European Mathematical Society Magazine

ISSN 2747-7894 / Issue 126 / December 2022

EMS Magazine

Richik Sengupta, Soumik Adhikary, Ivan Oseledets and Jacob Biamonte
Tensor networks in machine learning

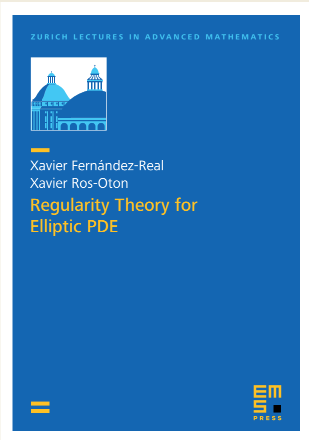
Marie E. Rognes
Waterscales: Mathematical and computational foundations for modelling cerebral fluid flow

Lajos Molnár
100 years of Acta Szeged

Kristine Bauer, Erin Chambers, Brenda Johnson, Kristin Lauter and Kathryn Leonard
Research Networks for Women

EUROPEAN MATHEMATICAL SOCIETY

New EMS Press titles



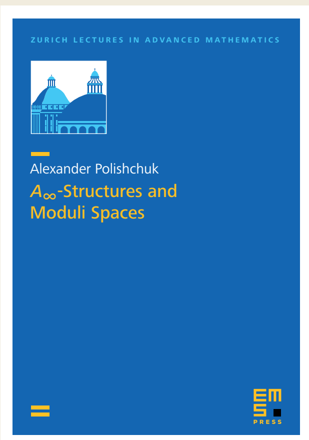
Xavier Fernández-Real (École Polytechnique Fédérale de Lausanne)
Xavier Ros-Oton (ICREA, Universitat de Barcelona, and Centre de Recerca Matemàtica)

Regularity Theory for Elliptic PDE

Zurich Lectures in Advanced Mathematics
ISBN 978-3-98547-028-0. eISBN 978-3-98547-528-5
December 2022. Softcover. 236 pages. €49.00*

This text aims to provide a self-contained introduction to the regularity theory for elliptic PDE, focusing on the main ideas rather than proving all results in their greatest generality. It can be seen as a bridge between an elementary PDE course and more advanced books.

The book starts with a short review of the Laplace operator and harmonic functions. The theory of Schauder estimates is developed next, but presented with various proofs of the results. Nonlinear elliptic PDE are covered in the following, both in the variational and non-variational setting and, finally, the obstacle problem is studied in detail, establishing the regularity of solutions and free boundaries.



Alexander Polishchuk (University of Oregon)

A_∞-Structures and Moduli Spaces

Zurich Lectures in Advanced Mathematics
ISBN 978-3-98547-026-6. eISBN 978-3-98547-526-1
December 2022. Softcover. 178 pages. €39.00*

This book discusses certain moduli problems related to A_∞-structures. These structures can be viewed as a way of recording extra information on cohomology algebras. They are useful in describing derived categories appearing in geometry, and as such, they play an important role in homological mirror symmetry.

The author presents some general results on the classification of A_∞-structures. For example, he gives a sufficient criterion for the existence of a finite-type moduli scheme of A_∞-structures extending a given associative algebra. He also considers two concrete moduli problems for A_∞-structures. The first is related to the moduli spaces of curves, while the second is related to the classification of solutions of an associative version of the Yang–Baxter equation.

The book will be of interest to graduate students and researchers working in homological algebra, algebraicgeometry, and related areas.

**20% discount on any book purchases for individual members of the EMS, member societies or societies with a reciprocity agreement when ordering directly from EMS Press.*

EMS Press is an imprint of the
European Mathematical Society – EMS – Publishing House GmbH
Straße des 17. Juni 136 | 10623 Berlin | Germany

https://ems.press | orders@ems.press

EUROPEAN MATHEMATICAL SOCIETY

EMS PRESS

European Mathematical Society Magazine

The views expressed in the *European Mathematical Society Magazine* are those of the authors and do not necessarily represent those of the European Mathematical Society (EMS) or the Editorial Team.

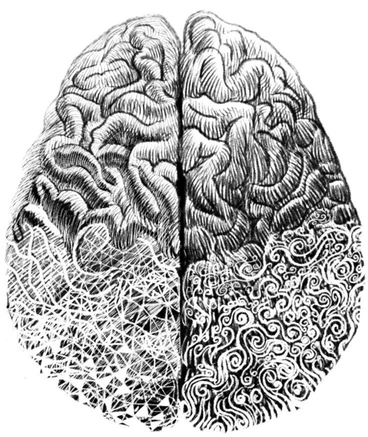
For advertisements and reprint permission requests please contact magazine@ems.press.

Published by EMS Press, an imprint of the
European Mathematical Society – EMS – Publishing House GmbH
Technische Universität Berlin, Institut für Mathematik,
Straße des 17. Juni 136, 10623 Berlin, Germany

https://ems.press

Typesetting: WisSat Publishing + Consulting GmbH, Fürstenwalde, Germany
Printing: Beltz Bad Langensalza GmbH, Bad Langensalza, Germany

ISSN (print) 2747-7894; ISSN (online) 2747-7908



The cover illustration by António B. Araújo is a reference to the modelling of cerebral fluid flow as described in the paper presented by Marie E. Rognes in the present issue.

Dear EMS members,

After four years as president of the EMS, I want to take a brief look back. When I presented the plans for my presidency at the council meeting in Prague in 2017, I was quite ambitious about what I thought would be important and what could be achieved in these four years.

I want to thank the members of the EC, the other committees, and the EMS office for their tremendous work in these difficult times, with the pandemic, a war in Europe, a website hack at the office in Helsinki, and many more obstacles. First, I would like to apologize for some of the late payments and replies from the office. We were not functioning well when the EMS secretary Elvira Hyvönen went on maternity leave. But fortunately, now she is back, and we have also decided to increase the office work capacity by an extra person who will be hired soon.

Although we sometimes felt that we would fail to achieve anything in this period, I think we can be optimistic about the future of the EMS.

First of all, the reorganization of the EMS Publishing House into EMS Press has been successfully completed. EMS Press has successfully implemented its new Subscribe-to-Open (S2O) publishing strategy and more than 20 of its journals are now published under this model. Despite this complex transition, EMS Press has established a solid financial basis, and the funds from the liquidation of the old Publishing House can and will be used for the support of scientific activities of the EMS.

The first call for nominations for an EMS Young Academy (EMYA) has been successful, and the selection committee will select up to 30 people who will be members of the EMYA for four years.

The first call for the creation of EMS Topical Activity Groups (EMS-TAG) to integrate the scientific cooperation across Europe in all mathematical fields has also been successful, and the evaluation committee will recommend the first set of TAGs to be approved by the executive committee in the next spring.

A first call for large, inclusive, and cross-institutional events can range from special semesters to interdisciplinary study groups and large showcase events, and such calls will be open on December 1, 2022.

A topic that will need further attention and a definite change of existing procedures is the sustainability and the CO_2 footprint in the EMS activities. This was easy during the pandemic years, where almost no in-person scientific or business meetings took place. Noticing, however, that it is not a good way to communicating and working together by solely looking into little rectangles on computer screens, we must think anew how to combine our need for in-person meetings and communication with the need of reducing the negative effects on the climate.

Another topic that will need further discussion is the way in which we operate big conferences like the ECM in the future. The new vice president Beatrice Pelloni will initiate a discussion of this soon.

A major concern is the ever-decreasing funding for mathematics in the European Research Council. Currently, discussions are ongoing at different levels on what are the reasons there are so few mathematics ERC proposals (with the effect that the budget for mathematics is shrinking). We call on the mathematical community to submit more excellent proposals on all levels to the ERC. Even though this is work and the success ratio may be small, writing such a proposal has a value by itself because it sharpens the view on one's own research and also helps the mathematical community.

In the last years, with all the difficulties, it became increasingly clear to me that the mathematical community is very small in the whole scientific world and that to prevent the reduction of our beloved Mathematics to an exotic niche we must stand and work together.

With this closing remark, I want to say good-bye to you. It was a great honor and pleasure to serve the EMS. And now I hand over the editorial to the new president Jan Philip Solovej.

Volker Mehrmann
President of the EMS

Tensor networks in machine learning

Richik Sengupta, Soumik Adhikary, Ivan Oseledets and Jacob Biamonte

A tensor network is a type of decomposition used to express and approximate large arrays of data. A given dataset, quantum state, or higher-dimensional multilinear map is factored and approximated by a composition of smaller multilinear maps. This is reminiscent to how a Boolean function might be decomposed into a gate array: this represents a special case of tensor decomposition, in which the tensor entries are replaced by 0, 1 *and the factorisation becomes exact. The associated techniques are called tensor network methods: the subject developed independently in several distinct fields of study, which have more recently become interrelated through the language of tensor networks. The tantamount questions in the field relate to expressability of tensor networks and the reduction of computational overheads. A merger of tensor networks with machine learning is natural. On the one hand, machine learning can aid in determining a factorisation of a tensor network approximating a data set. On the other hand, a given tensor network structure can be viewed as a machine learning model. Herein the tensor network parameters are adjusted to learn or classify a data-set. In this survey we review the basics of tensor networks and explain the ongoing effort to develop the theory of tensor networks in machine learning.*

1 Introduction

Tensors networks are ubiquitous in most areas of modern science including data science [11], condensed matter physics [19], string theory [22] and quantum computer science. The manners in which tensors are employed/treated exhibit significant overlap across many of these areas. In data science, tensors are used to represent large datasets. In condensed matter physics and in quantum computer science, tensors are used to represent states of quantum systems.

Manipulating large tensors comes at a high computational cost [28]. This observation has inspired techniques for tensor decompositions that would reduce computational complexity while preserving the original data that they represents. Such techniques are now known as tensor network methods.

Tensor networks have risen to prominence in the last fifteen years, with several European schools playing leading roles in their modern development, as a means to describe and approximate quantum states (see the review [14]). The topic however dates back much further, to work of Penrose [32] and in retrospect, even arose as special cases in work of Cayley [10]. Tensor networks have a rich modern history in mathematical physics [32], in category theory [44], in computer science, algebraic logic and related disciplines [1]. Such techniques are now becoming more common in data science and machine learning (see the reviews [12, 13]).

1.1 Basic material about multilinear maps

It might be stated that the objective of linear algebra is to classify linear operators up to isomorphism and study the simplest representative in each equivalence class. This motivates the prevalence of decompositions such as SVD, LU and the Jordan normal form. A special case of linear operators are linear maps from a vector space V to an arbitrary field like $\mathbb{C}$ or $\mathbb{R}$. These linear maps form the dual space (vector space of covectors) $V^\star$ to our vector space V.

A natural generalisation of linear maps is provided by the multilinear maps, i.e., maps that are linear in each argument when the values of other arguments are fixed. For a given pair of non-negative integers p and q, a type-(p,q) tensor T is defined as a multilinear map

$$T\colon \underbrace{V^* \times \cdots \times V^*}_{p \text{ copies}} \times \underbrace{V \times \cdots \times V}_{q \text{ copies}} \to \mathbb{K}, \tag{1}$$

where $\mathbb{K}$ is an arbitrary field. The tensor T is said to be of order (valence) $p+q$. Note that some authors refer to this as rank $p+q$, but we will never do that.

It is often more convenient to view tensors as elements of a vector space known as the tensor product space. Thus, the above (p,q)-tensor T in this alternative interpretation can be defined as an element

$$T \in \underbrace{V \otimes \cdots \otimes V}_{p \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{q \text{ copies}}.$$

EMS MAGAZINE 126 (2022) — DOI 10.4171/MAG/101

Moreover, the universality property of tensor products of vector spaces states that any multilinear map can be replaced by a unique linear map acting from the tensor product of vector spaces to the base field.

If we assume the axiom of choice, every vector space admits a Hamel basis. If e_i is such a basis in V, then the components of a tensor T are therefore the coefficients of T with respect to the basis e_i and its dual basis $\boldsymbol{\varepsilon}^j$ (basis of the dual space V^*), that is

$$T = T^{i_1\dots i_p}_{j_1\dots j_q}\mathrm{e}_{i_1} \otimes \dots \otimes \mathrm{e}_{i_p} \otimes \boldsymbol{\varepsilon}^{j_1} \otimes \dots \otimes \boldsymbol{\varepsilon}^{j_q}. \tag{2}$$

Adopting Einstein's summation convention, summation over repeated indices is implied in (2).

Returning to (1), given p covectors $(c^1, \dots, c^p)$ and q vectors $(v_1, \dots, v_q)$, the value of the tensor is evaluated as

$$T^{i_1\dots i_p}_{j_1\dots j_q}c^1(\mathrm{e}_{i_1}) \times \dots \times c^p(\mathrm{e}_{i_p}) \times \boldsymbol{\varepsilon}^{j_1}(v_1) \times \dots \times \boldsymbol{\varepsilon}^{j_q}(v_q) \in \mathbb{K}, \tag{3}$$

where $c^k(\mathrm{e}_{i_k})$ and $\boldsymbol{\varepsilon}^{j_l}(v_l)$ are numbers obtained by evaluating covectors (functionals) at the corresponding vectors. In quantum computation, the basis vectors e_{i_k} are denoted by $|i_k\rangle$ and the basis covectors $\boldsymbol{\varepsilon}^{j_l}$ are denoted by $\langle j_l|$. Using Dirac's notation, tensor products are written in compact form as

$$\mathrm{e}_{i_m} \otimes \mathrm{e}_{i_l} := |i_m i_l\rangle, \quad \boldsymbol{\varepsilon}^{j_m} \otimes \boldsymbol{\varepsilon}^{j_l} := \langle j_m j_l|, \quad \mathrm{e}_{i_m} \otimes \boldsymbol{\varepsilon}^{j_l} := |i_m\rangle\langle j_l|,$$

and the tensor T takes the form

$$T = T^{i_1\dots i_p}_{j_1\dots j_q}|i_1\dots i_p\rangle\langle j_1\dots j_p|.$$

Similarly, given a tuple of p covectors $(c^1, \dots, c^p)$ and q vectors $(v_1, \dots, v_q)$ we write them as $\langle c_1\dots c_p|$ and $|v_1\dots v_q\rangle$, as elements of the corresponding tensor product space(s).

In this notation the evaluation (3) takes the form

$$T^{i_1\dots i_p}_{j_1\dots j_q}\langle c_1\dots c_p|i_1\dots i_p\rangle\langle j_1\dots j_p|v_1\dots v_q\rangle.$$

Since in quantum computation the vector spaces under consideration as well as their duals are Hilbert spaces, Riesz's representation theorem for linear functionals implies that the evaluation $\langle * | * \rangle$ above can be seen as an inner product.

Finally, a tensor can be identified with the array of coefficients $T^{i_1\dots i_p}_{j_1\dots j_q}$ in a specific basis decomposition. This approach is not basis independent, but is useful in applications. Henceforth in this review we will fix the standard basis, which will establish a canonical isomorphism between the vector space and its dual. In the simplest case $p = q = 1$, for example, this gives us the following equivalences:

$$T_{ij} \cong T_{ji} \cong T^{ij} \cong T^{ji} \cong T^i_j \cong T^j_i.$$

In the more general case this leads to the equivalence between the components $T^{i_1\dots i_p}_{j_1\dots j_q}$ and $T_{j_1\dots j_q i_1\dots i_p}$. Given a valence-$m$ tensor T, the total number of tensors in the equivalence class formed by raising, lowering and exchanging indices has cardinality $(1+m)!$ (see [4]). Recognising this arbitrariness, Penrose considered a graphical depiction of tensors [32], stating that "it now ceases to be important to maintain a distinction between upper and lower indices." This convention is widely adopted in the contemporary literature.

1.2 Tensor trains aka matrix product states

Consider a tensor T with components $T_{j_1 j_2\dots j_n}$ with $j_k = 1, 2, \dots, d$. Hence, T has d^n components, a number that can exceed the total number of electrons in the universe when d is as small as 2 and $n \approx 300$. Clearly, storing the components of such a large tensor in a computer memory and subsequently manipulating it can become impossible. The good news is that for most practical purposes, a tensor typically contains a large amount of redundant information. This enables factoring of T into a sequence of "smaller" tensors.

Tensor trains (see [31]) and matrix product states (MPS) [30, 33] arose in data science and in condensed matter physics, where it was shown that any tensor T with components $T_{j_1 j_2 \cdots j_n}$ admits a decomposition of the form

$$T_{j_1 j_2\dots j_n} = \boldsymbol{\alpha}^\dagger A^{(1)}_{j_1} A^{(2)}_{j_2} \cdots A^{(n)}_{j_n}, \boldsymbol{\beta} \tag{4}$$

where $A^{(k)}_{j_k}$ is an $(r_{k-1} \times r_k)$-dimensional matrix, and $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as r_0- and r_n-dimensional vectors, respectively.

Likewise, an n-qubit state $|\psi\rangle \in (\mathbb{CP}^2)^{\otimes n}$, written in the computational basis as $|\psi\rangle = \sum_{j_1 j_2 \cdots j_n} T_{j_1 j_2\dots j_n}|j_1 j_2\dots j_n\rangle$, $j_k \in \{0, 1\}$, can equivalently be expressed as

$$|\psi\rangle = \sum_{j_1\dots j_n} \langle \alpha|A^{(1)}_{j_1} A^{(2)}_{j_2} \cdots A^{(n)}_{j_n}|\beta\rangle |j_1 j_2\dots j_n\rangle. \tag{5}$$

Here $A^{(k)}_{j_k}$ is a $r_{k-1} \times r_k$ dimensional matrix and $|\alpha\rangle$, $|\beta\rangle$ are r_0 and r_n dimensional vectors, respectively. Note that here we are adhering to the braket notation, as is customary in quantum mechanics. The representation of $|\psi\rangle$ in (5) is called the matrix product state representation with an open boundary condition (OBC-MPS). See Figure 1 for a graphical representation.

Yet another useful MPS decomposition that a state might admit is the MPS with periodic boundary condition (PBC-MPS) [30]. The PBC-MPS representation of an n-qubit state

$$|\psi\rangle = \sum_{j_1 j_2\dots j_n} T_{j_1 j_2\dots j_n}|j_1 j_2\dots j_n\rangle, \quad j_k = 0, 1,$$

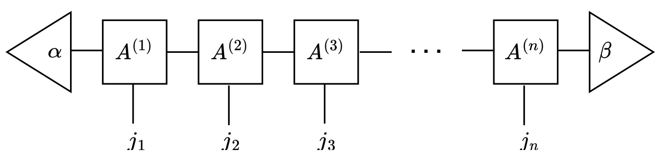


Figure 1. Graphical representation of tensor trains (open boundary condition—matrix product state representation). See (4), (5). Each index in the tensor $T_{j_1\dots j_n}$ is represented in the diagram by an open wire pointing downwards. We call these wires physical bonds. The horizontal wires represent extra indices which are summed over. Such internal wires are known as virtual bonds.

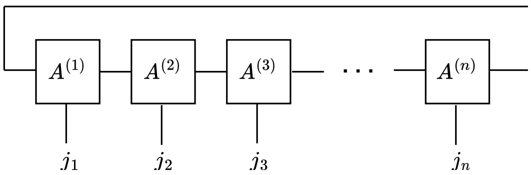


Figure 2. Graphical representation of the matrix product state in (6).

is given by

$$|\psi\rangle = \sum_{j_1 j_2 \ldots j_n} \mathrm{Tr}(A^{(1)}_{j_1} A^{(2)}_{j_2} \cdots A^{(n)}_{j_n})\, |j_1 j_2 \ldots j_n\rangle, \tag{6}$$

where $A^{(k)}_{j_k}$ is an $r \times r$ matrix. The graphical representation of a PBC-MPS is shown in Figure 2.

An n-qubit state $|\psi\rangle = \sum_{j_1\ldots j_n} T_{j_1\ldots j_n} |j_1 \ldots j_n\rangle$ has 2^n independent coefficients $T_{j_1\ldots j_n}$. The MPS representation of $|\psi\rangle$, on the other hand, is less data intensive. If $A^{(k)}_{j_k}$ is an $r \times r$ matrix for all k, the size of the representation becomes $2nr^2$, which is linear in n for a constant r. The point of the method is to choose r such that a good and compact approximation of $|\psi\rangle$ is obtained. The number r is often also referred to as the virtual bond dimension. Data compression becomes even more effective if the MPS is site independent, that is, if $A^{(k)}_{j_k} = A_{j_k}$ for all k. It has been shown that a site-independent representation of a PBC-MPS always exists if the state is translation invariant [33]. Note that MPS is invariant under the transformation $A_j \to P A_j P^{-1}$ for any invertible P; this follows from the cyclicity of the trace operator. Therefore, it is often customary to impose an additional constraint here, viz. $\sum_j A_j A_j^\dagger = \mathbf{1}$, in order to fix the gauge freedom [14] (see also the connections to algebraic invariant theory [5]).

2 Machine learning: classical to quantum

2.1 Classical machine learning

At the core of machine learning is the task of data classification. In this task, we are typically provided with a labelled dataset $\mathcal{S} = \{(\mathbf{x}_j, \mathbf{y}_j)\}_{j=1}^{M}$, where the vectors $\mathbf{x}_j \in \mathbb{R}^N$ are the input data (e.g., animal images) and the vectors $\mathbf{y}_j \in \mathbb{R}^d$ are the corresponding labels (e.g., animal types). The objective is to find a suitable machine learning model F with tunable parameters $\boldsymbol{\theta} \in \mathbb{R}^k$ that generates the correct label for a given input $\mathbf{x} \in \mathbb{R}^N$. Note that our model can be looked upon as a family of functions parameterised by $\boldsymbol{\theta}$: F takes a data vector as an input and outputs a predicted label; for an input datum $\mathbf{x}_j$, the predicted label is $F(\mathbf{x}_j, \boldsymbol{\theta})$. To ensure that our model generates the correct labels, it needs to be trained; in order to accomplish this, a training set $\mathcal{T} \subset \mathcal{S}$ is chosen, the elements of which serve as input data to train F. Training requires a cost function,

$$\mathcal{L}(\boldsymbol{\theta}) = \sum_{(\mathbf{x}_j, \mathbf{y}_j) \in \mathcal{T}} D(F(\mathbf{x}_j, \boldsymbol{\theta}), \mathbf{y}_j), \tag{7}$$

where $D(\cdot\,,\cdot)$ measures the mismatch between the real label and the estimated label. Typical choices for D include, e.g., the negative log-likelihood function, mean squared errors (MSE), etc. [37]. By minimising (7) with respect to $\boldsymbol{\theta}$, one obtains the value $\boldsymbol{\theta}^\star \in \arg\min_{\boldsymbol{\theta}} \mathcal{L}(\boldsymbol{\theta})$, which completes the training. After the model F has been trained, we can evaluate its performance by feeding it inputs from $\mathcal{S} \setminus \mathcal{T}$ (often referred to as the validation set) and checking for classification accuracy.

For a more formal description, let us assume that the dataset $\mathcal{S}$ is sampled from a joint probability distribution with a density function $p(\mathbf{x}, \mathbf{y})$. The role of a classifier model is to approximate the conditional distribution $p(\mathbf{y}|\mathbf{x})$. The knowledge of $p(\mathbf{x}, \mathbf{y})$ allows us, in principle, to establish theoretical bounds on the performance of the classifier. Consider the generalisation error (also called risk), defined as $\mathcal{G}(\boldsymbol{\theta}) = \mathbb{E}_{p(\mathbf{x},\mathbf{y})}(D(F(\mathbf{x}, \boldsymbol{\theta}), \mathbf{y}))$. A learning algorithm is said to generalise if $\lim_{M' \to \infty}(\mathcal{L}(\boldsymbol{\theta})/M') = \mathcal{G}(\boldsymbol{\theta})$; here M' is the cardinality of the training set. However, since in general we do not have access to $p(\mathbf{x}, \mathbf{y})$ we can only attempt to provide necessary conditions to bound the difference of the generalisation error and the empirical error by checking certain stability conditions to ensure that our learning model is not too sensitive to noise in the data [8]. For example, we can try to ensure that our learning model is not affected if one of the data points is left out during training. The technique of regularisation prevents overfitting.

Several different types of machine learning models F exist, which range from fairly elementary models, such as perceptrons, to highly involved ones, such as deep neural networks (DNNs) [24, 38]. The choice of F is heavily dependent on the classification task, the type of the dataset, and the desired training properties. Consider a dataset $\mathcal{S}$ with two classes (a binary dataset) that is linearly separable. That is, (i) $\mathbf{y}_j \in \{-1, 1\}$ and (ii) one can construct a hyperplane that separates the input data belonging to the different classes. Finding this hyperplane, aka the decision boundary, is therefore sufficient for data classification in $\mathcal{S}$. This task can be accomplished with a simplistic machine learning model—the perceptron—which is in fact a single McCulloch–Pitts neuron [26]. The algorithm starts with the candidate solution for the hyperplane $\mathbf{w}^\top \cdot \mathbf{x} + b = 0$, where $\mathbf{w}, b$ are tunable parameters and play the role of $\boldsymbol{\theta}$. It is known from the perceptron convergence algorithm that one can always find a set of parameters $\mathbf{w} = \mathbf{w}^\star, b = b^\star$, such that for every $\mathbf{x}_j$, if $\mathbf{y}_j = -1$, then $\mathbf{w}^{\star\top} \cdot \mathbf{x}_j + b^\star \le 0$, while if $\mathbf{y}_j = 1$, then $\mathbf{w}^{\star\top} \cdot \mathbf{x}_j + b^\star > 0$.

Most datasets of practical importance, however, are not linearly separable and consequently cannot be classified by the perceptron model alone. Assuming that $\mathcal{S}$ is a binary dataset which is not linearly separable, we consider a map $\Lambda\colon \mathbb{R}^N \to \Gamma$, $\dim(\Gamma) > N$, with the proviso that Λ is nonlinear in the components of its inputs [6]. In machine learning Λ is called a feature map and the vector space Γ is known as a feature space. Thus Λ nonlinearly maps each input datum $\mathbf{x}_j$ to a vector $\Lambda(\mathbf{x}_j)$ in the feature space. The significance of this step follows from Cover's theorem on separability of patterns [16], which suggest that the transformed dataset $S' = \{(\Lambda(\mathbf{x}_j), \mathbf{y}_j)\}_{j=1}^{M}$ is more likely to be linearly separable. For a good choice of Λ, the data classification step now becomes straightforward, as it is sufficient to fit a hyperplane to separate the two classes in the feature space. Indeed, the sought-for hyperplane can be constructed, using the perceptron model, provided the feature map Λ is explicitly known. Actually, a hyperplane can still be constructed even when Λ is not explicitly known. A particularly elegant way to accomplish this is via the support vector machine (SVM) [15], by employing the so-called kernel trick [45].

Consider again the binary dataset $S' = \{(\Lambda(\mathbf{x}_j), \mathbf{y}_j)\}_{j=1}^{M}$. The aim is to construct a hyperplane that separates the samples belonging to the two classes. In addition, we would like to maximise the margin of separation. Formally, we search for a set of parameters $\mathbf{w} = \mathbf{w}^\star, b = b^\star$ such that for every $\mathbf{x}_j$, if $\mathbf{y}_j = -1$, then $\mathbf{w}^{\star\top} \cdot \Lambda(\mathbf{x}_j) + b^\star \le -1$, while if $\mathbf{y}_j = 1$, then $\mathbf{w}^{\star\top} \cdot \Lambda(\mathbf{x}_j) + b^\star \ge 1$. As in the perceptron model, the SVM algorithm starts with the candidate solution $\mathbf{w}^\top \cdot \mathbf{x} + b$ and the parameters $\mathbf{w}, b$ are tuned based on the training data. An interesting aspect of the SVM approach, however, is the dependence of the algorithm on a special subset of the training data called support vectors, namely, the ones that satisfy the relation $\mathbf{w}^{\star\top} \cdot \Lambda(\mathbf{x}_j) + b^\star = \pm 1$. We assume that there are S such vectors $\Lambda(\mathbf{x}_j^{(s)})$, $j = 1, \ldots, S$. With some algebra, which we omit here, it can be shown that the decision boundary is given by the hyperplane

$$\sum_{j=1}^{S} a_j^\star \mathbf{y}_j [\Lambda(\mathbf{x}_j^{(s)})^\top \Lambda(\mathbf{x})] + b^\star = 0, \tag{8}$$

where

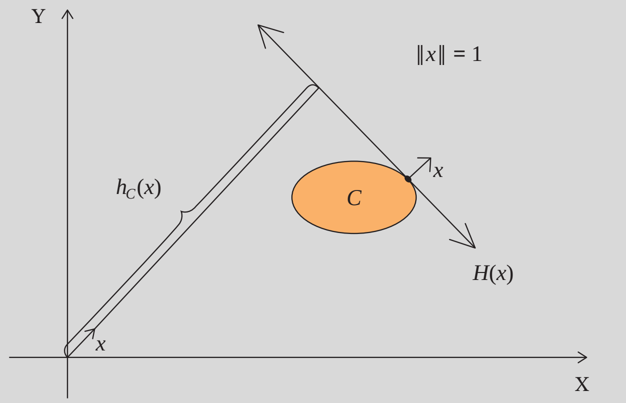
$$b^\star = \frac{1}{S} \sum_{j=1}^{S} \Big(\mathbf{y}_j - \sum_{k=1}^{S} a_j^\star \mathbf{y}_j [\Lambda(\mathbf{x}_j^{(s)})^\top \Lambda(\mathbf{x}_k^{(s)})]\Big), \tag{9}$$

$$a^\star = \arg\max_{a} \Big(\sum_j a_j - \frac{1}{2} \sum_j \sum_k a_j a_k \mathbf{y}_j \mathbf{y}_k \Lambda(\mathbf{x}_j)^\top \Lambda(\mathbf{x}_k)\Big) \tag{10}$$

with the additional conditions $\sum_j a_j \mathbf{y}_j = 0$ and $a_j \ge 0$. All summations in (10) are over the entire training set. We make a key observation here: from (8), (9) and (10) we see that the expression of the decision boundary has no explicit dependence on the feature vectors $\Lambda(\mathbf{x}_j)$. Instead, the dependence is solely on the inner products of the form $\Lambda(\mathbf{x}_j)^\top \Lambda(\mathbf{x}_k)$. This allows us to use the kernel trick.

Support functions in optimisation

If $C \subset \mathbb{R}^n$ is a nonempty closed convex set, the support function $h_C \colon \mathbb{R}^n \to \mathbb{R}$ of C is given by $h_C(x) = \sup\{\langle x|c\rangle : c \in C, x \in \mathbb{R}^n\}$. The hyperplane $H(x) = \{y \in \mathbb{R}^n : \langle y|x\rangle = h_C(x)\}$ is called a supporting hyperplane of C with outer unit normal vector x. The function $h_C(x)$ outputs the signed distance of $H(x)$ from the origin. The data points that lie on $H(x)$ are called support vectors in the machine learning literature.



Graphical representation of the support function and supporting hyperplane.

Kernel functions

The concept of the kernel function originates in the theory of Reproducing Kernel Hilbert Spaces (RKHS). Consider a Hilbert space H consisting of real-valued functions defined on an arbitrary set X. We can define an evaluation functional $E_x(f) = f(x)$ that maps each function to a real number. If for every $x \in X$ the linear functional E_x is bounded on H, we call H an RKHS. Applying Riesz's theorem mentioned in the introduction, we have the representation $E_x(f) = \langle f|K_x\rangle$, where $K_x \in H$. It follows that $E_y(K_x) = \langle K_x|K_y\rangle = K_x(y) = K(x,y)$.

The function $K(x,y) \colon X \times X \to \mathbb{R}$ is called the reproducing kernel of the space H. Briefly, in an RKHS the evaluation of a function at a point can be replaced by taking an inner product with a function determined by the kernel in the associated function space.

Now given a feature map $\Phi \colon X \to \Gamma$, where Γ is a Hilbert space, we define a normed space $H_\Phi = \{f \colon X \to \mathbb{C} : \exists g \in \Gamma, f(x) = \langle g|\Phi(x)\rangle\ \forall x \in X\}$ with the norm $\|f\|_\Phi = \inf\{\|g\|_\Gamma : g \in G, f(x) = \langle g|\Phi(x)\rangle\ \forall x \in X\}$. One can show that H_Φ is a RKHS with the kernel $K(x,y) = \langle \Phi(x)|\Phi(y)\rangle$.

Formally, a kernel $k(\mathbf{x}_j, \mathbf{x}_k)$ is defined as a function that computes the inner product of the images of $\mathbf{x}_j, \mathbf{x}_k$ in the feature space, i.e., $k(\mathbf{x}_j, \mathbf{x}_k) = \Lambda(\mathbf{x}_j)^\top \Lambda(\mathbf{x}_k)$. Accordingly, all the inner products in (8), (9) and (10) can be replaced by the corresponding kernels. We can then use the kernel trick, that is, assign an analytic expression for the kernel in (8), (9) and (10), provided that the expression satisfies all the required conditions for it to be an actual kernel [39]. Indeed, every choice of a kernel can be implicitly associated to a feature map Λ. However, in the current approach we do not need to know the explicit form of the feature map. In fact, this is the advantage of the kernel trick, as calculating $\Lambda(\mathbf{x})$ is less efficient than using the kernels directly.

Most modern applications in machine learning, however, involve deep neural networks (DNNs) [43]. A DNN can also be regarded as a collection of perceptrons arranged in a definite fashion. The architecture of a DNN is best understood and visualised in the form of a graph. Specifically, a DNN is composed of an input layer, an output layer, and several intermediate hidden layers. Each layer consists of several nodes. Each of these nodes represents a neuron. Edges are allowed to exist between nodes belonging to adjacent layers only: nodes in layer j share edges with nodes in layer $j+1$ and nodes in layer $j-1$. For the sake of simplicity, we consider the case where all the nodes in the j-th layer are connected to all the nodes in the $(j \pm 1)$-th layers—a fully connected neural network.

A DNN takes a data vector $\mathbf{x}$ as an input (at the input layer). This input is subsequently manipulated by the neurons in the next layer (the first hidden layer) to output a transformed vector $\mathbf{x}^{(1)}$, and this process is repeated till the last layer (output layer) is reached. Consider the k-th neuron in the j-th layer; for convenience, we denote this neuron by (k, j). It receives an input vector $\mathbf{x}^{(j-1)}$ whose components are the outputs of the neurons in the $(j-1)$-th layer, and then transforms $\mathbf{x}^{(j-1)}$ by the rule

$$\mathbf{x}^{(j-1)} \to \Psi\big((\mathbf{w}_k^{(j-1)})^\top \cdot \mathbf{x}^{(j-1)} + b_k^{(j)}\big) = \mathbf{x}_k^{(j)}, \qquad (11)$$

where $\mathbf{w}_k^{(j-1)}$ are weights associated with the edges that connect the neuron (k, j) to the neurons in the previous layer, $b_k^{(j)}$ is the bias corresponding to the neuron (k, j), and $\Psi(\cdot)$ is a differentiable nonlinear function known as the activation function. This is the fundamental mathematical operation that propagates data in a DNN, layer by layer, in the forward direction (input layer to output layer), through a series of complex transformations. The training component of the algorithm is however accomplished through the back-propagation step [35]: a cost function is calculated by comparing the signal at the output layer (the model predicted label for the data $\mathbf{x}$) and the desired signal (the actual label for the data $\mathbf{x}$), based on which the weights and the biases are adjusted so that the cost function is minimised. Apart from supervised learning, DNNs are routinely used for unsupervised learning, including generative learning. In generative learning, given access to a training dataset, a machine learning model learns its underlying probability distribution for future sample generation. To formulate this mathematically, consider a dataset $\mathcal{S} = \{\mathbf{x}_j\}$, whose entries $\mathbf{x}_j \in \mathbb{R}^N$ are independent and identically distributed vectors and are sampled according to a distribution $q(\mathbf{x})$. The purpose of a generative model is to approximate $q(\mathbf{x})$, given the access to training data from the dataset $\mathcal{S}$. To achieve this, a machine learning model (with tunable parameters $\boldsymbol{\theta}$) is trained so that the model generated distribution, $p(\mathbf{x}, \boldsymbol{\theta})$, mimics the true distribution. The standard practice in generative learning is to minimise the negative log-likelihood with respect to the model parameters, which is tantamount to minimising the Kullback–Leibler divergence $D_{\mathrm{KL}}(q(\mathbf{x})||p(\mathbf{x}, \boldsymbol{\theta}))$ between the two distributions.

2.2 *Variational algorithms and quantum machine learning (QML)*

Variational quantum computing has emerged as the preeminent model for quantum computation. The model merges ideas from machine learning to better utilise modern quantum hardware.

Mathematically, the problem in variational quantum computing can be formulated as follows: given (i) a variational quantum circuit (aka ansatz) $U(\boldsymbol{\theta}) \in \mathrm{U}_{\mathbb{C}}(2^n)$ which produces an n-qubit variational state $|\psi(\boldsymbol{\theta})\rangle = U(\boldsymbol{\theta})|0\rangle^{\otimes n}$; $\boldsymbol{\theta} \in [0, 2\pi)^{\times p}$, (ii) an objective function $\mathcal{H} \in \mathrm{herm}_{\mathbb{C}}(2^n)$, and (iii) the expectation $\langle\psi(\boldsymbol{\theta})|\mathcal{H}|\psi(\boldsymbol{\theta})\rangle$, find

$$\boldsymbol{\theta}^\star \in \underset{\boldsymbol{\theta} \in [0,2\pi)^{\times p}}{\arg\min} \; \langle\psi(\boldsymbol{\theta})|\mathcal{H}|\psi(\boldsymbol{\theta})\rangle.$$

Then $|\psi(\boldsymbol{\theta}^\star)\rangle$ will approximate the ground state (eigenvector corresponding to the lowest eigenvalue) of the Hamiltonian $\mathcal{H}$. The operator $\mathcal{H}$, often called the problem Hamiltonian, can suitably treat several classes of problems so that the solution to a problem is encoded in the ground state of $\mathcal{H}$. The variational model of quantum computation was shown to be universal in [3].

Quantum machine learning, both discriminative and generative, emerged as an important application of variational algorithms with suitable modifications to the aforementioned scheme. Indeed, by their very design, variational algorithms are well suited to implement machine learning tasks on a quantum device. The earlier developments in QML came mainly in the form of classification tasks [18, 27].

Classification of a classical dataset $\mathcal{S} = \{(\mathbf{x}_j, \mathbf{y}_j))\}_{j=1}^M$ on quantum hardware typically involves four steps. First, the input vector $\mathbf{x}_j$ is embedded into an n-qubit state $|\psi(\mathbf{x}_j)\rangle$. The effect of data encoding schemes on the expressive power of a quantum machine learning model was studied in [42]. What is the most effective data embedding scheme? Although there are several interesting candidates [25, 34], this question remains largely unanswered. In the second step, a parameterised ansatz $U(\boldsymbol{\theta})$ is applied to $|\psi(\mathbf{x}_j)\rangle$ to output $|\psi(\mathbf{x}_j, \boldsymbol{\theta})\rangle$. A number of different ansatzes are in use today, including the hardware efficient ansatz, the checkerboard ansatz, the tree tensor network ansatz, etc., which are chosen according to the application and implementation specifications under consideration. The third step in the process is where data is read out of $|\psi(\mathbf{x}_j, \boldsymbol{\theta})\rangle$: expectation values of certain chosen observables (Hermitian operators) are calculated with respect to $|\psi(\mathbf{x}_j, \boldsymbol{\theta})\rangle$ to generate a predicted label $F(\mathbf{x}_j, \boldsymbol{\theta})$. The measured operators are typically the Pauli strings, which form a basis in $\mathrm{herm}_{\mathbb{C}}(2^n)$. In the final step, a cost function is constructed as in (7) and minimised by tuning $\boldsymbol{\theta}$. This approach was used in several studies to produce successful classifications in practical datasets (see, e.g., [40]).

An interesting variation of the approach described above was shown in [21, 41] to implement data classification based on the kernel trick. In this method the Hilbert space is treated as a feature space and the data embedding step, $\mathbf{x}_j \to |\psi(\mathbf{x}_j)\rangle$, as a feature map. A quantum circuit is used directly to compute the inner product $\langle\psi(\mathbf{x}_j)|\psi(\mathbf{x}_k)\rangle$, using, e.g., the swap test, which is then employed for data classification by means of classical algorithms such as SVMs.

Quantum machine learning has also been used to classify genuine quantum data. Some prominent examples of such applications include: classifying phases of matter [47], quantum channel discrimination [23], and entanglement classification [20]. Other machine learning problems with quantum mechanical origins that have been solved by variational algorithms include quantum data compression [36] and denoising of quantum data [7]. Both of these applications use a quantum autoencoder. A quantum autoencoder, much like its classical counterparts, consists of two parts: an encoder and a decoder. The encoder removes the redundant information from the input data to produce a minimal low-dimensional representation. This process is known as feature extraction. To ensure that the minimal representation produced by the encoder is efficient, a decoder is used which takes the output of the encoder and tries to reconstruct the original data. Thus, in an autoencoder, both the encoder and the decoder are trained in tandem to ensure that the input at the encoder and the output at the decoder closely match each other. While in the classical case the encoders and the decoders are chosen to be neural networks, in the quantum version of an autoencoder neural networks are replaced by variational circuits.

Considerable advances were made on the front of quantum generative learning as well. In [2] it was shown that generative modelling can be used to prepare quantum states by training shallow quantum circuits. The central idea is to obtain the model generated probability distribution $p(\boldsymbol{\theta})$ by performing repeated measurements on a variational state $|\psi(\boldsymbol{\theta})\rangle$. The state $|\psi(\boldsymbol{\theta})\rangle$ is prepared on a short-depth circuit with a fixed ansatz and parameterised with the vector $\boldsymbol{\theta}$. The target distribution q is also constructed in much the same manner, by performing repeated measurements on the target state. The measurement basis (preferably informationally complete positive operator-valued measures), as expected, is kept to be the same in both cases. The training objective therefore is to ensure that $p(\boldsymbol{\theta})$ mimics q so that the variational circuit learns to prepare the target state. The same task in an alternate version can be looked upon as a machine-learning assisted quantum state tomography [9].

3 Tensor networks in machine learning

3.1 Tensor networks in classical machine learning

Recently tensor network methods have found several applications in machine learning. Here we discuss some of these applications with a focus on supervised learning models. We return to our labelled dataset $\mathcal{S} = \{(\mathbf{x}_j, \mathbf{y}_j)\}_{j=1}^{M}$, where $\mathbf{x}_j \in \mathbb{R}^N$. As mentioned earlier, there are several machine learning models F to choose from to perform a classification on the dataset $\mathcal{S}$.

However, in more abstract terms, a classifier F can be expressed as a function of the form

$$F_{\mathrm{W}}(\mathbf{x}) = \sum_{j_1, j_2, \dots, j_N \in \{0,1\}} W_{j_1 j_2 \dots j_N} x_1^{j_1} x_2^{j_2} \cdots x_N^{j_N}, \tag{12}$$

in the polynomial basis [29]. Here $\mathbf{x} \in \mathbb{R}^N$ is an input datum and $x_k \in \mathbb{R}$ is the k-th component of $\mathbf{x}$. The tensor $W_{j_1 j_2 \dots j_N}$ is what we call as the weight tensor, which encodes the tunable parameters in F. Going back to the case of binary classification, that is, $\mathbf{y}_j \in \{1, -1\}$, $F(\mathbf{x})$ can be regarded as a surface in $\mathbb{R}^{N+1}$ that can be tuned (trained) so that it acts as a decision boundary between the two classes of input data. Indeed, the training can be accomplished by the minimisation

$$\min_{\mathrm{W}} \left(\sum_{j=1}^{M} |\operatorname{sgn}(F_{\mathrm{W}}(\mathbf{x}_j)) - \mathbf{y}_j|^2 \right),$$

where $\operatorname{sgn}(F_{\mathrm{W}}(\mathbf{x}_j))$ is the predicted label. However, in practice we run into a bottleneck when we compute (12), since this involves 2^N components of the weight tensor. One way to circumvent this bottleneck is to express the weight tensor as a MPS. Following the observation in Section 1.2, for a suitable choice of the virtual bond dimension r, the MPS representation of the weight tensor W would involve only $O(\operatorname{poly} N)$ components, thus making the computation of (12) less resource intensive [46]. Here it is worth noting that we could alternatively have opted for any other basis in the decomposition of the function F, depending on the optimisation problem at hand.

Yet another application of tensor networks in machine learning can be seen in DNNs. Consider the transformation in (11). For most practically relevant DNNs, this transformation is highly resource intensive. This is due to the fact that the vectors $\mathbf{x}^{(j-1)}$ are typically very large and hence computing the inner products $(\mathbf{w}_k^{(j-1)})^\top \cdot \mathbf{x}^{(j-1)}$ is difficult. This computation can be made efficient by using MPS. In order to do this, the vectors $\mathbf{w}_k^{(j-1)}, \mathbf{x}^{(j-1)}$ are first reshaped, converting them into tensors and then expressing them as an MPS. When we express a vector as a MPS we need to keep track of much fewer components compared to the original representation. This makes the computation of (11) tractable.

3.2 The parent Hamiltonian problem

Consider the quantum state preparation problem using a variational algorithm. Given a variational circuit $U(\boldsymbol{\theta})$ and an n-qubit target state $|t\rangle$, tune $\boldsymbol{\theta} \to \boldsymbol{\theta}^\star$ such that $U(\boldsymbol{\theta}^\star)|0\rangle^{\otimes n}$ approximates $|t\rangle$. To accomplish this task one needs to construct a Hamiltonian $\mathcal{H} \in \operatorname{herm}_{\mathbb{C}}(2^n)$ with $|t\rangle$ as its unique ground state, which will serve as the objective function of the algorithm. Constructing such a Hamiltonian for a given target state is known as the parent Hamiltonian problem. The simplest recipe is to set $\mathcal{H} = \mathbf{1} - |t\rangle\langle t|$. This construction is however not always useful, because expressing $\mathcal{H}$ in the basis of Pauli strings—the basis of measurement—may require an exponential number of terms. Thus estimating the expectation of $\mathcal{H}$ in polynomial time becomes unfeasible.

Ideally, we want the sought-for Hamiltonian to enjoy the following properties:

1. The Hamiltonian is non-negative.
2. The Hamiltonian has a non-degenerate (unique) ground state $|t\rangle$.
3. The Hamiltonian is gapped. An n-qubit Hamiltonian $\mathcal{H}(n) \geq 0$ is said to be gapped if
$$\lim_{n\to\infty} [\dim\ker \mathcal{H}(n)] = 1. \tag{13}$$
Validity of (13) ensures that $\mathcal{H}(n)$ is gapped for all finite n.
4. The Hamiltonian is local. An n-qubit Hamiltonian $\mathcal{H}(n)$ is said to be local if it can be expressed as
$$\mathcal{H}(n) = \sum_{j\in 2^V} h(j),$$
where V is the set of n symbols (qubits) and $h(j) = \bigotimes_{k\in j} P_k \in \mathrm{herm}_{\mathbb{C}}(2^n)$, where $P \in \mathrm{herm}_{\mathbb{C}}(2)$. The Hamiltonian $\mathcal{H}(n)$ is said to be k-local if none of the $h(j)$'s operates on more than k symbols (qubits) nontrivially; here a trivial operation refers to the case when P_k is the identity for some index k.
5. The Hamiltonian must have $O(\mathrm{poly}\, n)$ terms when expressed in the Pauli basis. The number of terms in a Hamiltonian when expressed in the Pauli basis is also known as the cardinality of the Hamiltonian, $\|\mathcal{H}\|_{\mathrm{card}}$ (see [3]).

Hamitonians with such properties can indeed be constructed if $|t\rangle$ admits a matrix product state, albeit with the additional requirement that $|t\rangle$ must satisfy the injectivity condition. For the parent Hamiltonian construction consider the following setting. Let $|t\rangle$ be an n-qubit state written as a translation-invariant and site-independent MPS with periodic boundary conditions:
$$|t\rangle = \sum_{j_1\dots j_n} \mathrm{Tr}(A_{j_1}\cdots A_{j_n})|j_1\dots j_n\rangle,$$
where $A_{j_k} \in \mathrm{Mat}_{\mathbb{C}}(r)$. For the sake of brevity we will call these matrices Kraus operators.[1] Consider the map
$$\Gamma_L : X \to |\psi^{(L)}\rangle_X = \sum_{j_1\dots j_L} \mathrm{Tr}(XA_{j_1}\cdots A_{j_L})|j_1\dots j_L\rangle,$$
where $X \in \mathrm{Mat}_{\mathbb{C}}(r)$. We say that the state $|t\rangle$ is injective with injectivity length L if the map Γ_L is injective. Several corollaries follow from this definition. A particularly useful one connects the notion of injectivity to the rank of reduced density matrices. It asserts that for an L-qubit reduced density matrix, $\rho^{(L)}$, of $|t\rangle$, we have $\mathrm{rank}(\rho^L) = r^2$ if injectivity holds. It has been shown that in the large-n limit, $\rho^{(L)}$ is given by
$$\rho^{(L)} = \mathrm{Tr}_{n-L}(|t\rangle\langle t|) = \sum_{\alpha,\beta=1}^{r} \Lambda_\alpha |\psi^{(L)}_{\alpha\beta}\rangle\langle\psi^{(L)}_{\alpha\beta}|, \tag{14}$$
with $|\psi^{(L)}_{\alpha\beta}\rangle = \sum_{j_1\dots j_L}\langle\alpha|A_{j_1}\cdots A_{j_L}|\beta\rangle|j_1\dots j_L\rangle$, $|\alpha\rangle, |\beta\rangle \in \mathbb{C}^r$ and $\Lambda_\alpha \in \mathbb{R}_+$. Alternatively, this would mean that $\{|\psi^{(L)}_{\alpha\beta}\rangle\}_{\alpha\beta}$ is a linearly independent set.

The form of the reduced density matrix in (14) is particularly telling and allows us to construct the parent Hamiltonian of $|t\rangle$: $\mathcal{H} \geq 0$. We formally write our parent Hamiltonian as
$$\mathcal{H} = \sum_{j=1}^{n} h_j^{(L)}, \tag{15}$$
where $h_j^{(L)}$ operates nontrivially over L-qubits from j to $L+j$ and obeys the condition $\ker h_j^{(L)} = \mathrm{span}\,\{|\psi^{(L)}_{\alpha\beta}\rangle\}_{\alpha\beta}$. The latter condition combined with (14) ensures that $\mathrm{Tr}(h_j^{(L)}\rho_j^{(L)}) = 0$ for all j, which in turn implies that $|t\rangle \in \ker\mathcal{H}$. In fact, $|t\rangle = \ker\mathcal{H}$, provided $|t\rangle$ is injective, and so condition 1 for $\mathcal{H}$ is satisfied. Conditions 3 and 4 are satisfied naturally due to the form of $\mathcal{H}$ in (15). In addition, $\mathcal{H}$ can also seen to be frustration free, that is, $\langle t|\mathcal{H}|t\rangle = 0 \Rightarrow \langle t|h_j^{(L)}|t\rangle = 0$ for all j. Finally, it was shown in [17] that if $|t\rangle$ is injective, then $\mathcal{H}$ is gapped.

4 Conclusion

The importance of matrix product states in physics is due to the ease with which one could calculate and verify important quantities or properties, such as two-point functions, thermal properties, and more. This is also true in machine learning applications. For example, images of size 256×256 can be viewed as rank-one tensor networks on $\mathbb{R}^{256}$. Departing from this linear (train) structure results in tensors with potentially much greater expressability at the cost of many desirable properties being lost.

References

[1] J. Baez and M. Stay, Physics, topology, logic and computation: A Rosetta Stone. In *New Structures for Physics*, Lecture Notes in Phys. 813, Springer, Heidelberg, 95–172 (2011)

[2] M. Benedetti, D. Garcia-Pintos, O. Perdomo, V. Leyton-Ortega, Y. Nam and A. Perdomo-Ortiz, A generative modeling approach for benchmarking and training shallow quantum circuits. *npj Quantum Inf.* **5**, 1–9 (2019)

[3] J. Biamonte, Universal variational quantum computation. *Phys. Rev. A* **103**, L030401 (2021)

[4] J. Biamonte and V. Bergholm, Tensor networks in a nutshell, preprint, arXiv:1708.00006 (2017)

[5] J. Biamonte, V. Bergholm and M. Lanzagorta, Tensor network methods for invariant theory. *J. Phys. A* **46**, 475301 (2013)

[1] There is indeed there a connection between the matrices A_{j_k} and completely positive trace-preserving (CPTP) maps from which the A_{j_k} derive their name. For the purpose of this paper, however, we will skip the detailed explanation.

[6] C. M. Bishop, *Pattern recognition and machine learning*. Information Science and Statistics, Springer, New York (2006)

[7] D. Bondarenko and P. Feldmann, Quantum autoencoders to denoise quantum data. *Phys. Rev. Lett.* **124**, 130502 (2020)

[8] O. Bousquet and A. Elisseeff, Stability and generalization. *J. Mach. Learn. Res.* **2**, 499–526 (2002)

[9] J. Carrasquilla, G. Torlai, R. G. Melko and L. Aolita, Reconstructing quantum states with generative models. *Nat. Mach. Intell.* **1**, 155–161 (2019)

[10] A. Cayley, On the theory of groups as depending on the symbolic equation $\theta^n = 1$. *Philos. Mag.* **7**, 40–47 (1854)

[11] A. Cichocki, Tensor networks for big data analytics and large-scale optimization problems, preprint, arXiv:1407.3124 (2014)

[12] A. Cichocki, N. Lee, I. Oseledets, A.-H. Phan, Q. Zhao and D. P. Mandic, Tensor networks for dimensionality reduction and large-scale optimization. I: Low-rank tensor decompositions. *Found. Trends Mach. Learn.* **9**, 249–429 (2016)

[13] A. Cichocki, A.-H. Phan, Q. Zhao, N. Lee, I. Oseledets, M. Sugiyama and D. P. Mandic, Tensor networks for dimensionality reduction and large-scale optimization. II: Applications and future perspectives. *Found. Trends Mach. Learn.* **9**, 431–673 (2017)

[14] J. I. Cirac, D. Pérez-García, N. Schuch and F. Verstraete, Matrix product states and projected entangled pair states: Concepts, symmetries, theorems. *Rev. Modern Phys.* **93**, 045003 (2021)

[15] C. Cortes and V. Vapnik, Support-vector networks. *Mach. Learn.* **20**, 273–297 (1995)

[16] T. M. Cover, Geometrical and statistical properties of systems of linear inequalities with applications in pattern recognition. *IEEE Trans. Electron. Comput.* **14**, 326–334 (1965)

[17] M. Fannes, B. Nachtergaele and R. F. Werner, Finitely correlated states on quantum spin chains. *Comm. Math. Phys.* **144**, 443–490 (1992)

[18] E. Farhi and H. Neven, Classification with quantum neural networks on near term processors, preprint, arXiv:1802.06002 (2018)

[19] C. Fernández-González, N. Schuch, M. M. Wolf, J. I. Cirac and D. Pérez-García, Frustration free gapless Hamiltonians for matrix product states. *Comm. Math. Phys.* **333**, 299–333 (2015)

[20] E. Grant, M. Benedetti, S. Cao, A. Hallam, J. Lockhart, V. Stojevic, A. G. Green and S. Severini, Hierarchical quantum classifiers. *npj Quantum Inf.* **4**, 1–8 (2018)

[21] V. Havlíček, A. D. Córcoles, K. Temme, A. W. Harrow, A. Kandala, J. M. Chow and J. M. Gambetta, Supervised learning with quantum-enhanced feature spaces. *Nature* **567**, 209–212 (2019)

[22] A. Jahn and J. Eisert, Holographic tensor network models and quantum error correction: A topical review. *Quantum Sci. Technol.* **6**, 033002 (2021)

[23] A. Kardashin, A. Pervishko, D. Yudin, J. Biamonte et al., Quantum machine learning channel discrimination, preprint, arXiv: 2206.09933 (2022)

[24] Y. LeCun, Y. Bengio and G. Hinton, Deep learning. *Nature* **521**, 436–444 (2015)

[25] S. Lloyd, M. Schuld, A. Ijaz, J. Izaac and N. Killoran, Quantum embeddings for machine learning, preprint, arXiv:2001.03622 (2020)

[26] W. S. McCulloch and W. Pitts, A logical calculus of the ideas immanent in nervous activity. *Bull. Math. Biophys.* **5**, 115–133 (1943)

[27] K. Mitarai, M. Negoro, M. Kitagawa and K. Fujii, Quantum circuit learning. *Phys. Rev. A* **98**, 032309 (2018)

[28] A. Novikov, D. Podoprikhin, A. Osokin and D. P. Vetrov, Tensorizing neural networks. *Adv. Neural Inf. Process. Syst.* **28** (2015)

[29] A. Novikov, M. Trofimov and I. Oseledets, Exponential machines, preprint, arXiv:1605.03795 (2016)

[30] R. Orús, Advances on tensor network theory: Symmetries, fermions, entanglement, and holography. *Eur. Phys. J. B* **87**, 280 (2014)

[31] I. V. Oseledets, Tensor-train decomposition. *SIAM J. Sci. Comput.* **33**, 2295–2317 (2011)

[32] R. Penrose, Applications of negative dimensional tensors. In *Combinatorial Mathematics and its Applications (Proc. Conf., Oxford, 1969)*, Academic Press, London, 221–244 (1971)

[33] D. Perez-Garcia, F. Verstraete, M. M. Wolf and J. I. Cirac, Matrix product state representations. *Quantum Inf. Comput.* **7**, 401–430 (2007)

[34] A. Pérez-Salinas, A. Cervera-Lierta, E. Gil-Fuster and J. I. Latorre, Data re-uploading for a universal quantum classifier. *Quantum* **4**, 226 (2020)

[35] R. Rojas, The backpropagation algorithm. In *Neural Networks*, Springer, 149–182 (1996)

[36] J. Romero, J. P. Olson and A. Aspuru-Guzik, Quantum autoencoders for efficient compression of quantum data. *Quantum Sci. Technol.* **2**, 045001 (2017)

[37] L. Rosasco, E. De Vito, A. Caponnetto, M. Piana and A. Verri, Are loss functions all the same? *Neural Comput.* **16**, 1063–1076 (2004)

[38] F. Rosenblatt, The perceptron: A probabilistic model for information storage and organization in the brain. *Psychol. Rev.* **65**, 386 (1958)

[39] B. Schölkopf, A. J. Smola, F. Bach et al., *Learning with kernels: Support vector machines, regularization, optimization, and beyond*. MIT Press (2002)

[40] M. Schuld, A. Bocharov, K. M. Svore and N. Wiebe, Circuit-centric quantum classifiers. *Phys. Rev. A* **101**, 032308 (2020)

[41] M. Schuld and N. Killoran, Quantum machine learning in feature Hilbert spaces. *Phys. Rev. Lett.* **122**, 040504 (2019)

[42] M. Schuld, R. Sweke and J. J. Meyer, Effect of data encoding on the expressive power of variational quantum-machine-learning models. *Phys. Rev. A* **103**, 032430 (2021)

[43] H. Schulz and S. Behnke, Deep learning. *KI-Künstliche Intelligenz* **26**, 357–363 (2012)

[44] P. Selinger, A survey of graphical languages for monoidal categories. In *New Structures for Physics*, Lecture Notes in Phys. 813, Springer, Heidelberg, 289–355 (2011)

[45] J. Shawe-Taylor, N. Cristianini et al., *Kernel methods for pattern analysis*. Cambridge University Press, Cambridge (2004)

[46] E. Stoudenmire and D. J. Schwab, Supervised learning with tensor networks. *Adv. Neural Inf. Process. Syst.* **29** (2016)

[47] A. V. Uvarov, A. S. Kardashin and J. D. Biamonte, Machine learning phase transitions with a quantum processor. *Phys. Rev. A* **102**, 012415 (2020)

Richik Sengupta is a research scientist at Skolkovo Institute of Science and Technology.

r.sengupta@skoltech.ru

Soumik Adhikary is a research scientist at Skolkovo Institute of Science and Technology.

s.adhikari@skoltech.ru

Ivan Oseledets is full professor, director of the Center for Artificial Intelligence Technology, head of the Laboratory of Computational Intelligence at Skolkovo Institute of Science and Technology.

i.oseledets@skoltech.ru

Jacob Biamonte is full professor, head of the Laboratory of Quantum Algorithms for Machine Learning and Optimisation at Skolkovo Institute of Science and Technology.

j.biamonte@skoltech.ru

Waterscales: Mathematical and computational foundations for modelling cerebral fluid flow

Marie E. Rognes

1 Introduction

On 18 October 2013, I was sitting on my living room sofa scrolling through the news, when one article caught my attention. The headline read something along the lines of "Sleep cleans the brain of toxins", and described a recent original research study from the lab of neuroscientist Prof. Maiken Nedergaard [39]. In a series of experiments, Nedergaard and her team had injected a fluorescent dye (a so-called tracer or contrast agent) into the fluids surrounding the brain of mice, and observed that the tracer moved into (and out of) the brain many times faster in sleeping mice than in awake mice. Their findings revealed a fundamental interplay between sleep and brain clearance, but also highlighted how far from complete our understanding of the brain's waterscape (Definition 1) is.

Definition 1. The *brain's waterscape* is the circulation, flow and exchange of tissue fluid and associated solute transport through and around the brain.

Our brains are composed of very soft tissue consisting of neurons, glial cells, and interstitial space filled with interstitial fluid (ISF), penetrated by blood vessels, and surrounded by a bath of cerebrospinal fluid (CSF). Its well-being crucially relies on the transport of solutes: the influx of oxygen and nutrients, and clearance of metabolic waste. Due to the barrier layers between blood, tissue and CSF in the brain, the fluid exchange between these spaces is limited and regulated. Tracer experiments act as a proxy for quantifying this fluid flow and exchange, in lack of more direct medical techniques for measuring water movement in-vivo (Figure 1)[1]. In the nearly ten years since 2013, the brain's waterscape has been attracting substantial and increasing interest from the neurocommunity; more about that to follow.

A year before the striking sleep study [39], Nedergaard had launched a new theory for describing the brain's waterscape: the *glymphatic system* [22]. In the original glymphatic concept, CSF enters the brain via perivascular spaces surrounding brain arteries, mixes with ISF and flows in bulk and convectively through the tissue, and ISF is cleared from the brain along perivascular spaces surrounding brain veins. With my background in analysis of numerical methods for partial differential equations describing biological tissue, I was immediately intrigued by this setting. I also quickly realized that this was an underdeveloped area for applied mathematics, and that even fundamental concepts for multiscale modelling of the brain's waterscape were missing. Fortunately, the European Research Council Mathematics Panel agreed, and funded my project proposal in the summer of 2016.

In the next few years, we very much realized that the brain's waterscape was even less understood and more discussed that what I had originally thought, and that its mechanisms were surrounded by controversy. It seemed as though almost every piece of the glymphatic theory was disputed with different groups arguing about, e.g., the role of diffusion versus convection, the existence of any convective flow and, if existent, its directionality, the importance vs. nonimportance of aquaporin channels and glial cells, the pathways for clearance, the anatomy of the compartments involved, etc. The list could go on and on. For us, this made the need for mathematical and computational foundations even more obvious to, e.g., quantify observations, bridge between species, and test, reject or support hypotheses.

2 Quantifying brain solute transport

Transport of solutes is often described via the usual convection-diffusion-reaction equation: in a domain $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$) and over a time interval $(0, T]$, find the concentration $c = c(x, t)$ such that

$$c_t - \operatorname{div} D \operatorname{grad} c + \operatorname{div}(c\varphi) + rc = 0, \qquad (1)$$

where the subscript t denotes the time derivative, div and grad are the divergence and gradient respectively, D is the diffusion tensor, φ is a convective velocity field and r is a kinetic reaction rate. In addition, c may be prescribed on the boundary $\partial\Omega$ for a.e. $0 < t \leqslant T$, and initially $c(x, 0) = c_0(x)$ for $x \in \Omega$. Numerical

[1] All clinical data presented and visualized here courtesy of P. K. Eide and G. Ringstad, Oslo University Hospital – Rikshospitalet.

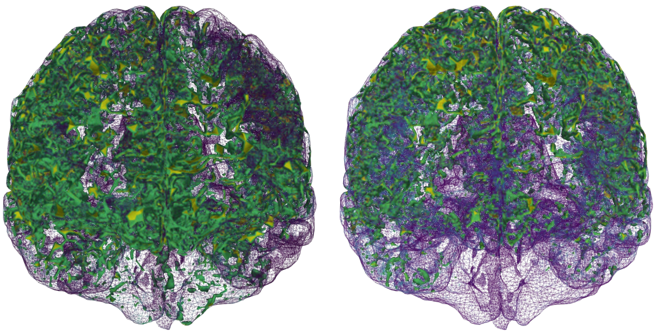


Figure 1. Contrast agent concentrations c_1, c_2 in a human brain at 6 (left) and 24 (right) hours after injection in the lower spine.

approximations of (1) are readily computed using finite difference and finite element methods, via, e.g., the FEniCS software [2].

2.1 Modelling the glymphatic theory via stochastic fields

Now, in our first step towards modelling human brain solute transport [7], we took a just-published clinical study as a starting point [32]. Ringstad, Vatnehol and Eide [32] had injected a contrast agent into the fluid-filled spaces surrounding the spine, and then imaged its whereabouts within the brain 1–48 hours later (Figure 1). They concluded that *it seems unlikely that diffusion alone explains the brain-wide distribution*. We asked: "is it?" and were curious as to whether mathematical modelling could support, quantify and/or add to this statement.

Stochastic diffusion. First, in order to quantify the unlikeliness of diffusion as a sole mechanism, a key question was how to capture the uncertainty associated with the brain's diffusivity D as a random field. Importantly, we aimed to ensure that diffusivity samples were positive, with a literature-based expected value D_g^*, and appropriate variability. To this end, we represented D as a continuous random field

$$D^*(x, \omega) = 0.25 D_g^* + D_f^*(x, \omega),$$

where for each fixed $x \in \Omega$, $D_f^*(x, \omega)$ is a gamma-distributed random variable with a prescribed shape ($k = 3$) and scaling ($\theta = 0.75 D_g^*/k$). To also enforce continuity and to effectively sample the diffusion field, we drew samples of D_v^* by first sampling a Matérn field (with a given smoothness and correlation length), and then mapping it onto a gamma random field via a copula (see, e.g., [7] and references therein for further details). Next, we pieced together inputs from several sources: a human brain finite element mesh with nearly two million vertices and ten million cells, and a upwards-travelling contrast agent distribution on the mesh boundary based on an eye-norm estimation of the published data [32].

In short, we found that the uncertainty in the diffusion field magnitude had a substantial impact on the amount of contrast agent both in the grey and white matter. While the clinically observed distribution of contrast in the grey matter was well within the expected variability of diffusion, its distribution into the white matter was not. We could thus support the claim that diffusion was likely not sufficient as a sole mechanism, but at the same time highlight that diffusion was likely nearly sufficient. Further, we demonstrated that local variations (i.e., heterogeneity) in the diffusion field had little impact on expected values, and thus that contrast agent distribution in larger brain regions could be well approximated by average diffusion coefficients.

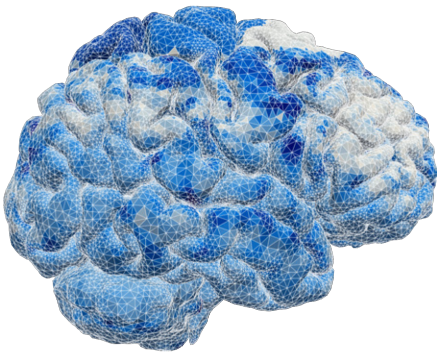
Stochastic convection. We next asked how we could represent the convective velocity $\varphi = (\varphi_1, \varphi_2, \varphi_3)$ described by the glymphatic theory in a stochastic setting at the brain scale? Well,

- φ varies more after a distance proportional to the mean distance between arterioles and venules ($\lambda \approx 1$ mm);
- the vasculature is random and independent of space in the sense that the presence of paraarterial or paravenous spaces are equally likely at any point in space (hence the expected value of each φ_i is zero);
- φ is continuous and $\operatorname{div} \varphi = 0$; and
- older experimental studies indicate an expected velocity magnitude $E(\|\varphi\|) = v_{\text{avg}} = 0.17$ µm/s with some variability.

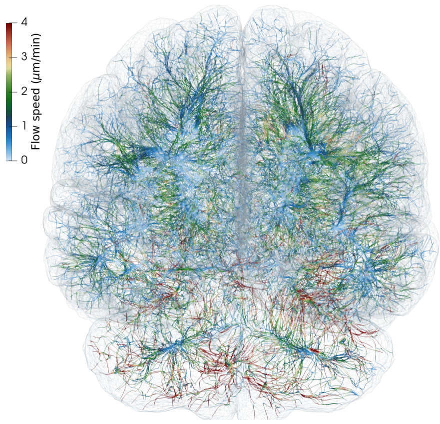
In turn, we defined the stochastic glymphatic circulation velocity field φ as the curl of three standard independent identically distributed Matérn fields with correlation length λ and scaled to align with the expected value and variability. Then, again using Monte Carlo simulations, we computed c samples via (4), but now with non-zero φs.

Surprisingly (or perhaps not), we found that the expected tracer distribution with or without the glymphatic velocity were nearly identical and with little variability. Thus, on average, small fluctuations in the CSF/ISF velocity did not increase (or decrease) the influx or clearance of contrast agent in the brain on a larger scale. Indeed, further model variations, such as including a directional flow at the macroscale and allowing for local fluid influx and/or efflux, were required for the convective terms to effectively contribute to the solute transport [7].

Remark 2. Despite its limitations, this uncertainty quantification study felt like a true breakthrough: we had taken advantage of an original mathematical approach and produced new physiologically relevant insights. Importantly, it was the first study by my team that had been accepted and published in Fluids and Barriers of the Central Nervous System, a domain-specific journal read by everyone and anyone interested in brain mechanics, including neurosurgeons and neuroradiologists, neuroscientists, biophysicists, bioengineers (and mathematicians).



(a) Subject-specific mean diffusivity $\bar{D}^*$ extracted from diffusion tensor imaging.



(b) Optimal convective flow field φ (streamlines) between 6 and 24 hours post contrast injection (cf. Figure 1), estimated by high-dimensional inverse modelling [40].

Figure 2

Remark 3. Another, more mathematical, question is how to sample Matérn field efficiently, especially in the context of more advanced Monte Carlo methods over non-trivial geometries. As such, this physiological problem setting also guided us to develop new sampling and Monte Carlo algorithms [8].

2.2 *Identifying convective flow: An inverse problem*

With these insights, we understood more about the roles and properties of diffusion and convection via CSF/ISF flow within the brain. We also understood that in order to get much further, we needed a closer collaboration with the clinicians and better access to the raw data. Luckily, Prof. P.-K. Eide and Dr. G. Ringstad at Oslo University Hospital were very willing to share their data and expertise with us and very enthusiastic about exploring how computational modelling could contribute to quantify and interpret their clinical findings.

We then found ourselves in the unprecedented position of having multi-modal magnetic resonance (MR) data, including T1- and T2-weighted structural images, T1-maps, diffusion tensor images and contrast MR images, available for several individual in different cohorts. MR data are characteristically at high spatial resolution, but only available at a sparse set of discrete time points, e.g., $\{1, 6, 24, 48\}$ hours after contrast injection. These images combined allowed us to segment and construct subject-specific finite element meshes, interpolate subject-specific diffusion tensor fields (Figure 2 (a)), and map MR contrast signal intensities onto finite element concentration fields for each subject and each time point (Figure 1).

Remark 4. In fact, to handle this data set, we developed and published an open-source pipeline going from magnetic resonance brain images to finite element simulations, accompanied by an open-access introductory booklet [25].

An optical flow problem? Now, given this data set, could we *identify and quantify the convective flow of ISF/CSF within the brain?* Or the task in more mathematical terms: given non-negative scalar fields c_1, c_2, identify a transport field φ that maps c_1 onto c_2 in a (to be defined) suitable sense.

This is a classical problem setting, known in computer vision as the *optical flow problem* [21]. The idea is that given $c_1, c_2 \colon \Omega \to \mathbb{R}$ at t_1, t_2 for $0 \leqslant t_1 < t_2$, find an optimal $\varphi \colon \Omega \times [t_1, t_2] \to \mathbb{R}^d$ that minimizes an objective functional J,

$$\min_{\varphi} J \equiv \min_{\varphi} \int_{t_1}^{t_2} \int_{\Omega} f^2 \,\mathrm{d}x \,\mathrm{d}t + \alpha^2 R^2. \tag{2}$$

In the original optical flow setting, it is assumed that φ maps c_1 to c_2 by convective transport alone – corresponding to

$$f = c_t + \operatorname{div}(\varphi c), \tag{3}$$

and the initial and final conditions $c(t_1) = c_1$, $c(t_2) = c_2$. The choice of regularization term R (and also f for that matter) gives variations of the method; one option is to minimize the kinetic energy:

$$R^2 = \|c\varphi \cdot \varphi\|_{L^1([t_1,t_2]) \times L^1(\Omega)} \equiv \int_{t_1}^{t_2} \int_{\Omega} c\varphi \cdot \varphi \,\mathrm{d}x \,\mathrm{d}t.$$

This formulation yields the L^2-Monge–Kantorovich mass transfer problem, analysed across three centuries (see [3] and references therein), and used by Tannenbaum, Ratner et al. [30] early on to study mouse brain transport via in-vivo imaging.

However, when we applied this method to our human image data set, we quickly faced several challenges. First, by considering transport by convection alone (3), we ignore the substantial contribution from diffusion. Second, we found the method sensitive to the choice of the regularization parameter $\alpha > 0$. And third, $c \approx 0$ locally, for which (2) is not obviously well posed. We therefore needed to develop a more physiologically accurate and more numerically robust approach.

Bilinear flow control. Still targeting the task of "given $c_1, c_2 \colon \Omega \to \mathbb{R}$ at t_1, t_2, identify a convective flow field φ defined over the time interval $[t_1, t_2]$", we instead considered the bilinear optimal control problem of finding minimizers

$$\min_{c,\varphi} \big(\|c(t_2) - c_2\|^2 + \alpha^2 R^2\big), \tag{4}$$

subject to $c(t_1) = c_1$, and such that (1) holds (with $r = 0$) over $(t_1, t_2]$. We let $R = \|\varphi\|_{L^2} + |\varphi|_{H^1}$, though that choice can easily be reevaluated. Note that we deliberately allow for non-divergence-free velocity fields in light of our previous results (Section 2.1) indicating that local fluid influx and efflux may be key for effective convective transport. This problem setting is very similar to that studied in terms of well-posedness and existence by Glowinski et al. [20] modulo choice of regularization and incompressibility (divergence-free) constraint (or lack thereof).

There are two main approaches to solving the constrained minimization problem (4): either (i) introducing a Lagrange multiplier for the PDE-constraint and solving the resulting nonlinear Euler–Lagrange problems directly (the *all-at-once* approach), or (ii) iteratively solving the optimization problem by solving the PDE-constraint (1) with c_1 as initial condition for a series of φ by means, e.g., a quasi-Newton method (the *reduced* approach). After numerical experiments and comparisons, we found that the latter approach (using an L-BFGS algorithm) was more robust for our uses in the sense that this method successfully converged for most patients and time intervals between brain scans and yielded consistent results for different mesh resolutions and values of the regularization parameters.

Remark 5. Our previous work on automated solution of high-dimensional inverse problems and accompanying software dolfin-adjoint [2, 18] proved to be very useful here.

Ultimately, we thus decided to estimate net (time-averaged) velocity fields $\varphi \colon \Omega \to \mathbb{R}^d$ in the contrast agent influx phase $(t_1, t_2) = (6, 24)$ h and clearance phase $(t_1, t_2) = (24, 48)$ h for each patient by solving (4) (using subject-specific finite element meshes, diffusion tensors and concentration observations) via a reduced approach using a fixed maximal number of iterations (Figure 2 (b)). Interestingly, for all subjects and time intervals, we identified persistent velocity fields with brain-averaged velocity magnitudes (i.e., flow speeds) of $\sim$1–8 µm/min. These flow speeds correspond to bulk flow rates of 0.1–0.8 µL/(g min) – a range which compares fascinatingly well with estimated bulk flow rates of 0.1–0.3 µL/(g min) from early (1970s–80s) tracer experiments in cats [1]. Interestingly, we can also estimate the net fluid influx/efflux by computing, e.g., the brain-wide average of $\operatorname{div} \varphi$, and find that water dominantly flows into the brain (e.g., by filtration from the blood stream) at a rate of $\sim$1–4 min^{-1}.

Open problem 6. Our findings suggest that high-dimensional inverse modelling offers a powerful avenue of investigation for the brain's waterscape. However, a question that remains is how well-posed the optimization problem is with respect to the uncertainty in the input data and choice of regularization functional. Moreover, clearly there are "terra incognita" (or perhaps rather "mare incognita") for which there is little information in the medical images, i.e., little or no contrast agent present, and where velocity field estimates intuitively would be associated with substantial uncertainty. Quantifying the uncertainty and information level required for reliable flow estimates in this setting is a open problem.

So, using stochastic and inverse modelling as described, we have quantified the relative contributions of diffusion and convection in brain solute transport, suggesting that these modes coexist and co-contribute. The next question is: given that there is a persistent convective fluid flow within the brain parenchyma, what could be the mechanisms and the brain mechanics allowing for such a flow?

3 The brain as a pulsating poroelastic medium

From the viewpoint of mechanics, the brain is an almost surprisingly soft elastic medium, enclosed and protected by the stiffer meninges and much stiffer skull, and permeated by a number of fluid networks including the blood-filled vasculature (arteries, capillaries and veins), the ISF-filled extracellular space between brain cells, and potential CSF- or ISF-filled perivascular spaces surrounding the vasculature. With every heart beat (the cardiac rhythm) and breath (the respiratory cycle), your brain expands and contracts with displacements of the order of 1 mm and volume changes of the order of 1 cm^3. The precise interplay and transfer of forces and sources between these compartments, and how these change with age and disorders, still lacks understanding and quantification however.

In pioneering work, Tully and Ventikos [34] introduced the multiple-network poroelastic theory (MPET), a theory originating in fractured geological reservoirs, in the context of modelling brain solid and fluid mechanics. Specifically, the quasi-static MPET equations enforce balance of momentum and mass and read as: for a given number of networks $J \in \mathbb{N}$, $\Omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$), and a time interval I find the displacement field $u \colon \Omega \times I \to \mathbb{R}^d$ and the j-th network pressure field $p_j \colon \Omega \times I$ for $j = 1, \dots, J$ such that

$$\begin{aligned} -\operatorname{div}(\sigma(u)) + \textstyle\sum_j \alpha_j \operatorname{grad} p_j &= f, \\ \partial_t(s_j p_j + \alpha_j \operatorname{div} u) - \operatorname{div} \kappa_j \operatorname{grad} p_j + T_j(p) &= g_j. \end{aligned} \tag{5}$$

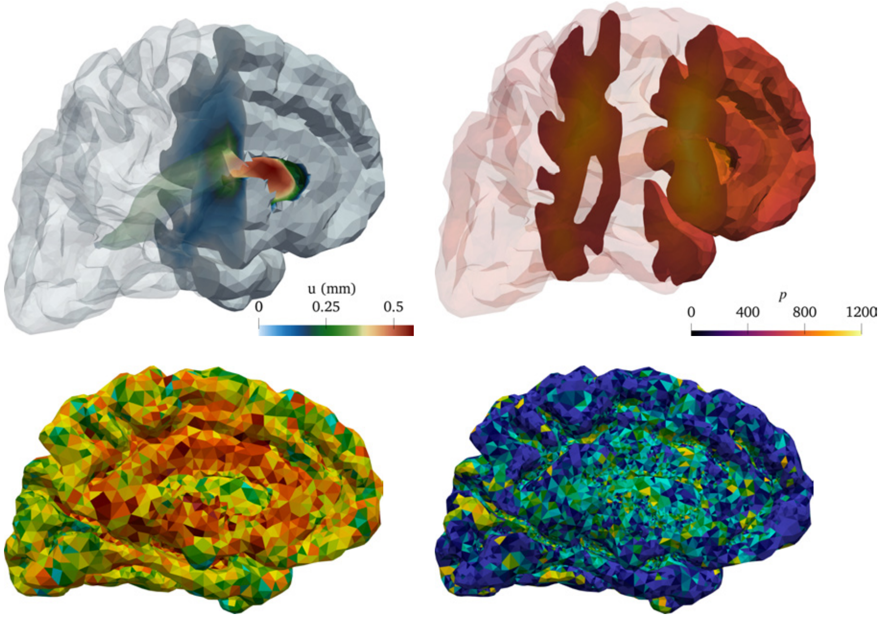


Figure 3. Top: Brain displacements and arterial blood network pressure predicted by a pulsatile fluid influx and the MPET theory (5) (snapshot in time). Below: A posteriori error indicators (sagittal view) on initial (left) and adaptively refined final (right) mesh [33].

Here $\sigma(u)$ is the elastic stress-strain relationship and T describes the transfer of mass between fluid networks. In the linear and isotropic case,

$$\begin{aligned} &\sigma(u) = 2\mu\varepsilon(u) + \lambda \operatorname{div}(u) I, \quad \varepsilon(u) \equiv \frac{1}{2}(\operatorname{grad} u + \operatorname{grad} u^{\mathrm{T}}), \\ &T_j(p) = \textstyle\sum_i \xi_{ji}(p_j - p_i). \end{aligned}$$

with Lamé parameters $\mu > 0$ and $\lambda > 0$. Each network j is equipped with its Biot–Willis coefficient $\alpha_j \in (0,1]$ with $\sum_j \alpha_j \leqslant 1$, its storage coefficient $s_j > 0$, and its hydraulic conductivity tensor κ_j, while the network interactions are described via the exchange coefficients ξ_{ji}. We also note that the fluid (Darcy) velocity v_j in network j is defined by

$$v_j = -\kappa_j \operatorname{grad} p_j. \tag{6}$$

For simplicity, consider the basic boundary conditions $u = 0$, $p_j = 0$ on $\partial\Omega$ and initial conditions $p_j(0) = p_{j,0}$ here.

In the case $J = 1$, the MPET equations reduce to Biot's equations, whose properties have been studied for decades (and still actively are). The general MPET equations however seemed to have received much less attention from the mathematical community.

3.1 *MPET as an elliptic-parabolic problem*

To analyse approximations of the MPET equations (Figure 3), we realized that the framework of coupled elliptic-parabolic problems as introduced by Ern and Meunier [17] provided an ideal starting point. Specifically, this framework considers variational problems in space and time of the form "given bilinear forms a, b, c, and d and input data f and g, find $u \in H^1(I; V_a)$ and $p \in H^1(I; V_d)$ such that, for almost every $t \in I$,

$$\begin{aligned} a(u,v) - b(v,p) &= \langle f, v\rangle_* \quad \forall v \in V_a, \\ c(p_t, q) + b(u_t, q) + d(p,q) &= \langle g, q\rangle_* \quad \forall q \in V_d, \end{aligned} \tag{7}$$

with the initial condition $p(0) = p_0$". Crucially, under natural assumptions on the bilinear forms and the underlying (Hilbert) spaces, and subsequently spatial and temporal discretizations, Ern and Meunier [17] derived optimal stability estimates, a priori error estimates and a posteriori error estimates for such problems.

Introducing $V_a = H_0^1(\Omega; \mathbb{R}^d)$, $V_d = H_0^1(\Omega; \mathbb{R}^J)$ and denoting $p = (p_1, \ldots, p_J)$ and analogously for α, the MPET equations (5) take the form of a coupled elliptic-parabolic problem (7) via the identifications

$$\begin{aligned} a(u,v) &= \langle \sigma(u), \varepsilon(v)\rangle, \\ b(u,p) &= \langle \alpha \cdot p, \operatorname{div} u\rangle, \\ c(p,q) &= \sum_{j=1}^{J} \langle s_j p_j, q_j\rangle, \\ d(p,q) &= \sum_{j=1}^{J} \langle \kappa_j \operatorname{grad} p_j, \operatorname{grad} q_j\rangle + \langle T_j, q_j\rangle, \end{aligned} \tag{8}$$

Clearly, V_a and V_d as defined are Hilbert spaces, dense in $L_a = L^2(\Omega; \mathbb{R}^d)$ and $L_d = L^2(\Omega; \mathbb{R}^J)$, respectively, a is symmetric, continuous and coercive if $\mu > 0$, $2\lambda + d\mu > 0$), b is continuous for

$a_j \in (0,1]$ and c is symmetric, continuous and coercive for $s_j > 0$. By assuming that the exchange coefficients are positive, symmetric $\xi_{ji} = \xi_{ij} > 0$, and bounded, and that the conductivities are bounded (including from below: $\kappa_j > \kappa_{\min} > 0$), shuffling terms and the Poincaré inequality gives that d is symmetric, coercive and continuous, then stability and uniqueness follows for (weak) solutions of the MPET equations (5) [17].

Introducing a conforming finite element discretization of V_a and V_d relative to a mesh $\mathcal{T}_h$ with mesh size h and a first-order implicit time discretization with time steps τ_n to define discrete approximations $u_{h\tau}$ and $p_{h\tau}$, a priori and a posteriori error estimates (Figure 3) also follow.

Theorem 7 (Eliseussen, Rognes and Thompson [12]). *For solutions (u, p) and approximations $(u_{h\tau}, p_{h\tau})$ of the* MPET *equations, and for each time step t_n, $n \in \{0, 1, \dots, N\}$,*

$$\begin{aligned}\|u - u_{h\tau}\|_{L^\infty(0,t_n;H^1)} + \|p - p_{h\tau}\|_{L^\infty(0,t_n;L^2)} + \|p - p_{h\tau}\|_{L^2(0,t_n;H^1)} \\ \lesssim \eta_1 + \eta_2 + \eta_3 + \eta_4 + \mathcal{E}_{h0}(u_0, p_0) + \mathcal{E}_h(f, g),\end{aligned}$$

where $\eta_1, \eta_2, \eta_3, \eta_4$ are a posteriori computable quantities involving appropriately weighted cell and facet residual contributions, $\mathcal{E}_{h0}$ and $\mathcal{E}_h$ are determined by the approximation of the initial data and source terms, respectively.

3.2 The incompressible MPET equations

As the human brain is composed of $\approx 80\,\%$ water, it is widely considered (nearly) incompressible ($\lambda \to \infty$ in (5)). It therefore seems highly relevant to ask if the variational formulation (7) with forms given by (8) and the estimates of Theorem 7 are robust with respect to λ? The answer is no: the continuity of the linear elasticity form a (and thus the error estimates) degenerate as $\lambda \to \infty$. Indeed, it is illuminating to take a look at the structure of the MPET equations in this scenario. Set $s_j = 0$ to reveal the extreme cases. As $\lambda \to \infty$, $\operatorname{div} u \to 0$, the MPET equations decouple into a coupled Darcy flow system for $p = (p_1, \dots, p_J)$ and an elliptic equation for u,

$$\begin{aligned} -\operatorname{div} \kappa_j \operatorname{grad} p_j + T_j(p) &= g_j, \\ -\operatorname{div} 2\mu\varepsilon(u) &= f - \textstyle\sum_j \alpha_j \operatorname{grad} p_j. \end{aligned} \tag{9}$$

We can thus expect finite element discretization of the MPET equations in the incompressible limit to be wrought with analogous challenges as for the linear elasticity equations.

How should we remedy this situation? An appealing solution, first introduced in the context of Biot's equations, is to introduce the *total pressure* p_0 as a new variable, $p_0 = \lambda \operatorname{div} u - \alpha \cdot p$. Then the action of the MPET operator transforms into

$$\begin{pmatrix} -\operatorname{div} 2\mu\varepsilon & -\operatorname{grad} & 0 \\ \operatorname{div} & \lambda^{-1} & \lambda^{-1}\alpha \\ 0 & \lambda^{-1}\alpha^{\mathsf{T}}\partial_t & \tilde{C} + \lambda^{-1}\alpha\alpha^{\mathsf{T}}\partial_t \end{pmatrix} \begin{pmatrix} u \\ p_0 \\ p \end{pmatrix},$$

This formulation is indeed robust in the incompressible limit with energy estimates uniform in λ and (subsequently a priori error estimates for stable discretizations):

Theorem 8 (Lee et al. [23]). *Given sufficiently regular f, g_j and initial conditions I_0, solutions u, p_j to system* (5) *satisfy an energy estimate of the form*

$$\begin{aligned}\|\varepsilon(u)\|_{L^\infty(I,L^2(\Omega))} + \textstyle\sum_j (\|p_j\|_{L^\infty(I,s_jL^2(\Omega))} + \|p_j\|_{L^2(I,\kappa_j H_0^1(\Omega))}) \\ \lesssim I_0 + \|f, \dot{f}, g_j, \dot{g}_j\|,\end{aligned}$$

with inequality constant independent of the parameter λ.

Preconditioning by congruence. When next turning to the efficient solution of the MPET equations via preconditioned iterative methods, an interesting puzzle appeared with a charming solution motif [27, 28]. To illustrate, consider a coupled Darcy flow with exchange system like (9) written as

$$Ap = (-K\Delta + E)p = b,$$

where $K = \operatorname{diag}(\kappa_1, \dots, \kappa_J)$ and

$$E_{ij} = -\xi_{ij} \quad \text{for } i \neq j, \qquad E_{ii} = \xi_i, \quad \xi_j = \textstyle\sum_i \xi_{ji}.$$

Block-diagonal preconditioners B of this system easily yield condition numbers that grow with the ratio between the exchange ξ_{ji} and conductance κ_j coefficients [28]. But, can the task of constructing block-diagonal preconditioners be simplified by a linear change of variables $p \mapsto \tilde{p}$ with a matrix P ($p = P\tilde{p}$)?

By definition, a matrix A is diagonalizable by congruence if and only if there exists a matrix P such that $P^{\mathsf{T}}AP$ is diagonal. Further, two matrices A and B are simultaneously diagonalizable by congruence if there exists a matrix P such that both $P^{\mathsf{T}}AP$ and P^TBP are diagonal. Now, if A is a real, symmetric and positive definite and B is a real symmetric matrix, then A and B are simultaneously diagonalizable by congruence. In our case, K is symmetric and positive definite as long as $\kappa_j > 0$ and E is symmetric; therefore, K and E are simultaneously diagonalizable by congruence.

Next, in general, if $C = A^{-1}B$ has J distinct eigenvalues with eigenvectors v_j, then the matrix $P = \{v_j\}_{j=1}^J$ constructed from these eigenvectors diagonalizes A and B by congruence. This observation gives a direct construction for P in terms of the eigenvectors of $K^{-1}E$. The transformed system is then block-diagonal and easily preconditioned with condition numbers that are uniform for a range of exchange and conductance parameters [28]. Interestingly, the construction extends to the MPET system [27].

3.3 Brain-CSF interactions

Remark 9. The total pressure formulation of the MPET equations is also advantageous when considering the fluid-structure interaction between the poroelastic brain and the CSF that surrounds it and

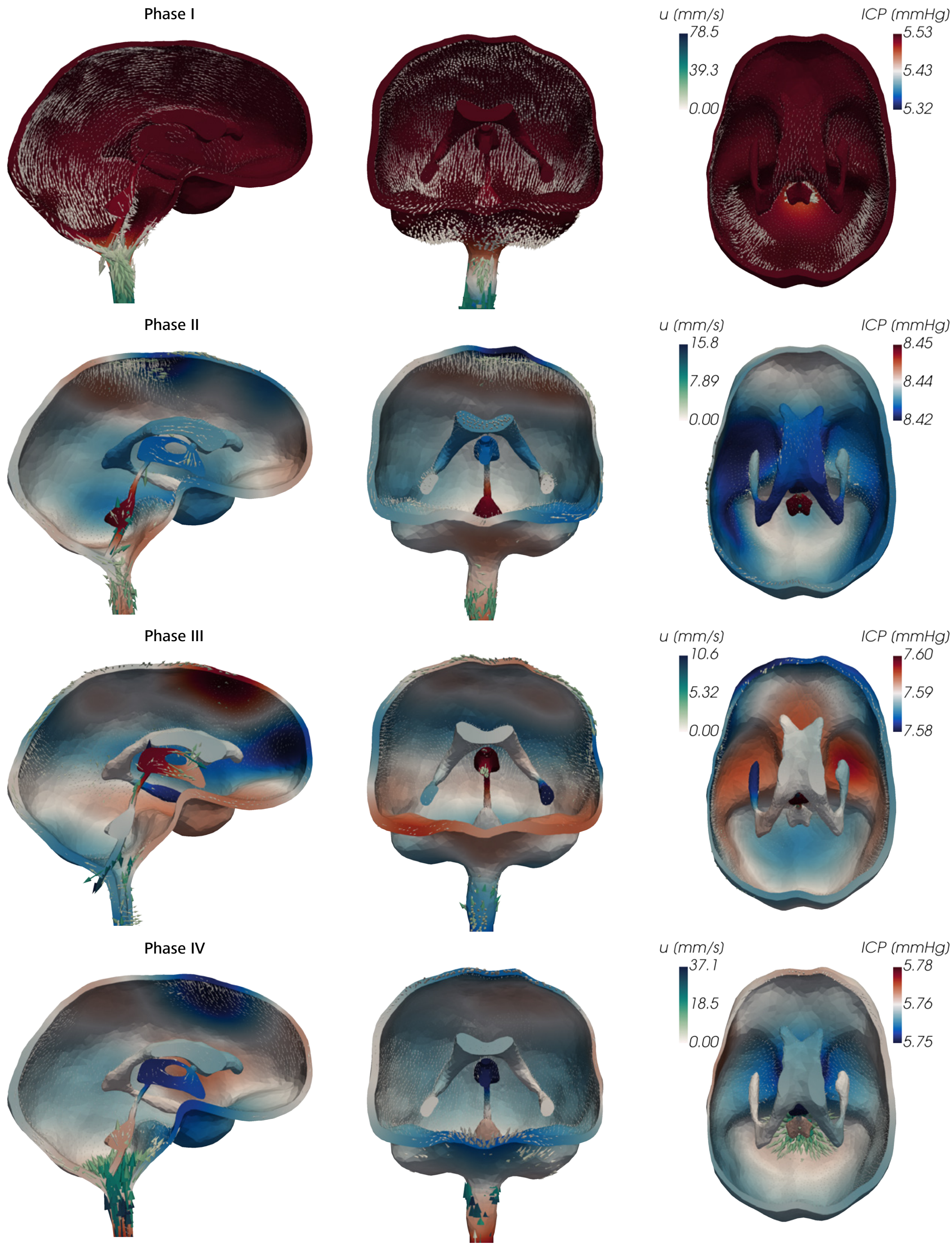


Figure 4. Intracranial pressure and fluid flow in the brain and surrounding CSF subject to a pulsatile fluid (blood) influx in the brain – a fluid-structure interaction problem [6, Figure 4].

clinical applications (Figure 4) [6]. The intracranial pressure, a key clinical measure, can be interpreted as the total pressure in the parenchyma and the hydrostatic pressure in the CSF.

Open problem 10. In-vivo imaging options for brain pulsatility (deformations, stresses) are currently limited, but new techniques such as, e.g., magnetic resonance encephalography (MREG) are coming into play. The quantification of MREG images remains somewhat enigmatic however, and we envision that computational fluid-structure poroelastic brain simulations could provide new interpretations.

3.4 Impermeability and minimal Biot–Stokes stability

The permeability of the brain's extracellular space is estimated to be of the order $10\,\mathrm{nm}^2$, which is many orders of magnitude lower than, e.g., the permeability of the blood circulation. Thus, another limit of interest for studying brain fluid flow is the impermeable case $\kappa \to 0$. This is a particularly interesting case, often attempted addressed by introducing the Darcy velocity (as defined by (6)) as a separate field.

Taking the (time-discrete) Biot equations (i.e., (5) with $J = 1$) and $s_1 = 0$ as an example; these form a generalized saddle-point system that in the limit as $\kappa = 0$ reduces to the incompressible Stokes equations in terms of u and p. On the other hand as noted previously (cf. (9)), in the limit as $\lambda \to \infty$, the Biot equations decouple into an elliptic equation for u and a (mixed) Darcy equation for (z and) p. These observations hint at close relations between Stokes, Darcy, Biot and MPET equations, and suggest that, for combinations of finite element spaces $U \times W \times P$, that the pairing $U \times P$ is Stokes stable and $W \times P$ Darcy stable in the discrete Babuška–Brezzi sense. However, in the limit at $\kappa \to 0$, it is highly non-trivial to ensure uniform Darcy stability [24].

Fortunately, we could show that a revised notion, coined *minimal Stokes–Biot stability* is sufficient.

Definition 11 (Minimal Stokes–Biot stability [24]). A family of discrete spaces $\{U_h \times W_h \times Q_h\}_h$ with $U_h \subset U$, $W_h \subset W$ and $Q_h \subset Q$ is called minimally Stokes–Biot stable if and only if

1. the bilinear form a is continuous and coercive on U_h;
2. $\{U_h \times Q_h\}_h$ are Stokes stable in the discrete Babuška–Brezzi sense;
3. $\operatorname{div} W_h \subseteq Q_h$ for each h.

Theorem 12 (Mardal, Rognes and Thompson [24]). *Minimally stable Stokes–Biot discrete solutions at each time step m converge,*

$$\|u^m - u_h^m\|_1 + \|z^m - z_h^m\|_{K^{-1},\tau;\mathrm{div}} + \|p^m - p_h^m\| \lesssim h^c M_1 + \tau M_2$$

for $c \in \mathbb{N}$ (depending on regularity, order, etc.), where

$$M_1, M_2 \lesssim (\|u\|, \|z\|, \|p\|, \|\partial_t u\|, \|\partial_{tt} u\|, \|\partial_t p\|, \|\partial_t z\|).$$

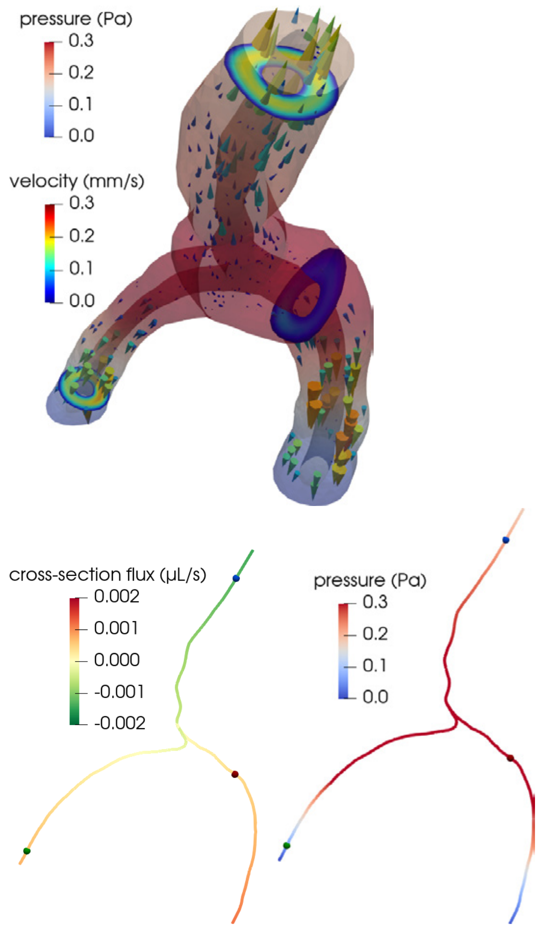


Figure 5. Brain blood vessels are surrounded by fluid-filled perivascular spaces. Are these dual networks amenable for geometrical model reduction? Pulsatile Stokes flow in perivascular spaces can accurately and efficiently be represented by geometrically reduced models over the centreline geometry [9, Figure 6].

Now, returning to the physiology at hand, the moderate pressure differences observed clinically and the nearly vanishing permeability of the extracellular space suggest that extracellular fluid flow is negligible. Could there be alternative, more permeable pathways within the brain?

4 Perivascular fluid flow and model reduction

As the brain lacks lymphatic vessels, fluid-filled spaces surrounding blood vessels – the *perivascular spaces* (PVSs) – are conjectured to act as higher-permeability proxy pathways; "highways" for fluid efflux and metabolic solute clearance [22, 31]. At this mesoscale (≈ mm, s), the vasculature and perivasculature form slender dual networks, running along the brain's surface and diving into the brain. While experimental studies indicate that solutes move rap-

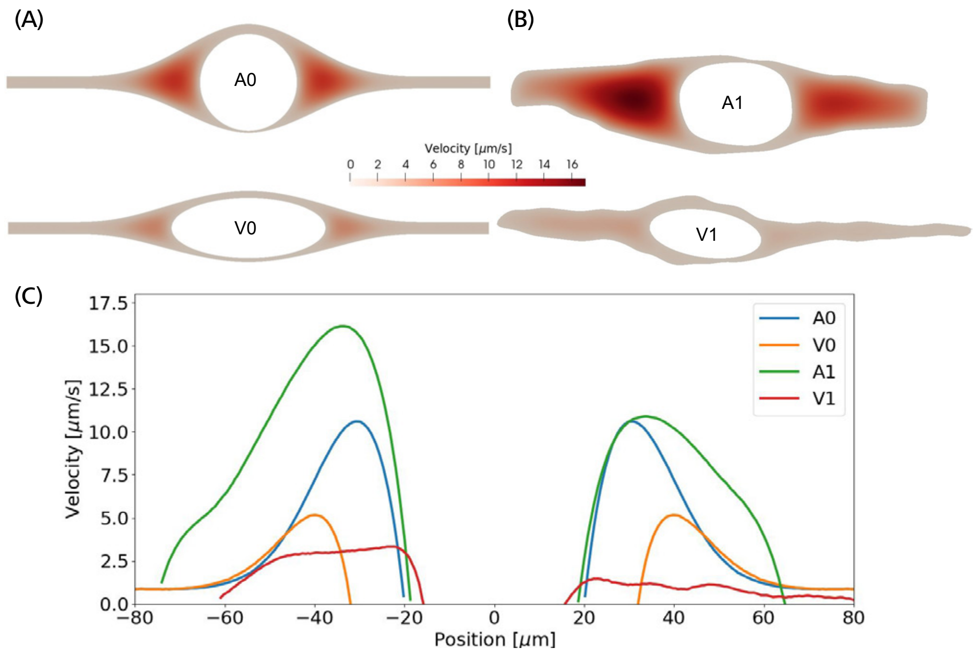


Figure 6. Fluid flow velocity profiles in idealized (A) and image-based (B) periarterial and perivenous geometries [36, Figure 4]. The slow perivenous flow results in delayed tracer transport around veins.

idly in PVSs, a lingering question is "what are the mechanisms, characteristics and forces underlying PVS fluid flow"?

Flow in perivascular spaces appears synchronized with the cardiac cycle. This observation has led to the widespread notion that perivascular flow is driven by arterial wall motion. However, can pulsatile wall motions be a driver for net flow at this scale? To investigate, we created computational models of an annular perivascular segment surrounding an image-based bifurcating blood vessel (Figure 5) and induced laminar (Stokes) flow in the moving PVS by (i) travelling pulse waves on the (inner) blood vessel wall, (ii) pulsatile pressure differences at inlet and outlets, and (iii) static pressure differences. Wall pulsations induce substantial pulsatile flow, but only negligible net flow (due to their long wavelength of $\sim$100 mm) [11]. On the other hand, even a small static hydrostatic pressure difference can induce net flow of relevant magnitudes. Such hydrostatic pressure differences could be induced by experimental procedures [37], but are also of comparable magnitude as transient pressure differences measured in the human brain [38].

Another disputed point is whether there is a net efflux of ISF along perivenous spaces. The original glymphatic theory emphasizes this concept, as injected tracers were observed along arteries, but not around veins at early time points, and along veins at later time points [22]. However, could an alternative explanation for this be connected to the geometry of brain surface arteries and veins? We created pressure-driven Stokes flow models as well as convection-diffusion models of tracer transport from optical coherence tomography images of perivascular spaces surrounding arteries and veins to investigate [36]. PVS flow speeds were 2–6 times higher in the periarterial geometries than in perivenous geometries (Figure 6). Interestingly, these differences in flow speeds due to geometrical differences (area, shape) lead to delayed tracer transport by about 25–30 min, in agreement with the originally observed delayed tracer distribution surrounding veins.

4.1 Perivascular spaces as topologically 1d-networks

Modelling the interplay between blood vessels and tissue via geometrically reduced models has been an active and important research topic for decades, cf., e.g., [5] and references therein. In the brain, the characteristic dual networks formed by the cerebral vasculature and perivasculature define a new setting for geometrically reduced fluid flow modelling [9].

Consider a perivascular domain Ω defined as the union of generalized annular cylinder branches Ω^i. Assume that each such cylinder has a well-defined, oriented, and topologically one-dimensional centreline Λ^i with coordinate s, and let $\Lambda = \bigcup_i \Lambda^i$ (Figure 5). The set of bifurcation points i.e., the points at which the centrelines of branches meet is denoted $\mathcal{B}$. Finally, assume that the boundaries pulsate with a wall speed w in the normal direction.

Under a set of assumptions on the moving geometries (axial symmetry, radial boundary movement and fixed centreline) and flow and pressure fields (axial symmetry, constant cross-section pressure, axial velocity profile), we can derive a reduced system of equations for the cross-section flux q and average cross-section pressure p over time $t > 0$ on each centreline Λ^i with inner and outer radius R_1, R_2 (denoting $q|_{\Lambda^i}$ by q^i and $p|_{\Lambda^i}$ by p^i),

$$\begin{aligned} \frac{\rho}{A^i} q^i_t - \frac{\mu}{A^i} q^i_{ss} + \mu \frac{\alpha^i}{A^i} q^i + p^i_s = 0 \quad &\text{on } \Lambda^i, \\ q^i_s = f^i \quad &\text{on } \Lambda^i, \end{aligned} \tag{10}$$

where

$$f^i(s) \equiv 2\pi \big(R^i_1(s,t) w(R_1, s, t) + R^i_2(s,t) w(R_2, s, t) \big).$$

In (10), ρ is the density of the fluid and μ its viscosity, $A^i = A^i(s,t)$ denotes the cross-section area, while $\hat{\alpha}^i = \hat{\alpha}^i(s,t)$ is a lumped flow parameter that depends on the domain geometry and the choice of axial velocity profile. We also define the (one-dimensional) normal stress induced by $\hat{q}$ and $\hat{p}$ as

$$\sigma \equiv \frac{\mu}{A} q_s - p,$$

which corresponds to an average of the axial (s-)component of the normal stress in each cross-section.

At the bifurcation points $b \in \mathcal{B} \subset \Omega$, we impose conservation of flux and continuity of normal stress,

$$\begin{aligned} q^p(s^p) &= q^{d_1}(s^{d_1}) + q^{d_2}(s^{d_2}), \\ \sigma^p(s^p) &= \sigma^{d_1}(s^{d_1}) = \sigma^{d_2}(s^{d_2}), \end{aligned}$$

where Λ^p and Λ^{d_1}, Λ^{d_2} represent centrelines of the parent and daughter branches meeting at the bifurcation point b, and $s^{\cdot}$ the respective coordinates of b.

This system is particularly amenable for a variational finite element formulation imposing the flux conservation condition weakly using a Lagrange multiplier. Relative to a finite element mesh $\mathcal{T}$ of the centreline Λ composed of mesh segments $\mathcal{T}^i$ (one for each centreline branch Λ^i), we define the

- flux space V_h as the space of continuous piecewise quadratics over $\mathcal{T}^i$ for each i,
- pressure space Q_h as the space of continuous piecewise linears on $\mathcal{T}$, and
- Lagrange multiplier space $R_h = \mathbb{R}^{|\mathcal{B}|}$.

The discrete problem then reads as follows: for each discrete time t^k and time step τ, find $q_h \in V_h$, $p_h \in Q_h$ and $\lambda_h \in R_h$ solving

$$a\big((q_h, p_h, \lambda_h), (\psi, \varphi, \xi)\big) = L^k\big((\psi, \varphi, \xi)\big)$$

for all $\psi \in V_h$, $\varphi \in Q_h$, and $\xi \in R_h$. Here, a is defined by

$$\begin{aligned} &a\big((q,p,\lambda), (\psi, \varphi, \xi)\big) \\ &\quad = \sum_{i \in I} \int_{\Lambda^i} \Big(C^i q^i \psi^i + \frac{\tau\mu}{A^i} q^i_s \psi^i_s + q^i_s \varphi^i - \tau \psi^i_s p^i \Big)\, \mathrm{d}s \\ &\qquad + \sum_{b \in \mathcal{B}} \lambda^b [\psi]^b + \xi^b [q]^b, \end{aligned}$$

where $C^i = A_i^{-1}(\rho + \tau\mu\alpha^i)$, and λ^b (and ξ^b) is simply the λ (or ξ) corresponding to the point b, and the natural jump is

$$[\psi]^b = \psi^p(b) - \psi^{d_1}(b) - \psi^{d_2}(b).$$

This formulation provides an inexpensive and reasonably accurate framework for estimating pulsatile perivascular fluid flow in large networks, and thus establishes a mathematical foundation for future computational studies of perivascular flow and transport (Figure 5) [9].

Open problem 13. Originally, I was planning on coupling the vasculature and perivascular spaces with the tissue via topologically one-dimensional dual networks embedded in the three-dimensional volume, see e.g., [5]. Due to the controversy regarding the existence of flow in brain PVSs and extracellular space, we did not go further with this approach, and as such it remain as (at least) an interesting mathematical problem.

5 Bridging electrochemistry and fluid mechanics

When you are thinking about the brain, the first thing that comes to mind is probably not brain fluid flow. Indeed, neuroscience is possibly foremost occupied with the electrical signalling of the brain. These signals are however generated by electric potential differences, which in turn are induced by ion concentration differences [29]. On the other hand, differences in ion concentrations also induce water movement by osmosis. Thus, up till now, we have only considered one (mechanical) piece of the puzzle, while ignoring electrical and chemical effects. The challenge is that

> *the coupling between electrochemical effects, fluid transport and elastic deformation is particularly difficult and is only little understood, specifically in the brain* [19].

By a pure serendipity, just as we were getting ready to address this challenge, Mori [26] published an elegant and comprehensive multidomain tissue-level model ionic electrodiffusion and osmotic water flow in biological tissue. In this framework, the tissue domain Ω is composed on R coexisting compartments (for instance neuronal, glial, and extracellular spaces) with $|K|$ interacting ionic species (such as sodium Na^+, potassium K^+, chloride Cl^-). The coupled, time-dependent, nonlinear system of PDEs describes the volume fractions α_r, ion concentrations $[k]_r$ for $k \in K$, electric potentials φ_r, mechanical pressures p_r, and fluid velocities u_r for each compartment $r = 1, \dots, R$ (see [13, 26]). For instance, in compartments $r = 1, \dots, R-1$, the movement of each ion concentration $[k]_r$ is governed by $\partial_t(\alpha_r [k]_r) + \operatorname{div} J^k_r \propto J^k_{rR}$, where J^k_r is the *ion flux density* and J^k_{rR} is the *transmembrane water flux* between the compartments r and R. Coupling (chemical) diffusion, drift due to the electric field, as well as fluid flow within each compartment, yields

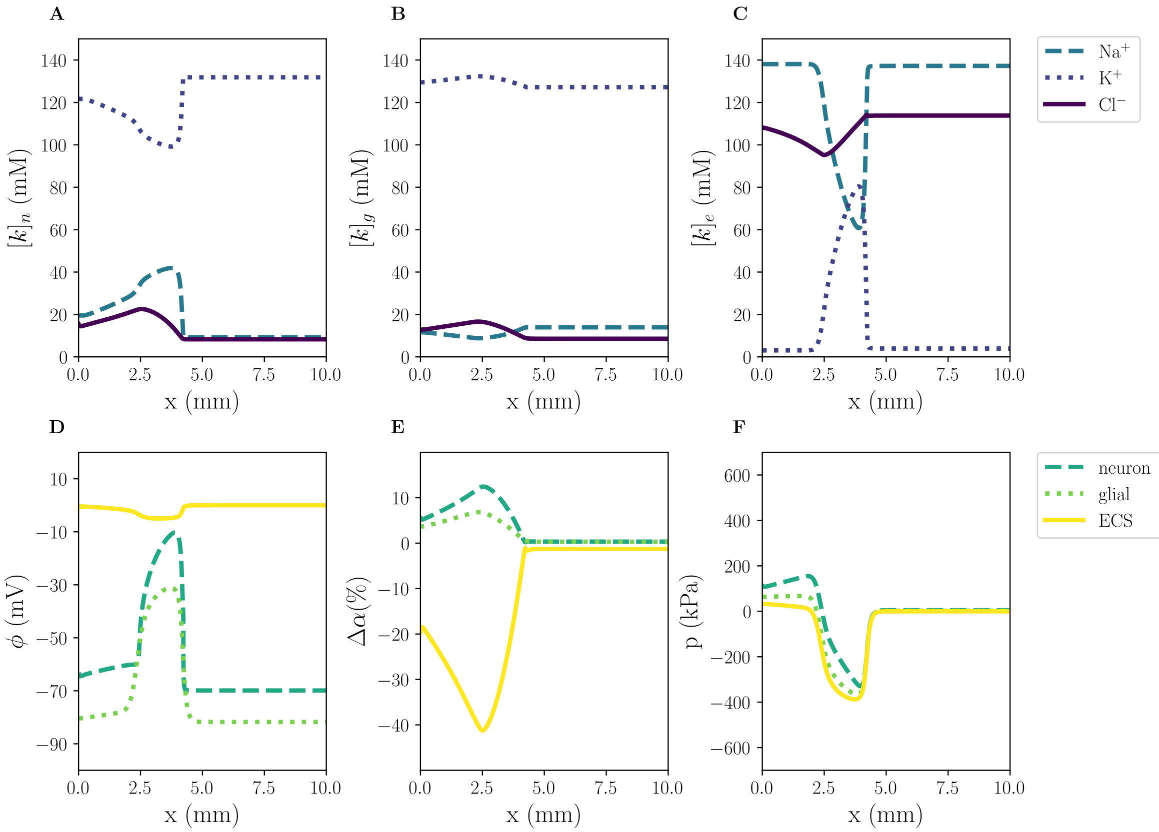


Figure 7. Accurate methods for simulating the interplay between chemical, electrical and mechanical quantities of interest (here during wave propagation through a 10 mm line of brain tissue) is an essential foundation for studying computationally the role of the brain's waterscape in brain signalling and vice versa [13].

ion fluxes of the form

$$J_r^k = -D_r^k [k]_r - \frac{D_r^k z^k}{\psi} [k]_r \operatorname{grad} \varphi_r + \alpha_r u_r [k]_r;$$

see the aforementioned references for a complete description.

The combination of diffusion, nonlinear reaction and convection dynamics makes this a highly interesting system to approximate numerically using, e.g., finite element or spectral methods [13]. Moreover, the underlying physiology induces sharp wave fronts that require a very fine spatial and fine temporal resolution for all current methods (Figure 7). As such, the numerical simulation of ionic electrodiffusion and water movement remains a real challenge, and solution approaches that retain accuracy at lower computational expense could enable further uptake in the community.

Remark 14. As the Waterscales project nears its end, it has been rewarding to observe its trajectory from roots in numerical analysis, rapid impact of poroelasticity and fluid dynamics in brain mechanics, and stretching toward impact in neuroscience in the long run. Our most recent work [15] took us back to the project's origins, by realizing a collaboration with medical experimentalists at the GliaLab at the University of Oslo involved in the development of the original glymphatic theory, while its publication in eNeuro points towards a closer relation with the neuroscience community.

Remark 15. Another question is whether the tissue level is the appropriate scale for modelling the interplay between electrical, chemical and mechanical effects. Indeed, we can envision a paradigm for modelling excitable tissue at the level of cells instead, with explicit representation of cell membranes and spaces. This is however a story for another day [14, 35].

6 Computational abstractions and algorithms

Over the last decades, we have witnessed a tremendous increase in the physical, mathematical, numerical and computational complexity of models for physiological processes. To keep up, there has been a corresponding change in how algorithms and software frameworks for the numerical solution of (partial) differential equations are designed. In the context of finite element methods, the FEniCS Project provides a computational platform combining

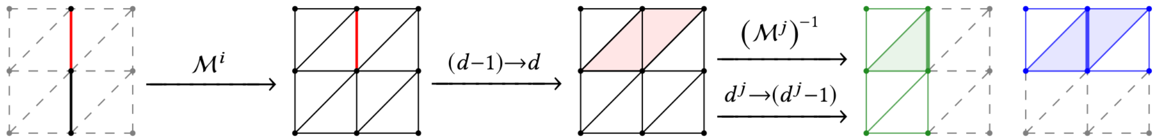


Figure 8. Computational representations of mixed-domain and mixed-dimensional problems take advantage of mappings within (simplicial) complexes such as meshes and submeshes, faces and stars [10, Figure 4].

high-level abstractions for the specification of discrete variational formulations with automated code generation of optimized kernels [2, 18]. However, FEniCS support for multidomain, multiscale and/or multiphysics problems has been sorely lacking.

A core Waterscales aim was to remedy this situation by designing numerical algorithms and software abstractions that allow for high-level specification and high-performance forward and reverse solution of models with multiscale features. We addressed this goal by designing and introducing mathematical software concepts together with lower-level algorithms for expressing, representing, and solving systems of PDEs coupled across interfaces or subdomains (Figure 8) [10]. These tools enable automated assembly and solution of a wide range of mixed finite element variational formulations, such as, e.g., the finite element spaces and formulations involved in the reduced perivascular flow models (10), interactions across the cell membrane in geometrically resolved models of excitable tissue [16, 35] or fluid-structure interfaces [6]. All algorithms are publicly and openly available via the FEniCS Project software [2, 10].

Open problem 16. High-level computational specification and automated solution of, e.g., coupled $n \times (n-2)$-dimensional problems ($n \geqslant 2$), such as to represent interactions between the (peri)vasculature and CSF or tissue, comes with an additional set of challenges including, e.g., non-conforming averaging and extension operators and computational geometry classification problems, and remains an open problem.

7 Brain clearance and neurodegeneration

All-in-all, why are we really so interested in solute transport in the human brain? Well, there are two intertwined clinical reasons – either to transport treatment drugs into the brain, or to understand how metabolic waste clears from the brain – in order to better comprehend and hopefully treat neurodegenerative diseases such as, e.g., Alzheimer's disease.

With new and quantitative physiological insights at hand, we could now mathematically describe the role of dynamical clearance in the propagation of toxic proteins across the brain. Representing the brain's connectome (i.e., fiber bundle pathways) by a graph $(\mathcal{V}, \mathcal{E})$, we study the distribution of the protein concentrations p_i and regional clearance values λ_i ($i \in \mathcal{V}$), evolving via the graph Laplacian L that describes anisotropic diffusion and nonlinear reactions as

$$\begin{aligned} \partial_t p_i &= -\rho \sum_{j=1}^{N} L_{ij} p_j + (\lambda_{\mathrm{crit}} - \lambda_i) p_i - \alpha p_i^2, \\ \partial_t \lambda_i &= -\beta_i p_i (\lambda_i - \lambda_i^\infty), \end{aligned} \tag{11}$$

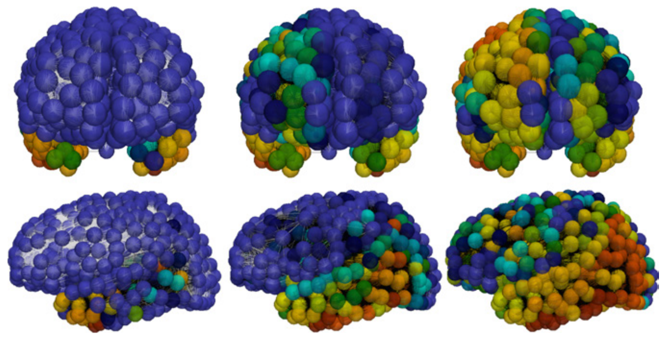


Figure 9. Stages of toxic protein concentrations (coronal view upper row, sagittal view lower row) governed by (11). Courtesy of G. Brennan, Mathematical Institute, University of Oxford.

with initial conditions on p_i, λ_i; here α_i, β_i are kinetic parameters, and $\lambda_i^{\mathrm{crit}}$ and λ_i^∞ critical and minimal clearance values, respectively [4].

The local ($|\mathcal{V}| = 1$) dynamical system (11) admits a class of "healthy" fixed points $(p, \lambda) = (0, \lambda)$ for any $\lambda \geqslant \lambda^\infty$, stable for $\lambda > \lambda^{\mathrm{crit}}$, and also unconditionally stable, "unhealthy" fixed points $(\alpha^{-1}(\lambda_{\mathrm{crit}} - \lambda^\infty), \lambda^\infty)$. The full system describes intriguing pathways of toxic protein propagation and clearance decay (Figure 9) in which the dynamic clearance waterscapes alter disease progression [4].

8 Concluding remarks and outlook

In my original project proposal, I wrote that Waterscales would be *bridging the fluid mechanics across scales and electrophysiology of the brain – with ample opportunities for further mathematical, numerical and more applied study.* Indeed, the brain's waterscape has proven to be a rich field for applied and computational mathematics, which we have only begun to explore. We have many more questions, both in terms of mathematics and physiology, now than when we started. An important interdisciplinary lesson has been that medicine is not mathematics: uncertainties prevail and a single published experiment or clinical study is not "proof" nor settles a case once and for all. However, we have successfully created new mathematical models, new numerical methods, new computational technology and new physiological insights, and importantly, a vibrant research environment targeting ground-breaking research in computational brain electromechanics ready for new challenges.

References

[1] N. J. Abbott, Evidence for bulk flow of brain interstitial fluid: Significance for physiology and pathology. *Neurochem. Int.* **45**, 545–552 (2004)

[2] M. Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M. E. Rognes and G. N. Wells, The FEniCS project version 1.5. *Archive of Numerical Software* **3**, 9–23 (2015)

[3] J.-D. Benamou and Y. Brenier, A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem. *Numer. Math.* **84**, 375–393 (2000)

[4] G. S. Brennan, T. B. Thompson, H. Oliveri, M. E. Rognes and A. Goriely, The role of clearance in neurodegenerative disease. *bioRxiv*, DOI 10.1101/2022.03.31.486533 (2022)

[5] L. Cattaneo and P. Zunino, Computational models for fluid exchange between microcirculation and tissue interstitium. *Netw. Heterog. Media* **9**, 135–159 (2014)

[6] M. Causemann, V. Vinje and M. E. Rognes, Human intracranial pulsatility during the cardiac cycle: A computational modelling framework. *Fluids Barriers CNS* **19**, 863–874 (2022)

[7] M. Croci, V. Vinje and M. E. Rognes, Uncertainty quantification of parenchymal tracer distribution using random diffusion and convective velocity fields. *Fluids Barriers CNS* **16**, 1–21 (2019)

[8] M. Croci, V. Vinje and M. E. Rognes, Fast uncertainty quantification of tracer distribution in the brain interstitial fluid with multilevel and quasi Monte Carlo. *Int. J. Numer. Methods Biomed. Eng.* **37**, Paper No. e3412 (2021)

[9] C. Daversin-Catty, I. G. Gjerde and M. E. Rognes, Geometrically reduced modelling of pulsatile flow in perivascular networks. *Front. Phys. (Lausanne)*, DOI 10.3389/fphy.2022.882260 (2022)

[10] C. Daversin-Catty, C. N. Richardson, A. J. Ellingsrud and M. E. Rognes, Abstractions and automated algorithms for mixed domain finite element methods. *ACM Trans. Math. Software* **47**, Article No. 31, DOI 10.1145/3471138 (2021)

[11] C. Daversin-Catty, V. Vinje, K.-A. Mardal and M. E. Rognes, The mechanisms behind perivascular fluid flow. *PLOS ONE* **15**, Paper No. e0244442 (2020)

[12] E. Eliseussen, M. E. Rognes and T. B. Thompson, A-posteriori error estimation and adaptivity for multiple-network poroelasticity, preprint, arXiv:2111.13456 (2021)

[13] A. J. Ellingsrud, N. Boullé, P. E. Farrell and M. E. Rognes, Accurate numerical simulation of electrodiffusion and water movement in brain tissue. *Math. Med. Biol.* **38**, 516–551 (2021)

[14] A. J. Ellingsrud, C. Daversin-Catty and M. E. Rognes, A cell-based model for ionic electrodiffusion in excitable tissue. In *Modeling excitable tissue: The EMI framework*, edited by A. Tveito, K.-A. Mardal and M. E. Rognes, Simula SpringerBriefs Comput., Rep. Comput. Physiol. 7, Springer, Cham, 14–27 (2021)

[15] A. J. Ellingsrud, D. B. Dukefoss, R. Enger, G. Halnes, K. Pettersen and M. E. Rognes, Validating a computational framework for ionic electrodiffusion with cortical spreading depression as a case study. *eNeuro* **9**, DOI 10.1523/ENEURO.0408-21.2022 (2022)

[16] A. J. Ellingsrud, A. Solbrå, G. T. Einevoll, G. Halnes and M. E. Rognes, Finite element simulation of ionic electrodiffusion in cellular geometries. *Front. Neuroinf.*, DOI 10.3389/fninf.2020.00011 (2020)

[17] A. Ern and S. Meunier, A posteriori error analysis of Euler–Galerkin approximations to coupled elliptic-parabolic problems. *M2AN Math. Model. Numer. Anal.* **43**, 353–375 (2009)

[18] P. E. Farrell, D. A. Ham, S. W. Funke and M. E. Rognes, Automated derivation of the adjoint of high-level transient finite element programs. *SIAM J. Sci. Comput.* **35**, C369–C393 (2013)

[19] A. Goriely, M. G. Geers, G. A. Holzapfel, J. Jayamohan, A. Jérusalem, S. Sivaloganathan, W. Squier et al., Mechanics of the brain: Perspectives, challenges, and opportunities. *Biomech. Model. Mechanobiol.* **14**, 931–965 (2015)

[20] R. Glowinski, Y. Song, X. Yuan and H. Yue, Bilinear optimal control of an advection-reaction-diffusion system. *SIAM Rev.* **64**, 392–421 (2022)

[21] B. K. Horn and B. G. Schunck, Determining optical flow. *Artif. Intell.* **17**, 185–203 (1981)

[22] J. J. Iliff, M. Wang, Y. Liao, B. A. Plogg, W. Peng et al., A paravascular pathway facilitates CSF flow through the brain parenchyma and the clearance of interstitial solutes, including amyloid β. *Sci. Transl. Med.* **4**, DOI 10.1126/scitranslmed.3003748 (2012)

[23] J. J. Lee, E. Piersanti, K.-A. Mardal and M. E. Rognes, A mixed finite element method for nearly incompressible multiple-network poroelasticity. *SIAM J. Sci. Comput.* **41**, A722–A747 (2019)

[24] K.-A. Mardal, M. E. Rognes and T. B. Thompson, Accurate discretization of poroelasticity without Darcy stability: Stokes–Biot stability revisited. *BIT* **61**, 941–976 (2021)

[25] K.-A. Mardal, M. E. Rognes, T. B. Thompson and L. M. Valnes, *Mathematical modeling of the human brain: From MRI to finite element simulation*. Springer (2022)

[26] Y. Mori, A multidomain model for ionic electrodiffusion and osmosis with an application to cortical spreading depression. *Phys. D* **308**, 94–108 (2015)

[27] E. Piersanti, J. J. Lee, T. Thompson, K.-A. Mardal and M. E. Rognes, Parameter robust preconditioning by congruence for multiple-network poroelasticity. *SIAM J. Sci. Comput.* **43**, B984–B1007 (2021)

[28] E. Piersanti, M. E. Rognes and K.-A. Mardal, Parameter robust preconditioning for multi-compartmental Darcy equations. In *Numerical mathematics and advanced applications—ENUMATH 2019*, Lect. Notes Comput. Sci. Eng. 139, Springer, Cham, 703–711 (2021)

[29] R. Rasmussen, J. O'Donnell, F. Ding and M. Nedergaard, Interstitial ions: A key regulator of state-dependent neural activity? *Prog. Neurobiol.* **193**, DOI 10.1016/j.pneurobio.2020.101802 (2020)

[30] V. Ratner, L. Zhu, I. Kolesov, M. Nedergaard, H. Benveniste and A. Tannenbaum, Optimal-mass-transfer-based estimation of glymphatic transport in living brain. In *Medical Imaging 2015: Image Processing*, SPIE (2015)

[31] M. L. Rennels, T. F. Gregory, O. R. Blaumanis, K. Fujimoto and P. A. Grady, Evidence for a 'paravascular' fluid circulation in the mammalian central nervous system, provided by the rapid distribution of tracer protein throughout the brain from the subarachnoid space. *Brain Res.* **326**, 47–63 (1985)

[32] G. Ringstad, S. A. S. Vatnehol and P. K. Eide, Glymphatic MRI in idiopathic normal pressure hydrocephalus. *Brain* **140**, 2691–2705 (2017)

[33] M. E. Rognes, T. B. Thompson and E. Eliseussen, A-posteriori error estimation and adaptivity for multiple-network poroelasticity (unpublished)

[34] B. Tully and Y. Ventikos, Cerebral water transport using multiple-network poroelastic theory: Application to normal pressure hydrocephalus. *J. Fluid Mech.* **667**, 188–215 (2011)

[35] A. Tveito, K.-A. Mardal and M. E. Rognes, *Modeling excitable tissue: The EMI framework*. Simula SpringerBriefs Comput., Rep. Comput. Physiol. 7, Springer, Cham (2021)

[36] V. Vinje, E. N. Bakker and M. E. Rognes, Brain solute transport is more rapid in periarterial than perivenous spaces. *Sci. Rep.* **11**, 1–11 (2021)

[37] V. Vinje, A. Eklund, K.-A. Mardal, M. E. Rognes and K.-H. Støverud, Intracranial pressure elevation alters CSF clearance pathways. *Fluids Barriers CNS* **17**, 1–19 (2020)

[38] V. Vinje, G. Ringstad, E. K. Lindstrøm, L. M. Valnes, M. E. Rognes, P. K. Eide and K.-A. Mardal, Respiratory influence on cerebrospinal fluid flow: A computational study based on long-term intracranial pressure measurements. *Sci. Rep.* **9**, 1–13 (2019)

[39] L. Xie, H. Kang, Q. Xu, M. J. Chen, Y. Liao, M. Thiyagarajan, J. O'Donnell, D. J. Christensen, C. Nicholson, J. J. Iliff et al., Sleep drives metabolite clearance from the adult brain. *Science* **342**, 373–377 (2013)

[40] B. Zapf, V. Vinje, G. Ringstad, P. K. Eide, M. E. Rognes and K.-A. Mardal, High-dimensional inverse modelling of transport in the human brain, personal communication (2022)

Marie E. Rognes is chief research scientist at the Department for Numerical Analysis and Scientific Computing at Simula Research Laboratory in Oslo, Norway, member of the Norwegian Academy of Technological Sciences, and of the FEniCS Steering Council. She is spending the 2022/2023 academic year as a visiting scholar at the Institute of Engineering in Medicine, University of California San Diego in sunny La Jolla, CA, USA.

meg@simula.no

100 years of Acta Szeged

Lajos Molnár

This year the journal Acta Scientiarum Mathematicarum, which is commonly called Acta Szeged, celebrates its 100th anniversary. In this article, we provide a brief overview of the past and the present of the journal, which is currently one of the 20 oldest and still active mathematical periodicals worldwide.

Alfréd Haar

Frigyes Riesz

1 Highlights from History

In the 1870s the idea of founding a third science university in the Hungarian Kingdom, besides the ones in Budapest and Kolozsvár (Cluj), was raised. The cities of Pozsony (Bratislava) and Szeged were strongly competing for hosting the new university. After the huge flood in 1879, which almost completely destroyed Szeged, the town was rebuilt from its ruins into a gorgeous city, and the public opinion, as well as a number of authorities, definitely supported the idea of the university. However, Pozsony, the former coronation city, won and the 3rd university was founded in 1912 there. Moreover, in the same year the 4th university was established in Debrecen.

So Szeged lost the fight for a university, but history soon changed the situation. After WWI, the Trianon Treaty of 1920 redefined Hungary's map and Kolozsvár became a part of Romania. The faculty of the university was unwilling to take an oath to the Romanian king, so the university had to move. Kuno Klebelsberg, Minister of Religion and Public Education at that time, urged the transition of the university from Kolozsvár to Szeged which had became a border town in post-war Hungary. The plan was carried out, and in 1921 the university of Kolozsvár moved to Szeged.

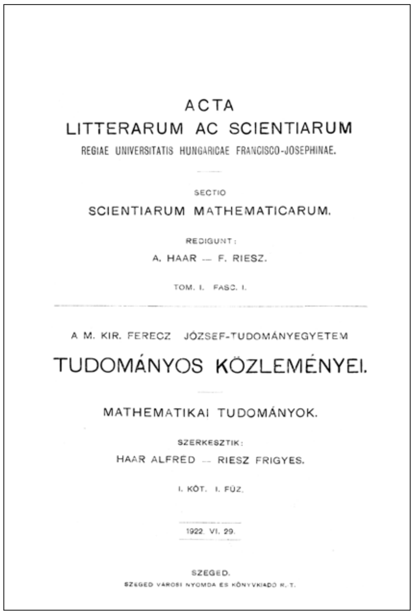
Alfréd Haar and Frigyes Riesz, at that time already world-famous scholars and professors of the university in Kolozsvár, also settled in Szeged and founded a mathematical school.

In order to create the conditions for high-quality research in their new place of work they wanted to build an adequate mathematical library, as the library in Kolozsvár had to be left behind. Besides using the available (limited) financial sources and occasional donations, among them a generous one by the Rockefeller Foundation, they decided to establish and run a respected mathematical journal that could be used in exchange for other periodicals. This way they would create a stable and firm background for the library, so in 1922 they founded the journal commonly called Acta Szeged[1], and their plan for exchanges turned out to work with high efficiency.

There exist documents concerning the initiations of those exchanges. We quote the following from [3].

> On 12 December 1922 Riesz wrote to Maurice Fréchet and sent him the first facsimile of Acta Szeged asking him to promote it (Riesz 1922). Fréchet's reply came soon. On 16 December 1922 Fréchet answered that "it would be a honour" to receive Acta Szeged and he offered Riesz in exchange the Publications de l'Institut de Mathematiques de l'Université de Strasbourg as well as copies of doctoral dissertations that have been presented in the Mathematical Institute of the University of Strasbourg. … In 1922, G. H. Hardy wrote to Riesz discussing the exchange of Acta Szeged (Hardy 1922). There exist also three letters by Gösta Mittag-Leffler (Sweden) on the same topic from between 1923 and 1924 (Mittag-Leffler 1923, 1924).

[1] Since its foundation the name of the journal was changed several times. First, it was Acta Literarum ac Scientiarum Regiae Universitatis Hungariae Francisco Josephinae, Sectio Scientiarum Mathematicarum; later it changed to Acta Universitatis Szegediensis, Sectio Scientiarum Mathematicarum; after WWII to Acta Scientiarum Mathematicarum (Szeged); and then it changed to Acta Universitatis Szegediensis, Acta Scientiarum Mathematicarum.



ACTA
LITTERARUM AC SCIENTIARUM
REGIAE UNIVERSITATIS HUNGARICAE FRANCISCO-JOSEPHINAE.

SECTIO
SCIENTIARUM MATHEMATICARUM.

REDIGUNT:
A. HAAR — F. RIESZ.

TOM. I. FASC. I.

A M. KIR. FERECZ JÓZSEF-TUDOMÁNYEGYETEM
TUDOMÁNYOS KÖZLEMÉNYEI.

MATHEMATIKAI TUDOMÁNYOK.

SZERKESZTIK:
HAAR ALFRED — RIESZ FRIGYES.

I. KÖT. I. FÜZ.

1922. VI. 29.

SZEGED.
SZEGED VÁROSI NYOMDA ÉS KÖNYVKIADÓ R. T.

The cover page of the first issue

The success of those early contacts was certainly due to Riesz's high scientific stature in the international mathematical community. Apparently, Haar and Riesz did not want to rely on this exclusively, but they wanted to make the journal itself attractive, hence they invited several leading mathematicians to submit some of their works to Acta Szeged. These attempts were very successful. On the list of authors in the first years, we find, among others, the names of mathematicians such as John von Neumann, Norbert Wiener, George D. Birkhoff, Henri Cartan, Antoni Zygmund, Lipót Fejér, György Pólya, Gábor Szegő. This fact contributed essentially to the achievement which E. R. Lorch described in his entertaining paper [2] as follows:

> Within a few years the Szeged Acta had a world-wide reputation. Every serious mathematics library receives it. So, Szeged is now known all over the world – at least to mathematicians.

In accordance with that description, Gyula Staar, a Hungarian scientific journalist, wrote in the collection [5] that

> Szeged soon became a research center in mathematics, and many people refer to this town even today as the "Hungarian Göttingen".

We remark that already Kuno Klebelsberg preferred to describe Pécs, where the university in Pozsony moved after the Trianon Treaty, as the "Hungarian Heidelberg" and Szeged as the "Hungarian Göttingen".

We quote an interesting story from [1] demonstrating how demanding Riesz was concerning the quality of the papers to be published in their journal.

> An anecdote about the Riesz brothers is told by both Szőkefalvi-Nagy and John Horvath. (Horvath was a long-time friend and colleague of Marcel Riesz.) It seems that Marcel once submitted a paper to the Szeged Acta, where Frigyes was founder and editor. It was certainly a good paper, but Frigyes wrote to his brother, "Marcel, you have written also better things." To be fair, Marcel did publish in the Szeged Acta.

Indeed, in the first two volumes of the journal he published four papers.

The exchange of Acta Szeged was a real success and, by the thirties the library of the mathematical institute had the most extensive collection of mathematical journals in Hungary, both in variety and quality. It should be pointed out that from the 1930s Acta Szeged not only initiated such exchanges, but it also received a number of similar inquiries and offers. A document shows that in the period 1948–1965 there were 178 exchanges. We mention the names of 10 journals from the long list: Acta Math., Amer. J. Math., Ann. Math., Canad J. Math., Duke J. Math., J. London Math. Soc., Math. Ann., Proc. Cambridge Math. Soc., Proc. London Math. Soc., Trans. Amer. Math. Soc.

In addition to the journal exchanges, several publishers sent books to Acta Szeged for review which, as a compensation for the review in the journal, became the library's possession.

Alfréd Haar passed away in 1933 and Frigyes Riesz moved to Budapest in 1946. From 1946 to 1981 Béla Szőkefalvi-Nagy (his surname is commonly shortened as Sz.-Nagy), a leading scholar in operator theory, was the editor-in-chief of the journal. He carried on the traditions that Haar and Riesz established, and handled the journal's affairs with very particular care and devotion. His daughter Erzsébet (Sz.-Nagy and his wife raised six children) told that in the family they referred to Acta Szeged as Sz.-Nagy's seventh child. During his long period of 35 years as editor-in-chief, Acta Szeged continued to be a highly respected periodical, it became an internationally leading journal in the fields of functional analysis and operator theory.

Besides Sz.-Nagy himself, who published 65 papers in the journal, many other outstanding researchers in these areas published high-impact works in Acta Szeged. We mention here T. Ando, H. Bercovici, A. Connes, Ch. Davis, J. Dieudonné, J. Dixmier, R. Douglas, N. Dunford, C. Foias, I. Gohberg, P. Halmos, W. Helton, M. G. Krein, H. Langer, M. A. Naimark, H. Radjavi, D. Sarason, D. Voiculescu, and the list could be continued much further.

After Sz.-Nagy's time at the helm, the next editors-in-chief were László Leindler (in the period 1982–1992) and László Kérchy (in the period 1993–2017).

In 1999, one year after Sz.-Nagy passed away, his daughter Erzsébet established the Béla Szőkefalvi-Nagy Medal in memory of her father. This prize is to recognize distinguished mathematicians who published significant works in Acta Szeged.[2]

2 Some high-impact papers from Acta Szeged

It should come as no surprise that the journal has published several important high-impact papers during its history. In this section, we present a dozen of them.

We start with a work by John von Neumann which appeared in the first volume.

J. von Neumann, Zur Einführung der transfiniten Zahlen. *Acta Sci. Math. (Szeged)* **1**, 199–208 (1922–23)

Ordinals as transfinite variants of non-negative integers were used from the very beginning of set theory, but they were external objects, which was not a convenient feature.

The basic requirement concerning ordinals is that every well-ordered set should be associated with an ordinal and exactly the isomorphic well-ordered sets should have the same associated ordinal.

In the above paper, John von Neumann, at the age of 19, introduced the modern concept of ordinals which is used even today. He called a set H an ordinal if H is transitive (i.e., every element of H is also a subset of H) and the relation $\in$ is a well-ordering on H. An ordinal K is said to be less than H if $K \in H$; this defines a well-ordering on the collection of ordinals such that every ordinal is the set of all ordinals less than the one in question. Moreover, von Neumann also defined cardinals as ordinals H with the property that whenever K is an ordinal less that H, then there is no surjective map from K onto H.

In this simple way, the complete theory of ordinals and cardinals can be built up.

From set theory, we move to the area of algebra and mention two papers.

M. Krasner and L. Kaloujnine, Produit complet des groupes de permutations et problème d'extension de groupes. III. *Acta Sci. Math. (Szeged)* **14**, 69–82 (1951)

In this paper, the authors introduced the concept of wreath product of groups and proved the following fundamental property. If G is an extension of the group A by the group B, meaning that G is a group having a normal subgroup N that is isomorphic to A and the factor group G/N is isomorphic to B, then the wreath product of A and B contains an isomorphic copy of G. This result is referred to as Krasner–Kaluzhnin or Kaluzhnin–Krasner theorem.

After its introduction, many group theorists applied the wreath product to construct various counterexamples and to prove embedding theorems. The wreath product became one of the main tools of group theory.

The other algebra paper concerns lattice theory.

G. Grätzer and E. T. Schmidt, Characterizations of congruence lattices of abstract algebras. *Acta Sci. Math. (Szeged)* **24**, 34–59 (1963)

This paper contains the so-called Grätzer–Schmidt theorem which says that the algebraic lattices (i.e., complete, compactly generated lattices) are exactly those that are isomorphic to the lattices of all congruence relations of abstract algebras. This solves a problem raised by G. Birkhoff in 1948 and became a fundamental theorem in universal algebra.

We proceed with a paper from geometry.

Á. Császár, A polyhedron without diagonals. *Acta Univ. Szeged. Sect. Sci. Math.* **13** 140–142 (1949)

This paper was inspired by the following incorrectly posed problem in a mathematical competition for students in Hungary in 1948: Prove that there is no polyhedron, beyond the tetrahedron, where any two vertices are connected by an edge. In other words, the assertion was that the tetrahedron is the only polyhedron without diagonals.

Indeed, this statement is valid for convex polyhedra. However, in this paper, Császár gave a nonconvex example that has no diagonal. Interestingly, the paper did not inspire many further studies until Martin Gardner's popular paper "Mathematical Games: On the remarkable Császár polyhedron and its applications in problem solving" which appeared in Scientific American in 1975. Currently, Google search displays thousands of hits to the expression "Császár polyhedron".

We now move to the area of analysis, namely potential theory.

O. D. Kellogg, Some notes on the notion of capacity in potential theory. *Acta Sci. Math. (Szeged)* **4**, 1–5 (1928–29)

This short paper contains three fundamental theorems in potential theory. The first one is the so-called generalized maximum principle for harmonic functions. The standard maximum principle says that if a (bounded) harmonic function is non-negative on the boundary of a domain G, then the same is true on the whole domain. Kellogg's observation is that for lower bounded functions non-negativity is not needed to hold at each point of the boundary, there can be a set of exceptional points which is of zero capacity.

[2] The list of former recipients of the medal can be found at https://en.wikipedia.org/wiki/Béla_Szőkefalvi-Nagy#Award_in_his_honour

For the second result, let G be a domain and let $K \subset G$ be a closed set. The problem of removable singularities for harmonic functions reads as follows: For which K is it true that any bounded harmonic function on $G \setminus K$ admits a harmonic extension to the whole domain G? In this paper, Kellogg proved that this is true exactly when K has zero capacity.

The third result says that if G is a domain and E is a subset of its boundary having positive capacity, then E necessarily contains a so-called regular point P; i.e., a point with the property that if the function in the boundary condition for the Dirichlet problem is continuous at P, then the Dirichlet solution is also continuous at P. In other words, the set of irregular points always has zero capacity.

We insert here a photo, one of the most precious relics of the mathematical institute in Szeged, which was taken during the visit of G. D. Birkhoff and O. D. Kellogg to Szeged in 1928.

Upper row: F. Riesz, B. Kerékjártó, A. Haar, D. Kőnig, R. Ortvay
Middle row: J. Kürschák, G. D. Birkhoff, O. D. Kellogg, L. Fejér
Lower row: T. Radó, I. Lipka, L. Kalmár, P. Szász

The next paper we mention here is an important work by F. Riesz from the first issue of the journal.

F. Riesz, Sur les valeurs moyennes du module des fonctions harmoniques et des fonctions analytiques. *Acta Sci. Math. (Szeged)* **1**, 27–32 (1922–23)

Here Riesz introduced the concept of subharmonic functions and verified some of their fundamental properties. The definition itself is simple: A real-valued continuous function defined on a domain G is called subharmonic if for any point P in G and for any closed disc centered at P which is contained in G, the value $f(P)$ is less than or equal to the integral mean of f along the boundary circle of the disc. Subharmonic functions are now central objects in modern potential theory.

Next, we list two papers from the area of measure theory/probability.

A. Prékopa, Logarithmic concave measures with application to stochastic programming. *Acta Sci. Math. (Szeged)* **32**, 301–316 (1971)

A. Prékopa, On logarithmic concave measures and functions. *Acta Sci. Math. (Szeged)* **34**, 335–343 (1973)

Currently, the latter one is the most cited paper that appeared in Acta Szeged – it has 162 citations in MathSciNet and 732 citations in Google Scholar. The former paper is also highly cited: 104 citations in MathSciNet, 552 citations in Google Scholar.

The author, A. Prékopa is one of the initiators of the field called stochastic programming. Some of his main results in that field concern the convexity theory of probabilistic constrained stochastic programming problems; i.e., the question of the convexity of the feasible sets of solutions of such problems). These two papers provide the most fundamental results of this theory. In them he introduced the concept of logarithmic concave measures and proved the following.

(1) If a Borel measure μ on $\mathbb{R}^n$ originates from a logconcave function (density), then the measure μ is logconcave, i.e., its satisfies the inequality $\mu(\lambda A + (1-\lambda)B) \geq \mu(A)^{\lambda}\mu(B)^{1-\lambda}$ for all Borel sets A, B and any real number $0 \leq \lambda \leq 1$.
(2) If we integrate a multivariate logconcave function with respect to some of its variables over their entire space, then the resulting function of the other variables remains logconcave.

These are breakthrough results that establish the convexity of the feasible set of solutions for a wide class of probabilistic constrained stochastic programming problems. Besides those consequences, the results found applications in convex geometry, mathematical analysis, and in some areas of physics, statistics, and economics.

Finally, we move to operator theory, the main focus of the journal during the period of B. Sz.-Nagy. We begin with some of his papers.

B. Sz.-Nagy, On uniformly bounded linear transformations in Hilbert space. *Acta Univ. Szeged. Sect. Sci. Math.* **11**, 152–157 (1947)

This is Sz.-Nagy's most cited paper, it has 88 citations in MathSciNet and 263 citations in Google Scholar.

The main result in the paper is the following fundamental theorem. An invertible bounded operator T on a Hilbert space is similar to a unitary operator if and only if the set of all integer powers of T forms a uniformly bounded family; i.e., it is a bounded set in the operator norm.

B. Sz.-Nagy, Sur les contractions de l'espace de Hilbert. *Acta Sci. Math. (Szeged)* **15**, 87–92 (1953)

This is the first part of a series of 12 papers on contractions, which are bounded operators with norm not greater than 1. The research was motivated by the aim to better understand the fine structure of general linear operators on Hilbert spaces (observe that any bounded linear operator is a scalar multiple of a contraction).

In this paper it is proved that to any contraction T on a Hilbert space H one can associate a larger Hilbert space K and a unitary operator U on K such that for any non-negative integer n, the nth power of T equals the nth power of U multiplied from the left by the orthogonal projection P of H onto K: $T^n = PU^n|_K$ (we also have $T^{*n} = PU^{-n}|_K$). This property is described by saying that every contraction admits a unitary power dilation. It is a fundamental result which says that we can reproduce (in some sense) general operators from two particular and well-understood types of operators: unitaries and orthogonal projections.

As mentioned above, this work was followed by 11 other papers with the same title, numbered from II to XII, which constitute the base of the following famous book in which the authors created a new important branch of operator theory. The papers from number III on were joint works with C. Foias, a young Romanian colleague of Sz.-Nagy.

> B. Sz.-Nagy and C. Foias, *Analyse harmonique des opérateurs de l'espace de Hilbert*. Masson et Cie, Paris; Akadémiai Kiadó, Budapest (1967)

The book was translated into Russian and English. We quote from the foreword to the Russian translation written by M. G. Krein (see [4]):

> In 1953, the famous Hungarian mathematician B. Szőkefalvi-Nagy published in the journal Acta Scientiarum Mathematicarum (Szeged) a theorem, now widely known, on the unitary dilation of contractions. This work was soon continued by the author and other researchers. In 1958, the young Romanian mathematician C. Foias joined in the elaboration of the theory of contractions. Since then a series of articles by B. Sz.-Nagy and C. Foias, under the common title On the Contractions of Hilbert Space, has appeared regularly in Acta Szeged. This research has evolved into a well-developed theory, which plays an important role in modern functional analysis.

Next, we refer to the following important paper by T. Ando which is closely related to the mentioned dilation theorem by Sz.-Nagy.

> T. Ando, On a pair of commutative contractions. *Acta Sci. Math. (Szeged)* **24**, 88–90 (1963)

According to MathSciNet, with its 136 citations, this is the most cited paper in operator theory published in Acta Szeged. Google Scholar displays 414 citations.

Ando proved in this paper that any two commuting contractions on a given Hilbert space admit commuting unitary power dilations. This answers a question raised by Sz.-Nagy. For curiosity, we mention that a similar result does not hold for triples of commuting contraction, as proved by S. Parrott in 1970.

We close our list with the following paper.

> N. Dunford, Resolutions of the identity for commutative B^*-algebras of operators. *Acta Sci. Math. (Szeged)* **12**, 51–56 (1950)

In this work the author developed an approach to the spectral theorem of normal Hilbert space operators that was built on the Gelfand–Naimark theorem for commutative C^*-algebras and on Kakutani's generalization of the Riesz representation theorem concerning bounded linear functionals on spaces of continuous functions. By now this approach became the most standard one, the majority of monographs and textbooks on the topic follow it.

The above list is, naturally, incomplete.

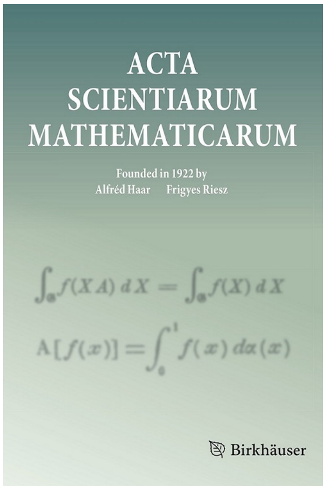
3 The 100th anniversary

Acta Szeged celebrates the 100th anniversary of its foundation in 2022. Indeed, it is the first international mathematical journal published in Hungary and one of the 20 oldest and still active periodicals in mathematics worldwide. On the occasion of the anniversary, we have made several changes around Acta Szeged. First of all, we have extended the editorial board. In the first 100 years, the board consisted of local members only. This year several outstanding mathematicians from abroad have joined the board.[3]

The publisher of the journal has also changed. From 2022, it is published by Springer Nature/Birkäuser (while the owner remains the Bolyai Institute of the University of Szeged).

We have already pointed to the fact, from the period when Sz.-Nagy was the editor-in-chief, the main focus of the journal was on functional analysis and operator theory. This motivated us to publish a related special issue (the double issue 1–2 of volume 88) dedicated to the 100th anniversary. We invited renowned researchers from those areas who once published papers in Acta Szeged to submit current works of theirs. We received a number of valuable contributions and the quite thick special issue will come out soon.

[3] The webpage of the journal is https://www.springer.com/journal/44146 and the current list of the members of the editorial board is given at https://www.springer.com/journal/44146/editors



The new cover page

Besides publishing a special issue, we also organized an online scientific meeting for researchers from those areas to celebrate the memorable event. The meeting took place on May 5–6, 2022 and over 200 people registered for it.[4]

4 Afterword

Having such a long and bright history of the journal, what can we say about the future? Very briefly: we try to keep walking in the steps of the great predecessors and, at the same time, address the challenges of modern times.

Although functional analysis and operator theory are focus areas of the journal, we follow the original tradition and consider Acta Szeged a broad-in-scope mathematics journal. High-quality submissions from other areas of analysis, algebra, and geometry are always very welcome.

The reorganized journal starts the next period with some fresh momentum and we will do our best to make it prosper further in the long run.

[4] The talks at the meeting were recorded and are available on the YouTube channel of Springer: https://www.youtube.com/playlist?list=PLLe3TzudRmf3SgBAHRJy6nlA-YRfSmQTZ

References

[1] R. Hersh and V. John-Steiner, A visit to Hungarian mathematics. *Math. Intelligencer* **15**, 13–26 (1993)

[2] E. R. Lorch, Szeged in 1934. *Amer. Math. Monthly* **100**, 219–230 (1993)

[3] L. Rodríguez, Frigyes Riesz between the two world wars. *BSHM Bull.* **32**, 235–245 (2017)

[4] B. Sz.-Nagy, C. Foias, H. Bercovici and L. Kérchy, *Harmonic analysis of operators on Hilbert space*. 2nd ed., Universitext, Springer, New York (2010)

[5] P. Weibel, *Beyond art: A third culture: A comparative study in cultures art and science in 20th century Austria and Hungary*. Springer, https://doi.org/10.1007/3-211-37846-4 (2005)

Lajos Molnár is the editor-in-chief of the journal Acta Szeged since 2018. Besides, he is a professor of mathematics at the University of Szeged. His research areas are in functional analysis and operator theory, especially in so-called preserver problems.

Interview with Caroline Series

Ulf Persson

Caroline Series and Ulf Persson met as graduate students at Harvard in the autumn of 1972 and have kept in intermittent but sustained contact since. Here, UP interviews CS to find out more about her life and career.

UP: When did you first discover mathematics?

CS: I must have been twelve or thirteen, when we started Euclidean Geometry. One day we were given an especially hard geometry problem for homework. That evening I struggled with it for hours and eventually solved it. The next day it turned out I was the only one to have done it, and I was asked to go to the blackboard and explain. The process of working on it and then the rush of satisfaction at finding the solution made a deep impression on me. At that moment I resolved to solve every maths problem we were given. Not long afterwards, I made it my secret ambition to go to Cambridge and get a first class degree in mathematics.

UP: So you went on to Oxbridge, can you tell me about it?

CS: I applied to both Oxford and Cambridge (you can't do this now) and took the exams in the autumn of 1968, when I was seventeen. Actually I always intended to go to Cambridge – we lived in Oxford where my father was an academic, a physicist, and I wanted to go away. As it turned out, I did very well in the Oxford exams and my interview with the Somerville mathematics tutor Anne Cobbe made a deep impression on me. Somerville (one of the women's colleges at Oxford, colleges were all single sex at that time) offered me their top scholarship, which definitely settled the matter.

I discovered by chance that having won a scholarship I did not need to do any A-levels (the British school leaving exams), so I could skip the last two terms of high school. I left school at a couple of weeks notice and had nine months of freedom. I spend three months as an au pair girl in Heidelberg learning rudimentary German, and after that I worked with a group of young women scanning bubble chamber pictures for some Oxford physicists.

Photo credit: Michelle Tennison

UP: So you grew up in Oxford and your father was a member of the University. Then for you to have gone up to Oxford (or Cambridge for that matter) would not have been such a big deal as it would have been for most students.

CS: Of course that is true. But I had very little to do with the University as a child, and as a student I lived in Somerville College, so it was a new stage of life. I studied very hard and made sure to do all the exercises we were given. The course was tough.

UP: What other interests did you have besides mathematics? And what about social life?

CS: I didn't really have time for anything extra-curricular. As for social life, I met my boyfriend Robin shortly before starting at University. He had been to a boys' school in Oxford and was much more advanced than me in mathematics. He helped me a lot and we spent most of our spare time together. At the end of my second year we got married.

UP: How was life at Oxford? Whom did you meet, who were your teachers?

CS: In the British system, university courses are quite specialised and so I only studied mathematics. At Oxford there are large university lectures for everyone, and then you have tutorials with only a few other students. This is where you get you main contact and feedback. I was fortunate to have wonderful tutors, especially Anne Cobbe whom I have already mentioned. Unfortunately, she developed cancer during my first year. She became too ill to do any teaching, and then sadly died. Because of this, I was taught by some other excellent mathematicians, among them Brian Davies, Peter Neumann and Graeme Segal.

UP: What mathematics interested you?

CS: At least in the first two years there was not much latitude for personal choice, but when I had the chance I chose pure mathematics. In the first year I loved semi-philosophical things like the uncountability of the reals, and once I got started I liked analysis. There was very little classical geometry, which I missed, and I do not recall any algebraic geometry. Algebraic topology was very formal, but I did like functional analysis.

UP: How did you come to apply to Harvard for graduate school?

CS: That is a story. While I had always planned to study mathematics as an undergraduate, I had really no plans beyond that. My vision of the future was very vague and high school teaching was my only concrete idea.

UP: Still your father was an academic, at Oxford to boot, the idea of going on to graduate school must have been very natural to you.

CS: Not at all. That we should do well academically was tremendously important to my parents but I don't think they really envisioned an academic career for me. My father probably thought I would mainly be a housewife like my mother.

UP: So what made you take the step?

CS: It was my husband Robin who was being encouraged to go to graduate school. He had a reputation as a brilliant student and his tutors assumed he would continue to a PhD. Michael Atiyah, who was connected with his college, suggested he might go to Harvard, to work with Raoul Bott. That meant I would have to go to the States also, but it didn't make much sense to train as a school teacher there. I thought the only way I could go to the US would be to do a PhD too.

Robin had applied for a scholarship from Harvard so I couldn't very well do so as well. I applied for all the other scholarships I could find, but only to go to Harvard, MIT, or Princeton. The information sent by Princeton seemed to me very elitist, I thought they would never want anyone like me there, and the material from MIT was sent by sea-mail, so by the time I got it, it was much too late. As it turned out, I got a very nice scholarship, a Kennedy scholarship, to study for a year at Harvard.

UP: What was your first impression of Harvard?

CS: That our graduate class in mathematics was so small! I had imagined everything in the States would be huge, I was very ill informed. Also that the professors were so informal and wore jeans. That would have been unthinkable at Oxford.

UP: And the mathematics, did you feel lost at first?

CS: Yes indeed. Both Robin and I took the qualifying exam the first thing we did and passed right away. After your qualifying exam, as you know, you are on your own. No more course requirements and you don't really know what you are supposed to do, we weren't given any advice. In the first vacation, we were assigned J. P. Serre's famous article *Faisceaux Algébriques Cohérents* to read. We both struggled with it and Robin began to get depressed as up till then everything had come so easily to him. I just persevered as best I could.

UP: We hear about Robin, what happened to him?

CS: He gradually lost his interest in mathematics and eventually he dropped out of Harvard. Things between us became very strained: I remained at Harvard but decided to try to get a Masters degree in Statistics, which I thought would be an easier way to make my living. I actually nearly renounced my scholarship but at the last minute my father convinced me to pull back. After a while I settled down and began to work seriously on mathematics again. Robin and I got divorced and we went our separate ways. He did eventually go back to Harvard and finish a thesis, but that was long after we had split up.

UP: So now you had free sailing?

CS: But I needed to settle on a topic and an advisor. At Harvard at the time, almost all the professors seemed to be doing algebraic geometry or number theory. I was very ill prepared for either, and felt I wanted to do analysis. This narrowed down the choice of an advisor to Loomis, Gleason and Mackey.

UP: Loomis was about to retire …

CS: … and for some reason I was terrified of Gleason …

UP: … and that left George Mackey.

CS: Right. Despite people having told me that Mackey was famous for his low opinion of women mathematicians, he was very gracious when I approached him and said he would take me on a trial basis. Nothing more was ever said: later I realised that he must have looked up my records to check me out.

UP: And so it worked out.

CS: It worked very well and got me started on independent work in mathematics. Mackey was a good advisor. Group representations, Mackey's subject, seemed to me very interesting. He was writing a book on the many aspects of the subject, drafts of which he gave me to read. I liked it because it gave a broad overview of the subject without getting bogged down in details.

UP: Did he give you a problem to work on?

CS: No, he told me he preferred his students to come up with their own problems. But anything I came up with seemed to have already been done. However Mackey was really helpful when it came to suggesting papers to read, so in that way I learned a great deal.

UP: So this was the first time you learned mathematics outside a formal course?

CS: Yes, more or less. In fact I concentrated mainly on dynamical systems, which Mackey was interested in at the time. He had some original ideas about groups acting on measure spaces and the associated unitary representations. However I was getting worried towards the end of my third year that I still didn't have a proper project to work on, while all the other students seems to know what they were doing.

UP: So did you worry about not writing a thesis?

CS: Yes of course, a great deal. However, in 1975, the year before I got my PhD, Harvard gave me a small grant to go to a summer school in Kingston, Ontario at which Alain Connes was the main speaker – that was before he won the Fields medal. By chance I had just the right background to understand at least part of his lectures, and I realised that I was following just as well as most other people in the audience. It gave me the self-confidence and determination to come back and finish my thesis, which I did. It was on a problem Mackey had suggested concerning his theory of 'virtual groups', closely related to what is known as orbit equivalence. I spent from September to March working and writing up my thesis. You had to apply for positions for the following year the previous autumn, so I had to get something on paper quickly. I recall that Mackey rejected the first draft of my thesis because he said it was much too vague. The second time around I started out with a long array of definitions and Mackey did not like that either, he said it was boring. He was quite right. Luckily on the third go he was satisfied.

UP: So now you were about to enter the third stage of your career, but before turning to that let us dwell a little bit more on the social side of graduate school. Were there any fellow graduate students, specifically other students of Mackey, whom you engaged with?

CS: Not really. I had very little contact with his student Bob Zimmer, who was a few years ahead of me, though I did study his thesis. In fact it was one of the BPs (Benjamin Pierce Fellows, or assistant lecturers) Troels Jørgensen who first showed an interest in what I was doing, for which I was very grateful. He also told me about his work, which subsequently became one of the inspirations for Thurston's theory of hyperbolic 3-manifolds. Although I didn't understand much at the time, later I found it was closely connected to my own work.

Some of the main social interactions I had were with other women graduate students. There weren't many of us, at most one in each year. That is how I met Linda Ness, like you a student of Mumford. Then for two years I shared an apartment with Terry Myers whom you certainly remember, she was a graduate student at Boston University and the wife of another Mumford student, Jerry Myers, who had graduated and had a job in Albany in up-state New York. And of course, for a year your then wife Mindy shared with us also, while you had your first job at Columbia in New York.

UP: Indeed, how could I forget that. You must also have met Ragni Piene, you both got your PhD in the spring of 1976.

CS: Yes, I did meet Ragni at about that time, although she was at MIT so I didn't see her often. I think it was through her that I met Dusa McDuff, at that time a lecturer in York but visiting Princeton. Both of them have remained lifelong friends.

UP: Let us talk about the third stage in your mathematical career. The post-doc stage. How did that play out?

CS: I applied for lots of jobs. My first choice was to go to Berkeley, and to my great joy I was offered a two-year lectureship. But I was also offered a Research Fellowship in Newnham College, Cambridge. I felt I couldn't give that up, as I knew I wanted eventually to return to the UK. Newnham kindly allowed me to postpone for a year, and I spent a year at Berkeley which I extended as long as possible by not returning to England until the autumn of 1977.

UP: Why did you want to go to Berkeley? What was it like?

Berkeley campus 1977. C. Series, private collection

CS: What interested me was the group there in dynamical systems. It was a comparatively new subject, and there were great people there – Calvin Moore (another Mackey student), Jack Feldman and Rufus Bowen, not to mention Don Ornstein and his group of ergodic theorists in Stanford.

UP: Dynamical system has origins both in topology and in hard analysis.

CS: I would rather say measure theory and probability than hard analysis. Ergodic theory is about measurable transformations on a measure space, and that is what my thesis was about. Dynamics in terms of topology goes back to Poincaré, while the probabilistic approach was mainly developed by the Russian school led by Kolmogorov. The two strands were just coming together in the 1960s and 70s.

UP: You wrote a paper with Bowen, how did that come about?

CS: After some time working on the abstract parts of ergodic theory, I began to look for more concrete examples and went back to the beautiful geometry and dynamics of the geodesic flow on a surface of constant negative curvature, which had been studied in the 1930s by Hedlund and Hopf. I wanted to reconcile their geometrical ideas with Bowen's more abstract method of constructing what are known as Markov partitions for Anosov flows. Markov partitions had also been developed by Yakov Sinai in the Soviet Union, and ideas of the Russian school were brought to Berkeley by Sinai's student Marina Ratner.

I discussed these ideas with Rufus and we agreed to start a collaboration which turned out to be crucial to my subsequent career. We settled on a joint project to try to make the geometrical approach from the 1930s work for all hyperbolic surfaces. Although by that time I was in Cambridge, I visited in Berkeley again in summer 1978. Not long after I had arrived, I had an idea which I believed should solve our problem. It was the weekend, but I was very excited and called Rufus at home. A voice I didn't recognise answered and when I asked to speak to Rufus the answer was 'I am very sorry, he is dead'. As you can imagine, this came like a thunderbolt. Rufus was only thirty-one, very healthy and athletic, how could this be?

After the initial period of shock and grief, I pulled myself together and checked that my idea did indeed work. I did the only thing I felt I could in the circumstances, and wrote up the solution as a joint paper.

Rufus' influence was also posthumous. Dennis Sullivan was very interested in what Rufus had been doing, he was making connections between dynamics and hyperbolic geometry and hence to the ideas of William Thurston which were just beginning to emerge at that time. So he naturally got in contact with me to see what we had been doing. Not long afterwards Dennis invited me to IHES and his ideas and lectures had a huge influence on me.

UP: So what did this lead to?

CS: The geometrical coding I had discovered turned out to be related to many other things, for example the word problem in the fundamental group of a surface, and to what are now called automatic groups, so it became an important tool. I wrote quite a few papers on various aspects and applications.

UP: By the way, you mentioned Cambridge as part of your post-doc years, but you did not collaborate with anyone at Cambridge?

CS: Cambridge didn't really work out for me. I felt quite isolated, as hardly anyone there was at all interested in dynamics and ergodic theory. One person I did talk to was S. J. (Paddy) Patterson. We realised our work had approached the same problem from different angles. I learnt a lot from him and we have remained in contact ever since.

By chance I met Dusa McDuff again, who by this time was a lecturer at Warwick, about to leave for a position in the US. She told me that there were several positions coming up in Warwick. At that time Warwick was the only place in the UK with a substantial group in dynamical systems, led by Bill Parry. I thought they wouldn't want any more dynamicists, but Dusa encouraged me to apply.

UP: And you did and secured one of the jobs. You were in Warwick in the fall of 1979, I recall it very well, I went to England for a short visit that December, just back from the States.

CS: Yes, that's right. You do have a remarkable memory for dates, in fact I started at Warwick in autumn 1978. So in the end I only spent one year in Cambridge.

UP: And you would stay on at Warwick for the rest of your career.

CS: That is true, with some longer breaks for visits of course. I was very happy there.

UP: Tell me about some of your other collaborations.

CS: I had a long and fruitful collaboration with Joan Birman at Columbia, she was known for her work on braids and the mapping class group. She approached me with a problem about simple curves on surfaces and I was able to contribute my expertise. Subsequently, I had a long collaboration with Linda Keen at City University in New York. Linda had been a student of Lipman Bers and was an expert on Teichmüller theory. We decided to try to understand some of the new ideas of Thurston on 3-dimensional hyperbolic geometry which we could approach from different angles. We settled on a problem which involved interpreting the intriguing computer pictures produced by David Mumford and David Wright, at that time a graduate student at Harvard. We spent the best part of a year in fruitless attempts before we began to understand what was going on. It involved going into 3-dimensions and using some of Thurston's wonderful ideas. That led to the discovery of what we called pleating rays, which allow one to understand families of Kleinian groups.

UP: You wrote a book called *Indra's Pearls* with David Wright and David Mumford, when was that?

CS: It came out in 2002, but it took the best part of ten years to write. As I mentioned earlier, David M. and David W. had embarked on a project to explore computer images of the effect of iterating a pair of Möbius maps in the complex plane. They plotted the places where the orbit of a point under such an iteration accumulated, the so-called limit set of the group generated by the two transformations. These were the pictures that Linda and I had wanted to understand – especially what happens as you vary the group.

The pictures the two Davids created were so spectacular that they wanted to write a coffee-table book about them, similar to books which had appeared on fractals and the Mandelbrot set. The book idea wasn't making much progress. Because of my interest in the pictures, they invited me to join them. In fact Mumford's idea was really more ambitious, he wanted to explain the pictures so that anyone with a good background in high school maths could understand what they were about. He had already drafted the beginning of the book, and Wright had pictures and some text for the last part. So as Mumford said when we got together, all we had to do was to fill in the middle. Easier said than done! It turned out to be a much longer-term project than any of us had bargained for, it spanned over a decade.

David Mumford awarded an honorary degree at Warwick, 1983.
Miles Reid on left. C. Series, private collection

UP: So how did the collaboration proceed?

CS: We met up periodically in one or other of our home universities. David M. had lots of ideas and David W. was brilliant at making computer pictures. I did a lot of writing, which I have always loved. The main difficulty was finding a way to express what we wanted to say in terms accessible to the intended audience. This meant getting rid of jargon and using as little notation as possible. If you think about it, so much notation is superfluous, for example, mathematicians often write something like 'Take a group G', and then they never actually use the notation G. It is not only laymen who get put off by formulae: even mathematicians can find them daunting and appreciate getting to the basic ideas in simple prose. So it was a very good exercise for me to to write mathematics in this way.

UP: It was not a research project.

CS: Definitely not. This doesn't mean we didn't present new results, but these were mainly experimental and we didn't feel constrained to make anything formal. I learned a lot from writing it, besides getting a much deeper appreciation of the mathematics involved. In the end the book was a lot more ambitious than initially envisioned, but we did get a very good response from all sorts of people, both amateurs and professionals. The beautiful pictures were part of it too, they appealed to many people even if they couldn't follow the maths. We called the book *Indra's Pearls* because the fractal pictures found remarkable analogies with an ancient Buddhist text, and this also sparked a lot of interest.

UP: So what came next after the book was published, did you continue doing mathematical research?

CS: I certainly didn't stop doing mathematics, with a variety of collaborators, but I also got engaged in other aspects of the mathematical community, which I have found very satisfying.

UP: Can you tell me some more about this?

CS: In the early 2000s, I organised a big programme on 3-dimensional hyperbolic geometry at the Isaac Newton Institute in Cambridge. Several major breakthroughs had just been made which completed most of Thurston's programme, so the timing was perfect. Then I took my turn at organising a year long symposium at Warwick, although this time I got several other people to run events on somewhat broader themes.

Around the same time I became involved in various national committees, for example of the London Mathematical Society and the Newton Institute. I enjoyed such work, even if it took time away from my research. I met many different people and it gave me insights into how the world works beyond mathematics.

UP: And you became President of the London Mathematical Society (LMS), which was very important to you. How did that come about?

CS: In 2016, just after I had retired, I was elected a Fellow of the Royal Society (FRS) and shortly afterwards I was asked if I would take on being the LMS President. The term is two years, in my case from November 2017 to November 2019. I was the 80th president but only the third woman to hold the post. I really enjoyed the experience and the opportunities it opened up. My involvement

Conference at Warwick 2011. C. Series, private collection

As LMS President, ICM Rio 2018. Left to right: Caucher Birkar, John Hunton, Caroline, Michael Atiyah, June Barrow-Green. Photo credit: LMS

with organisations for women in mathematics had given me some experience with leadership, but this was on a very different scale.

UP: It sounds like a full-time job.

CS: I did spend a great deal of time on it. There was always so much to be done: organising and chairing meetings, discussing and making decisions, initiating new projects, and occasions and travel representing the Society (including leading the LMS delegation to the ICM in Rio in 2018), so different from the regular life of a mathematician. The LMS has a wonderful staff. I got along very well with the Executive Secretary Fiona Nixon which made things much easier for me. One of the first things I did was to get videoconferencing equipment installed so that people didn't always need to come to meetings in person. At the time, although it is only a few years ago, this was something quite new.

UP: I believe Atiyah died on your watch.

CS: Yes, he died in January 2019. He was such a towering figure in British mathematics, I felt that the LMS should make an announcement without delay. Even though I might not have been the best qualified to do it, I spent the weekend writing a short obituary which went onto our official web-page. Later, we set up a big conference and a fellowship in his memory.

UP: So now let us come to the issue of women and mathematics. Has being a minority troubled you? I think that most men would welcome more women in mathematics. Would you actually want to actively promote more women in mathematics?

CS: Of course I am always pleased when a woman wants to study mathematics, I feel I have met a kindred spirit. On the other hand, I don't think there is any point in pushing unless the person is really interested. Indeed this applies to men also. However, there are still many obstacles for women wishing to become mathematicians.

UP: Do you think that women are mistreated in mathematics?

CS: Not any more, although in the past there were some shocking stories. It wasn't so very long ago that some people were wary of appointing a woman. The received wisdom was that she would marry and drop out. I was lucky that by the time I came along most institutions were really rather keen to hire women. When I arrived at Berkeley I felt completely accepted and I had a wonderful time. Cambridge (UK) was very different. Social life in the department was very much male dominated, and in addition I had the impression that no one expected a woman to be a serious mathematician. It seemed I didn't fit in. But as soon as I got to Warwick I felt immediately at home, even though I was for a long time the only woman. My opinions were taken seriously and it was a happy and supportive environment in which to pursue my life and research.

UP: Let us talk about your concrete activities for women and mathematics. How did they start?

CS: My first serious involvement started at the ICM in Berkeley 1986, where I was invited to sit on a panel organised by the Association for Women in Mathematics (AWM). The AWM was largely focused on the USA, and the five Europeans on the panel were inspired to organise something similar in Europe. That's how European Women in Mathematics (EWM) came about. It was an entirely bottom-up organisation. I was closely involved in setting up the basic structures; we set out to have a gathering every year and in 1988 I organised the third meeting in Warwick. It was run on a shoe-string, although we did have some limited funding from the LMS and a few other sources.

I worked extremely hard to make the Warwick meeting happen, from designing a more detailed 'constitution' for EWM, to all the planning of the event, inviting speakers, and so on. Those were the days before email, remember, so everything had to be done with snail-mail. To save money I even arranged for people to stay in the homes of colleagues who were away on vacation. There was a small group of female students helping, but I took on far too much, and had it not been for a very helpful and supportive departmental secretary I don't know how we would have managed.

I am proud that EWM continued its annual meetings, and indeed is still going strong, with a new generation of women behind it.

UP: So what did you get out of these meetings?

CS: The main thing for me was getting to know a whole group of women mathematicians, and the feeling of solidarity which we shared. Being in a mathematical setting with a roomful of other women was a new experience for all of us. I think that is what people valued most – some were much more isolated than I was. I also believe that EWM helped to initiate some quite profound changes, for example we were able to support women in Germany and Switzerland where things were extremely difficult; over time there have been tremendous improvements at the institutional level.

UP: More recently you have been involved in the IMU's Committee for Women in Mathematics (CWM). How did that happen?

CS: When Ingrid Daubechies was President of the IMU (she was its first female president), she had the idea of creating a section of the IMU website as a resource for women mathematicians, with information about all the initiatives and activities internationally. I was asked to be part of a small group gathering up the information, and then I got drawn into organising the material and designing the site. Then, together with Marie-Françoise Roy, one of the other founders of EWM, I approached Ingrid with a proposal that the IMU should create a specific committee for women mathematicians. With Ingrid's support, this is how CWM came about. Backed by generous funding from the IMU, we were able to support and encourage women to form groups in other parts of the world, particularly developing countries. I was the first Vice-Chair with Marie-Françoise as Chair; we worked together very well and I really enjoyed this. It also led me to further involvement with the IMU, for example getting involved with the makeover of the IMU website, which I found very interesting.

UP: Finally, let us turn to issues of so called human interest. What do you do when you are not doing mathematics?

CS: Now that I am retired I spend a lot of time working in my garden, growing herbs and so on, and I like cooking from scratch, which can take a lot of time. I do quite a bit of sewing – altering and what is now called repurposing. I also read a lot, fiction and non-fiction, my reading is very eclectic.

UP: I believe you are also concerned about the environment?

CS: Yes, this is something I care about deeply. I find it completely shocking that our society has been so slow to take effective action. In fact one of the things I did which I am most proud of, is that in 2000 I persuaded Warwick University to set up an Environment Committee to look into things like energy and water conservation, recycling, greener transport and so on on the campus, which is the size of a small town. It was a great struggle to get anything concrete done, but gradually the group became official and better resourced. Now environmental concerns are a major focus of the University.

On personal level, I try to live with as little waste of resources like food, energy, and possessions as is practically possible. It hurts me seeing how our world is steadily being degraded. The cause is both our unbridled consumer society, and also the ever increasing global population which barely seems to get discussed.

UP: The mantra you always hear is about more growth no matter what.

CS: Indeed, although many people are waking up to realise just how serious the situation is, none of us want to give up too much, and besides it is very hard to do this living in the society that we do. The mantra ought to be that we all should be doing everything within our power to alleviate the problems, after all, $\sum 1/n$ diverges.

UP: I can see this is another issue which could occupy us, but I think it is time to draw to a close. We have touched on many subjects and it has been fascinating to catch up with you again after so many years. Thank you for agreeing to do this interview.

Ulf Persson is a professor emeritus at Chalmers Tekniska Högskola (Göteborg, Sweden) and an editor of the EMS Magazine.

ulfp@chalmers.se

Research Networks for Women

Kristine Bauer, Erin Chambers, Brenda Johnson, Kristin Lauter and Kathryn Leonard

Research Networks for Women is a recent effort led by the Association for Women in Mathematics to increase gender diversity in specific research areas of mathematics through a grassroots effort to build mentoring and collaborative relationships. In this article, we describe the history and success of this initiative, and share information about how to grow such a network.

1 Introduction

The gender gap is well studied in mathematics, and is particularly drastic in senior academic and research-focused positions. For example, a recent analysis of the percentage of publishing authors in mathematics journals shows that only about 27 % of authors are women; this drops to under 10 % when considering the most highly ranked journals.[1] Similarly, when considering editorial boards of math journals, women comprise only 8.9 % of all such positions [5]. While the exact percentage of women in faculty positions varies across countries, it seems clear there is a persistent problem. As a result, many initiatives have been set up to increase recruitment and retention, generally by enforcing quotas or attempting to reduce bias in the review process for grants or publications [6].

This article describes a recent initiative from the Association of Women in Mathematics, which instead proposes focused research collaboration workshops for women. Instead of events for women which only focus on networking and work-life balance panels, these events bring research to the forefront. The philosophy of the Research Networks program is to promote the creation of new communities of women supporting each other in research to change the ecosystem. Our model for making change is based on the observation that the math research community itself is a network with its own hierarchical structure. To integrate with that, we have created Research Networks for women in a scalable way which can help advance women's careers and then connect back to the existing hierarchical structure where women are minoritized and have less power.

[1] https://gendergapinscience.files.wordpress.com/2020/07/gendergapinscience_diapos.pdf

Several key elements of the structure that enable success:
- Women help each other in research through vertical integration – senior women mentoring junior women on research problems and in their careers.
- Groups of women collaborators help to promote their joint work in the broader community through giving research talks and co-organizing conferences with many women speakers.
- Junior women are empowered to co-organize conferences with men and invite their women colleagues, and to participate in the editorial and reviewing process at an earlier career stage.
- Junior and senior women participate in program committees to actively promote the inclusion of women speakers in major conferences and on editorial boards.

While Research Networks for Women represent a drastic change from the standard goal of making all workshops more diverse (which generally means about 20 % women participants), we propose it as an effective way to build community for minoritized groups and to bootstrap progress on the larger diversity goal. Of course, not all approaches work equally well for all groups, but it is worth considering such efforts on a broader scale, with possible modification, if they are successful with one group. Preliminary data on these Research Networks shows that they are extremely effective, as the number of women in top research universities in these research areas has drastically increased and overall visibility of women in research areas linked to networks has expanded. Indeed, many women cite their experiences with these workshops as a reason they have persisted and felt supported in their careers.

2 History

The Research Networks for Women idea started with a conversation between Kristin Lauter, Rachel Pries, and Renate Scheidler at a number theory conference in 2006. Noting the lack of women invited speakers at number theory conferences worldwide, they decided to organize a research conference for women in number theory. To build community and to support graduate students entering the "leaky pipeline", they decided to design it as a collaboration conference, a Research Collaboration Conference for

Figure 1. Women in Numbers (WIN) founders, Renate Scheidler, Rachel Pries, Kristin Lauter, at WIN 2008

Women (RCCW), with group leaders proposing research problems ahead of time, and open to students and junior researchers to apply to participate. Former Fields Institute director Barbara Keyfitz supported the idea and suggested applying to Banff International Research Station (BIRS) through the usual scientific process, which succeeded. The first Women In Numbers (WIN) conference was hosted by BIRS in 2008, and a volume of research papers written by the groups was published in the Fields Institute Series [3].

BIRS is a conference center in the Canadian Rocky Mountains which runs 50 week-long conferences per year, where 42 mathematicians are invited to spend five days presenting and listening to math research talks. Historically more than 90 percent of the participants were men. The WIN model was different because the participants were all women, students were encouraged to apply, research problems and groups were proposed and decided in advance, and almost all of the conference time was devoted to group work on the research problems. The WIN groups continued to work together after the conference to finish their results for publication in the volume.

During and after the 2008 WIN conference, the organizers and many of the participants worked together to form the WIN network: a research community for Women in Number Theory. Michelle Manes created an email distribution list and organized follow-up special sessions at AMS meetings; Katherine Stange created a website[2] highlighting women in number theory; the WIN organizers created a Steering Committee to plan future meetings and selected new organizers for the next conference at each successive WIN conference.

Since then, the WIN Network has run seven more conferences on this model, four at BIRS and three in Europe to broaden participation. These conferences involved more than 200 women in number theory from around the world, were organized by more than 20 different women, many of them participants in the first WIN conference, and produced more than 50 published research papers in eight proceedings volumes. When planning started for the first WIN conference in 2006, there were three women professors in number theory at top research universities in the US; now there are several dozen women faculty in number theory, most of whom participated in WIN. The success of the WIN collaboration model was palpable from the very first conference: the BIRS staff said they had never experienced such energy and excitement from workshop participants, although they had been running week-long workshops 50 weeks per year for many years.

Early adopters pushed the WIN model into other areas of mathematics. Maria Basterra, Kristine Bauer, Kathryn Hess, and Brenda Johnson launched WIT: Women in Topology at BIRS (WIT workshops were held in 2013, 2016, 2019). Kathryn Leonard and Luminita Vese launched the Women in Shape Modeling (WiSh) network at IPAM in 2013, securing funding from private sources and companies to launch it (WiSh workshops were held in 2013, 2015, 2017, and 2021). In 2012, Kristin Lauter collaborated with the then director of the Institute for Mathematics and its Applications (IMA), Fadil Santosa, to plan and fund one conference on the WIN model per year, for three years. The result was three RCCWs hosted by the IMA: WhAM! Women in Applied Math, Dynamical Systems in Biology (2013), WINASC: Women in Numerical Analysis and Scientific Computing (2014), and WinCompTop: Women in Computational Topology (2016).

Based on the preliminary success of the WIN model in number theory and these other areas, AWM received a 5-year National Science Foundation Advance grant (2015–2020) to spread the model to all areas of mathematics (grant oversight committee: AWM Presidents Ruth Charney, Kristin Lauter, and Kathryn Leonard, and AWM executive director Magnhild Lien). The goal of the program, "Career Advancement for Women through Research-focused Networks", was to build and sustain Research Networks for women in many areas of mathematics. The program has supported more than 2,000 women researchers in 25 Research Networks so far, with more new networks in formation. In addition to BIRS, RCCWs are being hosted annually by Institute for Computational and Experimental Research in Mathematics (ICERM), Institute for Pure and Applied Mathematics (IPAM), IMA, and American Institute of Mathematics (AIM). Several Research Networks have run conferences at institutes in Europe, including Luminy (CIRM), Lorentz Center, Nesin Village, Henri Lebesgue Institute, Hausdorff Center for Mathematics (HCM), University of Leeds, and Universität Trier.

To support the Research Networks, AWM runs follow-up workshops at the annual JMM and SIAM meetings. Each workshop showcases the work of one of the RNs, and serves to reunite the participants to continue collaboration and mentoring. A significant portion of the funds from the AWM ADVANCE grant were devoted to participant expenses to speak at these annual workshops and at the special sessions for Research Networks at the biennial AWM Research Symposia.

[2] https://womeninnumbertheory.org/

Figure 2. Women in Numbers–Europe2 Workshop (WINE2) Lorentz Center, Leiden, Netherlands, 2016

The AWM Research Symposia started with the AWM 40th anniversary celebration at Brown University and ICERM in 2011, co-organized by Georgia Benkart, Kristin Lauter, and Jill Pipher when Jill was president of AWM and founding director of ICERM. The Symposia are two-day weekend meetings, run on the model of the AMS sectional conferences, with high-profile Plenary Speakers and Special Sessions organized by the Research Networks. The Symposia aim to bring women mathematicians together to recognize and celebrate their research contributions and achievements, and to network and build community in order to advance their careers and improve working conditions. Professional development activities include a non-academic jobs panel, an exhibit hall, and networking opportunities. The Symposia were held every two years until 2021, when it was was delayed to 2022.

More information about the Research Networks program is available on the AWM website[3], which hosts webpages and email listservs for each Research Network and provides a framework and global information about the program.

3 RCCW and RN Committees

It is important to emphasize that a research network is grown by a group of women in that research area, so these are not external efforts. AWM's role in this has been to encourage groups to form, by pointing out opportunities and providing logistical support and advice. Currently, AWM has two committees that help in the formation and growth of the Research Networks. The first is the Research Collaboration Conferences for Women Committee (RCCW Committee), chaired for the first three years by Michelle Manes and now by Erin Chambers. The goal of this committee is to help research networks form in new areas by accepting proposals twice a year. The chair assigns each proposal to a committee member, who helps edit and match with one of the math institutes to submit through their competitive processes. The second is the Research Network Committee (RN Committee), which was chaired initially by Sigal Gottlieb, then Kathryn Leonard, and now by Kristin Lauter. This committee helps existing Research Networks grow and continue after the initial workshop.

Both committees advise the Research Networks to form some structure to ensure continuity. In particular, networks are encouraged to consider the following actions:

- form their own Steering Committee;
- appoint a web administrator to create and maintain a webpage;
- create a listserv to facilitate communication;
- organize follow-up events such as Special Sessions and AWM Workshops;
- publish a Proceedings volume in the AWM-Springer Series;
- publish accounts of their networks in the AWM Newsletter.

In addition, the RCCW and RN Committees have helped to balance the number of submissions for RCCWs at any given math institute, as well as to brainstorm new venues to be considered. The RN Committee has developed a set of materials to help new networks decide on structures and processes, which they call "How to Launch a Research Network".[4] Both committees have helped the networks strategize on how to organize a workshop that best fits their area, and continue to discuss the benefits of the workshops in terms of growing a stronger community of active women researchers.

Figure 3. Women in Computational Topology Workshop, Institute for Mathematics and its Applications, 2016
© Institute for Mathematics and its Applications, 2016

[3] https://awm-math.org/programs/advance-research-communities/research-networks/

[4] https://awm-math.org/wp-content/uploads/2020/11/AwmAdvance_howtorn038.pdf

4 Launching a Research Network – an example

If you've read this far, you may be starting to wonder "how can I start a research network for women in my community?" In addition to making use of the guidance provided by the AWM through its document "How to Launch a Research Network" and many services to support networks, it can be instructive to consider specific examples as well. In what follows, we describe the formation of the Women in Topology network.

New networks often start with a small group of individuals looking to make a change in the makeup or culture of their research community. For Women in Numbers, that group of individuals, Kristin Lauter, Rachel Pries, and Renate Scheidler, were concerned about the dearth of invited women speakers at conferences. Women in Topology started with a conversation between Maria Basterra, Kristine Bauer, Kathryn Hess, and Brenda Johnson at a BIRS workshop in 2011. By that time, the first WIN workshop had taken place and planning for the second one was underway. Kristine Bauer had heard about WIN from her colleague Renate Scheidler and was wondering if such a program would serve to address issues in the field of algebraic topology, especially the lack of senior women in the field at research universities, and a leaky pipeline in the transition from graduate student to permanent academic position. These conversations were taking place when a particularly strong cohort of women researchers was completing PhDs and starting postdoctoral positions in algebraic topology. After initial conversations at the BIRS workshop and the AWM's first research symposium the following fall, a proposal was submitted for a workshop at BIRS, using the WIN collaborative research conferences as a model. The proposal was accepted, and planning began in earnest for the first WIT workshop in 2013.

When forming a new Research Network for women, especially in a field in which women have traditionally been underrepresented, it can be a challenge to find a critical mass of participants. For a collaborative research conference, one needs to find both team leaders and team members.

Given that the field of algebraic topology had few senior women at research universities in 2012, recruiting team leaders involved recruiting more junior faculty members, some of whom had little or no experience in directing research projects, or were just entering tenure-track positions. For these leaders, the experiment of participating in WIT carried some risks and required rapid development of certain skills. A benefit of this approach was to provide training and experience for future PhD advisors, and to expose participants to leaders from a wide range of institutions. Team members were recruited by first contacting PhD advisors in the field to solicit recommendations, and then sending out invitations to recommended individuals. This process also served to inform leaders in the community about the WIT workshop and enlist their support. One sign of success for the WIT network is that the number of women in the field has grown to the point that leaders and participants now must apply for the limited number of spots available in collaborative research conferences, and the network has explored other models for meaningful research events to involve more participants.

The organizers of the first WIT workshop chose to publish a proceedings volume and encouraged all teams to submit a paper for the volume. Although this put more pressure on team leaders, it also helped to focus teams on completing concrete projects, and served a number of other purposes, which we address in the next section. The proceedings for the first workshop were published in the American Mathematical Society's *Contemporary Mathematics* series [1].

Going into the first workshop, no one knew what to expect. But testimonials posted after the workshop on the BIRS website enthusiastically describe the intensely productive and cooperative atmosphere that prevailed:[5]

> This was the most productive workshop I've attended in my career. We went from a few half-baked ideas to enough material for a research paper over the span of four (albeit rather long) days. I now have new collaborators with whom I'm eager to work in the future. I expect this week to pay dividends for a long time.

> I have no hesitation to say that the WIT workshop was the best and most successful and productive mathematical event that I have ever attended, and here is why: – One of the workshop's goals was to forge collaborations between women in topology, and it has been accomplished. In my case (and I believe many others too), the collaboration will be continuing well beyond the scope of the workshop.

The WIT network grew out of this first workshop. A second collaborative research conference (WIT2) was held at BIRS in 2016, again organized by Maria Basterra, Kristine Bauer, Kathryn Hess, and Brenda Johnson, with proceedings published as a special issue of *Topology and its Applications* [2]. This team also organized a shorter workshop at Mathematical Sciences Research Institute (MSRI) in 2017 that accommodated many more participants, highlighted the work of early career researchers, and was open to all genders, though the focus was on the work of women. The original organizers stepped down after the 2017 workshop, but continue to serve on the WIT steering committee, which is composed of current and past organizers. Subsequent activities have been planned by a number of WIT conference alumnae. A third collaborative research conference was held in 2019 at the HCM in Bonn, and planning is underway for another collaborative research conference at HCM in 2023.

[5] www.birs.ca/events/2013/5-day-workshops/13w5145/testimonials

Figure 4. Women in Topology Workshop hosted at Banff International Research Station, 2013
© Banff International Research Station, 2013

5 Publication

The importance of publishing a volume of research articles based on a RCCW cannot be overstated. Setting a concrete goal together with a deadline motivates research groups to continue working together after a workshop and to complete at least one paper based on their collaboration. The AWM Springer Series was launched in 2014 to create a home for the proceedings volumes of research articles produced by the collaboration groups at the RCCWs. The Founding Editor of the series is Kristin Lauter. The series also publishes the proceedings of the AWM Research Symposia, AWM Workshops, and other AWM, events such as the 50th Anniversary volume. A list of the 30 volumes published to date is available on the series webpage[6]. RCCWs have also published proceedings as special issues of journals and in other conference proceedings series, such as the *Contemporary Mathematics* series produced by the American Mathematical Society. These proceedings volumes provide a record of the research done by the groups, demonstrating the serious nature of the conferences. They also benefit the participants and RCCWs in many other significant ways.

Besides being able to add another paper to their CV, early-career participants gain valuable experience in the process of publishing and advertising research results. At early career stages, such opportunities for professional development can help women to persist in the research mathematics profession. In addition, group leaders often write recommendation letters for graduate students or postdoctoral group members based on their contributions to the paper. In cases where collaboration groups develop a longer-term research program, publishing a proceedings paper on preliminary results guarantees that everyone in the group gets credit for their contributions, before those group members who have time and interest form spin-off collaborations to develop further results.

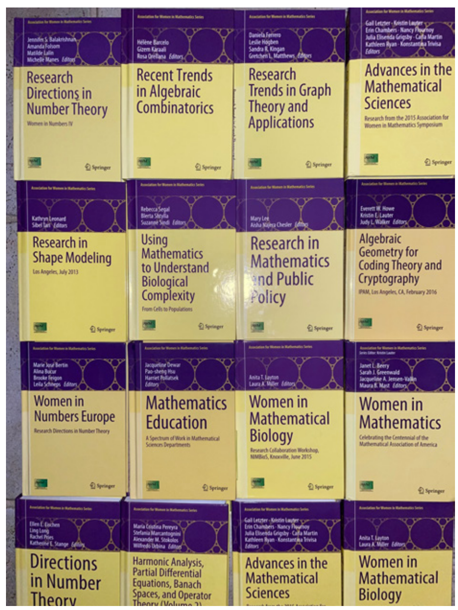


Figure 5. AWM-Springer Series volumes

[6] www.springer.com/series/13764

While there is an argument to be made for publishing results in journals rather than conference proceedings, the relatively short turn-around time for proceedings volumes can help to ensure that papers are appearing in time to be of use in the job search process.

The conference proceedings also benefit other participants and the RCCW movement as a whole. Many women organizers, editors, and reviewers gain valuable experience in the publication process through their work to produce these volumes. Editing the volume and reviewing the papers becomes a community-building experience, one that invites participants to join the publication process at an earlier career stage than is typical. Having artifacts such as volumes of published research papers shows the work done by women in the area and provides evidence to argue for the importance of funding subsequent conferences for women, which in turn supports further development and organization of the Research Networks. Finally, it is worth noting that for some volumes, the editors have invited contributions from women researchers who were not at the conference, so the volumes serve to highlight work by women in the research area more broadly.

6 Assessment

By many measures, the Research Network program has been a tremendous success. Funding from the NSF ADVANCE grant was allocated for an assessment team led by Erin Leahey, PhD, who surveyed RCCW and follow-on special session participants, and also performed a final impact survey to determine longer range effects of participants in the programs. During the funded period, AWM supported 46 research collaboration conferences, three research symposia, and 15 special sessions linked to AWM workshops at large meetings. Several other affiliated special sessions were held during the same time period, but they were not evaluated.

Participant surveys were hosted on SurveyMonkey and distributed with a link mailed to RCCW or special session participants shortly after the event. Response rates for all surveys exceeded 80 % and were often above 90 %. Responses to surveys were uniformly positive, with more than 90 % of respondents agreeing with the following statements about RCCWs:

- The group collaboration functioned well
- The project was exciting to work on
- The project has promise
- I expect to continue with my collaboration group
- I made connections outside of our collaboration group
- The workshops met or exceeded my expectations
- I would recommend the conference to a friend
- I would attend the conference again, either as a member or as a project leader.

Responses to the special session survey were also quite positive, with more than 90 % of participants agreeing with the following statements.

- I feel more confident working on a team.
- I feel more confident doing research in the mathematical sciences.
- I feel more confident about my professional opportunities.
- I feel more confident about my networking opportunities.

While most of this article focuses on the academic community, these workshops also affect industry, as the following comment shows:

> My Ph.D. was in pure algebraic geometry and I began a career in data science right after I finished graduate school. Exiting through the academic turnstile, I found making this switch to be disorienting and even a bit demoralizing. […] Even though I was no longer a practicing algebraic geometer, my desire to work at an edge of human understanding didn't vanish when I accepted a job at a tech company. In this age of gratuitous information exchange, I had most of what I needed for research at my fingertips except for the most essential piece – a research community! This workshop was exactly the bridge that I needed; it was the missing piece!! […] My engagement with the WiSDM group has added more focus and fervor to my data science work, even though compressed sensing doesn't obviously impact my company. Some of the different problem spaces I was exposed to at the workshop have even given me useful frameworks for thinking about seemingly unrelated problems at my company! A postscript that touches on the "W" in WiSDM: In my entire life, this was my very first exclusively female professional experience. I have always shied away from "women in" conferences and events because I just thought it was silly. Who cares if this is a male dominated field. Let's just all get along and move forward. We all have the same passion. Boy, did I ever learn some things about myself at WiSDM! I have never felt such a sense of ownership and responsibility in my entire life. It was super cool. While reviewing my past has caused some chagrin and frustration, going forward I am determined to own my ideas, push them forward, deliver value, and expect some credit. Thank you so much for providing me with this experience.
>
> *Women in the Science of Data and Mathematics (WiSDM), July 2017*

Following the end of the ADVANCE grant funding, Leahey and her team began a final impact study, sending surveys to all participants in RCCWs during the funded period. 302 women responded. Responses to the final impact survey were also strongly positive. Almost all respondents thought that participation in their research network had helped them professionally by improving the quantity and quality of their research (73 %), raising visibility (75 %), or increasing their access to opportunities ranging from speaking invitations to job opportunities. More than 90 % of respondents agreed

Figure 6. From Left to Right: Christina Osborne, Brenda Johnson, Kristine Bauer, Emily Riehl, Amelia Tebbe form Team BJORT at WIT II, 2016

that participation in a research network helped grow their professional network. Moreover, comments in response to an open-ended question show the effect on the mathematical community:

- I have seen many more conferences with a balanced number of women speakers.
- [My area] has seen a tremendous growth in women researchers.
- The number of keynotes delivered by women is notably higher than 6 years back.
- There are a lot of powerful women in [my area] because of [the research network].
- I have noticed a lot more [younger] women pop up in my field largely through work they do with more established women who then promote their younger collaborators.
- [In my area] there are far more papers published and more talks given by female speakers.
- There are still conferences with all-male speakers or program committees, but now the [research network] steering committee can point this out and direct organizers to the network for women speakers.

7 Testimonies

The Research Networks for Women are at their core a chain of grass-roots organizations which enhance both the communities who establish the networks and the broader research community. Perhaps the most profound measure of the success of these networks is the passionate way in which individual participants, with near uniformity, report the impact that participation in Research Networks has had on their own careers. Many of these testimonies come from researchers who participated at sensitive stages of their career, and were emboldened by their experience.

Amazing experience! Most of my time was with my group, and we progressed much more than we expected to. We have plans to continue the research and have discussed further projects as well. We really enjoyed each other, and I feel like I have new collaborators and mentors as I begin an Assistant Professorship – the timing couldn’t be better. Thank you for the wonderful experience.

Women in Math Biology (WIMB), June 2019

Overall, I really enjoyed this collaboration. My group worked well together and I am excited for this collaboration to continue. I am a new tenure-track faculty member at an institution with a small math department. It is especially important for me to find outside collaboration networks to help with publications for tenure and this workshop helped me to do so.

Women in Graph Theory and Applications (WIGA), August 2019

This workshop gave me an amazing opportunity, which previously (especially as a grad student) felt like an impossible dream. Being able to work with leaders in the field was incredibly uplifting and motivating. Furthermore, the programming, the schedule and atmosphere were one of a kind and reminded me of why and how much I enjoy mathematics.

Women in Symplectic and Contact Geometry and Topology (WiSCon), July 2019

… I made a lot of new connections with researchers that I expect to continue in the future, besides just future collaborations. One of my group members has told me that she would like me to come speak at her university sometime.

Women in Operator Algebras (WOA), November 2018

I am a graduate student, and this was the first workshop or conference that I felt was completely worth the time – I learned so much, made so many connections, and am excited to continue work on our project.

Women in Shape Modeling (WiSh), June 2021

WIN had a profound impact on my career. Before attending the first WIN workshop I was contemplating leaving the profession. I had started a great tenure-track job at Oregon State University two years earlier, but found that making the transition from graduate student to postdoc to tenure-track assistant professor was throwing me for a loop … the research network forged at WIN gave me the confidence to make research a crucial and nurtured part of my university experience. I often confess that I believe WIN saved my career. My WIN mentor, Dr. Ling Long, became a long-term collaborator, mentor, and friend. I worked with her on projects at WIN2 and WIN3 as well and we have coauthored seven papers. My work with WIN continued with remotely co-leading a project at WIN4 weeks after the birth of my child, and leading a (virtual) project at WIN5 that we just submitted. WIN has been a huge part of my research life and my growth as a researcher.

Holly Swisher, Professor, Oregon State University (WIN)

Many participants in research networks indicate that the benefits of belonging to a research network are not limited to advancing one's career goals, but rather that the workshops are personally gratifying in immeasurable ways. Some of these intangible benefits include being part of a community, a feeling of self-assuredness, personal growth, and resilience. Perhaps for this reason, many researchers return to participate in workshops in their networks again and again and are now able to report on their extensive experience.

I've participated in each of the first three editions of Women in Topology, and one of the unexpected rewards has been that it's given me a way to measure my growth as a mathematician. For instance, during the presentations of research projects at the most recent workshop, I was suddenly struck by the realization that I had a grasp of the context for all them: that is, I understand why one might want to pursue each of these questions. In contrast, chatting with some of my junior team members showed that many of these research directions were new to them, just as they had been for me during the first workshop in 2013. All of a sudden, I felt that I was en route to becoming a senior member of the topology community. It was an affirming and strengthening realization.

Anna Marie Bohmann, Assistant Professor, Vanderbilt University (WIT)

I am very grateful to have been involved for the past several years with three of the AWM research networks as either a workshop participant (WiSh 1, WinCompTop 1) or a co-organizer (WiSDM 1& 2, WinCompTop 2). In particular, thanks to the WiSh and WinCompTop workshops, I have continued to publish multiple papers with my team members as well as established new and lasting collaborations with others in the network. These research networks have been transformative for my career as they propelled me forward into new and diverse subfields of mathematics and computer science and enabled me to make exciting and meaningful connections with researchers at all levels.

Ellen Gasparovic, Associate Professor, Union College (Schenectady, NY, USA)

Together with Ayelet Lindenstrauss I was a team leader in three of the "Women in Topology" workshops and this had a huge influence on my career. Of course the closest bond is with the team members and with Ayelet. We had a follow up to the first WIT-meeting where we applied successfully for an AIM SQuaRE project. In total I share seven publications with Ayelet by now. I was deeply impressed with the mathematical abilities of some of the group members. But more generally I am more aware of the women whom I met in the WIT-programs. We meet each other now and then and I know what they work on. So I will for instance think of them when I'm asked to suggest speakers for seminars or conferences.

Birgit Richter, Hamburg University (WIT)

I am happy that the WIG 2 workshop gave me the opportunity to meet researchers from the Americas, specially from my home country, Mexico. Over the week that the workshop lasted, I was able to participate actively in a research project which involves both new and known topics of my own research. This interaction with colleagues whose research areas partially overlap mine has broadened my view on the connections between different areas of mathematics. I am glad to say that after a week of work at the CMO our team obtained its first results, and we have now created a plan to consolidate this effort in a paper and in future collaborations.

Ingrid Membrillo Solis, University of Southampton (WIG 2)

The first time I heard about WIN-E I was skeptical "a conference only for women?", but I gave it a try, and I just loved it. I had never felt myself that comfortable in a math conference and I had not realized before how much pressure I was feeling while attending regular conferences. I met incredible collaborators, colleagues, mentors, women. At that time I was going through a difficult moment finishing my thesis, and thanks to them and their support I found the motivation and the energy to pursue a career in academia. When I was given the opportunity to organize WIN-E3 I could just say "yes!". I was very happy about being able to give back to other women what I had received from this amazing community. Thanks to all of you, together we can make it!

Elisa Lorenzo García, participant in WIN-E1 and WIN-E2 and organizer and group co-leader of WIN-E3

The impact of the Research Networks is community-wide – indeed, the entire culture of an RCCW is different than most other research experiences. The schedule for most workshops is largely unstructured, dedicating the majority of the workshop's time to discussion and collaboration in small groups rather than prescribing a calendar of lectures for attendees. However unstructured the activity may appear from examining the schedule, it is in fact quite directed, with each team and each participant working towards a very specific workshop outcome. The fact that this works is completely due to the participants – participants show unusually high levels of energy and enthusiasm, dedication to their teams, and commitment to the completion of the project. Indeed, for many participants participation in a research network helped to relieve pandemic blues ("Zoom meetings to work on follow-up WIT projects were the highlight of my weeks during the pandemic lockdowns!" Anna Marie Bohmann, Vanderbilt University). The Women in Operator Algebras II team displayed heroic dedication to their task in order to hold a hybrid event, perhaps the first such event:

> The workshop [Women in Operator Algebras II] had forty-two participants from 15 countries (Australia, Canada, China, Czech Republic, France, Germany, India, Italy, Korea, Denmark, Netherlands, New Zealand, Norway, USA, UK). Most of the researchers participated online, while seven researchers from the USA and Canada were on location. The hybrid format worked well in facilitating collaborative work, with all participants actively involved in zoom or in person in discussions and exchange of ideas. [4]

Although hybrid formats are challenging, the organizers noted that the dedication of participants made the workshop successful.

> We had groups with people in 3–4 different time zones and they did their best to adapt, getting up very early or working way past their bedtime, to make sure their research group produced great work. It was a testament to the commitment of these women to their research work, their collaborators, and the research network in general.
>
> *Maria Grazia Viola, Lakehead University*

Figure 7. Women in Numbers group, Banff International Research Station, 2017

Figure 8. Group work at WinCompTop, IMA 2016

8 Continuing efforts

AWM continues to support both existing and new Research Networks, as well as associated events such as Special Sessions and the AWM Research Symposia. The Research Networks have also spawned several other related programs, some modeled on RCCWs and others blossoming out of them.

BIRS and AWM have launched a follow-on program to RCCWs, where small groups that have participated in an RCCW are funded to attend BIRS for up to two weeks of continued research collaboration.

Another exciting new program, launched by MSRI deputy director Hélène Barcelo to build on and support the success of the Research Networks program, is a Summer Research in Mathematics program for women, SRiM[7]. MSRI will fund follow-up collaboration meetings for groups of 4–6 researchers to visit MSRI in the summer to continue research on a project started at one of the RCCWs.

In summer 2022, the IMA will host the Roots of Unity Workshop[8] for graduate students in years 1–3 of their graduate programs, replicating the RCCW model, but with participants reading key research papers instead of working on research problems. The organizers, Christine Berkesch, Michelle Manes, Priyam Patel, Candice Price, Adriana Salerno, and Bianca Viray, have almost all been active members of Research Networks.

The Institut des Hautes Études Scientifiques (IHES) highlighted AWM at its 2021 fundraising gala and dedicated proceeds from that event to funding RCCW-like events at IHES. AWM President Kathryn Leonard was a member of a panel hosted by IHES discussing challenges facing women in math and discussing how Research Networks help address those challenges.

Finally, AWM has launched its first research journal, La Matematica, which publishes work in a broad range of mathematical areas

[7] www.msri.org/web/msri/scientific/summer-research-in-mathematics
[8] www.ima.umn.edu/2021-2022/SW6.12–16.22

and is dedicated to unbiased and constructive review practices. The founding Editors-in-Chief, Donatella Danielli, Kathryn Leonard, Michelle Manes, and Ami Radunskaya, have all participated in RCCWs and played organizational roles in their Research Networks. La Matematica's editorial board is more than 75 % women, compared to an average of 9 % across math journals [5] in the US, and more than 50 % from other minoritized groups. A substantial proportion of the editorial board has participated in RCCWs.

References

[1] M. Basterra, K. Bauer, K. Hess and B. Johnson, *Women in topology: collaborations in homotopy theory*. Contemp. Math. 641, American Mathematical Society, Providence (2015)

[2] M. Basterra, K. Bauer, K. Hess and B. Johnson, Virtual special issue – Women in topology II: Further collaborations in homotopy theory [Editorial]. *Topology Appl.* **235**, A1–A2 (2018)

[3] A.-C. Cojocaru, K. Lauter, R. Pries and R. Scheidler, *WIN – Women in numbers – Research directions in number theory*. Fields Inst. Commun. 60, American Mathematical Society, Providence (2011)

[4] S. Reznicoff and M. G. Viola, Women in operator algebras II, Final report. Technical report, BIRS, Banff, AB Canada, www.birs.ca/workshops/2021/21w5199/report21w5199.pdf (2021)

[5] C. M. Topaz and S. Sen, Gender representation on journal editorial boards in the mathematical sciences. *PLOS ONE* **11**, e0161357 (2016)

[6] Implicit bias in academia: A challenge to the meritocratic principle and to women's careers – and what to do about it. www.leru.org/publications/implicit-bias-in-academia-a-challenge-to-the-meritocratic-principle-and-to-womens-careers-and-what-to-do-about-it (2018)

Kristine Bauer is an associate professor at the University of Calgary. She earned her PhD at the University of Illinois in 2001. In 2011, she went for an animated walk in Banff, AB Canada with Maria Basterra, Brenda Johnson, and Kathryn Hess. Two years later, this group founded the Women in Topology (WIT) network. Dr. Bauer is an ELATES fellow, a co-director of the Pacific Institute for the Mathematical Sciences, and a mother of two.

bauerk@ucalgary.ca

Erin Wolf Chambers is a professor and department chair at Saint Louis University in computer science, with a secondary appointment in mathematics. She is active in organizing for both the Women in Computational Topology and the Women in Shape Analysis research networks, as well as serving on committees to support AWM's broader research networks initiatives. She currently serves as editor for several journals, on the board of trustees for the Society for Computational Geometry, and on the SafeToC organizing committee.

erin.chambers@slu.edu

Brenda Johnson is a professor of mathematics at Union College in Schenectady, New York. She completed her graduate work at Brown University, and has held visiting positions at Northwestern University and Johns Hopkins University. Her research interests are in algebraic topology, particularly homotopy theory. Throughout her career she has been concerned about the retention of historically underrepresented groups in STEM disciplines, and has sought ways to make her classes, research community, and institution more welcoming and supportive of all who are interested in such fields.

johnsonb@union.edu

Kristin Lauter is director of West Coast Labs for Meta AI Research, Seattle WA, USA. She co-founded WIN in 2008 and became president of the Association for Women in Mathematics in 2015 with the mission to spread and support Research Networks for Women in mathematics. She is an affiliate professor of mathematics at the University of Washington; elected fellow of the AMS, SIAM, AAAS, and AWM; and an honorary member of the Royal Spanish Mathematical Society.

klauter@fb.com

Kathryn Leonard is a professor of computer science at Occidental College. She received her PhD in mathematics in 2004 from Brown University. She founded two research networks, Women in Shape Modeling (WiSh) in 2013 and Women in the Science of Data and Mathematics (WiSDM) in 2017. She is currently the president of the Association for Women in Mathematics, director of the Center for Undergraduate Research in Mathematics, and co-editor-in-chief of the research journal La Matematica.

kleonard.ci@gmail.com

Increasing investment in mathematics in changing times

Masato Wakayama and Ivan Fesenko

We reflect on our experience in Japan and the UK in raising mathematics funding, from the public and private sectors, and establishing new mathematics institutes. We hope that some of this information and relevant examples will be of use for mathematicians in other countries.

1

Mathematics in a broad sense includes pure mathematics, statistics and applied mathematics and their interaction with computer science. It is a common language for sciences. Galileo wrote, "Mathematics is the language with which God has written the universe." The Nobel Prize winner E. Wigner talked about "the unreasonable effectiveness of mathematics in the natural sciences." Mathematics is an open innovation that transcends differences in nationality, language, culture, and generation. Such modern areas as cryptographic services, cybersecurity, internet data transmission, computational modelling, machine learning, deep learning, artificial intelligence, quantum computing, financial markets, banking system, insurance system, defence and security are impossible without mathematics.

It is crucial to appreciate that without fundamental mathematics further substantial achievements in modelling, quantum computing, AI, ML, etc., are not possible. Just one example: using fundamental mathematical insights, an interdisciplinary group, including pure mathematicians, led by A. Zhigljavsky (Cardiff University) produced a new type of epidemic modelling with longer term forecasts for the decision makers (links to three related papers are available at the bottom of this International Mathematical Union webpage[1]). Media articles emphasised the increasing role of mathematics in epidemic modelling.[2]

With the disruptive innovations of the current industrial revolution, the need for mathematical knowledge and abilities is increasing. Mathematics helps to develop and stimulate breakthrough innovation in many key fields of the industrial revolution. To be successful in the latter requires such abilities as taking a bird's-eye view on a problem, finding solutions in an integrated way, and generalising, all of which are provided by mathematical mindfulness. CEOs and politicians in several countries have stated that only those countries that invest now more in mathematics can continue to be prosperous and successful. Giant IT companies are proactively recruiting excellent mathematicians whose duties can be compared to those of university researchers. In comparison with many other areas, in mathematics one can move very quickly from investment to a productive phase once the latter is expected. Without increasing the mathematical mindfulness of the population, inevitable problems with employment in sectors where AI replaces humans may have very drastic consequences.

Serious problems affecting mathematicians these days are well known. The shortsighted race to a higher number of publications and a higher citation index often results in pressure to produce short-term work that essentially brings only minor improvements to known results. Some mathematicians are losing the enthusiasm and passion for long-term research and adopt the most pragmatic attitude concerning what and when to study in mathematics. They are forced to specialise narrowly, which leads to an emphasis on technical perfection as opposed to innovation, and on presentation rather than substance of work. Specialisation in a small area and lack of knowledge of even adjacent areas are becoming more and more typical. Following this path eventually makes it more arduous to think in broader terms, to learn new areas or concepts, to study new groundbreaking theories, to develop in new directions. Associated issues are lack of inventiveness, fear to look too far away or think non-linearly, more widely spread imitation, fear to stand alone in scientific endeavour, and consequently increasing dependence on other people's opinions without studying on one's own. Following this path by many directly harms the future of mathematics. These trends affect young researchers even stronger. New mathematical institutions and centres will apply efforts to address these problems. The proposed creation of an EMS Youth Academy may be useful. For another initiative reported to the EMS, see this talk[3]. In Japan,

[1] https://www.mathunion.org/corona

[2] https://www.nzz.ch/wirtschaft/wenn-mathematik-menschenleben-rettet-und-milliardenkosten-spart-ld.1552780?reduced=true

[3] https://euro-math-soc.eu/system/files/uploads/6.%20Fesenko_young_0.pdf

through programs such as CREST and the activities of IMI (see the next section), an increasing number of the younger mathematicians are trying to take a broader interest and perspective.

Most researchers in other sciences, including experimental quantum physics and quantum computing, are often not aware of developments in mathematics in the recent decades. At the same time, researchers in other sciences complain about the inability of some pure mathematicians to explain their work or just their novel ideas to them. Much needs to and can be improved in relation to the increasing inability of mathematicians to explain their research even to wider groups of mathematicians (see, e.g., this message[4] of the president of the EMS). Despite so many changes related to new forms of accessing information and communication, mathematics is taught in almost the same way as fifty and more years ago (even though the recent two years have brought some changes). The task of modernising the ways and forms of passing the mathematical knowledge to the diverse range of young people is complex and huge. By increasing funding of mathematics and at the same time improving the ways to teach mathematics, carry out research, and administer grant distribution, countries have the opportunity to stimulate and support long-term impact developments.

2

Mathematics is the foundation of plenty of advanced technologies driving modern society. There are several Japan Science and Technology Agency math programs[5] such as CREST, PRESTO, ACT-X; these are separate from the Japan Society for the Promotion of Science (JSPS) grants. The CREST[6] and PRESTO programs have a long history that begun in 1995 in other fields; the first CREST mathematical sciences programs only started in October 2007.

To meet the global demand in many scientific and technical fields in terms of research personnel skilful in mathematics, M. Wakayama worked for years on establishing the Institute of Mathematics for Industry (IMI)[7] at Kyushu University. It was founded in April 2011, becoming the first institute in Asia for industrial mathematics. Two years later, IMI was authorised as the Joint Usage/Research Center by the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan, thus becoming a national centre.

IMI was established

> by amalgamating and reorganizing pure and applied mathematics into a flexible and versatile form, with a view to serving as the foundation for developing future technologies. Problems that emerge from the requirements of industrial sectors are, though their targets are clear, in many cases yet to be formulated mathematically. History tells us that many useful technologies originate in mathematical theories being created by flexible and free ideas, without any intention of application. While helping to solve problems requiring short-term solutions and to develop technologies currently in demand in cooperation with industry and other fields, the IMI actively promotes basic and fundamental research in mathematics that will serve as seeds for future innovative, difficult-to-foresee technologies. […] Furthermore, it is one of the important missions of the IMI to use these outstanding research activities for education.[8]

In 2018 METI (the Ministry of Economy, Trade and Industry) initiated and together with MEXT hosted an Industry-Academia Round Table Discussion on the role of mathematics and science[9]. A summary of the report "The Coming Era of Mathematical Capitalism – How the Power of Mathematics Changes Our Future" includes the following:

> We believe that we have identified the top three science priorities in order for Japan to lead the fourth industrial revolution and to even go beyond its limits: mathematics, mathematics, and mathematics!
>
> It is not too much to say that "winners in the field of mathematics are winners in the fourth industrial revolution." Furthermore, as the end of the Moore's law era is coming into sight, businesses have been advancing a race of the development of computing technologies using new principles, such as quantum computers. In this trend, human resources who may manipulate such technologies should be those who have higher knowledge of mathematics and are able to make full use of such knowledge.
>
> What management or social system is suitable to the era of mathematical capitalism, a time in which industries foster human resources expertized in the fields of science and mathematics to bring out their higher ability of mathematics and connect it to innovation? Such a management or social system is expected to be a novel one, different from conventional systems. Despite this expectation, all countries and companies are still working to find specific solutions to this question. If any country or company uncovers an optimal solution to the question ahead of other countries and companies, it will be the winner in the era of mathematical capitalism.[10]

[4] https://euromathsoc.org/magazine/articles/49
[5] https://www.jst.go.jp/kisoken/en/index.html
[6] https://www.jst.go.jp/kisoken/crest/en/index.html
[7] https://www.imi.kyushu-u.ac.jp/eng/
[8] https://www.imi.kyushu-u.ac.jp/eng/pages/about.html?active=message
[9] https://www.meti.go.jp/english/press/2019/0326_004.html
[10] https://www.meti.go.jp/english/press/2019/pdf/0326_004a.pdf

Therein “mathematics” refers to mathematics in a broad sense, as above, and also including quantum theory and other fields that substantially use mathematical expressions, and “mathematical capitalism” is understood in the sense of mathematics becoming and being a crucial source of national wealth.

M. Wakayama also worked on the creation of a new NTT Institute for Fundamental Mathematics[11] in the Nippon Telegraph and Telephone Corporation. This institute opened in October 2021. About 80 % of the research effort of its members will be fundamental mathematics research. It will

> explore diverse and wide-ranging issues in modern mathematics and promote the search for mathematical truth through the development of the necessary language and concepts.

For example, the institute

> will accelerate research geared towards innovations in quantum technology that surpass the capabilities of digital technology, including clarifying the origins of the superior power of quantum computing, which are not yet clearly understood, and devising new cryptosystems that are guaranteed to be unbreakable even by quantum computers.

Mathematics was included in the Cabinet Office’s Moonshot Program[12] by emphasising its transversal/horizontal role. This was the outcome of work by M. Kotani (executive vice president of Tohoku University, next International Science Council president) when she served as a member of the Council for Science, Technology and Innovation, Cabinet Office.

3

For many years, UK mathematics had been hugely underfunded. One of the more recent reports is available at this webpage[13]. The issues with funding of UK mathematics had been reported at various levels, but that had not led to any substantial improvement.

Sharing and discussing some general articles about recent achievements, such as this text on the work of S. Mochizuki[14] and this article on the work of C. Birkar[15], with decision makers proved to be successful in attracting their attention and support to the needs of mathematics.

In August 2019 I. Fesenko was asked by D. Cummings, at that time the senior advisor to the Prime Minister, to “assemble a group of mathematicians … coordinate the effort … produce the best possible roadmap for funding mathematics” in the UK. Various observations by Cummings about mathematics and mathematicians can be found in his blog[16]. Here are three extracts:

> A visit to the classic Bell Labs of its heyday would reveal many things. One of the simplest was a sign posted randomly around: “Either do something very useful, or very beautiful”. Funders today won’t fund the second at all, and are afraid to fund at the risk level needed for the first. […]
> A […] risk aversion is present in the science funding process. Many scientists are forced to specify years in advance what they intend to do, and spend their time continually applying for very short, small grants. However, it is the unexpected, the failures and the accidental, which are the inevitable cost and source of fruit in the scientific pursuit. It takes time, it takes long-term thinking, it takes flexibility.

> To get things changed, scientists need mechanisms a) to agree priorities in order to focus their actions on b) roadmaps with specifics. Generalised whining never works. […]
> Scientists also need to be prepared to put their heads above the parapet and face controversy. Many comments amounted to ‘why don’t politicians do the obviously rational thing without me having to take a risk of being embroiled in media horrors’. Sorry guys but this is not how it works.

> Courses such as Politics, Philosophy and Economics […] do not train political leaders well. They encourage superficial bluffing, misplaced confidence (e.g. many graduates leave with little or no idea about fundamental issues concerning mathematical models of the economy […], and they do not train people to make decisions in complex organisations. […] Universities need new inter-disciplinary courses. […] It would be great if Oxford created alternatives to PPE such as *‘Ancient and Modern History, Maths for Presidents, and Coding’*.

A draft of I. Fesenko’s proposal was considered at the first meeting of a group of mathematicians with officials at 10 Downing Street in August 2019, for the final version of his proposal see this page[17]. Two further meetings at No. 10 discussed aspects of new additional funding and problems affecting effective support of mathematics research and produced executive summaries.

---

[11] https://group.ntt/en/newsrelease/2021/10/01/211001a.html
[12] https://www8.cao.go.jp/cstp/english/moonshot/top.html
[13] https://www.eu-maths-in.eu/wp-content/uploads/2018/05/EraOfMathematicsReport.pdf
[14] https://ivanfesenko.org/wp-content/uploads/2021/10/rpp.pdf
[15] https://www.thetimes.co.uk/magazine/the-times-magazine/caucher-birkar-from-asylum-seeker-to-fields-medal-winner-at-cambridge-xrz5t7ktj
[16] https://dominiccummings.com/
[17] https://ivanfesenko.org/wp-content/uploads/2021/10/mathsproposal-2.pdf

Members of the group included (not a complete list): J. Norris (University of Cambridge), D. van Dyk (Imperial College London), J. Greenlees (University of Warwick), I. Gordon (University of Edinburgh), J. Keating (Heilbronn Institute for Mathematics Research and University of Oxford).

At the end of January 2020, a new additional funding of 300 million GBP for the next 5 years was announced by the UK government[18]. Its volume increases the previous research math funding approximately 2.5 times. The new funding aims to support more PhD students and postdocs, to provide more grant funding, and to enlarge funding of the Isaac Newton Institute for Mathematical Sciences and the International Centre for Mathematical Sciences. The first 1/3 of the funding became available from the 2020/2021 academic year.

Masato Wakayama specialises in representation theory, number theory and mathematical physics. Recently, he has been working on the mathematical structure of quantum interactions such as asymmetric quantum Rabi models in relation to the Riemann hypothesis and problems in arithmetic geometry. His and N. Kurokawa's zeta function workshops in Okinawa, held since 2000, are famous worldwide. He is head and fundamental mathematics research principal of the NTT Institute for Fundamental Mathematics; principal fellow of Center for Research and Development Strategy of Japan Science and Technology Agency; chair of Mathematical Sciences Subcommittee of Moonshot Research and Development Program[19], Cabinet Office of Japan; scientific advisor & senior advisor of RIKEN Interdisciplinary Theoretical and Mathematical Sciences Program[20]; professor emeritus of Kyushu University. His previous positions include professor and distinguished professor in mathematics, dean of the Graduate School of Mathematics of Kyushu University, the first director of the Institute of Mathematics for Industry, executive vice president of Kyushu University, and vice president and professor of Tokyo University of Science. He was visiting fellow at Princeton University, visiting professor in the University of Bologna and the distinguished lecturer at Indiana University.

wakayama@imi.kyushu-u.ac.jp

Ivan Fesenko is a visiting professor at Research Institute for Mathematical Sciences, Kyoto University, and currently holds a part-time professorial position at the University of Warwick. Contributor to class field theory, higher class field theory, higher adelic theory, higher harmonic analysis, zeta functions and zeta integrals of arithmetic schemes, and interaction of modern number theory with various areas. Principal investigator of the Engineering and Physical Sciences Research Council programme grant "Symmetries and Correspondences"[21]. Contributor to the recent paper on effective abc inequalities and a new proof of Fermat's Last Theorem[22]. Supervisor and host of sixty PhD students and postdoctoral researchers. Co-organiser of over 40 workshops, conferences and symposia. His research visits include stays at Poincaré Institute, Newton Institute, Hebrew University, RIMS, Max Planck Institute for Mathematics, Caltech, Columbia University, University of Chicago, University of Toronto, Bogomolov's laboratory at the Higher School of Economics, the Institute for Advanced Study. His previous positions include chair in pure mathematics at the University of Nottingham.

ivan.fesenko@warwick.ac.uk

[18] https://www.gov.uk/government/news/boost-for-uk-science-with-unlimited-visa-offer-to-worlds-brightest-and-best

[19] https://www.jst.go.jp/moonshot/en/about.html

[20] https://ithems.riken.jp/en/members/#scientific-advisors

[21] https://ivanfesenko.org/wp-content/uploads/2022/02/scpage.pdf

[22] https://projecteuclid.org/journals/kodai-mathematical-journal/volume-45/issue-2/Explicit-estimates-in-inter-universal-Teichm%c3%bcller-theory/10.2996/kmj45201.short

Some catchy title, e.g., this one

Conversation with Christian Goichon about mediators of science

Vladimir Salnikov

These days we hear more and more discussions about the role of "popularization" of science and various general-audience activities. This article is inspired by several short conversations that I had with Christian Goichon on various occasions. Christian is doing this professionally (at least part-time), but his trajectory is different from the usual "researcher who decided to do some outreach." At some point I thought that telling his story could be useful and inspiring for young scientists and educators, so I told him: "let us sit down and talk". Below is the transcript of that conversation (or me listening to the story).

VS: Once I was giving a talk, a serious one, on mathematical physics, but doing it a bit my style, joking[1] from time to time (which was needed, otherwise after lunch people fall asleep). And then, as a feedback, a colleague of mine says: "Do you think that sometimes comedians after the show gather together and say, ah, I saw you on stage, I think you could be a good scientist?"

CG: I will not exactly answer this question, but I can tell you that I have never learned so much about science and research as since I became a comedian.

VS: Exactly, you kind of went the other way. Tell me this story.

CG: It often happens to me that, after a show where I have presented some totally incredible objects/subjects, people come to me and say, "Monsieur,[2] you could be a wonderful teacher, I would love to go to your classes." And another anecdote: I have a show called "Sea, Sex et Oursins" [sea urchins], that I was doing within the framework of a street festival, so oriented for a general audience. After the show a lady comes to me and says, "Oh, you are an excellent scientist. And this book you wrote, where could I get it?". And the book does not exist, it was part of the show. So, I not only learned some science around what I do, but also some "codes" of scientists, the way they function, the way they address the audience, some "mechanics" of the meeting.

Now I am old [smiles], but when I was young, I did a "Bac Science" [scientific class in a high school], and I did two years of Bachelor at the science department of Poitiers (when La Rochelle did not exist, not the city – the university). Actually, I did my first year twice, and I have a couple of memories from back then. One is a professor (or I do not exactly know his title), always the same jacket, yellowish lecture notes on plant biology, in 78–79, 1978.

VS: Thanks for specifying.

CG: And another one is a professor of mathematics who was also professor of philosophy. It is and was already no longer common. And this professor always started his lecture with two points: first he was asking for a number and was computing its square root immediately. And then he was saying: "If you do not understand what I say it's not your fault, it's mine." And I do not necessarily regret leaving, but in the last maybe 15 years, from time to time I do think back and say, ah, that could have actually worked, maybe in biology. For a simple reason, in my double identity of a comedian and an educator there is an important common point: you need to "go off the beaten path". The same in science – and follow the quote[3] from Einstein "Invention is not the product of logical thought".

When I do my shows, or organize events, or when I coach the participants of MT180[4], I work on this notion (I used to do it with a colleague who is now retired, so now I am alone): *"mise en sens et mise en scène"* [making sense and staging]. And, again, it does not apply exclusively to comedians, but to any person presenting in public. That is work on how you would tell a story, explain your subject, depending on the audience and the space where you will be. How to "play" with these two elements to communicate.

[1] The title of this article is inspired by https://arxiv.org/abs/math/0105080.

[2] The original conversation was in French, we hope that not much was lost in translation.

[3] "Inventer c'est penser à côté", in the usual French translation.

[4] Ma thèse en 180 secondes (https://mt180.fr/) is the French-speaking version of the "Three minute thesis" (https://threeminutethesis.uq.edu.au/), where PhD students present their work to a general audience in three minutes.

And also, what is your way to solicit the audience. I mean, what would you do if at your event there is a visitors' flow, and you are in the process of "popularization", that's probably the term. I don't like the frequently-used "vulgarization", I prefer rather "scientific mediation". And I am often inspired (in a good way) by commercials. I search for intervention strategies (show or event) that attract the audience. What is your opening? Because you have about 20 seconds to catch the audience. And I obviously think about the personality of the presenter, but also about the power of material objects. Sometimes people come because they are simply curious about the object: "oh, what's that thing?" And then the power of mediation is during the discussion to show the person in front of you that they also know something. It is not a typical teaching process, one side giving → one side taking, that's a two-way street. Sometimes researchers adopt the strategy "I know, I will help you understand". That's already not bad. But when it really works, on top of that you will try to extract something from the audience's knowledge. That gives a possibility to share information, and also helps to gain some trust in this relation. So, to sum up, it is not to transfer the knowledge, but to have an exchange (discussion) around the knowledge. When you talk about mathematics, you ask questions, not to judge the knowledge of a person, but rather to gauge and adapt your level of communication.

Christian Goichon as scientific mediator and coach.
Picture credits: Gautier Dufau / Université de Bordeaux.

Then there are also all types of provocation. You see even at CNRS they start talking about shows. After all, MT180 is one. At some point I was making an intervention in the libraries, to show how one can use "supports", in that case books, for presentation. And I am not talking about scientific books, but about fiction. Because there is always a person behind, a scientist, so how starting from a story one can explain science.

When I started meetings like that, I was saying "science", "research", etc., and immediately in the audience people got warned, were raising hands, saying "attention, we do humanities, or letters". I answered: "and I am a specialized educator", and it changes the meeting at once. I did quit the department of science, but that to go to the field of specialized education, to work with autistic people, to do musical psychology …

And then I realized an old dream to go to Quebec.

VS: Within your work?

CG: No, it was a teenage dream, to the point that when I was at school I was thinking of getting a helicopter license to travel through Canadian woods. And then in 95–97 I though it was a right time, so I went on a sabbatical leave from my job (Institut Médico-Éducatif in Niort). And that was one of unbeaten paths, as I said. Long story short, really sketchy, I was accompanied by the French Office for Youth in Quebec. There I organized a musical tournée and also something in the field of psycho-music. I thought of a collaboration with a similar institute there, but that did not advance. At some point I met a guy from the Society for Promotion of Science and Technology (SPST), and I could work with them because I dared to call Patrick Beaudin – the communication director of the youth office encouraged me to, saying "you are both equally crazy". I did, not necessarily to promote myself, but to participate in an event in an interesting fashion. And one of the first projects was on ecomobility, you see, already at that time. And we had an idea of a bus inside which the passengers go on foot.

VS: Ah, funny, I heard such a thing in the news recently, didn't know you invented it long ago. And right, often by ecologists.

CG: Yes, exactly. We had a whole animation around, a show, and I had drawn it way back then. Those ideas circulate in the air, and then there is a project call, and you dare, and then it is financed, and you end up working on it for three years, and one thing leads to another …

So I ended by working for them, on small contracts; in fact out of 7 years, I had at most six months of career breaks, having had left without any precise project. And even half of those six months I was following some educational program (where I was also lucky to be among the 12 selected candidates). After that I had an internship in a communication company. I was supposed to take the job after, but after having traveled a month back to France (for family reasons) I learned that there was no offer available. So back to Quebec to basically start over, and there again great luck. Actually before, during my internship I ran into Patrick B. at a metro station (Quebec, big city, zero probability), and I told him a bit about the company, the coming job offer. And exiting the station he said something like, if it does not work, give me a call. Which I did, and started the day after.

Then after 7 years I decided to come back [to France]. I confess, I regret a bit [laughs]. I actually learned a lot about scientific culture there [in Canada]. I ended up being responsible for programs in the province. I designed programs there for innovators in teaching, researchers in high schools … Created things around improvisation and science. And I am doing all that for about 30 years now.

VS: But did these things exist here [in France] back in the days? Because there is some impression that in North America it somehow developed earlier.

CG: I am not totally convinced about that. Some French people do say that Quebec is very advanced, etc., etc. And I confirm that for, say, natural parks, because it is more in their personality, and there were a lot of exchanges between Quebec and France too. But I am still not sure that it is one direction.

VS: I mean, obviously something existed on the level of individual researchers, since always. But that is like DIY improvised things, little scientific clubs. But the whole generalized "movement" like now, where there is some financial support, even job opportunities, is rather new.

CG: In Quebec in the Universities there are also people, and it is also not that old. For me in France (though I am not a specialist in history) the scientific culture came form "éducation populaire". Now this comes from associations, like here [in La Rochelle] Association for School of Sea[5]. The centers for scientific culture, they are all "associations" in the French legislation. But I still think that it went from schools, from a desire to make something – yes, as you say – out of what you find in pockets. And then it went to the universities, with the "responsables", the services for "culture scientifique", some associated really with culture, some with teaching and research too. Or from doctoral schools: there are places where now they have very good courses on scientific mediation. I think about Dijon (University of Burgundy) with "Experimentarium", or Nancy (University of Lorraine) with a "Sous-direction de la culture scientifique et technique".

VS: There, by the way, I have a question, why La Rochelle? Niort, I understand, like going home. But around there are bigger centers, Poitiers, Tours?

CG: Very simple, that is again a story of meeting people. Like Patrick B., who is clearly a key person in my professional life (we keep in touch even now). He is someone who convinced me that I can live from what I have in mind, from my ideas.

VS: That happens pretty often in the scientific community. It is sometimes enough to meet one person who would just believe in you. In contrast to other professions, where things are more formalized, with CV, interviews, etc. And here just "I hire you, but not because of your publication record, but because I feel you are able to … and I am interested in that".

CG: For me it was in North America, meeting people, learning things.

VS: Maybe tell me more about projects there?

CG: One of the first projects was to promote the literature on tectonic movement. It was a development by the association I told you about: SPST. The association no longer exists, because the head retired. I realized after that I was supposed to take the lead there (if I had not come back), but too late. So, the tectonic project, where we needed an idea, because the topic is less "cute" than the whales and the urchins. And we brainstormed to a "Petition against the movement of continents". I was surprised that it was even possible, and in fact yes: we made t-shirts, posters, invited people. (So to go to MT180 I reproduced that t-shirt).

And then there was "stop the light diffusion", with Hydro-Québec [The Quebec State Electric company] as partner. To explain with a pizza box why your pizza stays warm. That was around 1990.

VS: Right, you normally learn this in school, but not with a pizza box.

CG: Yes. And the petition, some people still remember it. And, back to that idea, I thought if we are allowed to talk about science like this, I definitely go for it.

It was very easy going. With Patrick we lived in different districts, but from time to time we were arriving to work in the same metro train, so walking to the office, like 10 minutes, exchanging ideas. And I explained one improvisation idea. He said, write me half a page about this, the council meeting is in the afternoon. By the evening it was decided. And that is where I got extremely motivated: you trust me blindly like this, so I will do my best to show you that it was a good decision. And I appreciate a lot those people who can just from an idea dare to say: OK, go for it. That was one of the features of this job.

We organized a conference "Science with capital L" to speak about literature and science. It was French speaking, I went to Montreal to invite speakers, to bring to Quebec. So it was really a place of experimenting, of trial. And it permitted me to create things in scientific culture.

VS: But you still went back to France.

[5] E.C.O.L.E de la mer – association La Rochelle

CG: Funny story, I was a bit motivated to make a decision to come back to France by a baker near my place. He said, after seven years you reached the top of a mountain, it is time to choose on which side you go. I went to France with this idea and this image of a mountain in mind, and I still remember the words and the voice of this guy. I went back to start over. And even in the personal relations, friends, when you are away you think that time is not advancing, so you expect to come back to seven years ago, but that all changed in the meantime.

When I came back, I was a bit lost. I even though of contacting my previous employers, but obviously the position was taken. And actually, I was not willing to be back there. I was now positioning myself in "scientific culture".

And after three months I decided to use what I invented: theater to speak about science, how to work on events. And created this company "Les brasseurs d'idées" [literally, "The ideas brewers", but also playing words "The brainstormers"].

VS: So "Les Brasseurs" it was that: "I use theater for something else"

CG: For science. Really to speak about science. All our events for, what, like 20 years now were about science. I ended up in the organization meeting of Fête de La Science [Science Festival] in Poitiers in the Scientific Culture Center (it is about 40 years old – one of the first).

VS: Just to put into context, back in the days La Rochelle University was less than 10 years old. Very young.

CG: And behind was Catherine Benguigui, she liked it, and got interested. And there were also some people from the regional delegation of the ministry for Education and Research. Catherine said: "I want someone like you in La Rochelle". I was a bit noticeable, also with my little Quebec accent. I got some meetings in the ministry, and for them it was new, theater, events … So I ended up a contractor in the University for the Festival of Science and some related projects. And the more it developed, the more I had work to do.

VS: I confess, the first time I heard about you, was in 2017, at the poll for proposals for Festival of Science. The first impression was, and this is the first reaction of colleagues usually, who is this CG? He is in this administration building, he will ask us for paperwork about things that we would do anyway. This changed for me, I understood that no-no-no, he IS the festival of science in La Rochelle, no exaggeration.

CG: Not surprised, actually working like 1/4 of my time here, also back in the days as an external contractor, I was not necessarily very visible. The same by heads of the Laboratories in the University. You come from outside, nobody knows you, you are neither a researcher, nor an official in scientific culture, no specific education, no CV in the usual sense of the word. Just experience of fieldwork in Quebec.

VS: No, the titles and degrees in principle, we do not have to care about. But that is true, at first one positions you like a secretary or something, or administration, but not immediately as a colleague. That is after, while talking to you, one realizes, oh, that is actually him who created all that, it is his project. Like the Street of Science that you made last year.

CG: Right, with some humble pride. All those things I constructed. I had a carte blanche, and I took all that. I coordinate Festival of Science from A to Z. And not only giving ideas, also technical work, information flow.

VS: And I could have said, behind the curtains, but also in front.

CG: Right. At first as a contractor, and since 2017 as a part-time job. Also the thing that made me "known" is the MT180, since 8 years.

VS: Aha, tell me how you got into that. This is not exactly the same principle, not exactly scientific mediation, it is something else.

CG: And for MT180 I don't remember, probably I suggested that to doctoral school.

VS: Because in the French-speaking world it comes from Quebec, but it is a coincidence, right? It is not because you learned it there?

CG: No, right. I didn't show my passport to start [laughs]. And in the world it comes from Australia. I do not remember exactly, I for sure wanted to do it with my "double identity". What I did probably is suggested individual coaching. Because it is done differently in various universities, depending on the means available, time, etc.

VS: Ah, so it is not a general rule? Because I discovered it with Oscar[6], who told me that we would be briefed and coached. I was relieved, otherwise, geometric numerical methods for general audience – good luck …

CG: Some combine, like three hours with a comedian and three hours with a scientific mediator. I am doing it in one package and individually. Because for me it is important to match the personality to the project, it is about human interaction. For instance, a student

[6] Oscar Cosserat is a PhD student of VS and Camille Laurent-Gengoux. He is working in Poisson geometry and got the second prize of MT180 in France in 2022.

who has just defended his thesis sent me an e-mail saying that this work helped him a lot, and this kind of feedback is very interesting. Or now a young researcher, who was in my second year of MT180, so 6 years ago, we are still in touch. She is now in Argentina, and doing a similar thing there.

VS: I understand. This kind of things, you try once and then you apply it everywhere. I like gong-shows for example – whenever I organize some conference or school, I also do a series of short talks by students. And is different from a classical talk, there is some animation.

CG: Exactly. And the same idea for commenting a poster for example. In MT180 what helps, again, is combining the two experiences: mediation and theater, I am like sitting between two chairs. For me when this competition comes, when I talk about it, these are very strong moments. You see how they start from scratch and develop. I remember a lot of them, and I saw like 75 students, so 75 subjects.

VS: So you have a pretty good sampling.

CG: There is one former PhD student who is now in Dijon running the "Experimentarium". His thesis was on implication of PhD students in scientific culture projects. MT180 also helped me to be noticed by, for example, by the directors of the laboratories and senior researchers. They first look at you with suspicion, but then when they see the result: a student who was shy to talk to people, who works in his corner, whom you barely understand when he talks, not because of science, but because of the manner, and who is now on the scene, there they start appreciating.

VS: Ah, here I insist again I started appreciating before. Before I saw the result of this year.

CG: Another thing, especially from this year, I had students of 8 different nationalities, obviously with different culture of oral communication, of human contact, and that is very interesting. You start from all this and see how you can shape it, clearly different for someone from Vietnam or Burkina Faso or Dijon. Also it does not always matter if the person wins in the end of the day. I had someone with a very complicated technical subject, who really had trouble communicating. And, no miracle, we did not produce something absolutely perfect for the audience. But after his speech I saw his eyes shining. I told him, look, you managed to live through it, to feel, to smile in the end. So comparing "before" and "after" it was an extraordinary thing.

VS: To change the subject a bit, the question that I wanted to ask in the very beginning (before the conversation totally deviated [both laugh]) … This profession, say scientific mediator or popularizer of science, is it something that can be taught and learned, or should one be sort of born with it? Because what happens often is a researcher who would spend some time doing general-audience or school-level teaching activities. Or even all of his time from some age (like Kolmogorov, who founded a high school when he was 60). But can one maybe produce Christians Goichons in some Bachelor or Master courses?

CG: In general, the "schools" of mediation for educators exist, at the university level. It starts with a professional bachelor. I was giving some lectures in one of them in Tours. Those who graduate from such a school will mostly go to some project management. There are also masters of scientific communication.

Then, in the idea of educating people, one still needs some trigger to realize the desire to contribute to such activities. When I was in the science department and when I was leaving it I had no idea that I would come back with a totally different activity [theater] and learn a lot from that activity. And that trigger often comes from some science professor, who will not necessarily teach you science, but give a taste of it. Because I think our main goal is not to teach people science, but to give a flavor of it, encourage curiosity. We need to be credible on some positions, but more importantly we need to transfer the desire to learn, to be curious. And sometimes people tell me, certainly you can do that because you've learned a lot. But actually, no, I do not consider myself as having more background than my audience. But I am open to many things. I think that for "scientific culture" one just needs to be curious, dare to look around even outside science and research. And again, go off the beaten path – that gives a different viewpoint. I don't know it probably does not totally answer the question.

VS: But to speak about background, your interventions, you clearly prepare them. On the other hand, you speak about improvisation. How does it come together? I mean a theater show can be something totally rehearsed, and fellow scientists talking in the corridor is absolutely improvised.

CG: Maybe one should not oppose preparation and improvisation. I am probably not capable of improvising completely unprepared. But then obviously in the role play there is room for improvisation, like when you play a professor you would need to answer questions staying in the role. And sometimes you even provoke the questions. An important "tool" in that is hearing – watching the person or understanding the group in front of you, interacting, being able to read fast whom we deal with. Which is tricky, especially estimating the intentions of your audience. For a show on stage it is easier, but for a live event with turning audience, even the basics: how much time do you have for that interaction. And you can simply ask, but then you need to have different templates or scenarios depending on the answer – how do objects appear in action, when is the punchline. You will have some optimal scenario which you repeat for a new audience, but you need to stay flexible.

VS: And also probably depending on the audience: parents who bring their kids to a Museum of Natural History (which is not the first thing you would do visiting a city) and people whom you catch at the entrance of a bakery in the city center are not the same, they may have different education background, etc.

CG: Yes, and also the environment is different. And this is exactly what these lectures I was mentioning are about: "mise en sens et mise en scène", and the notion of audience – explaining how you can work depending on the space and depending on people. I do not know exactly the contents of other courses, but there is obviously history of science, role play technique, exhibition production.

VS: Are these kind of things new?

CG: Well, some are. The one in Tours exists for 30 years already.

VS: And it became popular now because it reached some maturity? Or there are more investments?

CG: Difficult to say exactly, those who were interested in the subject knew before … And I should also say that there are exceptions, for example, I know someone who's only education is the law department, but who by a chain of coincidences became a great scientific mediator. So, if you have a taste for communication, for storytelling, for interaction … That person I would typically make the head of such a training program. So again, you can work on these skills, but you also need to be willing to.

Another thing that some people say is "I'd love to do that, but I do not feel confident, if there are questions, etc. …"

VS: Ah, here I totally share the feeling. The first time we did the Festival of Science, we prepared a bit in the last moment, we didn't know the audience at all, and it was tricky. Even while we had, as you say, several scenarios, anything can happen. Like, totally unrelated to what we do, someone says some phrase containing "pyramids" and "aliens". On the one hand you need to react, but also you should not lie.

CG: Yes, but anyway, whatever the context, when you face this type of statements you need to find some way out.

VS: But related to that, still a question. A theater show can be total fiction. A scientific mediation show is (preferably) truth. And I think what you usually present is true.

CG: [smiles] Yes, that is a different subject. I will give you an example. We will make a show, among others, about the sound of the sea in shells. And we start by describing a disappeared profession of sheller of sea sound – people who were putting this sound into the shells – and then we ask a serious question, could this actually be done, and with what tools. And around the table we behave as an R&D department contracted to do that.

VS: Funny thing, I was asked a very similar question a couple of days ago by my daughter, who is eight years old. And it is very tricky to answer not lying. By the way, do you have a favorite age for your audience?

Christian Goichon as author and comedian – show "Fouilles Sentimentales, comme les objets nous parlent". Picture credits: Doumé.

CG: Yes, beginning of school – 9 to 11 years old. That is very interesting audience. Since we mix truth and fiction, they should have some "critical thinking" or at least question what is going on. And in the end at debriefing we will separate fiction, because it is important. And they will be sometimes disappointed that it was fake, because it sounded so nice. But what we see, is that this is a fruitful approach, because it catches attention. And it also helps to see that there are extraordinary things in science, should it be biology or mathematics, which we should valorize. I think that history of science and technology should be mandatory in high school and should continue in the university. I do not know if it exists.

VS: For high school I doubt. In the university I had history and philosophy of science, in some senior years. And while it was not necessarily liked by everyone, there was always a group of people who appreciated it a lot. And since it was a historian or philosopher who knew he was talking to mathematicians, he could adapt and he was giving true details.

CG: For truth or lies we have an interesting experience: in addition to our realistic fiction, sometimes we present facts that are so incredible that people start doubting, so we need to convince them that we did careful fact-checking. And in any case it is important to debrief. Sometimes a show is announced like a show and sometimes it is announced like a lecture. But even when it is a show, some people believe. And there you realize that even while you are not trying to manipulate people, they are losing the critical thinking and are manipulated by all sorts of fake news.

VS: Yes, and there you are touching a very important subject. We all realized in the last couple of years the power of mass media in all its forms, and also the need for educators and scientific mediators.

CG: Exactly. A sensible subject to debate on: the media will not necessarily go to the most qualified specialist for comments, but rather to the most communicating specialist, appreciated by audience, and that is dangerous …

VS: Ok, to finish on a lighter topic, but something that probably cannot be taught. You are "a specialist of scientific mediation and humor".

CG: Yes, some newspaper wrote that. It was the first time I gave my "Sea, Sex et Oursins" lecture. I prefer another one, that comes from my colleague in Montreal: "General practitioner in specific curiosity". And I follow a 3S rule: "simple, souple, surprenant" [simple, flexible, surprising]. But you are right, I use different means of communication. I did not talk about it, but for two years I was doing humor chronicles for a Swiss radio station, a show named "Impatience". They saw me at some conference "Affaire louches: info ou inox" [untranslatable play of words involving cutlery and (dis)information] – means of transport and migrations of spoons", talked to me – there is an interview of me on the internet as a specialist of louches [shady things]. And then I suggested that show, as reports of a traveler, but to unusual places: like meet sugar in a sugar box. And we made 35 episodes, with the same Philippe Boisvert as personage. And again, it was supported with no hesitation.

VS: Nice that you mention Switzerland, Quebec. Is the sense of humor different? Do you feel it?

CG: Yes. And I more or less study that. I have a pass to a museum in Lausanne where I can watch these humoristic things, and watch reactions. Speak differently …

VS: The very last one, a little bit in the spirit of "what if". You work here in La Rochelle, but have you considered other options? A big city, or abroad?

CG: I did, and even applied to some places, some of which worked, and some did not, for various reasons. But here I like the balance between mediation, education, doctoral studies. And also in this discussion the word "carte blanche" often pops up – here I can do less conventional, more provocative projects. But I also have some from time to time elsewhere: Paris, or a recent one planned in Krakow around MT180.

VS: Ok I let you go learn Polish then :-)
Thank you!

Christian Goichon is actor, scientific coach and mediator, general practitioner in specific curiosity, CEO and founder of "Les Brasseurs d'Idées".

brasseursdidees@yahoo.fr, christian.goichon@univ-lr.fr

Vladimir Salnikov is a researcher at CNRS, La Rochelle University, France. His scientific interests are graded and generalized geometry, dynamical systems, applications to mechanics and theoretical physics. He is also involved in various science outreach activities, including Festival of Science and the Young Mathematicians' Column of the EMS Magazine.

vladimir.salnikov@univ-lr.fr

Sustaining the swMATH project: Integration into zbMATH Open interface and Open Data perspectives

Maxence Azzouz-Thuderoz, Moritz Schubotz and Olaf Teschke

1 Introduction

The swMATH database was launched in 2013 as the main result of a joint 2011–2013 project of FIZ Karlsruhe and Mathematisches Forschungsinstitut Oberwolfach supported by the Leibniz Association, aiming to increase the visibility of research software contributions in mathematics, as well as to provide quality measures by evaluating the software usage in peer-reviewed publications indexed in the zbMATH database [1, 2]. Since then, the service has been produced by FIZ Karlsruhe, further developed in collaboration with the Zuse Institute Berlin (ZIB) supported by the Forschungscampus MODAL [3–5]. Due to its history, the service has been, on the one hand, closely connected to zbMATH, and, on the other hand, it has been developed and maintained on a single, independent platform, which grew out of the initial project. This resulted in both advantages (some features could be implemented independently) and disadvantages (full, and ever-growing, functionalities of zbMATH could not be transferred easily to swMATH). We describe here the new version of swMATH fully integrated to the zbMATH Open framework, which has been made possible by the transition to an open service [6, 7].

2 swMATH as part of zbMATH Open transition

Since its inception, swMATH has been a free service, although its publication part has always been based on data contained in zbMATH, which was a subscription-based service until 2020. That led to some restrictions in designing swMATH at that time: as a general rule, only limited paywalled information was made available via swMATH. This resulted in a restriction of certain functions; e.g., swMATH was not interlinked with the author and journal databases of zbMATH, and the retrieval lacked the options of full logical combinations of the various search fields. Moreover, and perhaps even more unsatisfactory, swMATH was open access, but not open data. On the other hand, the much reduced data allowed for a rather lean, independent front-end implementation employing the Django framework. This facilitated the addition of some small extra features.

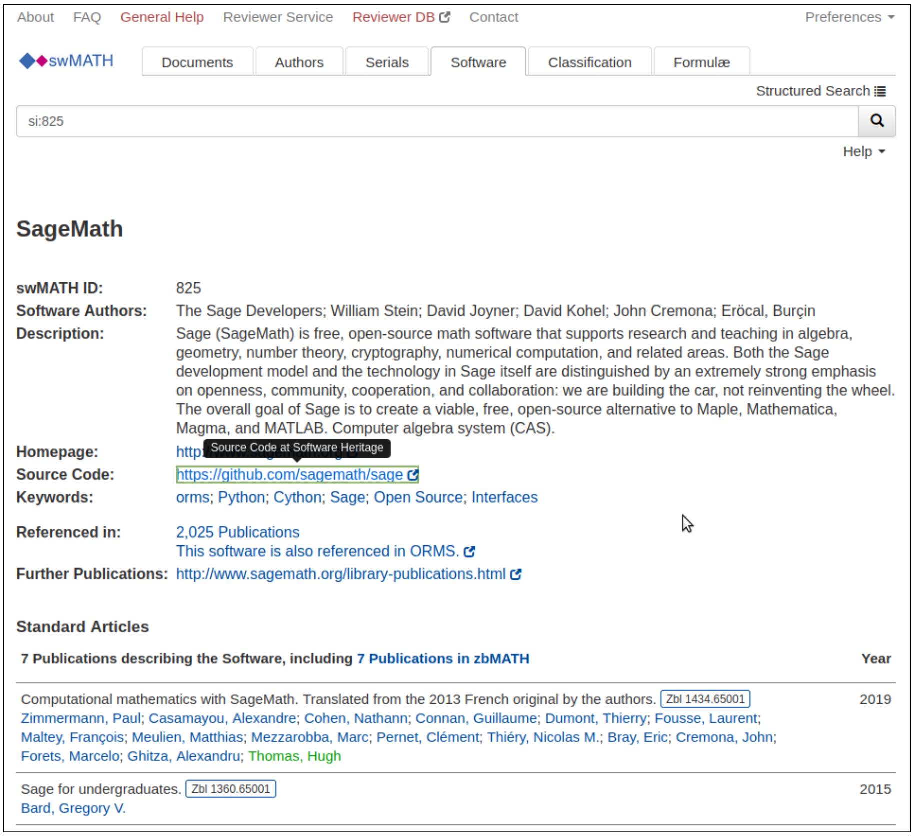
With the transition toward zbMATH Open, the main obstruction causing reduced swMATH functions became obsolete [6]. Simultaneously, in 2020/2021, a lot of internal preparatory work was completed that allowed a more flexible development, like the replacement of the indexing component or the migration of the code to Python 3 [7]. While before the system had been quite specialised, with a focus on bibliographic data, it became now much easier to add additional layers. At the same time, the life cycle of the Django-based software came to an end, making a replacement necessary.

The natural next step of swMATH at this point involved three directions of development: First, to make the swMATH data available through an API, adding a truly open data layer to the service and enabling its use in various interfaces. Second, to establish a fully integrated software facet within zbMATH Open. And third, to provide an independent platform for features incompatible with the integrated version. Here, we will mainly concentrate to report on the first two aspects.

3 Integrated functions available for software search

The search in the software layer of zbMATH Open allows now, as in the case of the other search facets, any logical combination of expressions in the various available search fields (software, name, authors, classification, keywords, and identifier are indexed) in the one-line search. Likewise, wildcard search with * is now available. Results are sorted in descending order by the number of articles referencing a software package (default), or alphabetically. The information for a single package is arranged in the detailed zbMATH Open profile standard, as shown in the figure below.

The profile information contains not just all information familiar from the old swMATH platform (though arranged in a different manner), but also a granular and interlinked breakdown of the publications using the software with respect to authors, journals, and subjects. Furthermore, while the old swMATH platform had just a static list of these documents, this is now directly interlinked to the dynamic result page in zbMATH Open, where the results can be further filtered, refined, or extended.



Profile of SageMath software

The only drawback of this version is that the classical swMATH contained also some additional references to documents not (yet) indexed in zbMATH Open, as, e.g., from arXiv, which do not fit into this format, since they are not yet indexed with respect to authors, journals, or MSC (Mathematics Subject Classification). This information is, however, not lost, but will be on display in the upcoming independent swMATH platform based on the MediaWiki framework.

4 swMATH and API

As part of the FAIRCORE4EOSC project, the new independent MediaWiki-based swMATH version will become an integrated component of the European open science cloud (EOSC). By standardised API specifications such as [9, 10] OAI-PMH and CodeMeta [8], swMATH data will become an integral component of the EOSC. This contributes to five of nine core components of FAIRCORE4EOSC.

1. The EOSC Metadata Schema and Crosswalk Registry (**MSCR**) aims to support publishing, discovery and access of metadata schemata and provides functions to operationalise metadata conversions by combining crosswalks. Through the new API, research software programs and their metadata will be easily accessible. This component will be exposed so that one can use it to convert metadata of mathematical research software to ease the querying process.
2. For each mathematical software program, the EOSC Research Software APIs and Connectors (**RSAC**) will ensure the long-term preservation of research software. We will demonstrate it easily, as the Software Heritage identifier SWHID of the archived project will be displayed on the new swMATH website.
3. Moreover, swMATH will integrate to the EOSC PIDGraph, a knowledge graph which improves the way of interlinking research entities across domains and data sources on the basis of PIDs (Persistent Identifiers); this can play a key role in helping applied mathematicians to identify the most convenient mathematical research software programs.
4. Standardised access to swMATH data is provided by integrating swMATH in the EOSC PID Meta Resolver (**PIMDR**).
5. Eventually, links between software and publications will also be discoverable via the EOSC Research Discovery Graph (**RDGraph**) which ensures that whenever a paper associated to a software that is indexed in swMATH is found in the Graph, the link to the software can be used in the EOSC context.

5 Conclusion and outlook

We outlined the new opportunities gained from the new integrated swMATH version, which first and foremost improves the situation for mathematicians who frequently use zbMATH Open. Software is now natively supported by the zbMATH Open website. At the same time, we outlined how we will improve the accessibility to swMATH data for researchers in Europe who are used to interact with the European open science cloud. Everyone using the EOSC can benefit from swMATH data without even knowing that swMATH exists or understanding the design of the swMATH website. Utilising standards and processes developed in EOSC, the data will also be an integral component of the common research data good within Europe. Also, swMATH will benefit from other EOSC data sources that can be used to improve the quality of the service further. We envision that the integration with the EOSC will also pave the ground for manifesting the role of swMATH, among many domain-specific aggregators for research software, as one that takes into account the particularities of mathematics, and at the same time takes advantage of the technology and methodological insights and achievements from aggregators in different domains.

References

[1] S. Böhnisch, G.-M. Greuel and W. Sperber, Building an information service for mathematical software – the SMATH project. *Eur. Math. Soc. Newsl.* **83**, 51–52 (2012)

[2] S. Böhnisch, G.-M. Greuel and W. Sperber, The software information service swMATH – release of the first online prototype. *Eur. Math. Soc. Newsl.* **87**, 48–49 (2013)

[3] H. Chrapary, W. Dalitz, Software products, software versions, archiving of software, and swMATH. In *Mathematical Software – ICMS 2018, Proceedings of the 6th International Conference (South Bend, 2018)*, edited by J. H. Davenport et al., Lect. Notes Comput. Sci. 10931, Springer, Cham, 123–127 (2018)

[4] H. Chrapary, W. Dalitz, W. Neun and W. Sperber, Design, concepts, and state of the art of the swMATH service. *Math. Comput. Sci.* **11**, 469–481 (2017)

[5] H. Chrapary, Y. Ren, The software portal swMATH: A state of the art report and next steps. In *Mathematical Software – ICMS 2016, Proceedings of the 5th International Conference (Berlin, 2016)*, edited by G.-M. Greuel et al., Lect. Notes Comput. Sci. 9725, Springer, Cham, 397–402 (2016)

[6] K. Hulek and O. Teschke, The transition of zbMATH towards an open information platform for mathematics. *Eur. Math. Soc. Newsl.* **116**, 44–47 (2020)

[7] K. Hulek and O. Teschke, The transition of zbMATH towards an open information platform for mathematics (II): A two-year progress report. *Eur. Math. Soc. Newsl.* **125**, 44–47 (2022)

[8] M. B. Jones, C. Boettiger, A. C. Mayes, A. Smith, P. Slaughter, K. Niemeyer, Y. Gil, M. Fenner, K. Nowak, M. Hahnel, L. Coy, A. Allen, M. Crosas, A. Sands, N. C. Hong, P. Cruse, D. S. Katz and C. Goble, CodeMeta: An exchange schema for software metadata. Version 2.0, KNB Data Repository, DOI 10.5063/schema/codemeta-2.0 (2017)

[9] M. Petrera, D. Trautwein, I. Beckenbach, D. Ehsani, F. Müller, O. Teschke, B. Gipp and M. Schubotz, zbMATH Open: API solutions and research challenges. In *Proceedings of the Workshop on Digital Infrastructures for Scholarly Content Objects (Online, 2021)*, edited by W.-T. Balke et al., CEUR Workshop Proceedings, 4–13 (2021)

[10] M. Schubotz and O. Teschke, zbMATH Open: Towards standardized machine interfaces to expose bibliographic metadata. *Eur. Math. Soc. Newsl.* **119**, 50–53 (2021)

Maxence Azzouz-Thuderoz joined FIZ Karlsruhe in 2022 to develop core components of the European Open Science Cloud after many years as a natural language processing engineer in the industry. His goal is to make the swMATH database a connected resource of the EOSC ecosystem. As a researcher, his main research interests cover natural language processing, open science for mathematical software and recent development in artificial neural networks.

maxence.azzouz-thuderoz@fiz-karlsruhe.de

Moritz Schubotz is a senior researcher for mathematical information retrieval and decentralised open science at zbMATH Open. He maintains the support for mathematical formulae in Wikipedia and is an off-site collaborator at NIST.

moritz.schubotz@fiz-karlsruhe.de

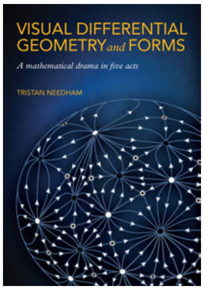
Olaf Teschke is managing editor of zbMATH Open and vice-chair of the EMS Committee on publications and electronic dissemination.

olaf.teschke@fiz-karlsruhe.de

Book reviews

Visual Differential Geometry and Forms by Tristan Needham

Reviewed by Frank Morgan

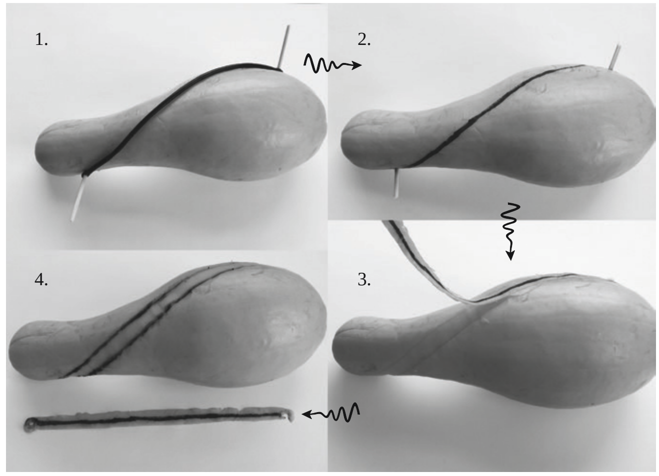


Needham proposes to provide a truly geometric "visual" explication of differential geometry, and he succeeds brilliantly. I know of nothing like it in the literature. The only account of comparable richness of exposition is Spivak's five-volume *Comprehensive Introduction to Differential Geometry*, which is, like all other texts, much more algebraic, and still much longer.

I've always loved geometric explanations. Needham's geometric derivation of the derivative of $\sin x$ is what I use in my *Calculus* book, but many of Needham's geometric explanations were wonderfully new for me. For example, Needham describes with meticulously hand-labeled pictures of summer squash how you can find a geodesic on a surface by pressing on a thin piece of adhesive tape (see his Figure 1.11 below). His modest figures, without computer technology, pay high dividends in geometric content. In another example of insightful geometric perspective, Needham explains the large circumference of circles centered at a saddle point by how far your hands move if you wave your arms up and down as you spin around. And most geometric concepts are perfectly presented to prepare for the ones that follow. Needham could not, however, resist using a geometric intrinsic definition of Gauss curvature as "angular excess per unit area," though it now takes most of the text for him to identify it with the usual, extrinsic definition as the product $\kappa_1\kappa_2$ of the principal curvatures, which he uses in the statement of Gauss's Theorema Egregium (which says that it actually is intrinsic).

As to whether his geometric explanations always qualify as proofs, Needham writes in the Prologue:

> … let me concede, from the outset, that when I claim that an assertion is "proved," it may be read as "proved beyond a reasonable doubt"!



Needham's text covers material from high school geometry through advanced graduate courses, *at leisure*. Needham explains in the Prologue:

> … I have made no attempt to write this book as a classroom textbook. … my primary goal has been to communicate a majestic and powerful subject to the reader as honestly and as lucidly as I am able, regardless of whether that reader is a tender neophyte, or a hardened expert.

And that's exactly what he does. The writing style is magnificent, and the scholarship is impeccable. The treatment of general relativity, perhaps the most striking application of differential geometry, is splendid. By not writing a textbook, the author will not be adequately compensated financially for his great contribution, but Princeton University Press can be proud to continue its noble mission of publishing the best.

Like any perfectionist, Needham does have his eccentricities. His new symbol "is ultimately equal to" may well be an improvement on the usual use of limits, but it might be moved from its overly prominent place in the Prologue to a follow-up section on preliminaries. His replacement of "Figure 1.1" with "[1.1]" (in brackets without "Figure") just makes the book harder to read.

In the third of his five "Acts," on Curvature, there is too much for my taste on the curvature of curves (as in many texts). On the other hand, I miss the interpretation of curvature and mean curvature as the rate of change of length or area under smooth deformations.

In my opinion the final Act V on Forms does not belong in this book, which is already long, despite the value of Cartan's moving frames. Needham himself writes:

> … our aim is to make Forms accessible to the widest possible range of readers, even if their primary interest is not Differential Geometry.

So perhaps give them a separate nice little volume. Actually I think that Act V is not only an unnecessary distraction, but also of lower quality than the first four Acts. In my opinion, it would better start with something familiar, such as the kind of integrand $f\,dx + g\,dy$ that occurs in line integrals and Green's Theorem, and explain it as a covector field. And I found it confusing the way Needham conflates covectors and covector fields. And I found the development too slow.

There are lots of nice exercises throughout, though no solutions. The index is very helpful.

Through it all flows a generous geometric spirit, a yearning after geometry fulfilled. This magnificent book, one of a kind, merits the close attention and tender appreciation of all true scholars and lovers of geometry.

Tristan Needham, *Visual Differential Geometry and Forms. A Mathematical Drama in Five Acts*. Princeton University Press, 2021, 584 pages, Hardback ISBN 978-0-6912-0369-0.

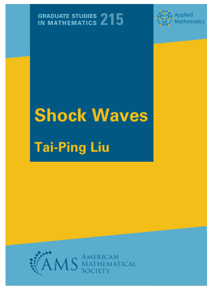
Frank Morgan is Atwell Professor of Mathematics, Emeritus, Williams College. Frank Morgan's proof with colleagues and students of the Double Bubble Conjecture is featured at the National Science Foundation Discoveries site. He has 200 publications, including six books: *Geometric Measure Theory*; *Calculus*; *Riemannian Geometry*; *The Math Chat Book*, based on his live, call-in Math Chat TV show and Math Chat column; *Real Analysis*; and *Real Analysis and Applications*. He has served as vice president of the Mathematical Association of America and of the American Mathematical Society and as editor-in-chief of *Notices of the American Mathematical Society*. He currently lives on the beach in Ocean City, New Jersey, and goes in the ocean every day, summer and winter.

fmorgan@williams.edu

DOI 10.4171/MAG/110

Shock Waves by Tai-Ping Liu

Reviewed by Denis Serre



Tai-Ping Liu, a prominent researcher in Hyperbolic Conservation Laws, offers his own view of the field in this 430-page long monograph. This text comes after several other ones that have been available on the market after the seminal *Shock waves and reaction-diffusion equations* by his advisor J. Smoller (Springer, 1983). Each one has its own merit, and Liu's one stands out with an original approach, driven by his research interests. It is not just another book on a well-known domain, but a rather personal account of the topic. The title itself singles out this text among the existing literature, by focusing on solutions and their qualitative aspects, rather than on the governing structures, conservation laws or PDEs. This is in tune with T.-P. Liu's perception of mathematical activity, which is closer to V. I. Arnold's[1] than to S. P. Novikov's[2].

Liu has been working for about fifty years about various topics, such as general Riemann problem, Glimm scheme, well-posedness of the Cauchy problem, time asymptotics, stability of nonlinear viscous waves, Boltzman shocks, relaxation and so on. The present book of course covers some of this material, but does not pretend to be exhaustive.

At first glance, the table of contents looks classical, with Chapters 3–5 covering scalar conservation laws and 6–9 devoted to systems, all this in one space dimension. Chapters 10–14 address important questions that could have deserved entire books of their own – let us think of multi-dimensional gas flow, which was the object of the famous Courant–Friedrichs *Supersonic flows and shock waves*. These latter chapters are useful introductions which can serve as starting points for further studies, though not being tailored to support some graduate course.

The originality of the approach is evident from the very beginning, in Chapter 3 devoted to scalar convex conservation laws. Instead of following a traditional strategy, where the equation is perturbed one way or another, Liu constructs exact solutions of the Cauchy problem when the initial data are step functions with finitely many jumps. This step involves repeatedly the analysis of wave interactions, one of Liu's trademarks. The corresponding global-in-time solutions remain piecewise smooth. A density argu-

---

[1] Arnold was sometimes provocative. He claimed that mathematics is a subset of physics; it is the part in which the experiments are cheap. Of course, Liu doesn't go that far.

[2] Novikov and his collaborators studied the first-order conservation laws, which they called *systems of hydrodynamic type*, from a geometrical point of view, ignoring the notion of weak solutions.

ment allows him to extend the well-posedness to rather general data. The strategy has some practical rewards; in order to check quantitative properties such as L^1-contraction or entropy dissipation – including that of a *generalized entropy functional*, – it is enough to verify them at the level of piecewise smooth solutions, hence to focus on shock waves and carry out some simple calculations. Somehow, everything must follow from the Rankine–Hugoniot condition and the associated entropy inequalities. That this method does not apply in higher space dimensions compels the author to come back to the more traditional vanishing viscosity method in the multi-D case, though only sketching the proof of the existence Theorem 6.3[3] of Chapter 5 (mind that the inequality (6.6) used to prove contraction is incorrect).

Liu's choice of considering piecewise smooth solutions and step functions data is of course motivated by the technique employed later in the study of the Cauchy problem for systems of conservation laws. The climax is achieved in the long and dense ninth chapter entitled *Well-posedness theory*. Approximate solutions are constructed through the Glimm scheme. As far as the existence is concerned, one only needs to control the BV-norm as the mesh size vanishes. This is done as usual with the help of the Glimm functional. The stability/uniqueness part however requires a finer tool, a functional elaborated by Liu and Yang (an alternate approach by homotopy was initiated by Bressan and his collaborators). The chapter culminates with the qualitative analysis of the solutions (asymptotic behaviour, N-waves, local regularity) involving the use of generalized characteristics. These materials are perhaps the most remarkable contributions of the author, among many other ones.

Overall this is a deep and quite technical book. It will not be easy to digest at the first reading. Some proofs are complete, while some others are only sketched, the reader being supposed to convert the ideas into details. This is a price to pay in order to keep such a huge topic within a reasonable length. Each chapter ends with historical notes, which put the results in perspective. The bibliography is a little narrow and the reader will sometimes like to consult that of C. Dafermos' book *Hyperbolic conservation laws in continuum physics*.

This book is recommended primarily to researchers and doctoral students. It is a unique reference about the well-posedness theory of the Cauchy problem for hyperbolic systems of conservation laws. It gives a unified presentation of the program carried out by Liu and his collaborators (often former students of him), which was so far disseminated into dozens papers, if not hundreds.

Tai-Ping Liu, *Shock Waves.* American Mathematical Society,
Graduate Studies in Mathematics 215, 2021, 437 pages, Paperback
ISBN 978-1-4704-6625-1.

Denis Serre is emeritus professor of mathematics at the École Normale Supérieure de Lyon. He wrote two books on hyperbolic systems of conservation laws, one of them in collaboration with S. Benzoni-Gavage. He translated into French D. Knuth's book *3:16, Bible Illuminated*.

denis.serre@ens-lyon.fr

DOI 10.4171/MAG/111

[3] The statements and remarks are numbered according to the sections, not the chapters. This makes the cross-references a bit odd.

European Mathematical Society

EMS executive committee

President

Volker Mehrmann (2019–2022)
Technische Universität Berlin, Germany
mehrmann@math.tu-berlin.de

Vice presidents

Betül Tanbay (2019–2022)
Bogazici University, Istanbul, Turkey
tanbay@boun.edu.tr

Jorge Buescu (2021–2024)
University of Lisbon, Portugal
jbuescu@gmail.com

Treasurer

Mats Gyllenberg (2015–2022)
University of Helsinki, Finland
mats.gyllenberg@helsinki.fi

Secretary

Jiří Rákosník (2021–2024)
Czech Academy of Sciences, Praha, Czech Republic
rakosnik@math.cas.cz

Members

Frédéric Hélein (2021–2024)
Université de Paris, France
helein@math.univ-paris-diderot.fr

Barbara Kaltenbacher (2021–2024)
Universität Klagenfurt, Austria
barbara.kaltenbacher@aau.at

Luis Narváez Macarro (2021–2024)
Universidad de Sevilla, Spain
narvaez@us.es

Beatrice Pelloni (2017–2024)
Heriot-Watt University, Edinburgh, UK
b.pelloni@hw.ac.uk

Susanna Terracini (2021–2024)
Università di Torino, Italy
susanna.terracini@unito.it

EMS publicity officer

Richard H. Elwes
University of Leeds, UK
r.h.elwes@leeds.ac.uk

EMS secretariat

Elvira Hyvönen
Department of Mathematics and Statistics
P.O. Box 68
00014 University of Helsinki, Finland
ems-office@helsinki.fi

Join the EMS

You can join the EMS or renew your membership online at euromathsoc.org/individual-members.

Individual membership benefits

- Printed version of the EMS Magazine, published four times a year for no extra charge
- Free access to the online version of the *Journal of the European Mathematical Society* published by EMS Press
- Reduced registration fees for the European Congresses
- Reduced registration fee for some EMS co-sponsored meetings
- 20 % discount on books published by EMS Press (via orders@ems.press)*
- Discount on subscriptions to journals published by EMS Press (via subscriptions@ems.press)*
- Reciprocity memberships available at the American, Australian, Canadian and Japanese Mathematical Societies

* These discounts extend to members of national societies that are members of the EMS or with whom the EMS has a reciprocity agreement.

Membership options

- 25 € for persons belonging to a corporate EMS member society (full members and associate members)
- 37 € for persons belonging to a society, which has a reciprocity agreement with the EMS (American, Australian, Canadian and Japanese Mathematical societies)
- 50 € for persons not belonging to any EMS corporate member
- A particular reduced fee of 5 € can be applied for by mathematicians who reside in a developing country (the list is specified by the EMS CDC).
- Anyone who is a student at the time of becoming an individual EMS member, whether PhD or in a more junior category, shall enjoy a three-year introductory period with membership fees waived.
- Lifetime membership for the members over 60 years old.
- Option to join the EMS as reviewer of zbMATH Open.