and/or to emphasize particular ones: from *Geometrical thinking*, through *Geometry*, and currently *Geometry teaching and learning*. Typically, around 25 participants from all over Europe, the Americas, Asia, Africa, and Australia attend the working group, and 15–20 papers/posters are presented and discussed per conference.

Research on geometry teaching and learning has several components that have been addressed in the conferences. Emphasis and interest have varied from one conference to another, depending on the papers presented in the working group. Four main topics can be identified: specific aspects of mathematical activity in geometry, including what it means to be "doing geometry" and how to characterize geometrical thinking; learning geometry, in terms of students' processes in solving geometrical tasks, with attention to visualization, language, argumentation, transition between different representations and use of tools; teaching geometry, from the point of view of curriculum, methodologies, tools, tasks and competencies; teacher education in geometry, referring to contexts, practices, content and perspectives. On the one hand, this richness of components shows the complexity of the geometry thematic; on the other hand, participants critically comment that some components seem to be less present in the range of papers for discussion because those components are within the scope of other working groups. In general, the papers discussed in TWG4 concern research carried out from kindergarten to secondary school and, at the university level, mainly prospective primary school teachers; there are no papers concerning other university students or the transition to tertiary geometry (which might be addressed in TWG14 on university mathematics).

As said above, a recurring topic of discussions has been to characterize what is meant by "doing geometry", i.e., geometrical thinking and its development. Since the first meeting of the TWG, the development of shared theoretical frameworks has been crucial to ground collaboration among the participants. Besides the always-mentioned van Hiele levels [6], other approaches and frameworks, such as the geometrical paradigms, the geometrical working space, the formulation of geometrical thinking in terms of (four) competencies have been discussed [5]. Other approaches have been added over the years, referring to spatial skills. The metaphor of space has also been used to articulate three facets of doing geometry – in the realm of physical space, geometrical space, and graphical space [4]. During CERME10 [3] and CERME11 [4], the discussions were about the characterization of these spaces and their mutual relationships, from the psychological and mathematics education points of view. The psychological point of view refers to the exploration of relationships between physical and graphical space, while from the mathematics education point of view, the focus is traditionally on the links between graphical and geometrical space. In this perspective, geometry consists of establishing relationships between these spaces, and to solve geometry problems one needs to "grasp space" and make the information usable in another kind of space [1]. In the papers presented in CERME12, some of these frameworks have become less prominent or even missing in their theoretical references, although they do emerge in the discussions. For instance, research on spatial skills and on spaces was not present, though questions on relationships between geometrical knowledge and spatial knowledge arose. Other frameworks, such as van Hiele levels, were critically discussed, especially in relation to the tests based on them and to the components of geometrical thinking/competencies that they do or do not help to grasp. Finally, two other theoretical elements have become relevant in recently discussed research: the embodied approach in experiments carried out with students at different school levels, especially in relation to the use of tools, and emotional and motivational aspects emerging-in and accompanying activities in geometry. In general, these new theoretical elements allow us to reexamine the questions "What does it mean to learn geometry?", "What geometry should our students know when they move from primary to secondary to tertiary education?" and "What skills (visual, reasoning, operational and figural) should students/pupils acquire/develop by the end of a given school level?".

Besides psychology, another area of shared interest between mathematics education and other disciplines concerns language, both in terms of the emergence of the geometrical lexicon and the construction of the meanings of words used in geometry (considering that it is a long and complex process that cannot be reduced to the matter of "vocabulary"). Many contributions in TWG04 pertain to argumentation, justification, reasoning or demonstration, topics in which geometry has a privileged status, because it is often the only context in which school students engage in proving.

In all CERMEs, research on the use of tools (both material and digital) in geometry teaching and learning have been discussed. Many contributions have focused on dynamic geometry software for 2D geometry, mainly at the secondary level, but with some contributions concerning teacher education. With respect to the previous CERMEs, in CERME12 we discussed a paper on the 3D environment of GeoGebra that demonstrated and discussed how in solving a geometrical construction, students intertwined the 2D procedure of construction, visualization of 3D objects, and relationships between procedures and representations. 3D geometry learning environment is an interesting topic yet to be developed in future CERMEs, also including Augmented and Virtual Reality for 3D geometry. In addition, the situation of online teaching, compelled by the current pandemic, has raised some new questions that need to be studied pertaining to teaching geometry online, its consequences for learning, and its influence on students' conceptualization.

In addition to digital tools, research on material tools has been presented and discussed at several CERMEs. For instance, in CERME10, a paper on the Pythagorean theorem proposed the analysis of an experiment on the use of material artifacts (called "mathematical machines" and related to one of the proofs of the theorem), intending to discuss the mediation of these artifacts in the construction of meanings in geometry. Regarding 3D geometry, at CERME12 there were two contributions on the use of construction kits and 3D pens, which allow the creation and exploration of solids in microspace and mesospace [2]. Research on 3D geometry with different types of tools could be a fruitful topic for further development, with implications for teacher education in geometry.

In this working group, the variety of nationalities of the participants (from different countries of the world) has always allowed a comparison of teacher training programs, mathematics curricula (in particular, in geometry) and teaching practices that have developed in the different countries, even when this was not the main object of investigation. In the perspective of the vertical development of geometric thinking, the link between primary and secondary school has emerged as a fundamental question.

At CERME12, the works on teacher education accounted for half of the accepted contributions. While there is a group at CERME dedicated to teacher education, the discussion in TWG4 allowed us to focus on the specificities of geometry and its unique mathematical processes. The papers presented in CERME12, on the one hand, have led to the question of which tasks are emblematic for prospective teachers; on the other hand, they have paid attention to teachers' beliefs relating to geometry itself and the "ideas" that prospective teachers have about the fundamental objects of geometry. This aspect is closely linked to the study of the motivational and emotional aspects in doing geometry, as mentioned above, from the students' perspective. Classification tasks in the plane and in 3D space have emerged as particularly interesting in terms of enabling geometric processes for primary-school prospective teachers; these are tasks in which, after analyzing the content knowledge, prospective teachers are asked to anticipate or analyze pupils' solving processes. With these tasks, it is possible to work simultaneously on content knowledge, which is often lacking in prospective teachers according to research reports, and on the specific competence for teaching geometry. Further studies are needed to characterize design principles and applications of emblematic tasks for teacher education in geometry.

Finally, during CERME12 some researchers proposed to explore new topics, such as non-Euclidean geometries, topology, and analytical geometry at various levels of education.

To summarize, TWG4 is an active group whose work reflects both long-standing and emerging trends in the field of geometry education and stimulates further research in this area. On all the topics of this TWG, contributions are welcome not only from researchers in mathematics education, but also from research mathematicians, to enrich the discussion on geometry teaching and learning even more. We leave this suggestion for the next CERME. The diversity and inclusiveness of our group embodies the spirit of communication and collaboration of CERME and contributes to our understanding of international perspectives on geometry education, to advance the teaching and learning of geometry.

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The transition of zbMATH towards an open information platform for mathematics (II): A two-year progress report

Klaus Hulek and Olaf Teschke

Two years ago, we outlined in this column [*Eur. Math. Soc. Newsl.* 116, 44–47 (2020)] *the vision of zbMATH Open as an open service for mathematics research. Here we give a report of the achievements since then.*

1 Introduction

Two years ago, we described in this column [3] the vision to transform zbMATH into an open service for mathematics research. This has now become reality. For this, we first received a special transformation grant from the German government, with the perspective that this could be made permanent after a successful evaluation after two years. In our original application we outlined several goals which are essential requirements of the mathematical community. We described and discussed these also in [3]. Naturally, the two-year period is not long enough to expect that these long-term goals could have been fully completed. In addition, the pandemic created its own problems, which needed to be addressed. In spite of this, we were able to achieve several important milestones, and the evaluation at the end of the transformation period confirmed that the grant should be made permanent. This now provides zbMATH Open with sustainable funding through the Leibniz Association. In this column, we report on the progress made since the beginning of 2020.

2 Preparatory work (I): Legal aspects

What has been known as the reviewing service *Zentralblatt*, or later the zbMATH database, has been, since its foundation by Springer Verlag in 1931, a commercial enterprise for many decades. Naturally, all legal documents, from the editorial contract to the indexing agreements with the many publishers active in mathematics, were based on the model of a subscription service, distributed to a well-defined and controllable set of customers (with their own licensing contracts of various terms). Transition to an Open Access, and beyond this, Open Data platform, required a complete replacement of these agreements, and related negotiations. One result of the new editorial contract among the editorial institutions of the European Mathematical Society, FIZ Karlsruhe – Leibniz Institute for Information Infrastructure, and Heidelberg Academy of Sciences was that the role of a commercial distributor, which had been faithfully fulfilled by Springer-Nature AG, became obsolete, leading to separation from a partner which had been very supportive of zbMATH, especially in difficult times. On this occasion, we would like to thank our Springer colleagues for their decades of commitment, which was concluded by enabling a coordinated transition from subscriptions to Open Access by the end of 2020. The role of the European Mathematical Society, as well as the Heidelberg Academy, is to ensure the scientific quality of the service and to further the involvement of the mathematical community.

Retrospectively, the amount of legal preparations achieved in 2020 is amazing. This includes a large number of renewed indexing contracts with a majority of mathematics publishers, a considerable fraction of which agreed also to Open Data services within the new zbMATH Open platform. While it had already been decided that all data generated within the zbMATH editorial process (such as reviews, author disambiguation data, or semantic and interlinking data) would be made available under the CC BY-SA license (https://creativecommons.org/licenses/by-sa/4.0/), not all publisher data would fit into this framework (e.g., abstracts which would usually come along with a different copyright). Nevertheless, this could be achieved for a considerable part of the information, and there is ongoing activity to expand this further.

Also, the terms and conditions, both for users and reviewers, needed to be adapted and agreed upon accordingly. Finally, also the interface was revised, with a special focus on minimizing storage and processing of user data. While subscriptions required extensive user tracking and detailed usage reports for libraries, Open Access allowed for a platform built upon the principle of data avoidance and data minimization. As of 2021, zbMATH Open is indeed one of the few complex sites that can be used completely without cookies (and consequently, without the need for cookie approvals), with only optional cookies needed to store user preferences if required.

With these issues being completed in the course of 2020, it was possible to start zbMATH Open at the beginning of 2020.

3 Becoming open: Usage and feedback

2021 being the first full year of zbMATH Open as a free service, it might be interesting to look at some experiences concerning usage figures and user feedback. Without doubt, the special situation of the still looming COVID-19 pandemic and its impact on working behaviour drove both the interest in and the willingness to contribute to open services. Historically, zbMATH had about 1,200 subscriptions, where access was often channelled via institutional proxies. This resulted in about 22 million customer searches per year. In an average month of 2021, more than 60,000 unique visitors used the site, with more than 32 million searches in 2021. This indicates that Open Access facilitated a much broader user base, which is still growing (2022 March figures point toward 40 million searches this year).

The user survey conducted mid-2021 supported this impression. While a lot of specific questions pertaining to zbMATH Open features, service and data quality confirmed a significant positive development in comparison with the already very good results of 2016 (see the report [1]), the main emphasis of the feedback was an unanimously enthusiastic appreciation of becoming open. Moreover, numerous ideas for the further improvement have been advanced, some of which could already be incorporated, such as an upgrade of the BibTeX output toward a more standardized format and the improvement of the site's accessibility. We experienced also a new overwhelming willingness to contribute reviews. Despite the universal limitations caused by the pandemic, almost 1,200 new reviewers joined the service (about twice as many as in previous years), and 13 % more reviews were contributed in 2021 compared to 2020.

Though it might be unfair to pick a single example, a good illustration of the increased reach is perhaps Peter Scholze's review of the *Publ. Res. Inst. Math. Sci.* volume containing Shinichi Mochizuki's work *Inter-universal Teichmüller theory* (https://zbmath.org/1465.14002). Openly available, is was within days distributed, linked, and discussed on a large variety of platforms (Reddit, Twitter, MathOverflow, …), and was retrieved within a few days more than 10,000 times, making it likely the most-read zbMATH review of all time.

4 Preparatory work (II): Backend upgrades

While the transition to an open service required also some accompanying developments, the main efforts were focused on new features that would be enabled by being an open data platform allowing to interlink with other free sources. While most of such features could not be deployed before 2021, there was a considerable amount of necessary preparations done in 2020. Among the essential upgrades in the backend was the replacement of the indexing software. Since going online in the mid-90s, zbMATH was based on

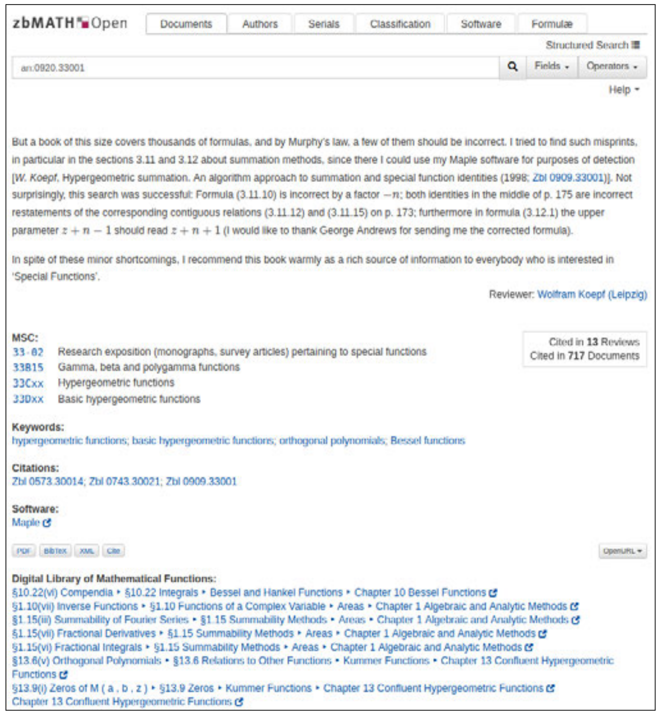

Figure 1. References to DLMF from the review of the Andrews, Askey and Roy's book *Special Functions*

an in-house code optimized for the specific data that traditionally formed the service. However, this came along with limitations as a growing interconnectedness lead to the import of heterogeneous data from various sources. Hence, the complete indexing code was replaced by Elasticsearch (www.elastic.co/de/elasticsearch/), which not just allowed for much more flexibility, but also led to a significant speed-up of update as well as search time. Moreover, the challenging part of retaining the traditional features of the service could be achieved.

Another key component was the development of a new reviewer backend that was also optimized with respect to the experiences of mobile work. It would have been impossible to handle the growth of the numbers of reviewers and reviews in 2021 without this system. Simultaneously, the backend and frontend components were recoded from Python 2 to Python 3, keeping up with the status of the programming language in which the system is developed.

5 New features

The first newly available open data facet even predated the open access of the interface: In September 2020, the first version of the zbMATH Open OAI-PMH API was released, based on the data of the *Jahrbuch über die Fortschritte der Mathematik* (JFM). This was possible thanks to the European Mathematical Society and SUB Göttingen, which had made these data already available under a CC BY-SA 4.0 license (more details about JFM as a part of zbMATH Open have been given earlier in this column, see [11]). The experience acquired in this initial version facilitated its expansion to zbMATH Open data in a stable version in 2021 [7, 9], which forms now a core component in distributing zbMATH Open data. Based on this interface, further APIs with the aim of supporting extended standards and interlinking specific services are being developed.

An immediate application was the development of a linking API enabling the interlinking of zbMATH Open and the NIST Digital Library of Mathematical Functions (DLMF). The system, which can be adapted in the future to interlink with further research data resources, was described in this column before [2], and the resulting interlinking data are now also visible in zbMATH Open.

The more traditional tasks of providing literature and author information also benefitted considerably from the open approach. The improved and extended zbMATcH API allowed for the interlinking with many more open fulltext sources, not just twice as many integrated arXiv links, but also almost 100,000 new links to Unpaywall and CiteSeer^X resources. Author profiles contain now identifiers from and links to as much as 15 external services. Vice versa, additional information from these sources can be included, enabling, e.g., the display of various non-ASCII spellings. Furthermore, the profiles differentiate now between various roles, like author, editor, or further contributions (like appendix authors). Likewise, additional information from the coauthor graph is displayed (both features were frequently asked for in the user surveys).

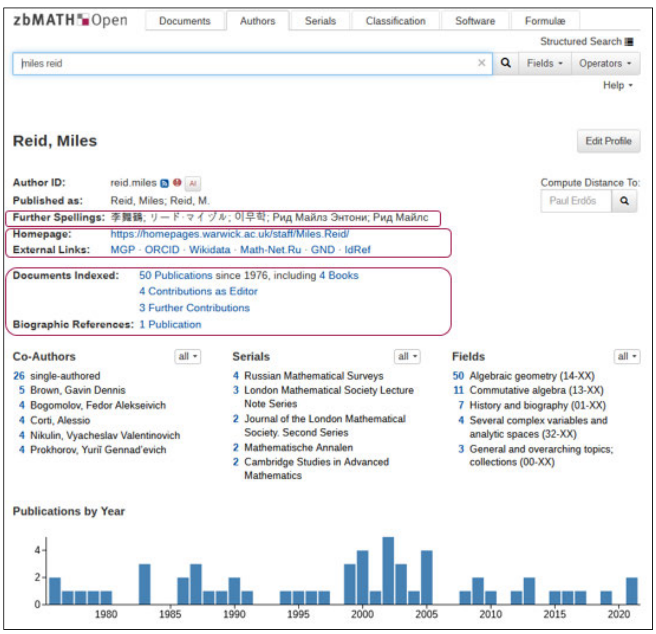

Figure 2. 2022 zbMATH author profile

It is perhaps also an interesting aspect that the underlying author disambiguation data have been further improved, with significant support by the community interface [5] as well as various external community platforms. The latter comprises such different examples as the correction of several mistakes in the attribution of works of Renée Peiffer (of the Peiffer identity) on Twitter [4], as well as the insight that the Rabinowitsch trick was most likely not, as commonly assumed, discovered by G. Y. Rainich [12].

Another frequently requested facet that is under active development is an affiliation information. There already exists an internal database of about 15,000 disambiguated entities of mathematical institutions, which is currently matched to publications and authors. When completed, this will allow us to release a transparent, open data institution facet of zbMATH Open – we will be happy to report on its progress soon!

6 Future developments and projects

The ongoing internal development to integrate and interlink further publications, author and affiliation information, research data and community platforms is only one side of the evolving network. The other is the use of zbMATH Open data in projects conducted worldwide. Currently, there are several projects which already make extensive use of zbMATH Open data, especially in semantic analysis and the history of mathematics. For example, Norbert Schappacher's book [8], commissioned on the occasion of the IMU centenary, contains a detailed analysis of ICM speakers, their networks and working fields based on zbMATH Open data. For already several years, the ISC project Gender Gap in Science (https://gender-gap-in-science.org/), led by the IMU, is supported by zbMATH Open data which are an integral part of its visualization platform Gender Publication Gap (https://gender-publication-gap.f4.htw-berlin.de/; see also [6] in this column).

Through zbMATH Open data, FIZ Karlsruhe, the institution which produces, develops, and maintains zbMATH Open, was able to engage in several Open Science projects. Perhaps the most important is the Mathematical Research Data initiative (MarDI) (https://mardi4nfdi.de), which has been approved as the mathematics consortium within the evolving German National Research Data Infrastructure and started to work by the end of 2021. Other projects, pertaining to the EOSC cloud and the development of a math-specific plagiarism detection system based on [10], have already been approved, while others are in preparation.

But, above all, it is the mathematics community that drives the further development of zbMATH Open by providing ideas and contributions. We highly appreciate the ever-increasing number of valuable reviews, as well as your suggestions and feedback to editor@zbmath.org!

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The Polyhedrists by Noam Andrews

Reviewed by Adhemar Bultheel

Euclid's *Elements* has been the book of choice for centuries to teach geometry. Three-dimensional geometry was only considered near the end in Book XI, and the regular or Platonic solids (tetrahedron, cube, octahedron, dodecahedron and icosahedron) appeared in the very last Book XIII. This was not on the regular curriculum of the quadrivium. Besides these five regular solids and the thirteen Archimedean solids that can be seen as semi-regular, it became fashionable among artists and craftsmen of the 15–17th century to use or depict these solids in their work. Soon several truncations, indentations, and stellations generated a plethora of so-called irregular solids. Irregular is a misleading term, since they are still very symmetric, but they do not satisfy the classical definition of the Platonic or Archimedean corpora. Note that it were not the mathematicians who published theory-free texts with plots and maps useful as manuals to produce these solids. With the invention of perspective, showing these objects in their pictures was a token of craftsmanship and it illustrated the supposed mathematical, or at least intellectual skills of the person being portrayed. With many examples and beautiful illustrations, the historian and architect Noam Andrews tells the story of this evolution in the wake of the Renaissance period.

The most iconic pictures involving nested polyhedra can be linked to Kepler, who associated the Platonic solids with the solar system. Also well known is Dürer's *Melencolia I*, featuring a polyhedron and a magic square. Some may know the portrait of Luca Pacioli attributed to Jacopo de' Barban that has a water-filled glass polyhedron hanging on a string and a solid polyhedron on the table. Pacioli is the author of *Divina proportione* (1509), a book illustrated by Leonardo Da Vinci in which he discusses the golden ratio and the use of perspective. The latter was invented less than a century ago and that was brought to a conclusion by Da Vinci.

Andrews explains how the polyhedral ideas evolved mainly throughout the 16th century. It all starts with Erhard Ratdolt's edition of Euclid's *Elements* (1482) that had some graphical illustrations. These illustrations are natural for us, but they were new then, in the early days of book printing. Graphics helped to understand the proofs, but it was a completely different matter to paint, etch or draw the three-dimensional geometric solids in proper perspective. At the universities, the study of perspective was part of optics. Drawing and constructing the solids was the concern of an artist and not so much of mathematical interest.

In the next chapters, the 16th century Bavarian city of Nuremberg is the historical scenery. This is where Dürer had access to the Regiomontanus-Walther library and where he published his book on geometry *Underweysung der Vermessung* (1525) [Instruction on measurement] treating among other things the Platonic, Archimedean, and other solids. That book contained a lot of accumulated knowledge, not only for painters and sculptors, but also for stonemasons, carpenters and goldsmiths. He places visualisation central, which was new for mathematical texts. Nets of planar developments of polyhedra could be cut out and folded and glued together to form a paper 3D version of the solid and vice versa, the solids could be developed into a printable 2D net.

Now *Lehrbücher* started circulating in Southern Germany. These were like very graphical instruction manuals, that were not generated at a university. They were manuals for artists and craftsmen that were devoid of theory and theorems. For example, Augustin Hirshvogel's *Geometria* (1543) was a very popular one. The 3D versions could be used as art objects but also as a teaching object to practice perspective. Placed in front of the student, mechanical devices were constructed that allowed to transfer the view of a solid to a planar perspective view. Theory and theorems were not involved.

Wenzel Jamnitzer was a famous goldsmith/mathematician who worked for the Holy Roman Emperors. He also came to Nuremberg where he published his *Perspectiva corporum regularium* (1568) providing 24 variations (6 per page) for each of the 5 Platonic solids, starting from the simple solid, modified by increasingly more complicate truncations, indentations, or stellations.

The next two chapters are about probably lesser known forms of polyhedrism. The first is polyhedral marquetry often for furniture in the form of cabinets. There is an exquisite example of such a cabinet in the Museum for Applied Arts in Cologne, richly decorated with polyhedral objects. The little known, somewhat enigmatic artist Lorenz Stöer was a very popular inspiration for Augsburger cabinets. He produced eleven surrealistic colour graphics showing deserted ruins of cities populated by all kinds of polyhedral structures. The last chapter is about the invention of sophisticated lathes or turn tables used by master turners to make ivory columns (*Säulen*) which had on top, or sometimes half way, some sphere-like dodecahedron or another solid with opened faces so that inside you can find another smaller one that had yet another one inside, like Russian matryoshkas. These contrefait spheres were fashionable at the Saxonian court in Dresden. Egidius Lobenigk and Georg & Hans Wecker were famous master turners.

In an epilogue, Andrews reflects on the role played by the polyhedrists. The popularisation did come from the application, and not from the theory. It was only by the end of the 16th century that gradually the subject became again absorbed by the academia. This movement with popular *Lehrbücher* has awakened geometry from almost two thousand year of frozen knowledge. Artists started to think outside the plane and the generalisation and endless variation of the classical solids made it possible for geometry to break loose from its static immobility and an incentive was given for a more evolved geometry as developed by Kepler, Monge, Descartes, and at a larger scale even Einstein.

This is a nicely illustrated and easy to read book on some less-known historical aspects of applied geometry. Most of the material covered is restricted to Southern Germany and its content is mostly historical and cultural, with little mathematics. It is however interesting to learn how this specific geometrical topic became aesthetically fashionable and how it evolved outside the universities, yet undoubtedly had an impact on the development of geometry.

Noam Andrews, *The Polyhedrists.* MIT Press, 2022, 316 pages, Paperback ISBN 978-0-2620-4604-6.

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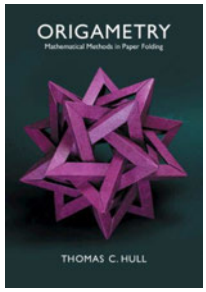
DOI 10.4171/MAG/93

Origametry – Mathematical Methods in Paper Folding
by Thomas C. Hull

Reviewed by Ana Rita Pires



Origami is the art of paper folding, with ancient origins: the classic Japanese paper crane was supposedly devised in the 6th century. The last hundred years have brought new interest in this art, with the creation of increasingly complex and beautiful origami models (such as the five intersecting tetrahedra on the cover of the book, created by the author Tom Hull and voted by the British Origami Society as one of the top ten origami models of all time), and also with the appearance of applications ranging from nano-robots for medical use to solar arrays for spacecraft. In parallel, rich mathematical theories related to origami were developed, at a particularly rapid pace in the last decade.

"Origametry" is the most comprehensive reference book on the connections between origami and mathematics. Its author, Tom Hull, is an associate professor of mathematics at Western New England University who has been studying the mathematics of origami for decades. He compiles and describes in one volume a truly impressive amount of material created by numerous researchers on a diverse array of the mathematical aspects of paper folding.

The book is divided into four parts.

Part I describes *Geometric Constructions*. It introduces the basic origami operations and shows how they can be used to trisect an angle, construct a regular heptagon, and more generally solve any cubic equation – all of which are famously impossible to achieve using a straightedge and compass. A complete classification of what constructions are possible with these basic origami operations is achieved by determining the field of origami numbers using Galois theory. Further avenues of research in this direction concern geometric constructions that can be achieved using multifolds (in which the paper is folded in a way that creates more than one crease at once) or curved creases.

If you unfold an origami model, you get a crease pattern, a pattern of line segments that represent valley folds and mountain folds and intersect at definite angles. The main question in Part II of this book, titled *The Combinatorial Geometry of Flat Origami*, is whether a crease pattern can be flat-folded, that is, folded into an origami model that lies flat in a plane once all the creases are folded (such as a paper crane before pulling out the wing flaps to make it three-dimensional). Maekawa's and Kawasaki's Theorems give conditions for a crease pattern to be locally flat-foldable around each of its vertices. Both results are easy to state and have short proofs. The first gives the necessary condition that the number of valley folds and the number of mountain folds at each vertex differs by two, and the second gives the necessary and sufficient condition that the alternating sum of consecutive angles at each vertex is zero. It turns out that the question of whether a crease pattern is globally flat-foldable is much harder; it is in fact NP-hard. The proof involves reducing this problem to the not-all-equal 3-satisfiability problem, an NP-complete version of the Boolean satisfiability problem, by creating origami "gadgets" whose flat-foldability requirements mimic the Boolean values of the variables and the clauses. This second part of the book also contains a variety of other foldability questions, of which two examples are the fold-and-cut problem (Given a two-dimensional shape, can you fold a piece of paper so that applying a single straight cut will produce that shape? Yes, for any shape.) and Arnold's rumpled rouble problem (Is it possible to increase the perimeter of a rectangle by folding it into a different shape? Yes, as much as one wishes.).

After looking at the geometry and combinatorics of flat origami, the book turns in Part III to connections with other branches of mathematics, namely *Algebra, Topology, and Analysis in Origami*. For algebra, group theory is used to relate the symmetries of a crease pattern with the symmetries of its flat-folded model. For topology, the notion of folding along straight lines on (a subset of) the Euclidean plane is extended not just to folding along geodesics on Riemannian surfaces, but further to "isometric foldings" of Riemannian manifolds in arbitrary dimension. An isometric folding is a continuous map from the crease pattern manifold to the origami model manifold that sends piecewise geodesic segments to piecewise geodesic segments. It turns out that even in this setting, suitable generalizations of Maekawa's and Kawasaki's Theorems exist. For analysis, it examines the problem of finding an isometric folding on Euclidean space that satisfies a given differential equation and boundary condition – a Dirichlet problem.

Part IV of the book is titled *Non-Flat Folding* and mostly examines the mathematical underpinnings of rigid origami, that is, three-dimensional origami models made of flat polygonal faces which remain rigid during the folding process. Rigid origami is the natural setting for applications in engineering, with objects whose faces are made of a rigid material such as metal or glass and are joined by hinges. This is an active area of research, with practical problems often driving the mathematical research. For example: a question coming from mechanics and robotics is whether a certain crease pattern will self-fold to its desired final state by applying forces in certain hinges.

This is a true maths book: with theorems, proofs, definitions, and examples. It also contains historical remarks, open problems, and diversions, which range from interesting and fun exercises to explore to straightforward parts of proofs that the reader is invited to complete. Between the diversions and the open problems, this book is bound to inspire several undergraduate, master's, and even PhD theses. It is a delightful and informative read for mathematicians curious about the mathematics behind origami, essential for researchers starting out in this area, and handy for educators searching for ideas in topics connecting mathematics, origami, and its applications. Even though it is not written with that goal specifically in mind, it could be used as a textbook for a graduate course or a reading course.

A final word of advice: have some paper at the ready, it is difficult to resist folding along while reading!

Thomas C. Hull, *Origametry – Mathematical Methods in Paper Folding.* Cambridge University Press, 2020, 342 pages, Paperback ISBN 978-1-1087-4611-3.

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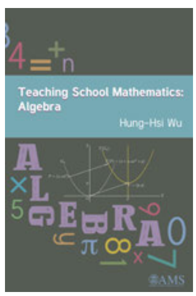
DOI 10.4171/MAG/98

Teaching School Mathematics: Algebra by Hung-Hsi Wu

Reviewed by António de Bivar Weinholtz



This is the third book in a series of six covering the K-12 curriculum, as an instrument for the mathematical education of schoolteachers. It follows two volumes entitled "Understanding Numbers in Elementary School Mathematics" and "Teaching School Mathematics: Pre-Algebra", and it completes the presentation of the mathematical topics included in the K-8 curriculum. With numbers and operations, finite probability and an introduction to geometry and geometric measurement covered in the previous two volumes, here the author deals with the topics that can be found in any middle school or high school introductory course on algebra: linear equations in one and two variables, linear inequalities in one and two variables, simultaneous linear equations, the concepts of a function, polynomial functions, exponents, and a detailed study of linear and quadratic functions. The volume also contains a very helpful appendix with a list of assumptions, definitions, theorems, and lemmas from the previous pre-algebra volume.

Repeating the advice given in the review of the second book in the series, I strongly recommend reading first the review of the first volume (António de Bivar Weinholtz, Book Review, "Understanding Numbers in Elementary School Mathematics" by Hung-Hsi Wu. *Eur. Math. Soc. Mag.* **122** (2021), pp. 66–67; https://ems.press/content/serial-article-files/15117). There, one can find the reasons why I deem this set of books a milestone in the struggle for a sound mathematical education of youths. I shall not repeat here all the historical and scientific arguments that sustain this claim. However, I wish to restate, regarding this third volume as well, that although it is written for schoolteachers, as an instrument for their mathematical education (both during pre-service years and for their professional development), and to provide a resource for authors of textbooks, its potential audience should be wider. Indeed, I believe that it should include anyone with the basic ability to appreciate the beauty of the use of human reasoning in our quest to understand the world around us and the capacity and will to make the necessary efforts, which are required here as for any enterprise that is really worthwhile.

As in the previous volumes, the author sets out to explain why, in his view, for the specific topics treated in each one of the books – here introductory algebra – the goal of getting students to properly learn these topics seems to have been so much out of reach, at least for the last few decades. He finds ample evidence, after such a long period of time of observing so many frustrated attempts to "renew" the teaching of school mathematics, that students fail to learn algebra not because they don't like the way it is taught, but because they find the core of what they are taught to be incomprehensible. As in previous volumes, the author calls "textbook school mathematics (TSM)" the content of what has generally been offered to students under the name of "mathematics", and in particular of "algebra", and argues that in fact it does not satisfy five fundamental principles of this subject:

1. Precise definitions are essential.
2. Every statement must be supported by mathematical reasoning.
3. Mathematical statements are precise.
4. Mathematics is coherent.
5. Mathematics is purposeful.

The precise implications of these principles, of course, depend on the grade we are dealing with, but if they are not constantly kept in mind by designers of curricula, textbook authors, and teachers, and if they are not progressively conveyed to students, no real learning of mathematics is possible. While being essential when dealing with any part of the school math curriculum, these principles are of particular importance when dealing with algebra; it is also in algebra that some of the most harmful misunderstandings have tainted TSM, along with the mistreatment of fractions that was widely analyzed in the previous books. With the explicit purpose of freeing school mathematics from these unfortunate mistakes, the author describes them with due detail, as far as school algebra is concerned, and he anticipates how the proper presentation of this subject in this volume makes possible to avoid all of them. Let us briefly review this list of "critical subjects".

(1) One of the more visible characteristics of algebra is the necessity to use a set of symbols that goes beyond those that represent specific numbers, basic operations and equality and order, extensively used in basic arithmetic since the first grades. The proper use of symbols is one of the important features of mathematics in general, and the learning of introductory algebra is one of the fundamental steps for the acquisition of this kind of skill. But the term "variable" that occurs naturally in this stage and in several other situations in mathematics is not in itself a mathematical concept, although it can occur in precise mathematical expressions like "a real function of two real variables" or it can be used in informal explanations, where "variable" is just a shorthand for "an element in a set". Nevertheless, TSM has risen "the understanding of the concept of variable" to the dignity of a crucial step in the learning of algebra, attempting definitions of "variable" like "a quantity that changes" and that, not having any precise sense whatsoever, only end up in confusing the minds of teachers and students.

(2) This confusion gets worse when one tries to use this so-called "concept of variable" to define what an equation is, as is common in TSM.

(3) As a geometric foundation of the properties of similar triangles is absent from TSM, any attempt to give a proper treatment of the concept and properties of the *slope* of a straight line is bound to be unsuccessful.

(4) This mistreatment prevents an adequate study of this most important relation between algebra and geometry that is the study of first-order linear equations in two variables and systems of two such equations and their graphs (or "sets of solutions", in their geometrical representation), namely why the graphs of these equations are exactly the non-vertical straight lines.

(5) For the same reason, it becomes impossible in TSM to properly learn the algebraic characterization of parallel and perpendicular lines. The "solution" often adopted in TSM to *define* parallelism and perpendicularity of lines by using the characteristic properties of slopes in each case (respectively equal slopes and the product of slopes equal to -1, except in the case of the perpendicularity of a pair of horizontal-vertical lines) is totally inadequate, as students by that time are already familiar with the concepts of parallel and perpendicular lines, and so they deserve an explanation on how the latter can be related to the aforementioned properties of slopes, not as new definitions, but as theorems to be proven.

(6) The concept of constant rate (in particular, constant speed) is one that students have met on several occasions when they reach the stage of an introductory algebra course, but this is the proper opportunity to clarify these concepts. However, they are never defined in TSM and instead TSM engages in an abstruse discussion of a "concept" called "proportional reasoning" which is supposed to be the basis of the understanding of rate, although it is hardly ever given a proper definition. Once again, the possible meanings of

this so-called "proportional reasoning" can and should be examined thoroughly so that much of the aforementioned abstruse approach can be eliminated from school curricula and the really fundamental concepts that it aims to replace can be put on a firm mathematical foundation.

(7) The graph of an equation (sometimes called the graphical representation of the set of its solutions) is also in severe want of a precise definition in TSM. This leads to a situation in which it is impossible to understand why the solution of two simultaneous linear equations is the point of intersection of the two lines that are the graphs of the equations.

(8) The same can be said about the graph of linear inequalities in two variables.

(9) The introduction of rational exponents is also often an occasion for the frequent confusion in TSM between definitions and theorems.

(10) Finally, the treatment of quadratic equations and functions is often chaotic in TSM, without the unifying proper use of their graphs.

Each one of these serious mathematical issues and others that are presented in due course in this volume are dealt with; this provides the tools to fix them and replace TSM by a sound mathematical treatment of introductory school algebra.

Also with respect to this topic, I have to state that, given the availability of this set of books, there is no excuse left for schoolteachers, textbook authors and government officials to persist in the unfortunate practice of trying to serve to students this fundamental part of school mathematics in a way that is in fact unlearnable …

As in the previous volumes of this series, on each topic the author provides the reader with numerous illuminating activities, and an excellent choice of a wide range of exercises.

Hung-Hsi Wu, *Teaching School Mathematics: Algebra.* American Mathematical Society, 2016, 274 pages, Hardback ISBN 978-1-4704-2721-4, eBook ISBN 978-1-4704-3019-1.

António de Bivar Weinholtz is a retired associate professor of mathematics of the University of Lisbon Faculty of Science, where he taught from 1975 to 2009. He was a member of the scientific coordination committee of the new mathematics curricula for all the Portuguese pre-university grades (published between 2012 and 2014 and recently abolished).

DOI 10.4171/MAG/106

Solved and unsolved problems

Michael Th. Rassias

The present column is devoted to Differential Equations.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

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Let $f\colon [0,\infty) \to \mathbb{R}$ be a C^1-differentiable and convex function with $f(0) = 0$.

(i) Prove that, for every $x \in [0,\infty)$, the following inequality holds:

$$\int_0^x f(t)\,dt \le \frac{x^2}{2} f'(x).$$

(ii) Determine all functions f for which we have equality.

Dorin Andrica ("Babeş-Bolyai" University, Cluj-Napoca, Romania) and Mihai Piticari ("Dragoş Vodă" National College, Câmpulung Moldovenesc, Romania)

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Let $y(x)$ be the unknown function of the following fractional-order derivative Cauchy problem:

$$\begin{cases} D^\alpha y = f(x,y), \quad 0 < \alpha < 1, \\ y(0) = y^*. \end{cases}$$

Find the solution of this problem by solving an equivalent first-order ordinary Cauchy problem, with a solution independent on the kernel of the fractional operator.

Carlo Cattani (Engineering School, DEIM, University "La Tuscia", Viterbo, Italy)

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Let $y(x)$ be the unknown function of the following Bernoulli fractional-order Cauchy problem:

$$\begin{cases} D^\alpha y = g(x) y^\beta, \quad 0 < \alpha < 1,\ \beta \neq 0, 1, \\ y(0) = y^*, \end{cases}$$

where $g(x)$ is a continuous function in the interval $I = [0,\infty)$.

Find the solution of this problem by solving an equivalent first-order ordinary Cauchy problem, with a solution independent on the kernel of the fractional operator.

Carlo Cattani (Engineering School, DEIM, University "La Tuscia", Viterbo, Italy)

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Let g be a real-valued C^2-function defined on $(0,\infty)$, strictly increasing, such that $g(x) > 1$ for all $x \in (0,\infty)$ and $g(2) < 4$. Consider the boundary value problem

$$y'' = -g(x) y, \quad y(0) = 1, \quad y'(0) = 0.$$

Prove that the solution y has exactly one zero in $(0, \pi/2)$, i.e., there exists a unique point $x_0 \in (0,\pi/2)$ such that $y(x_0) = 0$, and give a positive lower bound for x_0.

Luz Roncal (BCAM – Basque Center for Applied Mathematics, Bilbao, Spain, Ikerbasque Basque Foundation for Science, Bilbao, Spain and Universidad del País Vasco/Euskal Herriko Unibertsitatea, Bilbao, Spain)

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We propose an interesting stochastic-source scattering problem in acoustics. The stochastic nature for such problems forces us to deal with stochastic partial differential equations (SPDEs), rather than the partial differential equations (PDEs) which hold for the corresponding deterministic counterparts. In particular, we provide the appropriate variational formulation for the stochastic-source Helmholtz equation.

We consider the following boundary value problem (BVP) for the Helmholtz equation with a stochastic source:

$$\begin{cases} \Delta u + k^2 u = f & \text{in } D, \\ u = 0 & \text{on } \partial D, \end{cases} \tag{1}$$

where $f = \sum_a f_a H_a$ is a generalized stochastic source and

$$H_a(\omega) = \prod_{i=1}^{\infty} h_{a_i}(\langle \omega, \xi_{\delta^i} \rangle)$$

are stochastic Hermite polynomials with $\omega \in \Omega$, Ω being a probability space. The Hermite polynomials are denoted by h_{a_i}, whereas the tensor product is denoted by ξ_{d^j}. We also define the Hermite functions $\xi_n(x)$ as follows:

$$\xi_n(x) = \pi^{-\frac{1}{4}}((n-1)!)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} h_{n-1}(x), \quad n = 1, 2, 3, \ldots,$$

and we set $d^j \coloneqq (d_1^j, d_2^j, \ldots, d_m^j)$, where $d_i^j \in \mathbb{N}$ is related to the following tensor products:

$$\xi_{d^j} \coloneqq \xi_{d_1^j} \otimes \xi_{d_2^j} \otimes \cdots \otimes \xi_{d_m^j}, \quad j = 1, 2, 3, \ldots,$$

with $i < j \Leftrightarrow d_1^i + d_2^i + \cdots + d_m^i \leq d_1^j + d_2^j + \cdots + d_m^j$ and $|d_j| = d_1^j + d_2^j + \cdots + d_m^j$. In addition, we employ the countable index $I = \{a = (a_1, a_2, \ldots) \mid a_i \in \mathbb{N} \cup \{0\}\}$, and there only finitely many $a_i \neq 0$.

For the stochastic problem (1), we use the expansions

$$u = \sum_{a \in I} u_a H_a \quad \text{and} \quad f = \sum_{a \in I} f_a H_a$$

to get a hierarchy of deterministic BVPs

$$\begin{cases} \Delta u_a + k^2 u_a = f_a & \text{in } D, \\ u_a = 0 & \text{on } \partial D. \end{cases} \tag{2}$$

Assume that $u_a \in H_0^1(D)$ solves problem (2). Then prove that, for every $v \in H_0^1(D)$, the solution u_a satisfies

$$-\int_D \nabla u_a \cdot \nabla v \, dx + \int_D k^2 u_a v \, dx = \int_D f_a v \, dx.$$

George Kanakoudis, Konstantinos G. Lallas and Vassilios Sevroglou (Department of Statistics and Insurance Science, University of Piraeus, Piraeus 18534, Greece)

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For a Newtonian incompressible fluid, the Navier–Stokes momentum equation, in vector form, reads [3]

$$\begin{aligned} \rho\Big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\Big) &= -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}, \\ \mathbf{u} = \mathbf{u}(\mathbf{x}, t), &\quad \mathbf{u} \colon \mathbf{R}^n \times (0, \infty) \to \mathbf{R}^n. \end{aligned} \tag{1}$$

Here, ρ is the fluid density, $\mathbf{u}$ is the velocity vector field, p is the pressure, μ is the viscosity, and $\mathbf{F}$ is an external force field.

(i) Assuming that both the pressure drop ∇p and the external field $\mathbf{F}$ are negligible, it is easy to show that equation (1) reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u},$$

and finally to equation (2), where $\nu = \frac{\mu}{\rho}$ is the so-called kinematic viscosity [4].

(ii) Regarding the one-dimensional viscous Burgers equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t), \tag{2}$$

prove that an analytical solution can be obtained by means of the Tanh Method [1, 2, 4] as

$$u(x, t) = \lambda\Big[1 - \tanh\Big(\frac{\lambda}{2\nu}(x - \lambda t)\Big)\Big], \quad \lambda > 0.$$

M. A. Xenos and A. C. Felias (Department of Mathematics, University of Ioannina, Greece)

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II Open problems

(A) *Uniqueness of positive steady states for KPP equations in general domains*

by Henri Berestycki (Centre d'analyse et de mathématique sociales, EHESS-CNRS, Paris, France; Institute for Advanced Study, Hong Kong University of Science and Technology)

Reaction-diffusion equations

These arise ubiquitously in the modelling of population dynamics, and more generally in biology and ecology. Remarkably, various fields converge on these equations. In addition to modelling in the life sciences and, of course, nonlinear partial differential equations, they arise in probability theory (via branching particle systems) and statistical physics. These equations have witnessed remarkable progress in recent years. Yet, many basic problems remain open. The object of this note is to present a couple of such questions that are simple to formulate.

Reaction-diffusion equations of *homogeneous type* read in general as $\partial_t u - \Delta u = f(u)$ in $\mathbb{R}^N$. The nonlinear term f is called the reaction term and the Laplacian operator is associated with diffusion. This equation is termed *homogeneous* because it does not involve explicitly the location x (or time t) and also because it is set in all of space. The Fisher–KPP case (or strong KPP case) refers to the class of nonlinear terms f of class C^1 that satisfy

$$f(0) = f(1) = 0, \quad \text{and the function } s \mapsto \frac{f(s)}{s} \text{ is decreasing on } (0, 1]. \tag{1}$$

The archetypal example is $f(u) = u(1-u)$. These reactions terms were introduced and first studied by Fisher [9] and Kolmogorov, Petrovsky and Piskunov (KPP) [10]. I will discuss some questions related to the uniqueness of bounded positive stationary solutions, that is, bounded positive solutions of the semilinear elliptic equation $-\Delta u = f(u)$ with boundary conditions.

Heterogeneous equations

In recent years, many works have addressed *heterogeneous* versions of the equations introduced above. These arise in various guises. First, the reaction term f is allowed to vary in space and time: $f = f(t, x, u)$. Likewise, in various models, one wishes to consider more general second-order elliptic operators than the Laplacian:

$$\sum_{ij} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial u}{\partial x_i}.$$

My works with Hamel and Rossi [6] and Hamel and Nadin [4] are devoted precisely to this type of question. The interested reader will find in or infer from these papers open problems analogous to several that I describe here.

Another natural heterogeneity arises from the *geometry* of the domain of propagation when it is not the whole space. Given an open subset $\Omega \subset \mathbb{R}^N$ subject to Dirichlet boundary conditions, we are led to study the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ 0 < u \le 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Indeed, in many cases of interest, $f(s) < 0$ for all $s > 1$, and then one can show that any non-negative bounded solution (besides 0) satisfies $0 < u < 1$.

Existence

To discuss the existence of a positive solution of (2), we use the generalized principal Dirichlet eigenvalue in the domain Ω, defined as

$$\lambda(\Omega) \coloneqq \inf_{\phi \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla \phi|^2}{\int_\Omega \phi^2}.$$

This definition coincides in the present case with the notion of generalized principal eigenvalue introduced in [7] and applied to unbounded domains in [8]. We can then state the existence result in a more general framework of *weak KPP* class:

$$f(0) = f(1) = 0, \quad 0 < f(s) < f'(0)s \quad \text{for all } s \in (0,1). \tag{3}$$

Existence in (2) is conditioned by this eigenvalue.

Theorem 1. *Let f satisfy the weak KPP condition* (3)*. Then* (2) *admits a positive bounded solution if $\lambda(\Omega) < f'(0)$. Conversely, if $\lambda(\Omega) > f'(0)$,* (2) *has no positive bounded solution.*

This result from [2] is analogous to the one for variable-coefficient operators in $\mathbb{R}^d$ in [6] and is obtained with the same arguments.

Uniqueness. When the domain Ω is bounded and f satisfies the strong KPP assumption (1), the solution of (2) is unique when it exists [1]. This raises a natural question: is the same true in unbounded domains? Cole Graham and myself [2] have been working on this problem and our progress leads us to formulate the following.

Conjecture 2. *Consider an unbounded uniformly smooth (say $C^{2,\alpha}$) domain Ω. Under the strong KPP condition* (1)*, the solution of problem* (2) *is unique when it exists.*

Here, "uniformly smooth" means that there is a fixed $r > 0$ such that for any boundary point $p \in \partial\Omega$, its boundary neighbourhood $\partial\Omega \cap B_r(p)$ can be represented as the graph of some $C^{2,\alpha}$ function $\phi_p \colon D \to \mathbb{R}$, where D is the unit ball in $\mathbb{R}^{N-1}$ and $\|\phi_p\|_{C^{2,\alpha}}$ is bounded independently of the point p (see [5, Section 1.3]). One may be even more demanding and lift this uniform regularity condition.

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Open problem. *In a locally smooth domain Ω with f of strong KPP-type, is the solution of problem* (2) *unique when it exists?*

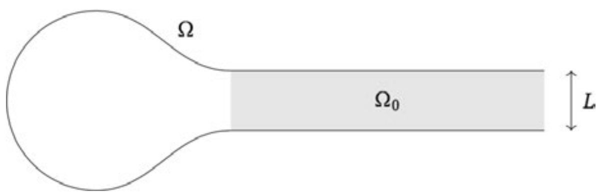
The conjecture in its full generality is open. In my work with Cole Graham [2], we prove uniqueness under a non-degeneracy condition. This result covers a large variety of cases and can be viewed as generic. Its statement requires the use of eigenvalues on various limits of Ω. We say that Ω^* is the *connected limit* of Ω along a sequence $(x_n)_{n \in \mathbb{N}} \subset \Omega$ if the following holds. There exists a uniformly $C^{2,\alpha}$ domain $\tilde{\Omega} \supset \Omega^*$ such that $\Omega - x_n \to \tilde{\Omega}$ locally uniformly in $C^{2,\alpha}$ as $n \to \infty$, and Ω^* is the connected component of $\tilde{\Omega}$ whose closure contains 0. We then define the *principal limit spectrum* as

$$\Sigma(\Omega) \coloneqq \{\lambda(\Omega^*) \mid \Omega^* \text{ is a connected limit of } \Omega\},$$

and we let $\overline{\Sigma}(\Omega)$ denote its closure. We refer to the elements of Σ as *(principal) limit eigenvalues*. One of our main results in [2] is the following.

Theorem 3. *Suppose Ω is uniformly smooth, f satisfies* (1)*, and $f'(0) \notin \overline{\Sigma}(\Omega)$. Then the solution of* (2) *is unique when it exists.*

An example. To illustrate Conjecture 2 and Theorem 3, consider the following domain in $\mathbb{R}^2$ that we call the "infinite light bulb".



We assume that the round portion is sufficiently large that $\lambda_1(\Omega) < f'(0)$. Then, by Theorem 1, we know that (2) admits at least one solution. We can show that $\Sigma(\Omega) = \{\lambda(\Omega), \pi^2/L^2\}$. Thus Theorem 3 applies when $L \neq \pi/\sqrt{f'(0)}$. The critical case $L = \pi/\sqrt{f'(0)}$ is not covered by our result. Nonetheless, in [2], we exploit the explicit structure of Ω to prove that the solution of (2) is still unique in this case. This supports Conjecture 2.

Robin type conditions

Other types of boundary conditions are of interest as well. We can consider the Robin problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \quad 0 < u \leq 1, \\ -\dfrac{\partial u}{\partial \nu} = \gamma u & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where ν is the unit outward normal vector field on the boundary $\partial\Omega$ and $\gamma \geq 0$ is a constant. More generally, one might consider a function $\gamma(x) \geq 0$ that varies on $\partial\Omega$.

Conjecture 4. *In a uniformly smooth domain Ω with f of strong KPP-type, the solution of problem* (4) *is unique when it exists.*

In our forthcoming work [2], we establish an analogue of Theorem 3 in the Robin case. This requires a suitable notion of the generalized principal *Robin* eigenvalue.

General positive and other reaction terms

In [2], we also consider the more general class of *positive* nonlinear terms f. This class is defined by the conditions

$$f(0) = f(1) = 0, \quad f'(0) > 0, \quad f(s) > 0 \quad \text{for all } s \in (0,1). \tag{5}$$

In all of space $\mathbb{R}^N$, uniqueness holds in the more general *positive* case. Indeed, under conditions (5), $u \equiv 1$ is the unique solution of (2) when $\Omega = \mathbb{R}^N$. For a proof, I refer the reader to the forthcoming book [3]. The presence of boundary changes matters significantly. In fact, in a proper subset $\Omega \subset \mathbb{R}^N$ with Dirichlet or Robin boundary, solutions of (2) (or (4)) need *not* be unique. However, uniqueness holds under *Neumann* boundary conditions.

Theorem 5. *In a uniformly smooth domain Ω with f of positive type* (5)*, the unique solution of the Neumann problem* (4) *with $\gamma = 0$ is $u \equiv 1$.*

This result is a generalization of one in my earlier work with Hamel and Nadirashvili [5]. This form is due to Rossi [11]. It naturally calls for the following.

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Problem. *Can the result of Theorem* 5 *be extended to locally smooth domains?*

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(B) *A problem in geometric analysis*

by Michael Struwe (Departement Mathematik, ETH Zürich, Switzerland)

The last 40 years have seen enormous progress in the application of variational methods to problems in geometric analysis, which in general are characterized by the possibility of "bubbling" and topological degeneration of sequences of approximate solutions obtained either by regularization of the problem, or as "Palais–Smale sequences" for the energy functional involved. In critical point theory therefore it is vital to understand the possible interaction of the problem at hand with its "cousins" that characterize the "bubbling", in particular, when the sought-after critical points are of "mountain-pass" type.

As an example consider the (by now classical) "Nirenberg problem" of finding conformal metrics of prescribed Gauss curvature on the standard 2-sphere, which has given rise to sophisticated analytic approaches and deep insights into the interplay of analysis and geometry, but which still poses a challenge, even though many partial answers have been obtained.

Nirenberg's problem

After the work of Berger [1] and Kazdan–Warner [4] on conformal metrics of prescribed Gauss curvature on closed Riemann surfaces, the particular case, proposed by Nirenberg, of finding conformal metrics $g = e^{2u}g_0$ on the sphere S^2 with its standard round metric g_0 having a given function f as Gauss curvature $K_g = f$ has attracted the attention of geometric analysts.

In view of the equation

$$K_g = e^{-2u}(-\Delta_0 u + 1)$$

relating K_g and u, where Δ_0 is the Laplace–Beltrami operator in the metric g_0, for given $f\colon S^2 \to \mathbb{R}$, we need to solve the nonlinear partial differential equation

$$-\Delta_0 u + 1 = fe^{2u} \quad \text{on } S^2. \tag{1}$$

The problem is variational. Indeed, introducing the Liouville energy

$$S(u) = \fint_{S^2} (|\nabla u|^2 + 2u)\, d\mu_0,$$

where $d\mu_0$ is the area element in the metric g_0 and $\fint_{S^2} = \frac{1}{4\pi}\int_{S^2}$ denotes the average, and setting

$$E(u) = S(u) - \log\Big(\fint_{S^2} fe^{2u}\, d\mu_0\Big) \tag{2}$$

for $u \in H^1(S^2)$, the standard Sobolev space of L^2-functions on S^2 with square-integrable weak derivatives, solutions of (1) may be characterized as critical points of E.

Via the Möbius group M of conformal diffeomorphisms of the sphere, for any point $p \in S^2$ the functional E may be compared with the functional

$$E_p(u) = S(u) - \log\Big(\fint_{S^2} f(p)e^{2u}\, d\mu_0\Big),$$

where f is replaced by the constant $f(p)$. Indeed, given any $p \in S^2$, any $t \geq 1$, letting $\Phi_p\colon S^2 \setminus \{-p\} \to \mathbb{R}^2$ be the stereographic projection from the point $-p \in S^2$ and letting $\delta_t\colon \mathbb{R}^2 \ni z \to tz \in \mathbb{R}^2$ be the standard dilation, we obtain the Möbius map

$$\Phi_{p,t} = \Phi_p^{-1} \circ \delta_t \circ \Phi_p \in M.$$

Letting $u_{p,t} = u \circ \Phi_{p,t} + \log|\Phi'_{p,t}|$, where we write $|\Phi'| = \sqrt{\det d\Phi}$ for brevity, we then have

$$S(u_{p,t}) = S(u)$$

(see for instance [2, Proposition 2.1]) and thus

$$\begin{aligned} E(u_{p,t}) &= S(u_{p,t}) - \log\Big(\fint_{S^2} fe^{2u_{p,t}}\, d\mu_0\Big) \\ &= S(u) - \log\Big(\fint_{S^2} (f \circ \Phi_{p,t}^{-1})e^{2u}\, d\mu_0\Big) \to E_p(u) \quad \text{as } t \to \infty. \end{aligned}$$

For large $t > 1$, it was shown by Chang–Yang [2] that the first and second variation of E at $u_{p,t}$ may be related to $\nabla f(p)$ and $\nabla^2 f(p)$, respectively. From this observation, they deduce the following existence result.

Theorem 6 (Chang–Yang [2], Theorem II′). *Suppose that $f > 0$ is a smooth function satisfying the non-degeneracy condition*

$$\Delta_0 f(p) \neq 0 \quad \text{at any } p \in S^2 \text{ with } \nabla f(p) = 0 \tag{3}$$

and the index count condition

$$\sum_{\nabla f(p) = 0,\, \Delta_0 f(p) < 0} (-1)^{\operatorname{ind}(p)} \neq 1. \tag{4}$$

Then there is a smooth solution u to (1).

Interpretation

Condition (4) in Theorem 6 may be interpreted in terms of the "last Morse inequality" related to the variational integral (2), that is, in terms of an equation identifying the "topological degree" $d = 1$ of the (contractible) set of admissible functions $H^1(S^2)$ with the sum of the topological degrees of all critical points of E, including the contributions of the degenerate variational problems related to the functionals E_p, $p \in S^2$. With what we remarked above, the latter contribution is given by the left-hand side of (4). Thus, if that term is different from 1, there has to be a further contribution to the total topological degree of all critical points, then necessarily coming from a solution u to (1).

Open problem

In [8] an example was given showing that condition (4) in Theorem 6 in general cannot be removed; thus, with the non-degeneracy condition (3), condition (4) is not only sufficient but also in general necessary for the existence of a solution to (1).

However, we are still lacking a precise characterization of *all* solutions of (1). In particular, we should be able to obtain the existence of multiple solutions in certain cases. A simple instance of such a case, where – hopefully – the problem is feasible, would be when the given function f is symmetric with respect to reflection in a plane and is a Morse function similar to the example studied in [8] but satisfying the Chang–Yang condition (4).

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Problem. *Let F be the set of functions* $0 < f \in C^\infty(S^2)$ *with*

$$f(x_1, x_2, x_3) = f(-x_1, x_2, x_3) \quad \textit{for } x = (x_1, x_2, x_3) \in S^2,$$

having a saddle point at the north pole $x_3 = 1$*, a minimum at the south pole* $x_3 = -1$*, and precisely two maxima as critical points, all of which are non-degenerate and satisfy* (3)*, and such that condition* (4) *holds. Find conditions for* $f \in F$ *such that there is more than one solution of* (1)*, and characterize the set of all solutions of* (1) *in the sense of Morse theory.*

Of course, the question may easily be widened to a larger class F of functions.

Related challenges

Note that Chang–Yang [2] showed that when $f \not\equiv 1$ solutions of (1) never are relative minima of the energy E.

The Nirenberg problem thus can be seen in the larger context of finding critical points of "mountain-pass" type for variational problems characterized by conformal invariance and "bubbling". A classic instance of such problems is in 4-dimensional gauge theory, in particular, in the question concerning the existence of 1-equivariant, non-minimal Yang–Mills connections in the trivial $SU(2)$-bundle over S^4, which remained open after Sibner–Sibner–Uhlenbeck [6] obtained m-equivariant, non-minimal Yang–Mills connections for any $m \geq 2$; see also Donaldson [3, pp. 309–310] for further details. Moreover, conformal invariance is responsible for many of the difficulties encountered by Rivière [5] in his recent work on "min-max" critical points for the Willmore energy related to sphere eversion.

Recall that Smale [7] famously showed that it is possible to "turn a sphere inside out" via a continuous path of C^2-immersions of S^2 into $\mathbb{R}^3$. Moreover, Bryant characterized all immersed Willmore spheres in $\mathbb{R}^3$ as being given by the images by inversions of simply connected, complete, non-compact minimal surfaces with planar ends, with Willmore energy given by $4\pi k$, where k is the number of ends, and index equal to $k - 3$. Finally, a topological result of Banchoff–Max shows that any path everting the sphere has to contain at least one immersion with a quadruple point and therefore, by a result of Li–Yau, with Willmore energy $\beta \geq 16\pi$. Combining these pieces of information, Rivière conjectured that the inversion of a simply connected, complete minimal surface with $k = 4$ planar ends, thus having index $m = 1$ and Willmore energy 16π, should give a "min-max Willmore sphere", achieving the least maximal Willmore energy along paths of immersions of S^2 into $\mathbb{R}^3$ that "turn the sphere inside out". But in the variational ansatz "bubbling" may occur, and many questions remain to be solved. See [5] for further details and references.

Similarly, in gauge theory the desired 1-equivariant, non-minimal Yang–Mills connections in the trivial $SU(2)$-bundle over S^4 should achieve the least maximal Yang–Mills energy along paths of connections beginning at a 1-equivariant Yang–Mills instanton and ending at a 1-equivariant anti-instanton. But again "bubbling" comes in the way.

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III Solutions

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Prove that the space of unordered couples of distinct points of a circle is the (open) Möbius band. More formally, consider

$$(S^1 \times S^1) \setminus \{(x, x) \mid x \in S^1\}$$

and the equivalence relation on this space $(x, y) \equiv (y, x)$; prove that the quotient topological space is the (open) Möbius band.

Costante Bellettini (Department of Mathematics, University College London, UK)

Solution by the proposer

The space of ordered pairs of points of a circle is the cartesian product $S^1 \times S^1$, hence a torus. This is the same as the unit square $[0, 1] \times [0, 1]$ with the following identifications: $[0, 1] \times \{0\}$ is identified with $[0,1]\times\{1\}$, both with orientation from left to right; $\{0\}\times[0,1]$ is identified with $\{1\}\times[0,1]$, both with orientation from bottom to top. Note that the four vertices of the square are the same point. We now need to remove couples of the type (x,x) (same point of the circle), which implicitly removes $(1,0)$ and $(0,1)$ as well. Hence, we are now looking at the square with the diagonal from $(0,0)$ to $(1,1)$ removed, and with $(1,0)$ and $(0,1)$ removed, keeping the identification we had earlier. Next, we need to identify couples (x,y) and (y,x) (since we want to study unordered couples). This amounts to removing one of the two triangles that have been obtained after removing the diagonal of the square; without loss of generality we assume that we remove the top-left triangle. What is left is the triangle with vertices $(0,0)$, $(1,0)$, $(1,1)$, with the longer side removed (the one that was the diagonal of the square), with the point $(1,0)$ removed, and with the following identification: any point $(x,0)$ was identified (in the original torus) with the point $(x,1)$, which has then been identified with the point $(1,x)$. Hence, the triangle has the horizontal side oriented left-to-right identified with the vertical side oriented bottom-to-top. We can check that this is the (open) Möbius band as follows: the point $(1,0)$ is not in the triangle (nor is the longer side of the triangle), so we can stretch the point $(1,0)$ until the triangle becomes a rectangle, with the stretched point that has become the side opposite to the side that was the diagonal of the square. The identification of the remaining two sides gives the (open) Möbius band.

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In the Euclidean plane, let γ_1 and γ_2 be two concentric circles of radius respectively r_1 and r_2, with $r_1 < r_2$. Show that the locus γ of points P such that the polar line of P with respect to γ_2 is tangent to γ_1 is a circle of radius r_2^2/r_1.

Acknowledgement. I want to thank the professors who guided me in the first part of my career for giving me the ideas for these problems.

Paola Bonacini (Mathematics and Computer Science Department, University of Catania, Italy)

Solution by the proposer

Let P be any point such that the polar line r of P with respect to γ_2 is tangent to γ_1. Then clearly P is external to γ_2. Let C be the centre of the two circles, $\{Q\} = \gamma_1 \cap r$ and $\{T_1, T_2\} = \gamma_2 \cap r$. Since r is tangent to γ_1 in Q, we can assert that the line CQ is orthogonal to r and, since r is the polar of P with respect to γ_2, the line CP is orthogonal to the line r. So, the points P, Q and C are collinear and the angles $\widehat{T_1QP}$, $\widehat{T_2QP}$, $\widehat{T_1QC}$ and $\widehat{T_2QC}$ are right. We also know that $\widehat{T_1CQ} = \widehat{T_2CQ} = \alpha$, where $0 < \alpha < \frac{\pi}{2}$. So we can assert that

$$\overline{QT_1} = \overline{QT_2} = \sqrt{r_2^2 - r_1^2},$$

and by looking at the triangle CQT_1, we see that

$$r_1 = r_2 \cos\alpha \implies \cos\alpha = \frac{r_1}{r_2}.$$

Clearly, $\widehat{T_1PQ} = \frac{\pi}{2} - \alpha$, and consequently,

$$\overline{QP} = \overline{QT_1}\cot\Big(\frac{\pi}{2} - \alpha\Big) = \frac{r_2^2 - r_1^2}{r_1}.$$

This implies that

$$\overline{PC} = \overline{QP} + \overline{QC} = \frac{r_2^2}{r_1},$$

which clearly shows that γ is a circle of centre C and radius r_2^2/r_1.

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Let $A \subseteq \mathbb{R}^3$ be a connected open subset of Euclidean space, and suppose that the following conditions hold:

(1) Every smooth irrotational vector field on A admits a potential (i.e., it is the gradient of a smooth function).

(2) The closure $\overline{A}$ of A is a smooth compact submanifold of $\mathbb{R}^3$ (of course, with non-empty boundary).

Show that A is simply connected. Does this conclusion hold even if we drop condition (2) on A?

Roberto Frigerio (Dipartimento di Matematica, Università di Pisa, Italy)

Solution by the proposer

The usual scalar product on $\mathbb{R}^3$ induces an identification between smooth vector fields and differential 1-forms, which identifies irrotational vector fields with closed forms, and fields admitting a potential with exact forms. Therefore, condition (1) may be restated as follows: every smooth 1-form on A is exact, i.e., the first de Rham cohomology group of A vanishes. By the de Rham Theorem, this is in turn equivalent to the fact that the *singular* homology module $H^1(A,\mathbb{R})$ vanishes.

Since any compact manifold with boundary is homotopy equivalent to its interior, we may thus assume that $H^1(\overline{A},\mathbb{R}) = 0$. A well-known consequence of the Poincaré Duality Theorem is that, for any compact orientable 3-manifold with boundary M, the dimension of $H^1(\partial M,\mathbb{R})$ is twice the dimension of $H^1(M,\mathbb{R})$. Since every codimension-0 submanifold of $\mathbb{R}^3$ is obviously orientable, we thus have $H^1(\partial\overline{A},\mathbb{R}) = 0$. Let $S_1,\dots,S_k$ be the components of the boundary ∂A. Since

$$H^1(\partial\overline{A},\mathbb{R}) = \bigoplus_{i=1}^{k} H_1(S_i,\mathbb{R})$$

and the 2-sphere is the only compact orientable 2-manifold without boundary with vanishing first cohomology group, we can conclude that S_i is diffeomorphic to the 2-sphere for every $i = 1,\dots,k$.

It is well known that every smooth sphere in $\mathbb{R}^3$ bounds a smooth closed disc (this is no longer true for non-smooth spheres; see below); hence, for every $i = 1, \dots, k$, we have $S_i = \partial B_i$, where $B_i \subseteq \mathbb{R}^3$ is a smooth disc. Since $\overline{A}$ is connected, it readily follows that there exists one of these closed discs, say B_1, such that

$$\overline{A} = B_1 \setminus (\operatorname{int}(B_2) \cup \dots \cup \operatorname{int}(B_k)).$$

In other words, $\overline{A}$ is a closed disc with some open discs removed, and in particular it is simply connected.

In order to prove that A is simply connected, the condition that A be the interior of a compact smooth manifold with boundary is essential. Indeed, let $S \subseteq \mathbb{R}^3 \subseteq S^3$ be the well-known Alexander horned sphere. Then S separates S^3 into two connected components: one of them, say A_1, is homeomorphic to an open ball; the other one, say A_2, is not simply connected. However, Alexander duality implies that

$$H^1(A_1, \mathbb{R}) \oplus H^1(A_2, \mathbb{R}) = H^1(S^3 \setminus S) \cong H_1(S, \mathbb{R}) = 0.$$

We thus have $H^1(A_2, \mathbb{R}) = 0$ while $\pi_1(A_2) \neq \{1\}$. By setting $A = A_2$, we thus get a non-simply connected open connected subset A of $\mathbb{R}^3$ such that every smooth irrotational vector field on A admits a potential.

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A *regulus* is a surface in $\mathbb{R}^3$ that is formed as follows: We consider pairwise skew lines $\ell_1, \ell_2, \ell_3 \subset \mathbb{R}^3$ and take the union of all lines that intersect each of ℓ_1, ℓ_2, and ℓ_3. Prove that, for every regulus U, there exists an irreducible polynomial $f \in \mathbb{R}[x, y, z]$ of degree two that vanishes on U.

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Proof

Let $\mathcal{P}$ be a set of 9 points that is obtained by arbitrarily choosing three points from each of ℓ_1, ℓ_2, and ℓ_3. We write

$$f(x, y, z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz + a_7x + a_8y + a_9z + a_{10}.$$

Asking f to vanish at a specific point is equivalent to a linear equation in the variables $a_1, \dots, a_{10}$. Thus, asking f to vanish at all points of $\mathcal{P}$ yields a system of 9 linear equations with 10 variables. Since the number of variables is larger, this system admits a nontrivial solution. Thus, there exists a nonzero polynomial $f \in \mathbb{R}[x, y, z]$ of degree at most two that vanishes on $\mathcal{P}$. Let $W \subset \mathbb{R}^3$ be the set of points at which f vanishes.

Let $f_1 \in \mathbb{R}[s]$ be the restriction of f to the line ℓ_1. Since f vanishes on at least three points of ℓ_1, the polynomial f_1 has at least three roots. Since $\deg f_1 \leq 2$ but this polynomial has more than two roots, we have that $f_1(s) = 0$. In other words, $\ell_1 \subset W$. By repeating the above argument, we get that $\ell_1, \ell_2, \ell_3 \subset W$. By definition, no plane contains a pair of skew lines, so W cannot contain a plane. This implies that f is irreducible of degree two.

Consider a line ℓ' that intersects ℓ_1, ℓ_2 and ℓ_3. Since these three lines are pairwise skew, the three intersection points are distinct, so $|\ell' \cap U| \geq 3$. By restricting f to ℓ' as above, we get that $\ell' \subset W$. Since U is the union of all such lines ℓ', we get that $U \subseteq W$. This proof is by Larry Guth, although it may have also existed earlier.

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(Enumerative Geometry). How many lines pass through 4 generic lines in a 3-dimensional complex projective space $\mathbb{CP}^3$?

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Introductory remarks

This is a problem in an over a century-old area of mathematics called enumerative geometry. Enumerative geometry is concerned with finding or counting geometric objects (mainly curves, i.e., 1-dimensional objects over the ground field) satisfying certain geometric conditions (e.g., passing through a specified set of objects or having a particular degree, genus, and types of singularities). Enumerative geometry was revolutionized in the mid-1990s by the novel predictions of mirror symmetry that led to the creation of Gromov–Witten theory and extensive study of such questions in complex algebraic geometry, symplectic geometry, and string-theoretic physics.

The most straightforward example in this area is the number of lines passing through two points, where the answer is 1. Here, one can interpret the word "line" as a real line in the real Euclidean space $\mathbb{R}^n$, a complex line in the complex Euclidean space $\mathbb{C}^n$, or a complex projective line (i.e., $\mathbb{CP}^1 \cong S^2$) in the complex projective space $\mathbb{CP}^n$. The answer is the same regardless of the context. The same is not true in most other questions. Gromov–Witten theory is mostly about counting complex curves in complex projective varieties or closed symplectic manifolds. The benefits of studying complex curves in compact complex/almost complex manifolds is two-fold. First, the compactness of the spaces involved results in finite counts. Second, working over complex numbers ensures that count of such objects does not depend on the choices involved. Recall that a degree-d polynomial over $\mathbb{C}$ has always d roots (when counted with multiplicities), but a degree-d polynomial over $\mathbb{R}$ has at most d roots.

Solution by the proposer

Before finding the answer, let us indeed argue that the expected answer is a finite number. As in linear algebra, this is done by computing degrees of freedom and the number of equations imposed by the constraints. As we mentioned above, there is exactly one line passing through two distinct points in $\mathbb{CP}^3$; the dimension of the space of pairs of such points is $3 + 3 = 6$. However, for each line, there is a $(1 + 1 = 2)$-dimensional family of pairs of points that yield that particular line. Therefore, assuming that the set of lines in $\mathbb{CP}^3$ is a nice geometric space, its dimension should be $6 - 2 = 4$. The reduction in the dimension caused by the condition of intersecting any of the given lines is 1. It follows that the reduction in the dimension caused by the condition of intersecting all given four lines is 4. Since $4 - 4 = 0$, the solution set should be discrete. Since we are working with compact spaces, it will indeed be finite. Bellow, using Schubert calculus on Grassmannians, we will compute this number. We challenge the reader to think about the following real affine version of the question using elementary techniques: How many lines pass through 4 generic lines in $\mathbb{R}^3$?

The n-dimensional complex projective space $\mathbb{CP}^n$ is the projectivization of $\mathbb{C}^{n+1}$ in the sense that each point in the former corresponds to a line in the latter. In one dimension higher, every projective line in $\mathbb{CP}^n$ is the projectivization of a plane in $\mathbb{C}^{n+1}$. Therefore, the space of lines in $\mathbb{CP}^3$ is the same as the space of planes in $\mathbb{C}^4$, which is known as the (complex) Grassmannian manifold $\mathrm{Gr}(2,4)$. More generally, the Grassmannian $\mathrm{Gr}(r,n)$ is a compact complex $(r \times (n-r))$-dimensional manifold that parametrizes the r-dimensional subspaces of $\mathbb{C}^n$. Let $\ell \subset \mathbb{CP}^3$ be a line that is the projectivization of a two-dimensional subspace $V \subset \mathbb{C}^4$. The subspace of lines in $\mathbb{CP}^3$ that intersect ℓ is a submanifold X_ℓ of $\mathrm{Gr}(2,4)$ with $\dim_{\mathbb{C}} X_\ell = 4 - 1 = 3$. The points of X_ℓ correspond to two-dimensional subspaces $V' \subset \mathbb{C}^4$ such that $\dim_{\mathbb{C}}(V \cap V') \geq 1$. Even though X_ℓ depends on ℓ, the homology class $A \in H_6(\mathrm{Gr}(2,4),\mathbb{Z}) \cong \mathbb{Z}$ of X_ℓ does not depend on ℓ. The homology groups of the Grassmannian are generated by a specific class of complex submanifolds known as Schubert cycles. All the odd degree homology groups are trivial.

Digression on Schubert calculus

Let $\lambda \overset{\text{def}}{=} (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r)$ be a sequence of non-negative integers between 0 and $n - r$, and define $|\lambda| = \sum \lambda_i$. Given a sequence of vector spaces $W \overset{\text{def}}{=} (0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_n = \mathbb{C}^n)$, the Schubert cycle $\sigma_\lambda = \sigma_\lambda(W)$, with Poincaré dual $\mathrm{PD}(\sigma_\lambda) \in H^{2|\lambda|}(\mathrm{Gr}(r,n),\mathbb{Z})$, is defined to be

$$\sigma_\lambda(W) = \{V \in \mathrm{Gr}(r,n) : \dim(V \cap W_{n-r+i-\lambda_i}) \geq i\}. \qquad (1)$$

The homology class of $\sigma_\lambda(W)$ does not depend on W. There is a geometric way of describing a non-decreasing sequence λ which helps with understanding the computations involving Schubert cycles. A Young diagram is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths weakly decreasing (each row has the same or shorter length than its predecessor). Listing the number of boxes in each row gives a sequence λ of non-negative integers, such that $|\lambda|$ is the total number of boxes of the diagram. Figure 1 shows the Young diagram of $\lambda = (5,4,1)$.

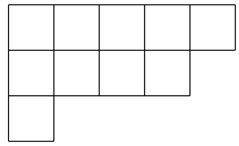

Figure 1. Young diagram of $\lambda = (5,4,1)$

A special case of the so-called Pieri formula states that

$$\sigma_{(1,0,\dots,0)} \cdot \sigma_\lambda = \sum \sigma_\nu,$$

where the left-hand side is the intersection of two cycles and the sum on the right-hand side is over all partitions ν which can be obtained by adding one box to the Young diagram of λ.

Going back to the counting of the proposed problem, it follows from (1) that $A = \sigma_{(1,0)}$. Therefore,

$$[X_{\ell_1}] \cdot [X_{\ell_2}] \cdot [X_{\ell_3}] \cdot [X_{\ell_4}] = \sigma^4_{(1,0)} \in H_0(\mathrm{Gr}(2,4),\mathbb{Z}) \cong \mathbb{Z}.$$

By Pieri's formula, we have

$$\sigma_{(1,0)} \cdot \sigma_{(1,0)} = \sigma_{(2,0)} + \sigma_{(1,1)} \implies \sigma^3_{(1,0)} = \sigma_{(2,1)} + \sigma_{(2,1)}$$
$$\implies \sigma^4_{(1,0)} = 2\sigma_{(2,2)} = 2.$$

Note that the Schubert cycle $\sigma_{(2,2)}(W)$ is the point $W_2 \in \mathrm{Gr}(2,4)$ (i.e., it generates $H_0(\mathrm{Gr}(2,4),\mathbb{Z})$). It is straightforward (but crucial) to show that for generic 4 lines $\ell_1, \ell_2, \ell_3, \ell_4$, the intersection $X_{\ell_1} \cap X_{\ell_2} \cap X_{\ell_3} \cap X_{\ell_4}$ is transverse. We conclude that the answer to the proposed problem is 2.

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I learned about the following problem from Shmuel Weinberger. It can be viewed as a topological analogue of Arrow's Impossibility Theorem.

(a) A group of n friends have decided to spend their summer cottaging together on an undeveloped island, which happens to be a perfect copy of the closed 2-disk D^2. Their first task is to decide where on this island to build their cabin. Being democratically-minded, the friends decide to vote on the question. Each friend chooses his or her favourite point on D^2. The friends want a function that will take as input their n votes, and give as output a suitable point on D^2 to build. They believe, to be reasonable and fair, their "choice" function should have the following properties:

- (*Continuity*) It should be continuous as a function $(D^2)^n \to D^2$. This means, if one friend changes their vote by a small amount, the output will change only a small amount.
- (*Symmetry*) The n friends should be indistinguishable from each other. If two friends swap votes, the final choice should be unaffected.
- (*Unanimity*) If all n friends chose the same point x, then x should be the final choice.

For which values of n does such a choice function exist?

(b) The friends' second task is to decide where along the shoreline of the island they will build their dock. The shoreline happens to be a perfect copy of the circle S^1. Again, they decide to take the problem to a vote. For which values of n does a continuous, symmetric, and unanimous choice function $(S^1)^n \to S^1$ exist?

These are special cases of the following general problem in topological social choice theory: given a topological space X, for what values of n does X admit a social choice function that is continuous, symmetric, and unanimous? In other words, when is there a function $A\colon X^n \to X$ satisfying

- A is continuous,
- $A(x_1, \dots, x_n)$ is independent of the ordering of $x_1, \dots, x_n$, and
- $A(x, x, x, \dots, x) = x$ for all $x \in X$?

Jenny Wilson (Department of Mathematics, University of Michigan, USA)

Solution by the proposer

Consider the general problem described in the last paragraph. The statement that $A(x_1, \dots, x_n)$ is independent of the ordering of $x_1, \dots, x_n$ is the statement that the function A factors through the *symmetric product*, the quotient of X^n by the action of the symmetric group Σ_n, endowed with the quotient topology. Elements of X^n/Σ_n are multisets of n (not necessarily distinct) points in X. The statement that $A(x, x, \dots, x) = x$ is the statement that the composition

$$X \xrightarrow{\Delta} X^n \to X^n/\Sigma_n \to X,$$
$$x \mapsto (x, x, \dots, x) \mapsto \{x, x, \dots, x\} \mapsto A(x, x, \dots, x) = x$$

is the identity function. Thus the problem is equivalent to the following: does there exist a retraction from the symmetric product X^n/Σ_n onto the image of the diagonal?

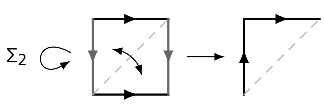
(a) An appropriate choice function exists for any n. Identify the island (up to homeomorphism) with the closed unit disk in $\mathbb{R}^2$. Since the disk is convex, we can (for example) let

$$A(x_1, x_2, \dots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

be the average value of the n points.

(b) Such a choice function only exists for $n = 1$. We first consider the case $n = 2$, since this case reduces to a problem that will be familiar to many algebraic topology students.

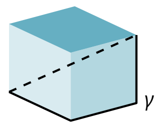
The symmetric product $(S^1 \times S^1)/\Sigma_2$ is the Möbius band and the image of the diagonal is its boundary, as pictured.



However, the boundary is not a retract of the Möbius band: the inclusion of the boundary induces the map $2\mathbb{Z} \hookrightarrow \mathbb{Z}$ on fundamental groups, which does not have a left inverse.

This argument generalizes for any $n \geq 2$. We can realize the n-torus $(S^1)^n$ as the unit cube in $\mathbb{R}^n$ with opposite faces identified. The 2^n corners are identified to a single point x, which we choose as basepoint.

Let γ be a path from the origin to the point $(1, 1, \dots, 1) \in \mathbb{R}^n$ along n mutually orthogonal edges of the cube, pictured here for $n = 3$. Construct (say, by straight-line homotopy) a based homotopy from the diagonal to γ.



The n orthogonal edges are identified in the symmetric product to a single loop. Thus, in $\pi_1((S^1)^n/\Sigma_n, \Sigma_n \cdot x)$ the image of the diagonal (which equals the image of γ) has an nth root. The image of the diagonal in $(S^1)^n/\Sigma_n$ cannot be a retract.

For the general problem, Eckmann–Ganea–Hilton and later (independently) Weinberger proved the following results; see Eckmann's survey [1].

Theorem. *Suppose that X is homotopy equivalent to a finite simplicial complex. If X is contractible, the function A exists for any n. If X is not contractible, it exists only for $n = 1$.*

Theorem. *Suppose that X is homotopy equivalent to a connected CW complex. Then the map A exists for all n if and only if X is a product of rational Eilenberg–MacLane spaces.*

Weinberger [2] notes that there exist other infinite CW complexes for which a choice function exists for (some) arbitrarily large values of n. For example, the infinite-dimensional real projective space $\mathbb{RP}^\infty$ admits a social choice function A for any odd value of n, but not for any even value.

References

[1] B. Eckmann, Social choice and topology: A case of pure and applied mathematics. *Expo. Math.* **22**, 385–393 (2004)

[2] S. Weinberger, On the topological social choice model. *J. Econom. Theory* **115**, 377–384 (2004)

We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions to Michael Th. Rassias by email to mthrassias@yahoo.com.

We also solicit your new problems with their solutions for the next "Solved and unsolved problems" column, which will be devoted to Number Theory.

Report from the EMS Council in Bled (25–26 June 2022)

Richard Elwes

The EMS Council is the governing body which meets every two years to take the decisions that will determine the society's future. After a condensed online meeting in 2020, the Council was able to hold an in-person meeting with a full agenda on the weekend of 25–26 June 2022, in the stunning surroundings of Bled (Slovenia). The meeting was generously and efficiently hosted by the University of Primorska, represented by rector Klavdija Kutnar. On Saturday evening, the meeting was addressed by Boštjan Kuzman from the Society of Mathematicians, Physicists and Astronomers of Slovenia (DMFA) who introduced the assembled company to Josip Plemelj (1873–1967) who grew up in Bled and went on to make important contributions to the theory of harmonic functions and related fields, as well as becoming the first chancellor of the University of Ljubljana. His beautiful villa in Bled is now managed by the DMFA and available to visit and as a venue for small meetings.

Officers' reports and finance

The Council was opened on Saturday morning by EMS Vice President Betül Tanbay with the unfortunate news that the President, Volker Mehrmann, had that morning tested positive for COVID and, despite being on site, would need to participate over Zoom. The President then presented his report, recalling several of the major developments during his tenure, which ends this year. Many of the items in his report were revisited later during the meeting. He reminded the Council that the EMS website[1] and database have been rebranded and rebuilt, with much of the expertise and leadership provided by EMS Press, while the *EMS Newsletter* has been transformed into the *EMS Magazine*. The EMS's 30th anniversary was celebrated (two years late) at a very enjoyable event in March 2022 at ICMS in Edinburgh. A celebratory brochure was distributed to mark the event (and is available online[2]).

[1] https://euromathsoc.org/
[2] *Thirty Years of EMS*, available at https://euromathsoc.org/about

Lake Bled (by Arne Müseler / arne-mueseler.com / CC-BY-SA-3.0)

The EMS has joined the European Open Science Cloud (EOSC) in order to raise the voice of mathematics in the developments and, if possible, to join major research proposals.

Following the success of the Caucasian Mathematical Conferences (the third of which took place in 2019 in Rostov-on-Don), the Executive Committee has decided to launch a series of Balkan Mathematical Conferences. The first will take place in July 2023 in Pitești jointly with the Congress of Romanian Mathematicians.

The President noted that funding for mathematics by the EU, particularly the European Research Council (ERC), has substantially reduced in recent years. A major reason for this is that the number of applications from the mathematics community is low. This is a major problem, and so far EMS initiatives aimed at improving this situation have not had a big impact, but it will be important to continue addressing this problem.

The Treasurer (Mats Gyllenberg) delivered his report next, stating that the EMS is in robust financial health. This, of course, is good news, but part of the reason for this is the pandemic, which has reduced opportunities to spend money on scientific activities. The Executive Committee is committed to spending money on scientific activities, a major part of the EMS's raison d'être. This is reflected in proposals for a new "EMS Young Academy" and "EMS Topical Activity Groups" (more details below). Recalling that for financial purposes the society is governed by Finnish law, the Council approved financial statements for 2020 & 2021, as well as the budgets for 2023 & 2024, and the appointment of both professional and lay auditors.

Membership and publicity

EMS Secretary Jiří Rákosník reported on the EMS membership, with individual members having recently exceeded 3,000. The Council regretfully approved the termination of the membership of the Mathematical Society of the Republic of Moldova, which has fallen behind on its membership fees and has been non-responsive to communications. Sadly, the Emmy Noether Research Institute for Mathematics has terminated its membership with effect from January 2023. On the positive side, the Mathematical Society of South Eastern Europe (MASSEE) has joined the EMS as an associate member, and the Council was pleased to agree an arrangement of reciprocal membership with the Indonesian Mathematical Society. There were no other applications for corporate membership.

EMS Publicity Officer Richard Elwes was absent, but delivered his report via video, updating the delegates on his activities, in particular the growth of the EMS social media platforms: the Twitter account @EuroMathSoc has recently passed 10,000 followers, the Facebook account (also @EuroMathSoc) is approaching 5,000, and a LinkedIn page has recently been launched. An EMS YouTube page was set-up in 2020 initially to host the EMS video "The Era of Mathematics" (created with the support of the EMS Education Committee). He offered the view that the EMS should aim to increase the use of its YouTube page in future. He also discussed other avenues of publicity including displaying new designs for EMS flyers and posters.

Elections to Executive Committee

The most exciting part of any Council meeting is the election of new officers. In previous years the election of the President has often been unopposed, but this time the Council was delighted to have three top quality candidates for the next EMS Presidency. After impressive presentations from all three, the Council was pleased to elect Jan Philip Solovej as President for the term 2023–26. Two unopposed elections followed: Beatrice Pelloni as Vice President, and Samuli Siltanen as Treasurer. This left one election for member-at-large of the Executive Committee: after strong presentations by four candidates, Victoria Gould was elected.

Changes to statutes and bylaws

The Council approved changes to the EMS's statutes and bylaws, several purely technical (for example to correct outdated or inconsistent terminology), but some substantive. Two changes will apply to future Council meetings. Firstly, the Executive Committee will be able to decide to enable delegates to participate virtually, and secondly a Nomination Committee will be set up to short-list candidates for future EMS elections. (As now, candidates will still be able to be proposed from the floor at the Council meeting.)

Most significantly, the Council approved the establishment of two major new initiatives: an EMS Young Academy for early career mathematicians, and EMS Topical Activity Groups. More details of these exciting developments will follow in future editions of the Magazine.

European Congress of Mathematics and other meetings

Following the great success (despite equally great setbacks) of the 8th European Congress of Mathematics in Portorož in 2021, Juan González-Meneses, chair of the Organizing Committee, reported on plans for the 9th ECM in Seville (15–19 July 2024). Preparations are going well, with support from all the Spanish mathematical societies. The organisers will provide 150 grants for young people and people from less advantaged countries, and satellite events are expected all over Spain.

In a significant increase in the EMS's offering, the Executive Committee has decided to open calls for proposals of large cross-institutional events. These might include special semesters, interdisciplinary study groups, or large showcase events targeting new theories or emerging problems. They might involve the interaction of more than one mathematical area, or the discussion between mathematics and other disciplines.

The goal will be for these events to be spread across Europe, complementing existing infrastructure for mathematical meetings. However, they will not provide funding for programmes and workshops organised in settings which are already well served. Instead, proposals will be required to deploy significant funding to support regions and communities that do not otherwise have the infrastructure or finances to organise such large-scale events.

Reports from EMS committees

The Council received reports regarding each of the EMS's ten standing committees, which carry out a great deal of the society's work. This began with the Education Committee, where Vice President Betül Tanbay relayed the regrettable news that the chair, Jürg Kramer, has recently resigned, and that the Executive Committee is searching for a replacement. She discussed the committee's many and varied activities, on both the theoretical and practical sides of mathematical education. These include the EU-funded INNOMATH project[3] to develop resources for gifted mathematics school-students, supporting the creation of the EMS video "The Era of Mathematics" in 2020 (available on the EMS YouTube channel), and undertaking a major survey and pedagogical research on the secondary-tertiary transition.

[3] https://innomath.eu/

EMS Secretary Jiří Rákosník conveyed to the meeting that the Meetings Committee has also undertaken a huge amount of work in the last two years evaluating more than 60 applications for EMS support for summer schools and other scientific events. Since this committee will become even busier in the future with the start of the Topical Activity Groups, the committee will be increased in size. He also reminded the Council that the deadline for applications has been moved from September to July to give the Meetings Committee more time for its deliberations.

Carola Schönlieb, chair of the Committee for Applications and Interdisciplinary Relations (formerly the Committee for Applied Mathematics), reported that the committee has been substantially renewed, both in its name, with nine new members in 2022, and a revised programme and mode of operation. It is a busy committee which has formed several individual working groups, including on prize nominations and relations with other bodies such as the European Research Council. The committee collaborates with other EMS organs, most notably the Meetings Committee.

Sophie Dabo reported via Zoom on the Committee for Developing Countries, which she chairs. The committee has 11 members and 17 associate members and cooperates with local organisations in developing countries towards several goals. These include developing mathematical curricula and MSc and PhD programmes, funding researchers to attend and organise conferences, helping to build libraries. The committee administers the designation *Emerging Regional Centre of Excellence (ERCE)* which is awarded to suitable institutions for a period of four years (renewable). Currently there are 7 ERCE centres. Since 2017, the committee has administered the *Simons for Africa* programme funded by the Simons Foundation, which supports researchers in Africa. To date

Council delegates and guests (photo courtesy of University of Primorska) more than 210 applications have been evaluated with almost 58 % success. An important source of funding for the committee's work is donations from EMS members through a link on the EMS website.

Adam Skalski, the chair of the ERCOM (European Research Centres on Mathematics) committee, delivered a report. This committee brings together the scientific directors of 30 mathematical research centres around Europe to work on issues of common interest. He discussed the committee's activities and invited the Council to view their website ercom.org.

Betül Tanbay (EMS Vice President) reported that the Ethics Committee is continuing to function. Its activities include maintaining the EMS Code of Practice[4], encouraging journals and publishers to respond to allegations of unethical behaviour in a conscientious manner, and providing a mechanism whereby individual researchers can ask the committee to help them pursue claims of unethical behaviour. The committee has also recently set up a webpage in cooperation with the Committee for Publications and Electronic Dissemination on predatory journals and publishers.[5]

Frédéric Hélein (Executive Committee) reported on the work of the European Solidarity Committee, which supports research-related activities for researchers from financially weaker countries (for example travel grants to young mathematicians). However, throughout the COVID pandemic, there has been little demand for this type of activity. The committee also handled the Kovalevskaja grants for the participation of young mathematicians at the ICM. Unfortunately, this large amount of work was in vain, once the ICM moved online. With many members' terms due to expire this year, this committee will need to be replenished.

Thierry Bouche, chair of the Committee for Publishing and Electronic Dissemination, reminded the Council that this committee has existed since April 2017, replacing the former Publications Committee and Committee for Electronic Publishing. The committee has carefully studied the evolving open access landscape and written strategy documents in response to the initiative *Plan S*, arguing for the importance of a diversity of models beyond "Gold" Open Access which is often favoured by aggressively commercial publishers. The committee also supervises the work of the European Digital Mathematics Library (see below), while a subgroup is working to evaluate the functioning of zbMATH Open from the perspective of a user. Committee member Tomaž Pisanski delivered a presentation on the problem of ranking and classifying mathematical journals, noting some discrepancy between Scimago and zbMATH Open.

Jorge Buescu (EMS Vice President) informed the meeting about the many activities of the Committee for Raising Public Awareness, including its involvement in the International Day of Mathematics[6], its on- and off-line outreach activities, and notably the POP MATH portal[7], an online calendar and map of popular mathematics events across Europe.

Alessandra Celletti, chair of the Committee for Women in Mathematics, reported on its work. The committee has proposed interviews of women mathematicians for the *EMS Magazine*, facilitated nominations for international prizes and EMS Lectures, and supported applications for summer schools at the Institut Mittag-Leffler jointly with the society of European Women in Mathematics. The chair and committee member Stanislawa Kanas published the article "Underrepresentation of women in editorial boards of scientific and EMS journals" in the *EMS Newsletter*[8] as part of their work on this topic. The committee monitors the gender gap in the editorial boards of EMS publications and helps to address such imbalances. On the 20th of May 2022, the committee organised the successful first *EMS Women in Mathematics Day*, within the broader initiative of "May 12th", a celebration of women in mathematics in memory of Maryam Mirzakhani.

Publications and projects

Fernando Pestana da Costa, editor-in-chief of the *EMS Magazine* since September 2020, reported on many recent improvements, with the name, format, and design having changed from issue 119 in March 2021. The EMS Press has constructed a new webpage[9] where articles can be read and downloaded either individually or as an entire issue. The EMS will appreciate the support of national societies in continuing to promote the *EMS Magazine* and acquire good contributions.

André Gaul, the managing director of the EMS Press, reported remotely. In 2019, a new company fully owned by EMS was established in Berlin, focused on independent, fair, and sustainable publishing using modern electronic tools. A completely new platform was built, and the Subscribe to Open (S2O) business model has been rolled out. EMS Press publishes 25 journals, with the new *Memoirs of the EMS* (first volume published in May 2022) catering for longer works than typical research articles. Book production amounts to approximately 12 titles per year. Unified EMS and EMS Press branding has been developed, and the EMS Press also provides significant technology to the EMS (e.g. website, document cloud, unified login system, submission forms, etc.).

Editor-in-chief Klaus Hulek of zbMATH Open delivered his report on what is now the world's biggest dedicated database on mathematical literature, with around 4.4 million documents and more than 39,000 software packages. It is published by FIZ Karlsruhe, the EMS, and Heidelberg Academy of Sciences and Humanities. He

[4] https://euromathsoc.org/code-of-practice

[5] https://euromathsoc.org/predatory-publishing

[6] https://www.idm314.org/

[7] https://www.popmath.eu/

[8] A. Celletti and S. Kanas, Underrepresentation of women in editorial boards of scientific and EMS journals, *EMS Newsl.* **118**, 60–63 (2020)

[9] https://euromathsoc.org/magazine

described the enormous preparations that were needed for the 2021 transition to open access, when *Zentralblatt MATH* became *zbMATH Open*, with financial support from the German government. The goal is to grow beyond an open access interface, to a fuller open data platform. Recently added features include enhanced author profiles, including non-ASCII scripts (e.g. Arabic, Chinese, Japanese, Cyrillic, …). The first year of open access shows very good results with around 60,000 unique visitors per month, and increased numbers of searches, completed documents, and reviewer commitments. Reviewers are rewarded for their efforts with reductions on EMS publications.

Thierry Bouche, chair of the European Digital Mathematical Library[10], delivered a report. The EuDML is a distributed library taking contents from several sources, and was created in a project partly financed by the European Commission in 2010–2013. Its service continues despite the lack of funding since then, and runs in a reduced capacity thanks to voluntary work from motivated partners.

A report was delivered by Zoltán Horváth, the president of the very active and successful project EU-MATHS-IN (European Service Network of Mathematics for Industry and Innovations). This was established by EMS and ECMI (European Consortium for Mathematics in Industry) in 2011, intended as a one-stop-shop at the European level, to facilitate exchanges between application-driven mathematical research and its use in innovations in industry, science, and society. He reported on plans for new tools and requested that the EMS help promote this network's important activities.

Russian war in Ukraine

On both days of the meeting, there was impassioned discussion of topics around the Russian invasion of Ukraine, and the measures the EMS has taken in response. The EMS President re-affirmed that the society stands in full solidarity with the people of Ukraine and especially our Ukrainian colleagues, but at the same time must continue to strive for the unity of the international mathematical community. He described the measures that have been taken so far, including the suspension of the EMS membership of the Euler Institute and the decision not to cooperate with Russian governmental organisations. Further possible steps were discussed, but as well as being politically sensitive, there are also many significant technical and legal complications here, considering the EMS's own regulations and its obligations under Finnish law.

The meeting closed with warm thanks to our Slovenian colleagues for once again hosting a major EMS event, following the ECM in 2021 in Portorož, with great accomplishment and kindness.

[10] https://eudml.org/

Richard Elwes is the EMS publicity officer and a senior teaching fellow at University of Leeds (UK). As well as teaching and researching mathematics, he is involved in mathematical outreach and is the author of five popular mathematics books.

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- Option to join the EMS as reviewer of zbMATH Open.

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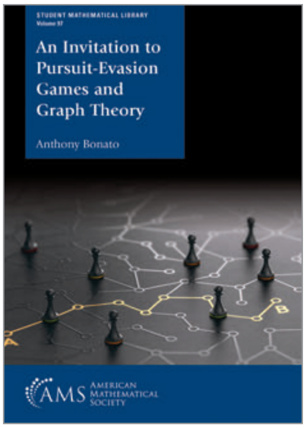

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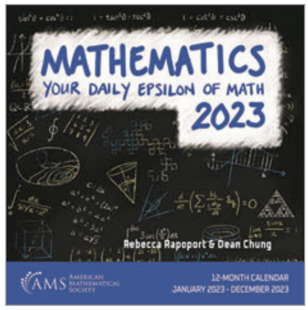

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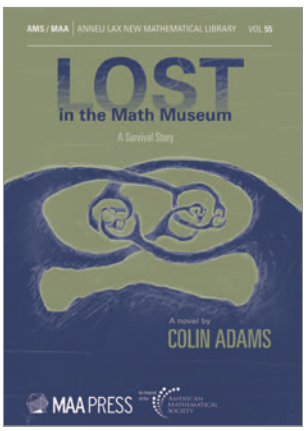

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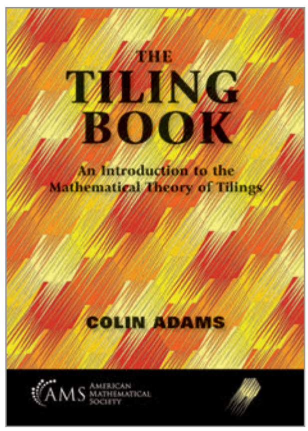

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