European Mathematical Society Magazine

ISSN 2747-7894 / Issue 127 / March 2023

EMS Magazine

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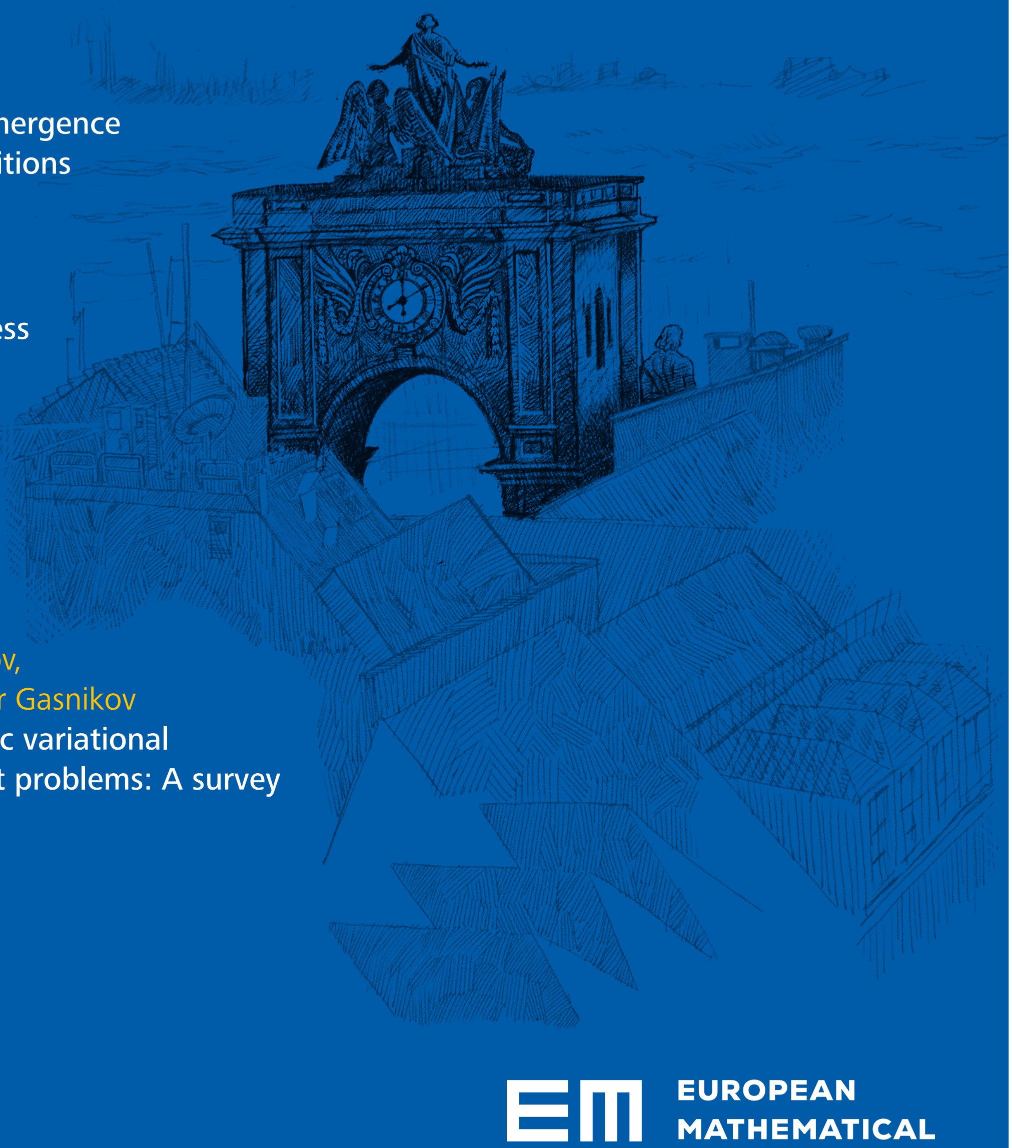

EUROPEAN MATHEMATICAL SOCIETY

European Mathematical Society Magazine

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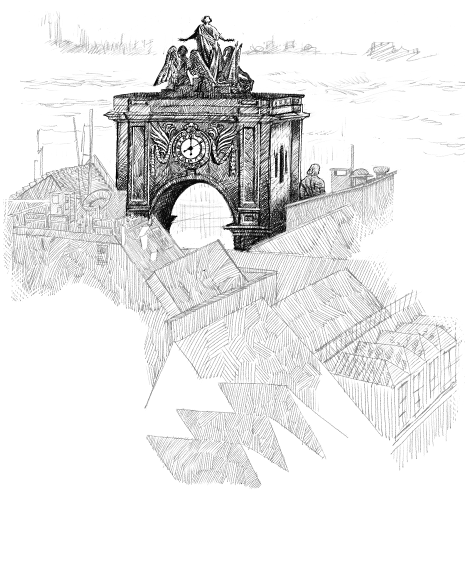
Published by EMS Press, an imprint of the
European Mathematical Society – EMS – Publishing House GmbH
Technische Universität Berlin, Institut für Mathematik,
Straße des 17. Juni 136, 10623 Berlin, Germany

https://ems.press

Typesetting: WisSat Publishing + Consulting GmbH, Fürstenwalde, Germany
Printing: Beltz Bad Langensalza GmbH, Bad Langensalza, Germany

ISSN (print) 2747-7894; ISSN (online) 2747-7908



The cover sketch by António B. Araújo depicts the triumphal arch of Rua Augusta in Lisbon, on occasion of the EMS executive committee meeting that took place in Lisbon in October last year. The arch is seen from above and behind, an unusual vantage point with a view to the Tagus river and its south bank.

Photo by Jim Høyer, University of Copenhagen

This is my first message as president of the European Mathematical Society since I took up office on January 1st, 2023. It is such a great honor, a great responsibility, and a great opportunity to work for the mathematical community in Europe. To introduce myself, I have been a professor of mathematics at the University of Copenhagen for over 25 years. Before that, I had positions at Aarhus University and at Princeton University. My research field is mathematical physics and, alongside my research and teaching, I lead the Centre for the Mathematics of Quantum Theory (QMATH) at the University of Copenhagen. I am also editor-in-chief of the Journal of Mathematical Physics and head of studies for the Master Program on Quantum Information Science offered jointly by the University of Copenhagen and the Technical University of Denmark. Throughout my career, I have continued to think of myself as a researcher and as an educator, and I have tried to put equal emphasis on both. I have had several administrative positions, both nationally and internationally: I have chaired the ERC Advanced Grant Mathematics Panel three times and been a Danish representative in Science Europe.

I have not previously had the opportunity to work for the European Mathematical Society, but I am very much looking forward to getting to know the society even better and to interacting with all of the many devoted people who work tirelessly for the EMS and for all the national societies that are institutional members of EMS. I strongly believe that EMS has been, and should continue to be, an important instrument for the promotion of mathematics in Europe in both the mathematical community itself as well as the outside world. The EMS has an important responsibility to facilitate the interaction between European mathematicians across national borders and scientific fields. In contrast to the national societies, it is indeed the role of the EMS to bridge cultural differences within European research and education – differences that may be seen as a fragmentation of the field, but should, to my mind, be perceived as a great resource for expanding the research and education we already do.

EMS should be the voice of mathematics in Europe making itself visible to the various political systems here, but also to the public at large. EMS should engage with politicians in Europe to ensure the best conditions and funding opportunities for mathematical research as well as for mathematical education. This can only be achieved by constantly emphasizing the importance of mathematics and continuing efforts to increase its visibility. Mathematicians often feel misunderstood or marginalized. I do not believe this is the case. I do not believe that anybody could sensibly question the importance of mathematics today. EMS should be a platform for the celebration of the importance and beauty of mathematics in all its diversity.

One of the most important roles a society such as EMS has is to disseminate research in a transparent and open way. The EMS should be actively involved in supporting open science and in ensuring that mathematics has a role in open science initiatives. In the fast changing world of scientific publishing, it is important that society publishers offer alternatives to commercial publishers, in particular, in the ways we choose to implement open access. EMS has taken a very important step in that direction with the creation of its publishing house, the EMS Press. The EMS Press has chosen the Subscribe to Open (S2O) route to open access. The idea is that journals become openly available when there are enough subscribers. Over the years, I have been very engaged in the discussion around open access within Europe. Open access to publications, and open science in general, are extremely important. It is, however, as important that there is equal opportunity for all researchers to participate in open science. I believe Subscribe to Open is an excellent and fair approach to open access and I hope the EMS Press will be successful in promoting it. I am certain that the EMS Press will continue to grow and I am very much looking forward to being involved in this development.

I would like to take this opportunity to thank the outgoing president Volker Mehrmann. As president, he had to struggle with a great many difficulties, such as the pandemic and a serious international conflict still taking place within Europe. Nevertheless, together with the Executive Committee he managed to initiate several very important initiatives aimed at increasing the engagement of individual members in the EMS. One initiative is the creation of topical activity groups (EMS-TAGs) intended to enhance scientific cooperation across all mathematical fields in all of Europe. Another initiative is the EMS Young Academy (EMYA). Junior mathematicians from all over Europe can be elected to EMYA for a 4-year term. The inaugural group of approximately 30 members will be announced this spring. The EMS hopes that EMYA will give the younger generations of mathematicians greater exposure and will help support their career perspectives. The EMS, however, also hopes that this will be an opportunity for the younger generations to get more involved with the EMS. I am very excited about all of these initiatives, and I personally hope that these are only the first in a series aimed at increasing the exposure of individual EMS members, as well as helping to promote a greater sense of involvement from members in the work of the EMS. I believe it is time for the EMS to move beyond its role as mainly a society of societies and for it to become also a society for individual members.

I am very much looking forward to being the president of the EMS. I am lucky to take over a thriving society with a well-established publishing house. I realize that there will be a lot of work and that there will be difficulties, but I am convinced that it will also be an exciting and fruitful experience. In particular, I am looking forward to the interaction with mathematicians from all over Europe and from all the diverse fields of the subject.

Jan Philip Solovej
President of the EMS

Brief words from the editor-in-chief

Dear readers of the EMS Magazine,

This issue of our Magazine marks a number of changes. The first and certainly the more relevant for the Society is the change of presidents. To the new president, Jan Philip Solovej, the Magazine editorial team wishes a very successful mandate. We hope the Magazine will continue to have in him a voice at least as encouraging and supportive as we received from the last president, Volker Mehrmann, to whom I would like to thank for all the ideas and useful suggestions, and, not least, for the invitation he addressed me at the beginning of 2020 to direct the Magazine.

The other main change in this issue concerns the editorial team: Vladimir Popov and Vladimir Salnikov completed their second four years term as editors at the end of 2022 and, as is the practice of the Magazine, they left the team. I thank both for all their good work for the Newsletter/Magazine over the last eight years. To reinforce the editorial team, two new editors are starting their duties with the current issue: Jason Cooper, who was already collaborating in the ERME Column and now becomes the editor responsible for Mathematics Education, and Youcef Mammeri, who will reinforce the Features and Discussion team. A short biographical note of each of them can be found at the end of the issue. I am grateful to both for accepting to dedicate part of their time to our Magazine.

Fernando Pestana da Costa
Editor-in-chief

Hydrodynamic limits: The emergence of fractional boundary conditions

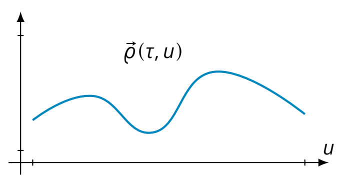
Patrícia Gonçalves

In these notes, I describe some recent developments concerning the hydrodynamic limit for some stochastic interacting particle systems that have been investigated by a group of researchers working under the research project funded by the ERC Starting Grant no. 715734*. The treatment is focused on stochastic systems with an open boundary, for which one can obtain partial differential equations with boundary conditions; or stochastic systems with long-range interactions, for which fractional equations appear in the scaling limits of those models. This is by no means an extensive review about the subject; the topics chosen reflect the personal perspective of the author.*

1 Introduction

The rigorous derivation of the evolution equations of classical fluid mechanics from the large-scale description of the conserved quantities in Newtonian particle systems is a long-standing problem in mathematical physics. More precisely, we are referring to the area of statistical mechanics dedicated to understanding the emergence of evolution laws from the kinetic description of the underlying system of particles. To attack this problem, we can assume that the motion of particles is random. We introduce two scales: a macroscopic scale, where the systems' thermodynamical quantities, such as, e.g., density, pressure, temperature, etc. (denote them by $\vec{\rho} \coloneqq (\rho_1, \dots, \rho_n)$) are analyzed. The other one, the microscopic scale, is the scale at which the particles of the system are analyzed as a whole. As a possible scenario, one can be interested in understanding the physical evolution of a gas confined to a finite volume. The number of molecules is of the order of Avogadro's number; therefore, one cannot give a precise description of the microscopic state of the system; rather, the goal is to describe the macroscopic behavior from the random movement of the molecules.

Understanding the connection between macro/micro-spaces is one of the goals in statistical mechanics. According to one of the creators of this area, Ludwig Boltzmann, first we should determine the stationary states of the system under investigation (denote them by μ), and then we should characterize these states in terms of the thermodynamical quantities of interest $\vec{\rho}$, resulting in $\mu_{\vec{\rho}}$. Finally, we can analyze the evolution of the system out of equilibrium. To formalize this problem from the mathematical point of view, consider a macroscopic space Λ and fix an arbitrary point u and a small neighborhood $\mathcal{V}_u$ around it, in such a way that it is macroscopically small, yet big enough to contain infinitely many molecules. Due to the strong interaction between molecules, we can assume that the system is locally in equilibrium so that its state at the point u should be close to $\mu_{\vec{\rho}(u)}$. Observe that this local equilibrium is characterized by the thermodynamical quantities $\vec{\rho}$ that now depend on the position u. We let time evolve, and we assume that the local equilibrium persists at a longer time. Later on, we stop the system at some time τ, and now the local equilibrium will be given in terms of $\vec{\rho}(\tau, u)$, depending both on time and space, i.e., the state of the system should be close to $\mu_{\vec{\rho}(t,u)}$. The function $\vec{\rho}(t, u)$ should then evolve according to some PDE, the so-called *hydrodynamic equation*.



As mentioned above, treating this problem from the mathematical point of view is challenging, and some simplifying assumptions are usually introduced. A possible approach is to consider that the dynamics of particles is random, which leads to the commonly known *stochastic interacting particle systems* (SIPS), which are random systems typically used in statistical mechanics to attack this sort of problems. Back in the 1970s, these systems were introduced in the mathematics community by Spitzer in [50], but were already known to physicists and biophysicists since the seminal article of MacDonald, Gibbs and Pipkin [47]. The dynamics of these systems conserves a certain number of quantities. At the micro-level one assumes that each molecule behaves as a continuous-time random walk evolving in a proper discretization of the macroscopic space Λ; this allows for a probabilistic analysis of the discrete system. For details on the formal definition of SIPS, we refer to the seminal book of Liggett [46]. The obtained evolution of molecules is Markovian, i.e., their future evolution conditioned to their past depends only on the knowledge of the present. We can discretize the volume Λ according to a scaling parameter $\varepsilon > 0$. At each site of the discrete set, we can place randomly a certain number of particles and repeat this independently of all the other sites. In this way, we have just fixed the initial state of the system. Each one of these particles waits an exponentially distributed time, after which one of them jumps to some other site if the dynamical rules allow for it. Once the dynamics is fixed, according to Boltzmann, one should find the stationary measures and characterize them in terms of the relevant thermodynamical quantities.

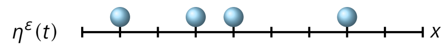
The goal, in the hydrodynamic limit, is to obtain the PDEs that govern the space-time evolution of each conserved quantity of the system studied [45, 51]. The macroscopic and microscopic spaces will be connected by means of the scaling parameter ε so that the typical distance between particles is of order ε. At the end, ε will be taken to 0. To observe a non-trivial macroscopic impact of the particles' motion, one has to look at the system on a longer time scale $\tau(\varepsilon)$, which depends on the scaling parameter ε and on the dynamical rules. If the dynamical rules allow a strong long-range interaction, then the time needed for a macroscopic effect is shorter compared to a dynamics that allows very short-range interactions.

2 Hydrodynamic limit

In order to exhibit PDEs that can be obtained for some SIPS, in the next subsections, we describe the hydrodynamic limit for a system with a single conservation law, and then we discuss the case with more conservation laws.

2.1 *A classical SIPS: The exclusion process*

The model. One of the most classical SIPS is the exclusion process, whose dynamics can be described as follows. Recall that ε is the scaling parameter connecting the macroscopic space Λ and the microscopic space Λ_ε. Assume that, at each site of Λ_ε, there can be at most one particle (the so-called exclusion rule) so that if η is a configuration, then $\eta_x(t)$ denotes the number of particles at site x and at time t, and $\eta_x(t) \in \{0,1\}$. To each bond $\{x,y\}$ of Λ_ε, there is attached a Poisson process of rate one. The trajectories of Poisson processes are discontinuous, and at each site where a discontinuity occurs, we say that there is a mark of the Poisson process. Poisson processes attached to different bonds are independent. This means that particles have to wait for a random time which is exponentially distributed with mean one, and when there is a mark of the Poisson process associated to a bond $\{x',y'\}$, the particles at that bond exchange positions at the rate $p(y'-x')$, where $p\colon \mathbb{Z} \to [0,1]$ is a transition probability. The jump occurs if and only if the exclusion rule is obeyed; otherwise, the particles wait for another mark of one Poisson process. The number of particles in the system is fixed by its initial state, and since this dynamics only exchanges particles along the microscopic space, the density is a conserved quantity.



The state space is $\{0,1\}^{\Lambda_\varepsilon}$, and when jumps are allowed only to nearest neighbors, the process is said to be simple. First, we explain phenomena observed in the case of nearest-neighbor jumps, and then we treat the extension to the long-jumps case. To that end, for now, we assume that $p(-1) = 1 - p(1)$ and $p(1) = p + E\varepsilon^\kappa$, where $p \in [0,1]$ and $E, \kappa \geq 0$. If $E = 0$ and $p = 1/2$, we obtain the extensively studied symmetric simple exclusion process (SSEP); if $E = 0$, but $p \neq 1/2$, we get the asymmetric simple exclusion process (ASEP); and if $E \neq 0$ and $p = 1/2$, we get the weakly asymmetric simple exclusion process (WASEP). Observe that the parameter κ rules the strength of the asymmetry. The infinitesimal generator of the described process is given on $f\colon \{0,1\}^{\Lambda_\varepsilon} \to \mathbb{R}$ and $\eta \in \{0,1\}^{\Lambda_\varepsilon}$ by

$$\begin{aligned}\mathcal{L}^{\mathrm{ex}} f(\eta) = \sum_{x \in \Lambda_\varepsilon} \{&p(1)\eta_x(1-\eta_{x+\varepsilon}) \\ &+ p(-1)\eta_{x+\varepsilon}(1-\eta_x)\} \nabla_{x,x+\varepsilon} f(\eta),\end{aligned}$$

where $\nabla_{x,x+\varepsilon} f(\eta) = f(\eta^{x,x+\varepsilon}) - f(\eta))$ and $\eta^{x,x+\varepsilon}$ is the configuration obtained from η by swapping the occupation variables at x and $x+\varepsilon$. We can think of $\mathcal{L}^{\mathrm{ex}}$ as a differential operator that, when testing functions defined on the state space of the process, gives a weight which is the product between the jump rate and the difference between the values of the function f at the configurations after and before the jump. This operator corresponds to the time derivative of the semigroup S_t of the process via the formula

$$\mathcal{L}^{\mathrm{ex}} f(\eta) \coloneqq \lim_{t \to 0} \frac{S_t f(\eta) - f(\eta)}{t}.$$

Now let us speed the system in the time scale $t\tau(\varepsilon) = t\varepsilon^{-a}$, where $a > 0$ will be chosen ahead in order to see a non-trivial macroscopic evolution. The system conserves a single quantity: the number of particles $\sum_{x \in \Lambda_\varepsilon} \eta_x$. Next, we should obtain the stationary measures of this process and parametrize them by a constant density ρ. By this, we mean that if we denote by ν_ρ a stationary measure of the process, then if the initial process has distribution ν_ρ, i.e., the law of η_0 is given by ν_ρ, then at any time t, the same holds, i.e., the law of η_t is given again by ν_ρ. For the exclusion processes defined above, the space-time invariant measures are Bernoulli product measures of parameter $\rho \in [0,1]$:

$$\nu_\rho(d\eta) = \prod_{x \in \Lambda_\varepsilon} \rho^{\eta_x}(1-\rho)^{1-\eta_x}, \tag{1}$$

and in fact, these measures are reversible for some choices of $p(\cdot)$. The latter means that the adjoint generator $(\mathcal{L}^{\mathrm{ex}})^*$ in the Hilbert space $\mathbb{L}^2(\nu_\rho)$ coincides with $\mathcal{L}^{\mathrm{ex}}$.

Hydrodynamic limit of exclusion processes. The empirical measure associated to the number of particles is given on $\eta \in \{0,1\}^{\Lambda_\varepsilon}$ by

$$\pi^\varepsilon(\eta, du) := \varepsilon \sum_{x \in \Lambda_\varepsilon} \eta_x \delta_x(du), \tag{2}$$

where δ_x is a Dirac mass at x. Observe that, for a given configuration η, the measure $\pi^\varepsilon(\eta, du)$ gives weight ε to each particle. We define the process of empirical measures as $\pi_t^\varepsilon(\eta, du) = \pi^\varepsilon(\eta(t\tau(\varepsilon)), du)$.

The rigorous statement of the hydrodynamic limit is that, given a measurable profile $\rho(0,u)$, if the process starts from a probability measure μ_ε for which a Law of Large Numbers (LLN) for $\pi_0^\varepsilon(du)$ holds, i.e.,

$$\pi_0^\varepsilon \to \rho(0,u)du \quad \text{as } \varepsilon \to 0,$$

then the same holds at any time t, i.e.,

$$\pi_t^\varepsilon \to \rho(t,u)du \quad \text{as } \varepsilon \to 0,$$

where $\rho(t,u)$ is the solution (in some sense) of the hydrodynamic equation. Observe that the assumption above says that the random measure $\pi_0^\varepsilon(du)$ converges weakly, as $\varepsilon \to 0$, to the deterministic measure $\rho(0,u)du$. This means that, for any given continuous function f, one has

$$\lim_{\varepsilon \to 0} \left| \int_\Lambda f(u) \pi_0^\varepsilon(\eta, du) - \int_\Lambda f(u) \rho(0,u) du \right| = 0.$$

But we still need to say in which sense the convergence holds because the left-hand side of the last display is still random. We will assume that the convergence is in probability with respect to μ_ε, i.e., for any $\delta > 0$, one has

$$\lim_{\varepsilon \to 0} \mu_\varepsilon\left(\eta : \left| \int_\Lambda f(u) \pi_0^\varepsilon(\eta, du) - \int_\Lambda f(u) \rho(0,u) du \right| > \delta\right) = 0.$$

And this will be a restriction on the set of initial measures for which the result will be derived.

Hydrodynamic equations. To provide an intuition of which equations can be derived from SIPS, we give now a heuristic argument for the exclusion processes defined above. Recall that, for these processes, the invariant measures are the Bernoulli product with marginals given in (1). Consider the discrete profile

$$\rho_t^n(x) = \mathbb{E}[\eta_t(x)].$$

From Kolmogorov's equation, we have that $\partial_t \rho_t^n(x) = \mathbb{E}[\mathcal{L}^{\text{ex}} \eta_x(t)]$, and a simple computation shows that

$$\mathcal{L}^{\text{ex}} \eta(x) = j_{x-1,x}(\eta) - j_{x,x+1}(\eta),$$

where $j_{x,x+1}(\eta)$ denotes the instantaneous current at the bond $\{x, x+1\}$. Assume now that the process at hand is the SSEP. Then

$$j_{x,x+1}(\eta) = \eta_x(1-\eta_{x+1}) - \eta_{x+1}(1-\eta_x) = \eta_x - \eta_{x+1}.$$

Since $j_{x,x+1}$ is the gradient of η_x, we get $\partial_t \rho_t^n(x) = \mathbb{E}[\Delta_n \eta_x]$, where Δ_n denotes the discrete Laplacian. Here the expectation $\mathbb{E}$ is with respect to the Bernoulli product measure given in (1), but with a parameter given by $\rho_t^n(\cdot)$. Now, if we assume that $\lim_{n\to\infty} \rho_t^n(x) = \rho_t(x/n)$ for all x, then the evolution of the density is given by the heat equation $\partial_t \rho_t(u) = \Delta \rho_t(u)$. Of course, we worked under the local equilibrium assumption made above, but this heuristic argument can be made rigorous by certain methods and for many different models.

For the exclusion process introduced above, we can get the following hydrodynamic equations [19, 43, 45]:

a. SSEP with $a = 2$, the heat equation

$$\partial_t \rho = \frac{1}{2} \Delta \rho; \tag{3}$$

b. WASEP with $\kappa = 1, a = 2$, the viscous Burgers equation

$$\partial_t \rho = \frac{1}{2} \Delta \rho + E \nabla F(\rho); \tag{4}$$

c. ASEP with $a = 1$, the inviscid Burgers equation

$$\partial_t \rho = E \nabla \rho(1-\rho).$$

For symmetric $p(\cdot)$, i.e., such that $p(z) = p(-z)$ for all $z \in \mathbb{Z}$, allowing long jumps with infinite variance, e.g.,

$$p(z) = c_\gamma |z|^{-(1+\gamma)} \mathbf{1}_{z \neq 0}. \tag{5}$$

we obtain a fractional heat equation, namely,

$$\partial_t \rho = -(-\Delta^{\gamma/2})\rho$$

for $\gamma \in (0,2)$; see [42]. Note that the infinite variance case corresponds to $\gamma \in (0,2)$ since, in this range, $\sum_z z^2 p(z) = \infty$. When $p(\cdot)$ is asymmetric, one can obtain an integro-PDE [49]. All these equations can be supplemented with several types of boundary conditions by superposing the dynamics described above with another one, for example, by

1. Considering the exclusion process evolving on the lattice

$$\Lambda_\varepsilon = \{0, \varepsilon, 2\varepsilon, \ldots, \varepsilon^{-1}\varepsilon = 1\}$$

and adding at the boundary points $x = 0$ and $x = 1$ a dynamics that injects particles (at rate $\alpha\varepsilon^\theta$ and $\beta\varepsilon^\theta$ at the left and right reservoir

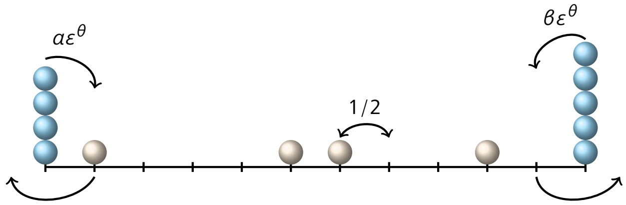


Figure 1. Symmetric exclusion with open boundary.

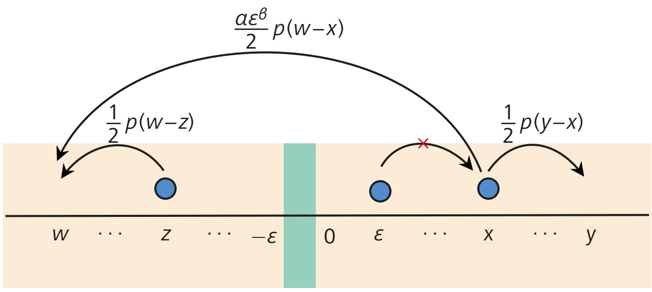


Figure 2. Long-jumps symmetric exclusion with a slow barrier.

respectively) or removes particles (at rate $(1-\alpha)\varepsilon^\theta$ and $(1-\beta)\varepsilon^\theta$ at the left and right reservoir respectively) in the system.

The parameters satisfy $\alpha, \beta \in [0,1]$ and $\theta \in \mathbb{R}$. Note that the conservation law is violated in this case, but inside the system, it still holds.

2. Considering the exclusion process with a dynamics that blocks the passage of particles between certain regions of the microscopic space Λ_ε (the conservation law is maintained in this case). For instance, assume that the exchange rate of particles in a certain number of bonds is given by a transition probability $p(\cdot)$, while in some other bonds, this rate is multiplied by a factor that makes it slower compared to the rate in all other bonds. In Figure 2, particles jump everywhere in $\Lambda_\varepsilon = \varepsilon\mathbb{Z}$, but the jump rate for bonds in $\varepsilon\mathbb{Z}_+$ or in $\varepsilon\mathbb{Z}_-$ is given by $p(\cdot)$, whereas the jump rate between sites in $\varepsilon\mathbb{Z}_+$ and $\varepsilon\mathbb{Z}_-$ is given by $p(\cdot)\alpha\varepsilon^\beta$, where now the parameters satisfy $\alpha > 0$ and $\beta \geq 0$.

Under this choice, we are creating a slow barrier at the macroscopic level, and the goal is to understand how these local microscopic defects propagate to the macroscopic level. Here we do not have a superposition of two dynamics; as in the previous case, we are just slowing down the dynamics in certain places of the microscopic space.

For recent results on 1., we refer to [3, 20, 22, 23, 26] for the SSEP in contact with slow/fast boundary reservoirs. In that case, the heat equation is supplied with boundary conditions of Dirichlet, Robin, or Neumann type, depending on the intensity of the reservoirs' dynamics. More precisely, we can get the heat equation (3) with the following boundary conditions:

(I) Dirichlet: $\rho_t(0) = \alpha$, $\rho_t(1) = \beta$ if $\theta < 1$.

(II) Robin: $\partial_u\rho_t(0) = \rho_t(0) - \alpha$, $\partial_u\rho_t(1) = \beta - \rho_t(1)$ if $\theta = 1$.

(III) Neumann: $\partial_u\rho_t(0) = \partial_u\rho_t(1) = 0$ if $\theta > 1$.

For the WASEP, one can get the viscous Burgers equation (4) with Dirichlet conditions as in (I) or with Robin boundary conditions, but in this case, the boundary conditions are nonlinear; see [14]. For the ASEP, the parabolic equations obtained above are replaced by hyperbolic laws with several types of boundary conditions [2, 54].

For the dynamics defined in 1., but in the case of long jumps, we refer to [4, 9, 10], where the authors consider the transition

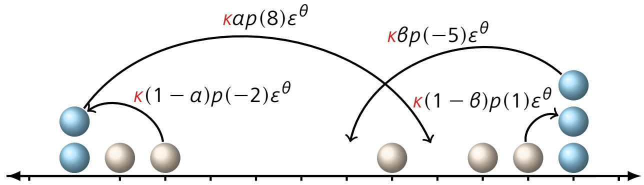


Figure 3. Long-jumps symmetric exclusion with a slow boundary.

probability (5), superposed with a dynamics that injects and removes particles in the system and that acts everywhere in Λ_ε with a strength regulated again by a parameter $\theta \in \mathbb{R}$; see Figure 3.

Depending on whether the variance of the transition probability $p(\cdot)$ is finite or not and on the strength of the Glauber dynamics, the variety of results for the hydrodynamic limit is extremely rich: indeed, different operators arise at the macro-level, and the corresponding equations come equipped with several types of boundary conditions of fractional form.

When the transition probability $p(\cdot)$ has finite variance, i.e., $\sum_z z^2 p(z) < +\infty$, which holds for $\gamma > 2$, the hydrodynamic equation for $a = 2$ is the heat equation with various boundary conditions. When $\gamma = 2$, the variance diverges as $\log(\varepsilon)$, and to compensate for this, we have to take the time scale $\varepsilon^{-2}/\log(\varepsilon)$ to obtain again the heat equation with several kinds of boundary conditions.

When $\gamma \in (0,2)$, the variance is infinite and the system becomes superdiffusive. Consequently, the resulting equation is written in terms of a fractional Laplacian operator rather than the ordinary Laplacian. Since the solutions of the equation are defined on the interval $[0,1]$, one deals, in fact, with a regional fractional Laplacian. Now the boundary conditions involve fractional derivatives. For a summary of the regimes where the boundary conditions are shown, see Figure 4.

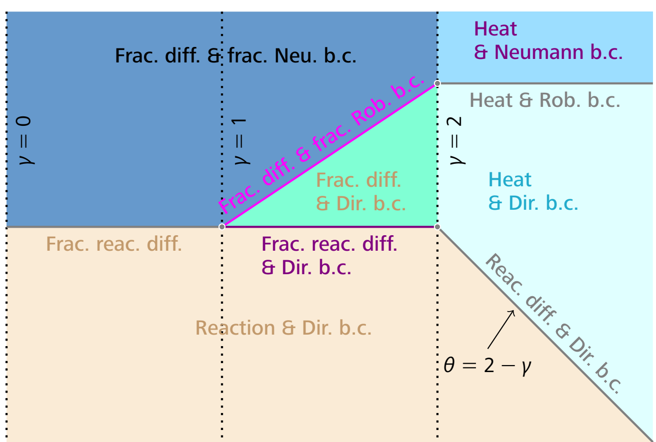


Figure 4. Variety of hydrodynamic limits.

For recent results on 2., we refer the reader to [27, 28] for the SSEP with a slow bond on the torus

$$\mathbb{T}_\varepsilon = \{0, \varepsilon, 2\varepsilon, \ldots, \varepsilon^{-1}\},$$

to [29] for the SSEP with a slow site on $\mathbb{T}_\varepsilon$, and to [16, 17] for the SSEP on $\varepsilon\mathbb{Z}$ with a slow barrier blocking the passage of particles.

We note that, in the case of a slow barrier, the variety of hydrodynamic limits is also very rich. When the intensity of the barrier is equal to $\alpha\varepsilon^\beta$ and slows down the passage of particles between negative and positive sites on $\varepsilon\mathbb{Z}$, an interesting behavior appears (contrarily to the slow bond case of [27]) when $\beta = 0$:

(i) For $\alpha = 1$, in [42], the author obtains the fractional heat equation.

(ii) For $\alpha \neq 1$, the fractional Laplacian is replaced by a regional fractional Laplacian, but defined on an unbounded domain. In this case, since there are infinitely many slow bonds at the microscopic level, the impact of their slowed dynamics (which differs from the dynamics of other bonds only by a constant) is felt at the macroscopic level.

(iii) For $\alpha > 0$ and $\beta = \gamma - 1$, one can get linear-fractional Robin boundary conditions.

(iv) For $\alpha > 0$ and $\beta > \gamma - 1$, one can get fractional Neumann boundary conditions.

Note that, while above we arrived at the heat equation or the fractional heat equation, it is possible to obtain a nonlinear version of those equations of the form $\partial_t \rho = \mathcal{P}\rho^m$, where $m \in \mathbb{N}$ and $\mathcal{P} = \Delta$ or $\mathcal{P} = -(-\Delta)^{\gamma/2}$, i.e., the porous medium equation and its fractional version. For details, we refer the reader to [13, 15, 21]. To arrive at these PDEs, one can simply start with an exclusion dynamics where the jump rate depends on the number of particles in the vicinity of the point where particles exchange positions; see [13, 15, 38].

2.2 Two conservation laws

In this subsection, we review the hydrodynamic limit for two different models with more than one conservation law. The analysis of the asymptotic behavior of the relevant quantities is much more intricate than for models with just one conserved quantity, such as the exclusion process described above.

2.2.1 The ABC model

The model. The ABC model consists of a system of particles of three species $\alpha \in \{A, B, C\}$, with exchanges only to neighboring sites on the torus $\mathbb{T}_\varepsilon$ and in the presence of a driving force, so the interaction rate depends on the type of particles involved. As in the exclusion process explained previously, at each site, there is at most one particle. The total number of particles of each species is conserved. This is a continuous-time Markov process with state space $\tilde{\Omega}_\varepsilon = \{A, B, C\}^{\mathbb{T}_\varepsilon}$. To properly define its hydrodynamic limit, we introduced the occupation number of the species α as $\xi^\alpha : \tilde{\Omega}_\varepsilon \to \{0, 1\}^{\mathbb{T}_\varepsilon}$, which acts on configurations by the rule $\xi^\alpha_x(\eta) = \mathbf{1}_{\{\alpha\}}(\eta_x)$. Its infinitesimal generator acts on functions $f\colon \tilde{\Omega}_\varepsilon \to \mathbb{R}$ as

$$\tilde{\mathcal{L}}_\varepsilon f(\eta) = \sum_{x \in \mathbb{T}_\varepsilon} c_x(\eta)[f(\eta^{x,x+\varepsilon}) - f(\eta)].$$

Here the rates $c_x(\eta)$ are defined by

$$c_x(\eta) = \sum_{\alpha,\beta} c_x^{\alpha\beta} \xi^\alpha_x \xi^{\alpha+1}_{x+1},$$

where a configuration (α, β) on the bond $\{x, x+\varepsilon\}$ is exchanged to (β, α) at the rate

$$c_x^{\alpha\beta} = 1 + \frac{\varepsilon^\gamma (E_\alpha - E_\beta)}{2},$$

for $\alpha, \beta \in \{A, B, C\}$, with $E_\alpha \geq 0$. The role of γ in this model is to tune the strength of the driving force. This model generalizes the one introduced in [24, 25]. The system will be considered in the diffusive time scale $a = 2$. We can think of this model as a two-species particle system, of species A and B, since the type C can be easily recovered from A and B.

We introduce the empirical measure (defined similarly to (2)) for each one of the conserved quantities ξ^A_x, ξ^B_x and ξ^C_x, i.e., for each $\alpha \in \{A, B, C\}$, we define

$$\pi^{\varepsilon,\alpha}(\eta_t, du) = \varepsilon \sum_{x \in \mathbb{T}_\varepsilon} \xi^\alpha_x(\eta_t)\, \delta_x(du).$$

Hydrodynamic limit of ABC. In the diffusive time scaling $a = 2$ and for $\gamma = 1$, for any t, the empirical measure

$$(\pi^{\varepsilon,A}(\eta_t, du), \pi^{\varepsilon,B}(\eta_t, du))$$

converges as $\varepsilon \to 0$ to the deterministic measure

$$(\pi^A_t(du), \pi^B_t(du)) = (\rho^A_t(u)du, \rho^B_t(u)du),$$

where the densities $(\rho^A_t(u), \rho^B_t(u))$ solve the following system of (parabolic) equations [12]:

$$\begin{cases} \partial_t \rho^A = \Delta\rho^A - \nabla[F(\rho^A)(E_A - E_C) - \rho^A\rho^B(E_B - E_C)], \\ \partial_t \rho^B = \Delta\rho^B - \nabla[F(\rho^B)(E_B - E_C) - \rho^A\rho^B(E_A - E_C)]; \end{cases} \tag{6}$$

here $F(\rho) = \rho(1-\rho)$, and for $\alpha \in \{A, B, C\}$, ρ^α denotes the density of particles of type α in the system. The equation for the species C can easily be obtained by using the identity $\rho^C = 1 - \rho^A - \rho^B$. In this case, the hydrodynamic limit is given by a system of coupled equations since the evolution of particles of one species is affected by the particles of the other species. One can also consider this model in contact with slow/fast reservoirs, extending the model defined above. Consider, for example, the dynamics described in Figure 5.

The rates satisfy $r_A + r_B + r_C = 1$ and $\tilde{r}_A + \tilde{r}_B + \tilde{r}_C = 1$, and can be interpreted as density reservoirs. For this model, the hydrodynamic equation is similar to (6), and it is supplemented with boundary conditions that can be of Dirichlet type or Robin type [39].

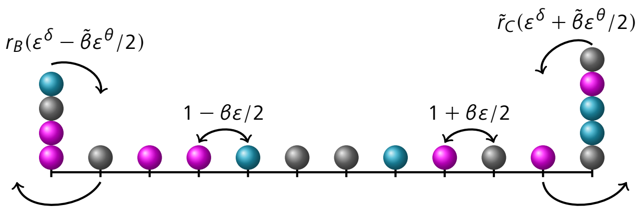


Figure 5. Dynamics of the ABC model with reservoirs at $x = 0$ and $x = 1$. Particles of species A, B and C.

2.2.2 Interface models

The models. Next, we describe another collection of models with two conservation laws. These systems were introduced in [11]; they consist of perturbations of Hamiltonian dynamics with a conservative noise and exhibit strong analogies with the standard chains of oscillators. The dynamics of these fluctuating interface models, denoted by $\{\eta_x(t)\}_{t\geq 0}$, depends on an interaction potential $V\colon \mathbb{R} \to [0, +\infty)$ and evolves in the state space $\hat\Omega_\varepsilon := \mathbb{R}^{\mathbb{T}_\varepsilon}$ (these variables now are continuous and unbounded). The dynamics conserves two quantities:

$$\text{energy} \sum_x V(\eta_x) \quad \text{and} \quad \text{volume} \sum_x \eta_x,$$

and in [11], it is proved that these are the only conserved quantities. Here η_x stands for the height of the interface at the site x.

There are some potentials that have been explored in the literature. Below, we focus on two of them, namely, the exponential potential and the quadratic potential; see [1, 5–8]. Fix a positive real parameter $b > 0$ and define the Kac–van Moerbeke potential $V_b\colon \mathbb{R} \to [0, +\infty)$ by

$$V_b(u) = e^{-bu} - 1 + bu.$$

The corresponding infinitesimal generator is given by

$$\hat{\mathcal{L}} = a\varepsilon^\kappa \mathcal{A}_b + \gamma \mathcal{S}, \tag{7}$$

where $\gamma, \kappa > 0$, $a \in \mathbb{R}$ and the operators $\mathcal{A}_b$ and $\mathcal{S}$ act on differentiable functions f by the rules

$$(\mathcal{A}_b f)(\eta) = \sum_{x\in\mathbb{T}_\varepsilon} \big(V_b'(\eta_{x+\varepsilon}) - V_b'(\eta_{x-\varepsilon})\big)(\partial_{\eta_x} f)(\eta),$$
$$(\mathcal{S} f)(\eta) = \sum_{x\in\mathbb{T}_\varepsilon} \big(f(\eta^{x,x+\varepsilon}) - f(\eta)\big).$$

The configuration $\eta^{x,x+\varepsilon}$ represents the swapping of particles as described above. For more details on the definition of these models, we refer to [5, 11, 53]. The parameter $a\varepsilon^\kappa$ regulates the intensity of the Hamiltonian dynamics in the system in terms of the scaling parameter ε. The role of the parameter γ is to regulate the intensity of the stochastic noise. Note that, when $\gamma = 0$ (i.e., in the absence of noise), this system is completely integrable. We will speed it up in the time scale $t\varepsilon^{-a}$ with $a > 0$.

As mentioned above, the system has two conserved quantities: energy $\sum V_b(\eta_x)$ and volume $\sum \eta_x$, but of course, since the generator is a linear operator, any linear combination (plus constants) of energy and volume is also conserved, e.g., $\sum_x \xi_x$ with $\xi_x = V_b'(\eta_x)$. Let us describe the space-time evolution of the relevant quantities of the system.

Hydrodynamic limit for interface models. We define the empirical measures associated with the energy and the volume as in (2) by

$$\begin{cases} \pi^{\varepsilon,e}(\eta, du) = \varepsilon \sum_{x\in\mathbb{T}_\varepsilon} V_b(\eta_x)\,\delta_x(du), \\ \pi^{\varepsilon,v}(\eta, du) = \varepsilon \sum_{x\in\mathbb{T}_\varepsilon} \eta_x\,\delta_x(du). \end{cases}$$

In [11], for $a = 1$ and in the strong asymmetric regime, it was proved that (before the appearance of shocks) the hydrodynamic equations (of hyperbolic type) are given by

$$\begin{cases} \partial_t e - ab^2\nabla(e - bv)^2 = 0, \\ \partial_t v + 2ab\nabla(e - bv) = 0. \end{cases}$$

As for the ABC model, the hydrodynamics is given by a system of coupled equations, but instead of parabolic equations, here we have hyperbolic equations.

We conclude by noting that, for the models described above, we obtained a variety of PDEs with several types of boundary conditions. The exploration of other types of boundary conditions and more general PDEs is certainly important and deserves attention. Moreover, we believe that, with the knowledge of the underlying SIPS, we can get information on the notion of weak solutions to some PDEs in a probabilistic way.

3 Equilibrium fluctuations

In the last section, we analyzed a Law of Large Numbers for the empirical measure in SIPS with one or more conservation laws. The limit considered in the hydrodynamic limit is deterministic, and we know what is the typical profile that we should observe at any time t. The question that we can address now is related to the corresponding Central Limit Theorem, i.e., providing a description of the fluctuations around the hydrodynamic limit. Typically, the study of non-equilibrium fluctuations is very intricate since it requires deep knowledge about the correlations of variables, and this can be quite challenging for the majority of the dynamics. What one is searching for, in the equilibrium scenario in e.g., exclusion processes, is the fluctuations around the constant hydrodynamical profile; see Figure 6.

We start by describing what can happen for systems with a single conservation law and then address the case of more conservation laws.

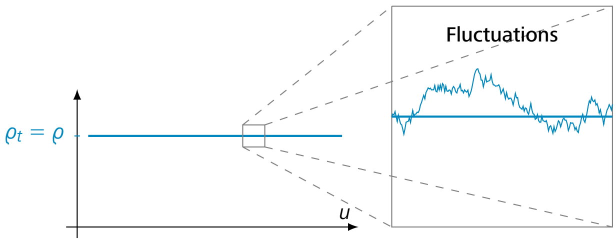


Figure 6. Fluctuations around the typical behavior.

3.1 Fluctuations for systems with a single conservation law: The exclusion process

As above, first we focus on a system with a single conservation law, the exclusion process, and from now on, we assume that it starts from the stationary state, the Bernoulli product measure of parameter $\rho \in (0,1)$ given in (2). We define the empirical field associated to the density, which is the linear functional defined on functions $f\colon \Lambda \to \mathbb{R}$ (belonging to a suitable space) as

$$\mathcal{Y}_t^\varepsilon(f) = \sqrt{\varepsilon} \sum_{x \in \Lambda_\varepsilon} f(x)\big(\eta_x(t\varepsilon^{-a}) - \rho\big). \tag{8}$$

This expression is obtained by first integrating the test function f with respect to the empirical measure in (2), then removing the mean with respect to (2), and finally dividing the result by $\sqrt{\varepsilon}$. The question that arises now is to understand the limit in distribution, as $\varepsilon \to 0$, of $\mathcal{Y}_t^\varepsilon$, denoted by $\mathcal{Y}_t$. For the exclusion processes introduced above, one can get several different limits.

A. For the SSEP and in the diffusive scaling $a = 2$, the Ornstein–Uhlenbeck (OU) process is given by

$$d\mathcal{Y}_t = \frac{1}{2}\Delta \mathcal{Y}_t\, dt + \sqrt{F(\rho)}\nabla \dot{\mathcal{W}}_t. \tag{9}$$

B. For the WASEP with a weak asymmetry, i.e., $\kappa > 1/2$ and in the diffusive scaling $a = 2$, one gets the same as (9), while for $\kappa = 1/2$, one gets the Kardar–Parisi–Zhang (KPZ) equation (introduced in [44]) or its companion, the stochastic Burgers (SB) equation, respectively, for the height field h_t or for the density field $\mathcal{Y}_t$,

$$dh_t = \frac{1}{2}\Delta h_t\, dt + 4E(\nabla h_t)^2 dt + \sqrt{F(\rho)}\dot{\mathcal{W}}_t,$$
$$d\mathcal{Y}_t = \frac{1}{2}\Delta \mathcal{Y}_t\, dt + 4E\nabla \mathcal{Y}_t^2 dt + \sqrt{F(\rho)}\nabla \dot{\mathcal{W}}_t.$$

Here $\dot{\mathcal{W}}_t$ stands for the standard space-time white noise.

The height field can be defined analogously to the density field, but the relevant quantity for this field is the net flux $J_{x,x+1}$ of particles through the bond $\{x, x+1\}$; the definition of the field is as in (8), but with η_x and its average replaced by $J_{x,x+1}$ and the corresponding average.

The results described above were obtained and analyzed in [18, 31–33, 36, 37, 40, 41] and were extended to many other stationary models in stationarity; recently, some of them have been extended to the non-equilibrium scenario; see [55].

C. For the ASEP, i.e., $E = 0$, $p \neq 1/2$ and in the hyperbolic scaling $a = 1$,

$$d\mathcal{Y}_t = (1-2\rho)(1-2p)\nabla \mathcal{Y}_t\, dt.$$

Note that if, in this expression, we take $\rho = 1/2$, we get a trivial evolution for the density field. The same is true if instead we redefine the field in a frame with the velocity $(1-2\rho)\varepsilon^{1-a}$. Therefore, to get a non-trivial behavior, we have to speed up the time, and for the choice $a = 3/2$, the limit field is given in terms of the so-called KPZ fixed point, which was constructed in [48]. In [30], it was proved that, up to the time scale $t\varepsilon^{4/3}$, there is no evolution of the density field, and its law coincides with the law of the initial field $\mathcal{Y}_0$. Nevertheless, beyond that time scale, the limit is not yet known, but it should be given in terms of the KPZ fixed point. The results of [30] applied to WASEP show that, below the line $a = (4/3)(\kappa+1)$, there is no time evolution, but in fact, the trivial evolution should go up to the line $a = (3/2)(\kappa+1)$; see the gray region on Figure 7.

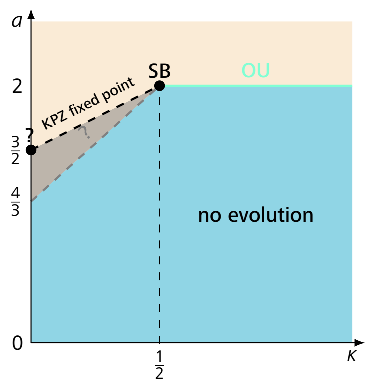


Figure 7. Fluctuations of the density in WASEP.

For a transition probability allowing long jumps, the limit behavior can be Gaussian, or given in terms of a fractional OU (when the symmetry dominates) or of the fractional SB equation (when symmetry and asymmetry have exactly the same strength); see [34, 35].

In the case of exclusion processes given by a general transition probability, we have already seen possible laws, given as solutions to stochastic PDEs (SPDEs) governing the fluctuations of the unique conserved quantity, the number of particles. The way to connect one solution to the other could be either by changing the nature of the tail of the transition probability or the symmetry/asymmetry dominance phase of the transition probability. The nature of the SPDE is very much related to the underlying SIPS, but the same equation can be obtained from a variety of different particle models, and in that sense, it is universal.

3.2 *Fluctuations for multi-component systems*

We observe that the results described in the last subsection are for systems with (only) one conservation law, and for these, there is no ambiguity concerning the choice of the fields that one should look at – the only choice is the field associated to the conserved quantity. When systems have more than one conserved quantity, and their evolution is coupled, as is the case for the ABC model or the interface models that we described above, we have to be careful when we define those fields. Moreover, a special feature of multi-component models is that different time scales coexist, which never occurs for systems with only one conserved quantity.

In [52], with a focus on anharmonic chains of oscillators, the nonlinear fluctuating hydrodynamics theory (NLFH) for the equilibrium time-correlations of the conserved quantities of that model was developed and analytical predictions were done based on a mode-coupling approximation. Roughly speaking, Spohn's approach starts at the macroscopic level, i.e., one assumes that a hyperbolic system of conservation laws governs the macroscopic evolution of the empirical conserved quantities. Then a diffusion term and a dissipation term are added to the system of coupled PDEs and one linearizes the system at second order with respect to the equilibrium averages of the conserved quantities. A fundamental role is played by the normal modes, i.e., the eigenvectors of the linearized equation. These modes evolve with different velocities and in different time scales. They might be described by different forms of superdiffusion or standard diffusion processes, and this description depends on the values of certain coupling constants. From this approach, many other universality classes arise, besides the Gaussian or the KPZ, already seen in systems with only one conservation law. Despite all the complications that one might face when dealing with multi-component systems, there is a choice of the potential V for the interface models described above, for which all the diagram for the fluctuations of its conserved quantities has been obtained. Now we quickly describe it.

The harmonic potential. Consider the generator given in (7), but with the quadratic potential $V(x) = x^2/2$, the harmonic potential. The invariant measures $\mu_{v,\beta}$ are explicitly given by

$$\mu_{v,\beta}(d\eta) = \prod_{x \in \Lambda_\varepsilon} \Big(\frac{\beta}{2\pi}\Big)^{1/2} \exp\Big\{-\frac{\beta}{2}\big(\eta_x - v\big)^2\Big\}\, d\eta_x,$$

where $v \in \mathbb{R}$ and $\beta > 0$. In this case, the system conserves two quantities, the energy $\sum_x \eta_x^2$ and the volume $\sum_x \eta_x$; note that the average with respect to $\mu_{v,\beta}$ of η_x and η_x^2 is equal to v and $v^2 + (1/\beta)$, respectively. According to NLFH, the quantities that one should analyze are now

$$\mathcal{U}_1 = \overline{\eta}_x \quad \text{and} \quad \mathcal{U}_2 = 2v\overline{\eta}_x + \overline{\eta_x^2}.$$

For a random variable X, we let $\bar{X}$ denote the centered random variable. Note that, for $v = 0$, we simply get $\mathcal{U}_1$ and $\mathcal{U}_2$ as the volume and energy, respectively. The corresponding fields should

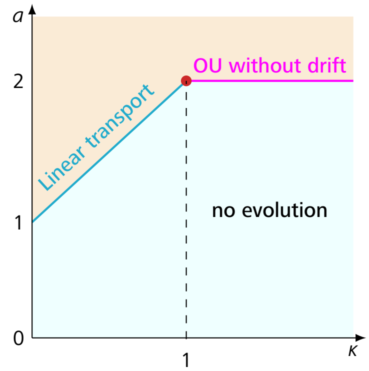


Figure 8. Fluctuations for $\mathcal{U}_1$.

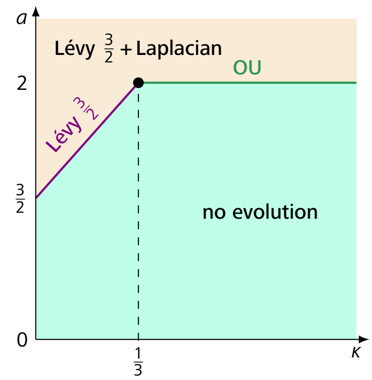


Figure 9. Fluctuations for $\mathcal{U}_2$.

be taken on a frame with velocity $v_1 := 2a_\varepsilon$ and $v_2 := 0$. According to NLFH, in the strong asymmetric regime ($\kappa = 0$), $\mathcal{U}_1$ should behave diffusively and $\mathcal{U}_2$ should behave as a Lévy process with exponent $3/2$. For the volume, i.e., the quantity $\mathcal{U}_1$, when we take the fluctuation field with velocity equal to 0, we get a process that is linearly transported in time (see the light-blue line in Figure 8), while if we take it with the velocity v_1, we get an OU process without drift (see the magenta line).

For $\mathcal{U}_2$ with velocity $v = 0$, i.e., the energy (recall that $v_2 = 0$), we have the results summarized in Figure 9.

In Figure 8, the light-blue line corresponds to $a = \kappa + 1$, while the purple line in Figure 9, where we see the Lévy process with exponent $3/2$, corresponds to $a = (3/2)(\kappa + 1)$. Note that this diagram is complete, but the method that was employed to derive these results relies heavily on the specific form of the dynamics.

There is still much work to do in this direction, and we believe that one should analyze the action of the generator on other relevant quantities and keep track of those that give a non-trivial contribution to the limit. There are several equations that one can obtain from this procedure by using many different microscopic forms of dynamics, and for this reason, they are said to be universal. Understanding how to connect universality classes is a major problem in the field of SIPS. There is much to do regarding this problem, and hopefully, in the next years, large steps will be made in this direction.

4 Final comments

Some of the problems described above were among the goals of the research project titled HyLEF, *Hydrodynamic Limits and Equilibrium Fluctuations: universality from stochastic systems*, one of the projects funded by the European Research Council (ERC) in the 2016 edition of the ERC Starting Grants. This is the first and so far the only ERC grant awarded in Portugal in the field of mathematics, and it is headed by the author of this article, Patrícia Gonçalves, now a full professor at the mathematics department of Instituto Superior Técnico (IST) of the University of Lisbon. It is a grant of nearly 1.2 million euros for 5 years (extended to 7 years due to the pandemic period) which started on the 1st of December, 2016.

The budget allowed creating a team composed of 4 post-doctoral researchers (2 years each), 2 Ph.D. students (4 years each), and 2 master students (1 year each). This was the first team in Portugal working in the field of SIPS. The budget also allowed organizing conferences and inviting external collaborators to work with the team at IST in Portugal.

I would like to thank the ERC and all the members of the Panel PE1 (Mathematics) who, by selecting my project for funding, have all contributed to a big change in my life and the lives of all the people involved in this project. If HyLEF was not funded by the ERC, the creation of this team under national funds would have been completely impossible.

The group of collaborators of this project includes several researchers, some of them working at the host institution and others working abroad, mainly at IMPA and at the Universities of Arizona, Juelich, Lyon, Nice, among others. Below is a photomontage of some of these members, to whom I am truly grateful for making the last years at IST extremely exciting, not only research-wise, but also personally. I will certainly remember them for a long time.

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Smooth monotone stochastic variational inequalities and saddle point problems: A survey

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This paper is a survey of methods for solving smooth, (strongly) monotone stochastic variational inequalities. To begin with, we present the deterministic foundation from which the stochastic methods eventually evolved. Then we review methods for the general stochastic formulation, and look at the finite-sum setup. The last parts of the paper are devoted to various recent (not necessarily stochastic) advances in algorithms for variational inequalities.

1 Introduction

In its long, more than half-century history of study (going back to the classical article [113]), variational inequalities have become one of the most popular and universal optimization formulations. Variational inequalities are used in various areas of applied mathematics. Here we can highlight both classical examples from game theory, economics, operator theory, convex analysis [6, 19, 106, 110, 113], as well as newer and even more recent applications in optimization and machine learning: non-smooth optimization [93], unsupervised learning [9, 22, 36], robust/adversarial optimization [11], GANs [47] and reinforcement learning [57, 100]. Modern times present new challenges to the community. The increase in scale of problems and the need to speed up solution processes have sparked a huge interest in *stochastic* formulations of applied tasks, including variational inequalities. This paper surveys stochastic methods for solving variational inequalities.

Structure of the paper. In Section 2, we give a formal statement of the variational inequality problem, basic examples, and main assumptions. Section 3 deals with deterministic methods, from which stochastic methods have evolved. Section 4 covers a variety of stochastic methods. Section 5 is devoted to the recent advances in (not necessarily stochastic) variational inequalities and saddle point problems.

2 Problem: Setting and assumptions

Notation. We use $\langle x, y\rangle \coloneqq \sum_{i=1}^d x_i y_i$ to denote the standard inner product of vectors $x, y \in \mathbb{R}^d$, where x_i is the i-th component of x in the standard basis of $\mathbb{R}^d$. It induces the ℓ_2-norm in $\mathbb{R}^d$ by $\|x\|_2 \coloneqq \sqrt{\langle x, x\rangle}$. We denote the ℓ_p-norm by $\|x\|_p \coloneqq \big(\sum_{i=1}^d |x_i|^p\big)^{1/p}$ for $p \in [1, \infty)$, and $\|x\|_\infty \coloneqq \max_{1\le i\le d}|x_i|$ for $p = \infty$. The dual norm $\|\cdot\|_*$ corresponding to the norm $\|\cdot\|$ is defined by $\|y\|_* \coloneqq \max\{\langle x, y\rangle \mid \|x\| \le 1\}$. The symbol $\mathbb{E}[\cdot]$ stands for the total mathematical expectation. Finally, we need to introduce the symbols $\mathcal{O}$ and Ω to enclose numerical constants that do not depend on any parameters of the problem, and the symbols $\tilde{\mathcal{O}}$ and $\tilde{\Omega}$ to enclose numerical constants and logarithmic factors.

We study variational inequalities (VI) of the form

$$\text{find } z^* \in \mathcal{Z} \text{ such that } \langle F(z^*), z - z^*\rangle \ge 0 \quad \forall z \in \mathcal{Z}, \tag{1}$$

where $F\colon \mathcal{Z} \to \mathbb{R}^d$ is an operator and $\mathcal{Z} \subseteq \mathbb{R}^d$ is a convex set.

To emphasize the extensiveness of formulation (1), we give a few examples of variational inequalities arising in applied sciences.

Example 1 (Minimization)**.** Consider the minimization problem

$$\min_{z\in\mathcal{Z}} f(z). \tag{2}$$

Let $F(z) \coloneqq \nabla f(z)$. Then, if f is convex, one can prove that $z^* \in \mathcal{Z}$ is a solution of (1) if and only if $z^* \in \mathcal{Z}$ is a solution of problem (2).

Example 2 (Saddle point problem)**.** Consider the saddle point problem (SPP)

$$\min_{x\in\mathcal{X}} \max_{y\in\mathcal{Y}} g(x, y). \tag{3}$$

Suppose that $F(z) \coloneqq F(x, y) = [\nabla_x g(x, y), -\nabla_y g(x, y)]$ and $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ with $\mathcal{X} \subseteq \mathbb{R}^{d_x}$, $\mathcal{Y} \subseteq \mathbb{R}^{d_y}$. Then, if g is convex-concave, one can prove that $z^* \in \mathcal{Z}$ is a solution of problem (1) if and only if $z^* \in \mathcal{Z}$ is a solution of problem (3).

The study of saddle point problems is often associated with variational inequalities.

Example 3 (Fixed point problem). Consider the fixed point problem

$$\text{find } z^* \in \mathbb{R}^d \text{ such that } T(z^*) = z^*, \tag{4}$$

where $T\colon \mathbb{R}^d \to \mathbb{R}^d$ is an operator. If we set $F(z) = z - T(z)$, then one can prove that $z^* \in \mathcal{Z} = \mathbb{R}^d$ is a solution of problem (1) if and only if $F(z^*) = 0$, i.e., $z^* \in \mathbb{R}^d$ is a solution of problem (4).

For the operator F from (1) we assume the following.

Assumption 1 (Lipschitzness). The operator F is L-Lipschitz continuous, i.e., for all $u, v \in \mathcal{Z}$, we have $\|F(u) - F(v)\|_* \leq L\|u - v\|$.

In the context of problems (2) and (3), L-Lipschitzness of the operator means that the functions $f(z)$ and $g(x, y)$ are L-smooth.

Assumption 2 (Strong monotonicity). The operator F is μ-strongly monotone, i.e., for all $u, v \in \mathcal{Z}$, we have $\langle F(u) - F(v), u - v\rangle \geq \mu\|u - v\|_2^2$. If $\mu = 0$, then the operator F is monotone.

In the context of problems (2) and (3), strong monotonicity of F means strong convexity of $f(z)$ and strong convexity-strong concavity of $g(x, y)$. In this paper we first focus on the strongly monotone and monotone cases. But there are also various assumptions relaxing monotonicity and strong monotonicity (e.g., see [55] and references therein).

We note that Assumptions 1 and 2 are sufficient for the existence of a solution to problem (1) (see, e.g., [37]).

Since we work on the set $\mathcal{Z}$, it is useful to introduce the Euclidean projection onto $\mathcal{Z}$,

$$P_{\mathcal{Z}}(z) = \arg\min_{v \in \mathcal{Z}} \|z - v\|_2.$$

To characterize the convergence of the methods for monotone variational inequalities we introduce the gap function,

$$\mathrm{Gap}_{\mathrm{VI}}(z) := \sup_{u \in \mathcal{Z}} [\langle F(u), z - u\rangle]. \tag{5}$$

Such a gap function, regarded as a convergence criterion, is more suitable for the following variational inequality problem:

$$\text{find } z^* \in \mathcal{Z} \text{ such that } \langle F(z), z^* - z\rangle \leq 0 \quad \text{for } z \in \mathcal{Z}.$$

Such a solution is also called weak or Minty (whereas the solution of (1) is called strong or Stampacchia). However, in view of Assumption 1, F is single-valued and continuous on $\mathcal{Z}$, meaning that actually the two indicated formulations of the variational inequality problem are equivalent [37].

For the minimization problem (2), the functional distance to the solution, i.e., the difference $f(z) - f(z^*)$, can be used instead of (5). For saddle point problems (3), a slightly different gap function is used, namely,

$$\mathrm{Gap}_{\mathrm{SPP}}(z) := \mathrm{gap}(x, y) = \max_{y' \in \mathcal{Y}} f(x, y') - \min_{x' \in \mathcal{X}} f(x', y). \tag{6}$$

For both functions (5) and (6) it is crucial that the feasible set is bounded (in fact it is not necessary to take the whole set $\mathcal{Z}$, which can be unbounded – it suffices to take a bounded convex subset $\mathcal{C}$ which contains some solution, see [95]). Therefore it is necessary to define a distance on the set $\mathcal{Z}$. Since this survey covers methods not only in the Euclidean setup, let us introduce a more general notion of distance.

Definition 1 (Bregman divergence). Let $\nu(z)$ be a function that is 1-strongly convex w.r.t. the norm $\|\cdot\|$ and differentiable on $\mathcal{Z}$. Then for any two points $z, z' \in \mathcal{Z}$ the Bregman divergence (or Bregman distance) $V(z, z')$ associated with $\nu(z)$ is defined as

$$V(z, z') := \nu(z') - \nu(z) - \langle \nabla \nu(z), z' - z\rangle.$$

We denote the Bregman diameter of the set $\mathcal{Z}$ w.r.t. the divergence $V(z, z')$ as $D_{\mathcal{Z},V} := \max\{\sqrt{2V(z, z')} \mid z, z' \in \mathcal{Z}\}$. In the Euclidean case, we simply write $D_{\mathcal{Z}}$ instead of $D_{\mathcal{Z},V}$. Using the definition of V, we introduce the so-called proximal operator as follows:

$$\mathrm{prox}_x(y) = \arg\min_{z \in \mathcal{Z}} \{\langle y, z\rangle + V(z, x)\}.$$

3 Deterministic foundation: Extragradient and other methods

The first and the simplest method for solving the variational inequality (1) is the iterative scheme (also known as the Gradient method)

$$z^{k+1} = P_{\mathcal{Z}}(z^k - \gamma F(z^k)), \tag{7}$$

where $\gamma > 0$ is a step size. Note that using the proximal operator associated with the Euclidean Bregman divergence this method can be rewritten in the form

$$z^{k+1} = \mathrm{prox}_{z^k}(\gamma F(z^k)).$$

The basic result asserts the convergence of the method to the unique solution of (1) for strongly monotones and L-Lipschitz operators F; it was obtained in the papers [19, 106, 110].

Theorem 1. *If Assumptions 1 and 2 hold and $0 < \gamma < 2\mu/L^2$, then after k iterations method (7) converges to z^* with a linear rate:*

$$\|z^k - z^*\|_2^2 = \mathcal{O}(R_0^2 q^k), \quad \textit{with } q = (1 - 2\gamma\mu + \gamma^2 L^2)$$

and R_0 denotes (here and everywhere in the sequel) the norm $\|z^0 - z^*\|_2$. For $\gamma = \mu/L^2$, we have $q = (1 - 1/\kappa^2)$, $\kappa = L/\mu$, thus *the upper bound on the number of iterations needed to achieve the ε-solution (i.e., $\|z^k - z^*\|_2^2 \le \varepsilon$) is $\mathcal{O}(\kappa^2 \log(R_0^2/\varepsilon))$.*

Various extensions of this statement (for the case when F is not Lipschitz, but with linear growth bounds, or when the values of F are corrupted by noise) can be found in [10, Theorem 1].

When F is a potential operator (see Example 1) method (7) coincides with the gradient projection algorithm. It converges for strongly monotone F. Moreover, the bounds for the admissible step size are less restrictive ($0 < \gamma < 2/L$) and the relevant complexity estimates are better ($O(\kappa \log(R_0^2/\varepsilon))$) than in Theorem 1; see [104, Theorem 2 in Section 1.4.2].

However, in the general monotone, but not strongly monotone case (for instance, for the convex-concave SPP, Example 2) convergence fails. The original statements on the convergence of Uzawa's method (a version of (7)) for saddle point problems [6] were wrong; there are numerous well-known examples where method (7) for F corresponding to a bilinear SPP diverges, see, e.g., [104, Figure 39].

There have been many other attempts to recover the convergence of gradient-like methods, not for VIs, but for saddle point problems. One of them is based on the transition to modified Lagrangians when $g(x, y)$ is a Lagrange function, see [45, 104]. However, we focus on the general VI case. A possible approach is based on the idea of *regularization*. Instead of the monotone variational inequality (1) one can deal with a regularized inequality, in which the monotone operator F is replaced by strongly monotone one $F + \varepsilon_k T$, where $T(z)$ is a strongly monotone operator and $\varepsilon_k > 0$ is a regularization parameter. If we denote by z^k the solution of the regularized VI, then one can prove that z^k converges to z^* as $\varepsilon_k \to 0$ (see [10]). However, usually the solution z^k is not easily available. To address this problem, an *iterative regularization* technique is proposed in [10], where one step of the basic method (7) is applied for the regularized problem. Step sizes and regularization parameters can be adjusted to guarantee convergence.

Another technique is based on the Proximal Point Method proposed independently by B. Martinet in [84] and by T. Rockafellar in [107]. At each iteration this methods requires the solution of the VI with the operator $F + cI$, where $c > 0$ and I is the identity operator. This is an implicit method (similar to the regularization method), however there exist numerous implementable versions of Proximal Point. For instance, some methods discussed below can be considered from this point of view.

The breakthrough in methods for solving (non-strongly) monotone variational inequalities was made by Galina Korpelevich [64]. She exploited the idea of extrapolation for the gradient method. How this works can be explained for the simplest example of a two-dimensional min-max problem with $g(x, y) = xy$ and $\mathcal{Z} = \mathbb{R}^2$. It has the unique saddle point $z = 0$, and in any point z^k the direction $F(z^k)$ is orthogonal to z^k; thus, the iteration (7) increases the distance to the saddle point. However, if we perform the step (7) and get the extrapolated point $z^{k+1/2}$, the direction $-F(z^{k+1/2})$ is attracted to the saddle point. Thus, the Extragradient method for solving (1) reads

$$\begin{aligned} z^{k+1/2} &= P_{\mathcal{Z}}(z^k - \gamma F(z^k)), \\ z^{k+1} &= P_{\mathcal{Z}}(z^k - \gamma F(z^{k+1/2})). \end{aligned}$$

Theorem 2. *Let F satisfy Assumptions 1 and 2 (with $\mu = 0$) and let $0 < \gamma < 1/L$. Then the sequence of iterates z^k generated by the Extragradient method converges to $z^\star$.*

For the particular cases of the zero-sum matrix game or the general bilinear problem with $g(x, y) = y^\top \mathbf{A} x - b^\top x + c^\top y$, the method converges linearly, provided that the optimal solution is unique (see [64, Theorem 3]). In this case, the rate of convergence is equal to $\mathcal{O}(\kappa \log(R_0^2/\varepsilon))$ with $\kappa = \lambda_{\max}(\mathbf{AA}^\top)/\lambda_{\min}(\mathbf{AA}^\top)$. More general upper bounds for the Extragradient method can be found in [119] and in the recent paper [87]. In particular, for the strongly monotone case the estimate $O(\kappa \log(R_0^2/\varepsilon))$ with $\kappa = L/\mu$ holds true (compare with the much worse bound $O(\kappa^2 \log(R_0^2/\varepsilon))$ for the Gradient method). An adaptive version of the Extragradient method (no knowledge of L is required) is proposed in [61].

Another version of the Extragradient method for finding saddle points is provided in [65]. Considering the setup of Example 2, we can exploit just one extrapolating step for the variables y:

$$\begin{aligned} y^{k+1/2} &= P_{\mathcal{Y}}(y^k + \gamma \nabla_y g(x^k, y^k)), \\ x^{k+1} &= P_{\mathcal{X}}(x^k - \gamma \nabla_x g(x^k, y^{k+1/2}), \\ y^{k+1} &= y^k + q(y^{k+1/2} - y^k), \end{aligned} \tag{8}$$

with $0 < \gamma < 1/(2L)$ and $0 < q < 1$. This method converges to the solution and if $g(x, y)$ is linear in y, then the rate of convergence is linear. If we set $q = 1$ in method (8), then $y^{k+1} = y^{k+1/2}$ and we get the so-called Alternating Gradient Method (alternating descent-ascent). In [123], it was proved that this method has *local* linear convergence with complexity $O(\kappa \log(R_0^2/\varepsilon))$, where $\kappa = L/\mu$.

L. Popov [105] proposed a version of extrapolation scheme (sometimes this type of scheme is referred to as *optimistic* or *single-call*):

$$\begin{aligned} z^{k+1/2} &= P_{\mathcal{Z}}(z^k - \gamma F(z^{k-1/2})), \\ z^{k+1} &= P_{\mathcal{Z}}(z^k - \gamma F(z^{k+1/2})). \end{aligned} \tag{9}$$

It requires the single calculation of F at each iteration vs two calculations in the Extragradient method. As shown in [105], method (9) converges for $0 < \gamma < 1/(3L)$. Rates of convergence for this method were derived recently in [41, 87], i.e., $O(\kappa \log(R_0^2/\varepsilon))$ with $\kappa = L/\mu$ for the strongly monotone case and $\kappa = \lambda_{\max}(\mathbf{AA}^\top)/\lambda_{\min}(\mathbf{AA}^\top)$ for the bilinear case. Note that in the general strongly monotone case this estimate is optimal [124], but for the bilinear problem the upper bounds available in the literature for both the Extragradient and optimistic methods are not tight [56]. Meanwhile, optimal estimates $O(\sqrt{\kappa} \log(R_0^2/\varepsilon))$ with $\kappa = \lambda_{\max}(\mathbf{AA}^\top)/\lambda_{\min}(\mathbf{AA}^\top)$ can be achieved using approaches from [4, 7].

An extension of the above schemes to an arbitrary proximal setup was obtained in the work of A. Nemirovsky [92]. He proposed the Mirror-Prox method for VIs, exploiting the Bregman divergence:

$$\begin{aligned} z^{k+1/2} &= \mathrm{prox}_{z^k}(\gamma F(z^k)), \\ z^{k+1} &= \mathrm{prox}_{z^k}(\gamma F(z^{k+1/2})). \end{aligned} \tag{10}$$

This yields the following rate-of-convergence result.

Theorem 3. *Let F satisfy Assumptions* 1 *and* 2 *(with* $\mu = 0$*), and let*

$$\hat{z}^k = \frac{1}{k}\sum_{i=1}^{k} z^{i+1/2}, \tag{11}$$

where $z^{i+1/2}$ *are generated by algorithm* (10) *with* $\gamma = 1/(\sqrt{2}L)$*. Then, after k iterations,*

$$\mathrm{Gap}_{\mathrm{VI}}(\hat{z}^k) = \mathcal{O}\Big(\frac{LD^2_{\mathcal{Z},V}}{k}\Big). \tag{12}$$

Numerous extensions of these original versions of iterative methods for solving variational inequalities were published later. One can highlight Tseng's Forward-Backward Splitting [120], Nesterov's Dual Extrapolation [95], Malitsky and Tam's Forward-Reflected-Backward [83]. All methods have convergence guarantees (12). It turns out that this rate is optimal [101].

4 Stochastic methods: Different setups and assumptions

In this section, we move from deterministic to stochastic methods, i.e., we consider problem (1) with an operator of the form

$$F(z) = \mathbb{E}_{\xi\sim\mathcal{D}}[F_\xi(z)], \tag{13}$$

where ξ is a random variable, $\mathcal{D}$ is some (typically unknown) probability distribution and $F_\xi\colon \mathcal{Z} \to \mathbb{R}^d$ is a stochastic operator. In this setup, calculating the value of the full operator F is computationally expensive or even intractable. Therefore, one has to work mainly with stochastic realizations F_ξ.

4.1 General case

The stochastic formulation (13) for problem (1) was first considered by the authors of [60]. They proposed a natural stochastic generalization of the Extragradient method (more precisely, of the Mirror-Prox methods):

$$\begin{aligned} z^{k+1/2} &= \mathrm{prox}_{z^k}(\gamma F_{\xi^k}(z^k)), \\ z^{k+1} &= \mathrm{prox}_{z^k}(\gamma F_{\xi^{k+1/2}}(z^{k+1/2})). \end{aligned} \tag{14}$$

Here it is important to note that the variables ξ^k and $\xi^{k+1/2}$ are independent and $F_\xi(z)$ is an unbiased estimator of $F(z)$. Moreover, $F_\xi(z)$ is assumed to satisfy the following condition.

Assumption 3 (Bounded variance). The unbiased operator F_ξ has uniformly bounded variance, i.e., for all $\xi \sim \mathcal{D}$ and $u \in \mathcal{Z}$, we have $\mathbb{E}\|F_\xi(u) - F(u)\|_*^2 \le \sigma^2$.

Under this assumption, the following result was established in [60].

Theorem 4. *Let* F_ξ *satisfy Assumptions* 1, 2 *(with* $\mu = 0$*) and* 3*, and let* $\hat{z}^k$ *be defined as in* (11)*, where* $z^{i+1/2}$ *are generated by algorithm* (14) *with* $\gamma = \min\{\frac{1}{\sqrt{3}L}, D_{\mathcal{Z},V}\sqrt{\frac{1}{7k\sigma^2}}\}$*. Then, after k iterations, one can guarantee that*

$$\mathbb{E}[\mathrm{Gap}_{\mathrm{VI}}(\hat{z}^k)] = \mathcal{O}\Big(\frac{LD^2_{\mathcal{Z},V}}{k} + D_{\mathcal{Z},V}\sqrt{\frac{\sigma^2}{k}}\Big). \tag{15}$$

In [17], the authors carried out an analysis of algorithm (14) for strongly monotone VIs in the Euclidean case. In particular, under Assumptions 1, 2 and 3 one can guarantee that after k iterations of method (14) one has that (here and below we omit numerical constants in the exponential multiplier)

$$\mathbb{E}\|z^k - z^*\|_2^2 = \tilde{\mathcal{O}}\Big(R_0^2\exp\Big(-\frac{\mu k}{L}\Big) + \frac{\sigma^2}{\mu^2 k}\Big). \tag{16}$$

Also in [17], the authors obtained lower complexity bounds for solving VIs satisfying Assumptions 1, 2 and 3 with stochastic methods. It turns out that the conclusions of Theorem 4 in the monotone case and estimate (16) are optimal and meet lower bounds up to numerical constants.

Optimistic-like (or single-call) methods were also investigated in the stochastic setting. The work [41] applies the following update scheme:

$$\begin{aligned} z^{k+1/2} &= P_{\mathcal{Z}}(z^k - \gamma F_{\xi^{k-1/2}}(z^{k-1/2})), \\ z^{k+1} &= P_{\mathcal{Z}}(z^k - \gamma F_{\xi^{k+1/2}}(z^{k+1/2})). \end{aligned} \tag{17}$$

For this method, in the monotone Euclidean case, the authors proved an estimate similar to (15). In the strongly monotone case, method (17) was investigated in the paper [54], but the estimates obtained there do not meet the lower bounds. The optimal estimates for this scheme were obtained later in [14].

The work [66] deals with a slightly different single-call approach in the non-Euclidean case:

$$z^{k+1} = \mathrm{prox}_{z^k}\big(\gamma_k F_{\xi^k}(z^k) + \gamma_k\alpha_k[F_{\xi^k}(z^k) - F_{\xi^{k-1}}(z^{k-1})]\big). \tag{18}$$

This update rule is a modification of the Forward-Reflected-Backward approach, namely, here α_k is a parameter, while in [83], $\alpha_k \equiv 1$. The analysis of method (18) gives optimal estimates in both the strongly monotone and monotone cases.

The theoretical results and guarantees discussed above rely in significant manner on the bounded variance assumption (Assumption 3). This assumption is quite restrictive (especially when the domain is unbounded) and does not hold for many popular machine learning problems. Moreover, one can even design a strongly monotone variational inequality on an unbounded domain such that method (14) *diverges* exponentially fast [26]. The authors of [48, 55] consider a relaxed form of the bounded variance condition and assume that $\mathbb{E}\|F_\xi(u) - F(u)\|_2^2 \le \sigma^2 + \delta\|u - z^*\|_2^2$ with $\delta \ge 0$ in the Euclidean case. Under this condition and Assumptions 1 and 2, it is proved in [48] that after k iterations of algorithm (14) (when $\mathcal{Z} = \mathbb{R}^d$) it holds that

$$\mathbb{E}\|z^k - z^*\|_2^2 = \mathcal{O}\Big(\kappa R_0^2 \exp\Big(-\frac{k}{\kappa}\Big) + \frac{\sigma^2}{\mu^2 k}\Big), \tag{19}$$

where $\kappa = \max\{\frac{\delta}{\mu^2}; \frac{L+\sqrt{\delta}}{\mu}\}$. The same assumption on stochastic realizations was considered in [67], where method (18) was used, yielding the estimate

$$\mathbb{E}\|z^k - z^*\|_2^2 = \mathcal{O}\Big(R_0^2 \exp\Big(-\frac{\mu k}{L}\Big) + \frac{\sigma^2 + \delta^2 D_{\mathcal{Z}}^2}{\mu^2 k}\Big). \tag{20}$$

Estimates (19) and (20) are competitive: the former is superior in terms of the stochastic term (second term), while the latter is superior in terms of the deterministic term (first term). However, none of these results deals completely with the issue of bounded noise, because the condition considered above is not general. The key to avoiding the bounded variance assumption on F_ξ lies in the way how stochasticity is generated in method (14). Method (14) is sometimes called Independent Samples Stochastic Extragradient (I-SEG). To address the bounded variance issue, K. Mishchenko et al. [86] proposed another stochastic modification of the Extragradient algorithm, called Same Sample Stochastic Extragradient (S-SEG):

$$z^{k+1/2} = z^k - \gamma F_{\xi^k}(z^k),$$
$$z^{k+1} = z^k - \gamma F_{\xi^k}(z^{k+1/2}).$$

For simplicity, we present the above method for the case when $\mathcal{Z} = \mathbb{R}^d$ ($F(x^*) = 0$), and refer the reader to [86] for a more general case of regularized VIs. In contrast to I-SEG, S-SEG uses the same sample ξ^k for both steps at iteration k. Although such a strategy cannot be implemented in some scenarios (streaming oracle), it can be applied to finite-sum problems, which have been gaining an increasing attention in the recent years. Moreover, S-SEG relies in significant manner on the following assumption.

Assumption 4. The operator $F_\xi(z)$ is L-Lipschitz and μ-strongly monotone almost surely in ξ, i.e., $\|F_\xi(z) - F_\xi(z')\|_2 \le L\|z - z'\|_2$ and $\langle F_\xi(z) - F_\xi(z'), z - z'\rangle \ge \mu\|z - z'\|_2^2$ for all $z, z' \in \mathbb{R}^d$, almost surely in ξ.

The evident difference between the I-SEG and S-SEG setups can be explained through the connection between the Extragradient and the Proximal Point (PP) methods [84, 107]. In the rest of this subs-section we assume that $\mathcal{Z} = \mathbb{R}^d$ ($F(z^*) = 0$). In this setup, PP has the update rule

$$z^{k+1} = z^k - \gamma F(z^{k+1}).$$

The method converges for any monotone operator F and any $\gamma > 0$. However, the update rule of PP is implicit and in many situations it cannot be computed efficiently. The Extragradient method can be seen as a natural approximation of PP that substitutes z^{k+1} in the right-hand side by one gradient step from z^k:

$$z^{k+1} = z^k - \gamma F(z^k - \gamma F(z^k)).$$

In addition, when F is L-Lipschitz, one can estimate how good the approximation is. Consider $z^{k+1} = z^k - \gamma F(z^k - \gamma F(z^k))$ (the Extragradient step) and $\tilde{z}^{k+1} = z^k - \gamma F(\tilde{z}^{k+1})$ (the PP step). Then $\|z^{k+1} - \tilde{z}^{k+1}\|_2$ can be estimated as follows [86]:

$$\begin{aligned}\|z^{k+1} - \tilde{z}^{k+1}\|_2 &= \gamma\|F(z^k - \gamma F(z^k)) - F(\tilde{z}^{k+1})\|_2 \\ &\le \gamma L\|z^k - \gamma F(z^k) - \tilde{z}^{k+1}\|_2 = \gamma^2 L\|F(z^k) - F(\tilde{z}^{k+1})\|_2 \\ &\le \gamma^2 L^2\|z^k - \tilde{z}^{k+1}\|_2 = \gamma^3 L^2\|F(\tilde{z}^{k+1})\|_2 \\ &\le \gamma^3 L^3\|\tilde{z}^{k+1} - z^*\|_2.\end{aligned}$$

That is, the difference between the Extragradient and PP steps is of the order $\mathcal{O}(\gamma^3)$ rather than $\mathcal{O}(\gamma^2)$. Since the latter corresponds to the difference between PP and the simple gradient step (7), the Extragradient method approximates PP better than gradient steps, which are known to be non-convergent for general monotone Lipschitz variational inequalities. This approximation feature of the Extragradient method is crucial for its convergence and, as the above derivation implies, the approximation argument significantly relies on the Lipschitzness of the operator F.

Let us go back to the differences between I-SEG and S-SEG. In S-SEG, the k-th iteration can be regarded as a single Extragradient step for the operator $F_{\xi^k}(z)$. Therefore, Lipschitzness and monotonicity of $F_{\xi^k}(z)$ (Assumption 4) are important for the analysis of S-SEG. In contrast, I-SEG uses different operators for the extrapolation and update steps. In this case, there is no effect from the Lipschitzness/monotonicity of individual $F_\xi(z)$s. Therefore, the analysis of I-SEG naturally relies on the Lipschitzness and monotonicity of $F(z)$ as well as on the closeness (on average) of $F_\xi(z)$ and $F(z)$ (Assumption 3).

The convergence of I-SEG was discussed earlier in this section. Regarding S-SEG, one has the following result [86].

Theorem 5. *Let Assumption* 4 *hold. Then there exists a choice of step size γ (see* [48]*) such that the output of S-SEG after k iterations satisfies*

$$\mathbb{E}\|z^k - z^*\|_2^2 = \mathcal{O}\Big(\frac{LR_0^2}{\mu} \exp\Big(-\frac{\mu k}{L}\Big) + \frac{\sigma_*^2}{\mu^2 k}\Big),$$

where $\sigma_*^2 = \mathbb{E}\|F_\xi(z^*)\|_2^2$.

This rate is similar to the one known for I-SEG, with the following differences. First, instead of the uniform bound on the variance σ^2, the rate depends on σ_*^2, which is the variance of F_ξ measured at the solution. In many cases, $\sigma^2 = \infty$, while σ_*^2 is finite. From this perspective, S-SEG enjoys a better rate of convergence than I-SEG. However, this comes at a price: while the rate of I-SEG depends on the Lipschitz and strong-monotonicity constants of F, the rate of S-SEG depends on *the worst* constants of F_ξ, which can be much worse than those for F. In particular, consider the finite-sum setup with uniform sampling of ξ: $F(x) = \frac{1}{n}\sum_{i=1}^n F_i(x)$, where F_i is L_i-Lipschitz and μ_i-strongly monotone, and $\mathbb{P}\{\xi = i\} = \frac{1}{n}$. Then Assumption 4 holds with $L = \max_{1\le i\le n} L_i$ and $\mu = \min_{1\le i\le n}\mu_i$ and these constants appear in the rate from Theorem 3. The authors of [48] tighten this rate. In particular, they prove that for S-SEG with different step sizes for the extrapolation and update steps one has that

$$\mathbb{E}\|z^k - z^*\|_2^2 = \mathcal{O}\Big(\frac{LR_0^2}{\overline{\mu}}\exp\Big(-\frac{\overline{\mu}k}{L}\Big) + \frac{\sigma_*^2}{\overline{\mu}^2 k}\Big),$$

where $\sigma_*^2 = \frac{1}{n}\sum_{i=1}^n \|F_i(z^*)\|_2^2$ and $\overline{\mu} = \frac{1}{n}\sum_{i=1}^n \mu_i$. Since $\overline{\mu}$ is (sometimes considerably) larger than μ, the improvement is noticeable. Moreover, when the constants $\{L_i\}_{i=1}^n$ are known, one can consider the so-called *importance sampling* [52]: $\mathbb{P}\{\xi = i\} = L_i/(n\overline{L})$, where $\overline{L} = \frac{1}{n}\sum_{i=1}^n L_i$. As the authors of [48] show, importance sampling can be combined with S-SEG by allowing the extrapolation and update step sizes at the k-th iteration to depend on the sample ξ^k. In particular, for the proposed modification of S-SEG they derive the estimate

$$\mathbb{E}\|z^k - z^*\|_2^2 = \mathcal{O}\Big(\frac{\overline{L}R_0^2}{\overline{\mu}}\exp\Big(-\frac{\overline{\mu}k}{\overline{L}}\Big) + \frac{\hat{\sigma}_*^2}{\overline{\mu}^2 k}\Big),$$

where $\hat{\sigma}_*^2 = \frac{1}{n}\sum_{i=1}^n \frac{\overline{L}}{L_i}\|F_i(z^*)\|_2^2$. The exponentially decaying term is always better than the corresponding one for S-SEG with uniform sampling. This usually implies faster convergence during the initial stage. Next, typically, a larger norm of $F_i(z^*)$ implies larger L_i, e.g., $\|F_i(z^*)\|_2^2 \sim L_i^2$. In this case, $\hat{\sigma}_*^2 \le \sigma_*^2$, because

$$\hat{\sigma}_*^2 \sim (\overline{L})^2 \quad \text{and} \quad \sigma_*^2 \sim \overline{L^2} = \frac{1}{n}\sum_{i=1}^n L_i^2 \ge (\overline{L})^2.$$

Moreover, one can allow other sampling strategies and cover the case when some μ_i are negative, see [48] for the details.

4.2 Finite-sum case

As noted earlier, when we deal with problem (13), it is often the case (especially in practical problems) that the distribution $\mathcal{D}$ is unknown, but nevertheless some samples from $\mathcal{D}$ are available. Then one can replace problem (13) by a finite-sum approximation:

$$F(z) = \frac{1}{n}\sum_{i=1}^n F_i(z).$$

This approximation is sometimes also called Monte Carlo approximation. For machine learning problems the term empirical risk is often encountered. Although calls of the full operator are now tractable, they remain expensive in practice. Therefore, it is worth avoiding frequent computation of F and mainly use calls to single F_i operators or small batches of them.

Before presenting the results, let us introduce the appropriate analogue of the Lipschitzness assumption.

Assumption 5 (Lipschitzness in the mean). The operator F is L_{avg}-Lipschitz continuous in mean, i.e., for all $u, v \in \mathcal{Z}$, we have

$$\mathbb{E}[\|F_\xi(u) - F_\xi(v)\|_*^2] \le L_{\text{avg}}^2\|u - v\|^2.$$

For example, if F_i is L_i-Lipschitz for all i and we draw the index $\xi = i$ with probability $p_i = L_i/\sum_j L_j$, then

$$L_{\text{avg}} = \frac{1}{n}\sum_j L_j.$$

The study of finite-sum problems in stochastic optimization is connected, first of all, with classical methods for minimization problems such as SVRG [59] and SAGA [29]. For the saddle point problems, these methods were adopted in [102] (in fact, these results are also valid for variational inequalities). The authors considered strongly convex-strongly concave saddles in the Euclidean case and proved the following estimates for SVRG and SAGA:

$$\mathbb{E}\|z^k - z^*\|_2^2 = \mathcal{O}\Big(R_0^2\exp\Big(-\min\Big\{\frac{1}{n}, \frac{\mu^2}{L_{\text{avg}}^2}\Big\}k\Big)\Big).$$

Since this last bound is not tight in terms of L_{avg}/μ, the authors proposed accelerating SVRG and SAGA via the Catalyst envelope [76]. In this case, they obtain the bound

$$\mathbb{E}\|z^k - z^*\|_2^2 = \mathcal{O}\Big(R_0^2\exp\Big(-\min\Big\{\frac{1}{n}; \frac{\mu}{\sqrt{n}L_{\text{avg}}}\Big\}\frac{k}{\log[L_{\text{avg}}/\mu]}\Big)\Big). \tag{21}$$

The same estimates for methods for saddle point problems based on accelerating envelopes were also presented in [118].

An important step in the study of the finite-sum stochastic setup was taken in the work [25], which is primarily focused on bilinear games. For this class of problems, the authors improved estimate (21) and removed the additional logarithmic factor. For general problems (saddle point and variational inequalities) the results of [25] are very similar to those in (21) and also include an additional logarithmic factor. The authors also considered the convex-concave/monotone case in the non-Euclidean setting and found that for their method after k iterations it holds that

$$\mathbb{E}[\text{Gap}_{\text{VI}}(\hat{z}^k)] = \tilde{\mathcal{O}}\Big(\frac{\sqrt{n}L_{\text{avg}}D_{\mathcal{Z},V}^2}{k}\Big). \tag{22}$$

The issue of the additional logarithmic factor was resolved in [2], where the following modification of the Extragradient method was proposed:

$$
\begin{aligned}
z^{k+1/2} &= P_Z(z^k + \tau(w^k - z^k) - \gamma F(w^k)),\\
\Delta^k &= F_{\xi^k}(z^{k+1/2}) - F_{\xi^k}(w^k) + F(w^k),\\
z^{k+1} &= P_Z(z^k + \tau(w^k - z^k) - \gamma \Delta^k) \qquad (23)\\
w^{k+1} &= \begin{cases} z^{k+1}, & \text{with probability } p,\\ w^k, & \text{with probability } 1-p.\end{cases}
\end{aligned}
$$

This algorithm is a combination of the extra step technique from the theory of VIs and the loopless approach [73] for finite-sum problems. An interesting ingredient of the method is the randomized negative momentum: $\tau(w^k - z^k)$. While for minimization problems it is usual to apply a positive/heavy-ball momentum, the opposite approach proves useful for saddle point problems and variational inequalities. This effect was noticed earlier [3, 42, 122] and is encountered now in the theory of stochastic methods for VIs. Also, in [2], the authors presented modifications for the Forward-Backward, Forward-Reflected-Backward as well as for the Extragradient methods in the non-Euclidean case.

As we noted earlier, the results of [2] give estimates (21) and (22), but without additional logarithmic factors. That is, to achieve

$$
\begin{aligned}
&\mathbb{E}\|z^k - z^*\|_2^2 \le \varepsilon \quad \text{in the strongly monotone case,}\\
&\mathbb{E}[\mathrm{Gap}_{VI}(\hat z^k)] \le \varepsilon \quad \text{in the monotone case,}
\end{aligned}
$$

the methods from [2] require

$$
\mathcal{O}\Big(\max\Big\{n; \frac{\sqrt{n}L_{\mathrm{avg}}}{\mu}\Big\} \log \frac{R_0^2}{\varepsilon}\Big) \qquad (24)
$$

and

$$
\mathcal{O}\Big(\frac{\sqrt{n}L_{\mathrm{avg}}D^2_{\mathcal{Z},V}}{\varepsilon}\Big) \qquad (25)
$$

stochastic oracle calls, respectively. It remains to discuss the effect of batching on the method from (23), i.e., see how the oracle complexity bounds change if instead a single sample F_{ξ^k} at each iteration we use but a batch size of b: $\frac{1}{b}\sum_{i\in S^k} F_i$, where $S_k \subseteq \{1,\dots,n\}$ is the set of cardinality b of indices in the mini-batch. In this case, the methods from [2] give estimates (24) and (25), but multiplied by an additional factor $\sqrt{b}$. This extra multiplier issue was resolved in [69] using the following method:

$$
\begin{aligned}
\Delta^k &= \frac{1}{b}\sum_{i\in S^k}\big[F_i(z^k) - F_i(w^{k-1})\\
&\qquad\qquad + \alpha(F_i(z^k) - F_i(z^{k-1}))\big] + F(w^{k-1}),\\
z^{k+1} &= P_Z(z^k + \tau(w^k - z^k) - \gamma\Delta^k),\\
w^{k+1} &= \begin{cases} z^{k+1}, & \text{with probability } p,\\ w^k, & \text{with probability } 1-p.\end{cases}
\end{aligned}
$$

The authors proved that in the strongly monotone case this method gives estimate (24), i.e., without additional logarithmic factors and without factors depending on b.

The only issue that remains to be understood is whether the current state-of-the-art methods with best complexities from [2, 69] are optimal. The lower bounds from [53] claim that under Assumptions 5 and 2, the methods above are optimal. However, under $L_{\max}$-Lipschitzness of all F_i, $i \in \{1,\dots,n\}$ and Assumption 2, the lower bound from [53] is

$$
\mathbb{E}\|z^k - z^*\|_2^2 = \Omega\Big(R_0^2 \exp\Big(-\min\Big\{\frac{1}{n}, \frac{\mu}{L_{\max}}\Big\}k\Big)\Big).
$$

The question whether this lower bound is tight remains open.

4.3 *Cocoercivity assumption*

In some papers, the following assumption is used instead of Assumption 1.

Assumption 6 (Cocoercivity)**.** The operator F is ℓ-cocoercive, i.e., for all $u, v \in \mathcal{Z}$, we have $\|F(u) - F(v)\|_2^2 \le \ell\langle F(u) - F(v), u - v\rangle$.

Cocoercivity is stronger than monotonicity + Lipschitzness, i.e., not all monotone Lipschitz operators are cocoercive. Note, for instance, that the operator for the bilinear SPP ($\min_x \max_y x^\top Ay$) is not cocoercive. However, if F is L-Lipschitz and μ-strongly monotone, then it is (L^2/μ)-cocoercive. Moreover, the operator corresponding to a convex L-smooth minimization problem is L-cocoercive.

There is no need to use an Extragradient method for cocoercive operators. One can apply the iterative scheme (7) and its modifications for the stochastic case. In spite of this, the first work on cocoercive operators in the stochastic cases used the Extragradient as the basic method [26]. In this paper, the authors investigated methods for finite-sum problems. The subsequent results from [15, 81] give an almost complete picture of stochastic algorithms based on method (7) for operators under Assumption 6. In particular, the work [15] provides a unified analysis for a large number of popular stochastic methods currently known for minimization problems [51].

4.4 *High-probability convergence*

Up to this point, we focused on convergence-in-expectation guarantees for stochastic methods, i.e., bounds on $\mathbb{E}[\mathrm{Gap}_{\mathrm{VI}}(\hat z^k)]$ and/or $\mathbb{E}\|z^k - z^*\|_2^2$. However, *high-probability convergence guarantees*, i.e., bounds on $\mathrm{Gap}_{\mathrm{VI}}(\hat z^k)$ and/or $\|z^k - z^*\|_2^2$ that hold with probability at least $1-\beta$ for a given confidence level $\beta \in (0,1)$, reflect the real behavior of the methods more accurately [50]. Despite this fact, high-probability convergence of stochastic methods for solving VIs is studied only in a couple of works.

It is worth mentioning that one can always deduce the high-probability bound from the in-expectation one via Markov's inequality. However, in this case, the derived rate of convergence will have a negative-power dependence on β^{-1}. Such guarantees are not desirable and the goal is to derive the rates that have a (poly-)logarithmic dependence on the confidence level, i.e., β should appear only in the $\mathcal{O}(\mathrm{poly}(\log(\frac{1}{\beta})))$ factor.

The first, and for many years the only high-probability guarantees of this type for solving stochastic VIs were derived in [60]. The authors assume that F is monotone and L-Lipschitz, the underlying domain is bounded, and F_ξ is an unbiased estimator with sub-Gaussian (light) tails of the distribution:

$$\mathbb{E}\Big[\exp\Big(\frac{\|F_\xi(x) - F(x)\|_2^2}{\sigma^2}\Big)\Big] \le \exp(1).$$

The above condition is much stronger than Assumption 3. Under the listed assumptions, the authors of [60] prove that after k iterations of Mirror-Prox with probability at least $1-\beta$ (for any $\beta \in (0,1)$) the following inequality is in force:

$$\mathrm{Gap}_{\mathrm{VI}}(\hat{z}^k) = \mathcal{O}\Big(\frac{LD_{\mathcal{Z}}^2}{k} + \frac{\sigma D_{\mathcal{Z}}\log(1/\beta)}{\sqrt{k}}\Big).$$

Up to the logarithmic factor this result coincides with in-expectation one and, thus, it is optimal (up to the logarithms). However, the result is derived under the restrictive light-tails assumption.

This last limitation was recently addressed in [49], where the authors derive the high-probability rates for the considered problem under just the bounded variance assumption. In particular, they consider the clipped-SEG for problems with $\mathcal{Z} = \mathbb{R}^d$:

$$z^{k+1/2} = z^k - \gamma \cdot \mathrm{clip}(F_{\xi^k}(z^k), \lambda_k),$$
$$z^{k+1} = z^k - \gamma \cdot \mathrm{clip}(F_{\xi^{k+1/2}}(z^{k+1/2}), \lambda_{k+1/2}),$$

where $\mathrm{clip}(x,\lambda) = \min\{1, \lambda/\|x\|_2\}x$ is the clipping operator, a popular tool in deep learning [46,103]. In the setup when F is monotone and L-Lipschitz and Assumption 3 holds, in [49] it is proved that after k iterations of clipped-SEG with probability at least $1-\beta$ (for any $\beta \in (0,1)$) the following inequality holds:

$$\mathrm{Gap}_{\mathrm{VI}}(\hat{z}^k) = \mathcal{O}\Big(\frac{LR_0^2\log(k/\beta)}{k} + \frac{\sigma R_0\sqrt{\log(k/\beta)}}{\sqrt{k}}\Big).$$

Up to the differences in logarithmic factors, the definition of σ, and the difference between $D_{\mathcal{Z}}$ and R_0, the rate coincides with the one from [60], but it was derived without the light-tails assumption. The key algorithmic tool that allows removing the light-tails assumption is clipping: with a proper choice of the clipping level λ the authors cut heavy tails without making the bias too large. It is worth mentioning that the result for clipped-SEG is derived for the unconstrained case and the rate depends on R_0, while in [60], the analysis relies on the boundedness of the domain, the diameter of which appears explicitly in the rate obtained. To remove the dependence on the diameter of the domain, the authors of [48] show that with high probability the iterates produced by clipped-SEG stay in the ball around x^* with a radius proportional to R_0. Using this trick, they also show that it is sufficient that all the assumptions (monotonicity and Lipschitzness of F and bounded variance) hold just on this ball. Such a degree of generality allows them to cover problems that are non-Lipschitz on $\mathbb{R}^d$ (e.g., for certain monotone polynomially growing operators) and also the situation when the variance is bounded only on a compact set, which is common for many finite-sum problems. Finally, [48] contains high-probability convergence results for strongly monotone VIs and VIs with structured non-monotonicity.

5 Recent advances

In this section, we report briefly on a few recent theoretical advances with practical impacts.

5.1 Saddle point problems with different constants of strong convexity and strong concavity

Saddle point problems with different constants of strong convexity and strong concavity started gaining interest a few years ago, see e.g., [4, 77]. However, even for the particular case

$$\min_{x\in\mathbb{R}^{d_x}} \max_{y\in\mathbb{R}^{d_y}} g(x,y) = f(x) + y^\top \mathbf{A}x - h(y),$$

where the function f is μ_x-strongly convex ($\mu_x > 0$) and L_x-smooth, and the function h is μ_y-strongly convex ($\mu_y > 0$) and L_y-smooth, optimal algorithms have been proposed only recently [58, 72, 116]. These algorithms have the convergence rates

$$\mathcal{O}\Big(\Big(\sqrt{\frac{L_x}{\mu_x}} + \sqrt{\frac{\lambda_{\max}(\mathbf{A}^\top\mathbf{A})}{\mu_x\mu_y}} + \sqrt{\frac{L_y}{\mu_y}}\Big)\log\frac{1}{\varepsilon}\Big)$$

and attain the lower bound, which was obtained in [56, 124] (here one needs to assume that $\lambda_{\min}(\mathbf{A}^\top\mathbf{A}) \le \sqrt{\mu_x\mu_y}$; without this assumption no optimal methods are known).

Note that the algorithm from [58] is built upon a technique related to the analysis of primal-dual Extragradient methods via relative Lipschitzness [28, 115]. As a by-product, this technique makes it possible to obtain Nesterov's accelerated method as a particular case of primal-dual Extragradient method with relative Lipschitzness [28].

For the non-bilinear SPP, optimal methods, based on the accelerated Monteiro–Svaiter proximal envelope, were developed only in the non-composite case [24, 71]. For the non-bilinear SPP with composite terms, there is a poly-logarithmic gap between the lower bound and the best known upper bounds [118]. A gap also appears for the SPP with stochastic finite-sum structure [58, 82, 118]. The stochastic setting with bounded variance was considered in [32, 85, 125].

Further deterministic "cutting-plane" improvements are connected with the additional assumptions about small dimension of the involved vectors x or/and y (see [43, 44, 91]) or with different structural (e.g., SPP on balls in 1- or ∞-norms) and sparsity assumptions, see e.g., [21, 111, 112] and references therein. Here lower bounds are mostly unknown.

In this subsection we mentioned many works dealing with (sub-)optimal algorithms for different variants of SPP. We note that, in contrast to convex optimization, where the oracle call is uniquely associated with the gradient call $\nabla f(x)$, for SPP we have two criteria: the number of $\nabla_x g(x, y)$-calls and that of $\nabla_y g(x, y)$-calls (and more variants for SPP with composites). "Optimality" in the most of the aforementioned papers means that the method is optimal according to the worst of the criteria. In [4, 118], the authors consider these criteria separately. However, the development of the lower bounds and optimal methods for a multi-criterion setup is still an open problem.

5.2 Adaptive methods for VI and SPP

Interest in adaptive algorithms for stochastic convex optimization mainly arose in 2011 after the development of the AdaGrad (adaptive gradient) [33] and Adam (adaptive moment estimation) [63] algorithms. For variational inequalities and saddle point problems, people became interested in adaptive methods only in the last few years, see, e.g., [8,40] (see also [61]). Currently, this area of research is well developed. One can mention here works devoted to both adaptive step sizes [5, 34, 35, 114, 117] and adaptive scaling/preconditioning [12, 31, 80]. Approaches from the second group are based on the idea of a proper combination of AdaGrad/Adam with Extragradient or its modifications. All of the mentioned adaptive methods have no better (typically the same) theoretical rates of convergence than their non-adaptive analogues, but require less input information or demonstrate better performance in practice.

5.3 Quasi-Newton and tensor methods for VI and SPP

Quasi-Newton methods for solving nonlinear equations (unconstrained VI) and SPP are proposed in [75, 121] and [79], respectively. In these papers, local superlinear rates of convergence are derived for the modifications of the Broyden-type methods for solving nonlinear equations with Lipschitz Jacobian and SPP with Lipschitz Hessian. Stochastic versions of these methods for VI and SPP still await to be developed.

Tensor methods for convex optimization problems are currently quite well developed. In particular, starting with [99] it has been shown that optimal second- and third-order methods can be implemented with almost the same complexity of each iteration as the Newton method [39, 89, 97]. Moreover, optimal p-order methods (which use p-order derivatives) significantly reduce the rate of convergence from k^{-2} to $k^{-(3p+1)/2}$ (see [23, 70]). For VI and SPP, the study was initiated in [88, 94] and optimal p-order methods reduce the rate of convergence from k^{-1} to $k^{-(p+1)/2}$ (see [1, 78]) (for k^{-1}, see Theorem 3). However, in contrast to convex optimization, the use of tensor methods for sufficiently smooth monotone VIs and convex-concave saddle point problems is not expected to be as effective. Note that in [1, 78] one can also find optimal rates for strongly monotone VIs and strongly convex-concave SPP. Stochastic tensor methods for variational inequalities and saddle point problems still await to be developed.

5.4 Convergence in terms of the gradient norm for SPP

Several recent advances in the development of optimal algorithms are based on accelerated proximal envelopes with proper stopping rules for inner loop algorithms [68, 70, 71, 109]. Such rules are built upon the norm of the gradient calculated for the target function of the inner problem.

For smooth convex optimization problems, Yu. Nesterov in 2012 posed the problem of making the gradient norm small with the same rate of convergence as a gap in the function values, i.e., proportional to k^{-2} (see [96]). To address this problem, in [96] he proposed an optimal (up to a logarithmic factor) algorithm. This question was further investigated, leading to optimal results without additional logarithmic factors [62, 98] (see also [30] for explanations and a survey). In the stochastic case, algorithms were presented in [38].

For smooth convex-concave saddle point problems an optimal algorithm with $\|\nabla_{x,y} f(x^k, y^k)\|_2$ proportional to k^{-1} was proposed in [122] (see also [30] and [71] for monotone inclusion). For the stochastic case, see [20, 27, 74].

5.5 Decentralized VI and SPP

In practice, in order to solve a variational inequality problem more efficiently and quickly, one usually resorts to distributed methods. In particular, methods that work on arbitrary (possibly time-varying) decentralized communication networks between computing devices are popular.

While the field of decentralized algorithms for minimization problems has been extensively investigated, results for broader classes of problems have only begun to appear in recent years. Such works are primarily focused on saddle point problems [16–18, 90, 108], but we note that most of these results can easily be extended to variational inequalities. Let us emphasize two works that were from the outset devoted to VIs. In [13], the authors proposed a decentralized method with local steps, and [69] presented optimal decentralized methods for stochastic (finite-sum) variational inequalities on fixed and varying networks.

Acknowledgements. The work was supported by Russian Science Foundation (project No. 21-71-30005).

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Interview with Elizabeth Gasparim

Ulf Persson

UP: Let us start from the beginning, not with your great grandparents who came to Brazil from Italy, but with your parents: what did they do?

EG: My father was a businessman, doing economics and finances, and my mother was a housewife, she did a lot of sewing. They were a very loving couple.

UP: So how did your interest in mathematics start?

EG: What a surprising question, no one has ever asked me that. I have not really thought of it at all, I guess I always found mathematics easy at school. We were supposed to do calculations. I just did them and it was always clear to me what I needed to do. In most subjects, what is to be done is not clear at all. People have different opinions about what is right or not, but in mathematics there is rigour with theorems and proofs, I enjoyed the strong concepts of truth.

UP: And that was important to you?

EG: Very much so.

UP: So when did you start to think that you should do mathematics?

EG: I did not, my father did so. I wanted to do music, play the piano, but I was not allowed to. One day my father locked the piano with the key, and decided I should do mathematics instead of continuing in the school of arts where I dedicated all my time to music.

UP: So you obeyed him.

EG: It was not just a question of obeying, my father had a lot of power. There was no choice.

UP: So how old were you then?

Elizabeth Gasparim, © Universidad Católica del Norte, Chile

EG: Seventeen, I had just started the universities, both the school of arts and the federal university.

UP: And why did your father want you to do mathematics?

EG: I have no idea. Maybe we should have asked him, but he died a long time ago.

UP: So this meant that you went to a science-oriented high school. Did you find mathematics interesting at all at that stage?

EG: Yes, I did find it interesting and I also liked chemistry.

UP: What else did you do at school?

EG: I studied languages. English, German and French.

UP: Not Spanish?

EG: That I picked up by myself. I also found French very easy, and of course Italian came naturally without any effort; but actually I mostly did swimming and dancing.

UP: And then you went to university, did you have to leave your home town?

EG: No, I did not have to move, my hometown Curitiba is the capital of the state of Paraná. Every state capital in Brazil has a Federal University. But actually, just before going to university, I made my first trip abroad. It is actually a curious story, would you be interested?

UP: Of course, and it meant a lot to you, I presume.

EG: Certainly. I told my mother that I was going to the United States. She shook her head and said: Betinha, do not live with such illusions, we are poor, we can never afford you travelling abroad. But I was confident and a few days later I won a best student award from the Inter-Americano Institute, that paid my expenses to go to several cities in the US. They wanted me to become an English teacher and work for them. But I did not wish to become an English teacher.

Many amazing things happened to me during the trip, more than I can tell you. Once when I was walking alone at night a man invited me to dinner and then took me to an amusement park. He was nice and handsome, but he warned me never to do such a foolish thing again, walking alone in New York city at night. I was so young, naive and innocent.

UP: And you did not come to grief?

EG: Not at all. It was all wonderful.

UP: So can you tell me about your time at the university?

EG: There is not much to say, at least not what concerns mathematics. I did not know much mathematics, but neither did the faculty. Just to give you an example. One teacher claimed that there was only one infinite cardinal; that all infinite sets had the same cardinality. I objected strongly, I tried to substantiate my claim by explaining that clearly even a piano with an infinite number of keys would never produce as many notes as the strings of a violin …

UP: … Let me stop the film here. Had you ever come across Cantor's notion of higher cardinalities before, or had you actually intuited it?

EG: No, I had never heard about infinite cardinals before that. In fact, the story went around and I got various offers to study with strong mathematicians doing research in set theory. They seemed to be very impressed.

UP: I bet they were. It is really impressive. So please go ahead, sorry for interrupting you.

EG: It is fine, you are supposed to bother me with questions. I often objected to things, giving counter-examples. I was known as the counter-example machine. I did not know about cardinals, but that claim about all infinities being equal revolted my stomach. To give you an idea of how weak my undergraduate education was: when I first started graduate school and a teacher wrote some partial derivatives on the blackboard, I thought she just wrote ordinary derivatives with very poor handwriting. A few courses were lacking in my undergraduate studies. At times faculty were on leave and could not be replaced; those courses, such as the second year calculus, never actually happened.

UP: How did you escape?

EG: I met with Newton Carneiro Afonso da Costa, a mathematician and philosopher, very well know for the creation of paraconsistent logic. I told him I wanted to do research in geometry. Newton gave me a booklet about the Axiom of Choice and told me to come back in a month to discuss with him. I was ready after a couple of days. So, he invited me to give a lecture at the University of São Paulo about what I had learned. Besides the three theorems presented in the booklet, I myself added seven more equivalent statements; needless to say, everything I said must have been well-known, as they were just simple exercises. But in the audience there was the head of mathematics of the University of Campinas and he was sufficiently impressed to invite me to become a graduate student at Unicamp.

UP: But most of all, how had this desire to do research been awakened at a backward university? Most people do not realise that people do mathematical research at all, for many it would be a contradiction in terms. Had you read about research mathematicians, such as Bell's Men of Mathematics?

EG: No. I read no mathematics book really, I read Plato and I loved it. I enjoyed the Phaedo very, very much, and while reading it I thought: What a pity that I was not there. I wanted so much to have been part of the discussion. I wanted to discuss new ideas with people who would carry through careful arguments. I learned to respect and value thinking about geometry by reading Plato. I also learned not to fear death based on the ideas of Socrates. But these were studies I did on my own, using books that my father had. They were not part of my school education.

UP: How was graduate school?

EG: Hell at first. I knew very little mathematics, my undergraduate school taught me close to nothing. I really had to struggle to catch up with my colleagues. For one entire semester I did not swim, which was not at all easy for me, and I slept only about two hours a night. By the end of the semester I was physically exhausted, but the effort paid off, I got all A's and from the second semester onwards I received best student awards.

UP: Impressive. What interested you most in mathematics? So far we have been touching upon the logic of mathematics, but what about the mainstream?

EG: Geometry. I had seen only some basic geometry at school, then in Campinas …

UP: … Campinas?

EG: Yes, at Unicamp, where I got my Master's degree, I encountered Riemannian geometry for the first time and I really enjoyed it. I never took a course in algebraic geometry, not even during my Ph.D. studies, but I read Hartshorne's book on my own.

UP: What is it in geometry that attracts you?

EG: It is visual and concrete. Even in higher dimensions when you cannot draw things, still you can, amazingly enough, have a visual intuition. Moreover, it gives room for imagination, for creativity.

UP: I agree very much, it has a visual, palpable, feel. What about physics and its connections to geometry? You feel that classical mechanics is enhanced geometry, as the notions of time and movement come into play? In the British tradition, mechanics is part of mathematics, and in the classical Tripos at Cambridge, the problems were mostly of what we now would call mathematical physics. Are you interested in physics?

EG: Yes, I am very much interested in physics and in science as a whole. Furthermore, I do not believe in 'physicsless' mathematics. Mathematics and physics go along together as partners in the scientific description of the world.

UP: Did you take physics courses?

EG: No, but I read physics books. I have learned some physics by myself and some from my collaborators. I enjoy talking to physicists. I find it very interesting that physicists and mathematicians think in very different ways about the same concepts. I also publish in physics journals sometimes, and the culture there is very different from that of mathematics, it is more argumentative. In physics you can get into discussions with the referees, and it is possible to sometimes win arguments during a submission process, which is impossible in a mathematics journal, even when the referee says something very wrong.

UP: One bone of contention between mathematicians and physicists is the issue of rigour.

EG: I have been very lucky to meet with physicists that are rigorous in their arguments. But, it is true, some physicists are very careless in their mathematical reasoning and that is a difficulty.

UP: … which may be a sign of impatience, the same way we mathematicians look at logicians as being overly pedantic to the point of timidity. Not all mathematicians are rigorous either, especially not in the past, Archimedes was a notable exception; but that does not prevent them from getting correct results. Sorry for interrupting you.

EG: It is OK. Integrating physics in mathematics leads to a broader view of science that provides a deeper understanding of geometric concepts. As to rigour in physics, I have an amusing anecdote about Atiyah wanting to sue me for an illegal application of the Atiyah–Bott localisation.

UP: Did he really sue you?

EG: Let us come back to that later, just be patient.

UP: On the other hand, one should not play down rigorous thinking in mathematics. It is a matter of hygiene after all, to spot mistakes. In fact what started me on mathematics was Euclidean geometry with rigorous proofs I encountered as I was about to turn fourteen, which is now phased out of most modern school curricula, which is a shame. One wonders whether mathematics is not so central in Brazilian culture. To be a bit vulgar: One associates Brazil with football, beaches, samba dancing, and carnivals.

EG: I would not say that mathematics is marginalised in Brazilian culture. There are Brazilians who are making significant contribution to mathematics, even though the general population is unaware of it, but this is also the case for most countries. Brazil is a country full of contrasts, there are children not going to school, but there are also strong universities. Yes, mathematics in my university was weak, but it has improved since then. The situation is much better for people in Rio de Janeiro and São Paulo, where the strong schools are concentrated.

UP: So after all, attending a provincial Brazilian school in the late seventies put you at a disadvantage.

EG: Yes, this is true.

UP: What else besides Euclidean geometry did you learn?

EG: Some basic point-set topology.

UP: No number theory?

EG: Just the elementary number theory concepts required for the exams, which happened at the end of the first year of Master studies. Then in the second year I wanted to do real research. The quest for originality motivated me more than anything else. In fact, it was suggested that I translate some articles and write detailed versions of a few classical theorem for a Masters's dissertation, but I refused, saying I would not just repeat and prolong texts that were already in the literature. Wanting to do original research was a complicated issue, it took some struggle to be able to do so.

UP: And they relented. So what did you do after you finished your course work and could resume swimming?

EG: I wrote my Master's thesis working with professor Ofelia Teresa Alas from Universidade de São Paulo. She was very happy to let me try to work on finding original results. In Campinas no one believed a student could get any original result in the short time allowed for the Master thesis. I ended up working on compactifications and products of spaces with high cardinalities, and I got more than one original result.

UP: This testifies to your commitment and initiative. Let me pose some provocative statements, which may cause some offence. Higher-order cardinalities is the kind of mathematics only logicians engage in. Does mathematical logic play an important role in Brazilian mathematical culture? You mentioned earlier how you got your foot in by being invited to give a lecture on the Axiom of Choice.

EG: There was a big fashion of logic in Brazil at the time when I was doing my undergraduate studies, but later on the logicians were expelled to the philosophy departments. It is also true that in my early mathematical education philosophy played an important part. I did participate in logic conferences and there was interesting content there, for example there were precise applications of logic to the legal system, such as in trials with large amounts of testimonies containing contradictory statements. There were also some interesting attempts of applications of logic to psychoanalysis, which might have originated in Brazil.

However, as you say, logic and set theory were looked down upon by other mathematicians, and in fact, in Campinas I was promised a higher scholarship if I fulfilled two requirements; first of course was to get the best grades and second not to do my thesis in logic. A clear bribe that worked out well.

UP: So your Master work was a diversion? What did you write your Ph.D. thesis on? And where did you go? You were not entirely satisfied with Campinas. Did you go to Universidade de São Paulo?

EG: No. I was offered a Brazilian scholarship to study at Lille in France, but after six months I requested to transfer to Stony Brook instead. My request was in principle approved, and I went to New York, but soon afterwards my Brazilian stipend was cancelled completely, so there I was with no money, and no return ticket, stuck in New York.

UP: A nightmare no doubt.

EG: Yes, but then I was saved in a way connected to my fight for originality during Master studies. Professor Wojciech Kucharz happened to be participating at a conference in Campinas and heard the discussions where I insisted to write an original result for my Master's degree. When he found out that I had been left without a Ph.D. scholarship, he talked to his former colleague Charles P. Boyer, who then gave me a scholarship for the University of New Mexico. So, I did my Ph.D. in Albuquerque, New Mexico, and wrote my thesis on moduli spaces. In fact, of moduli spaces of instantons, an inspiration from physics, which translated into working on moduli spaces of vector bundles.

UP: More specifically?

EG: I did concrete calculations of moduli spaces. I enjoyed the fact that the concrete calculations presented indisputable facts, as opposed to personal interpretations. My results also gave counterexamples to some presumed results (certainly wrong) by a mathematician that was very well respected by the faculty at UNM. So, it was difficult to get people to hear me at first, because they assumed I was wrong and would not listen to my arguments.

UP: … You were after all the counter-example kid …

EG: Right, the counter-example machine (laughing). One difficulty at UNM was that the library had very few books. So, I was dependent on Hartshorne's book, which I bought myself. I also had Okonek, Schneider and Spindler's vector bundles book, which was given to me by my supervisor, since he had bought a newer edition. I read those two books carefully and they both had water stains because I used to take them for reading by the swimming pool. Then, several times, when I had serious questions about algebraic geometry, I emailed Hartshorne himself.

UP: That was brave, but what could you do?

EG: Yes, at that time one could not just google it. Other students were aghast saying that I must be asking trivialities and boring him, but he was extremely nice. Almost all of my questions got replies such as: this is a result of Grothendieck, this is a result of Serre. He would reply with long emails and it was wonderful. But in case other students were right, in my next email to Hartshorne I apologised if I had been asking silly questions, but I also said, that no matter what, I was likely to ask more. To which he replied: You are welcome! His reply made me very happy, and of course later I sent more questions. Eventually I got to meet him because I found an error in his book. It was a great visit. I should tell you about that.

UP: By all means, but in due time. Hartshorne's book is very algebraic. It takes a general approach and includes finite characteristics. Griffiths and Harris' book, on the other hand, takes a more classical, some would say old-fashioned approach, by only considering complex numbers and thus being more analytic. There are no integrals in Hartshorne's book, but plenty in Griffiths and Harris' one. Would that have been more congenial to you?

EG: In fact, once I met with Nigel Hitchin at a conference and he told me to read Griffiths and Harris' book, this is how I first heard of it. Several complex variables turned out to be highly interesting and I was very surprised by Hartog's theorem. In general complex analytic geometry is more natural to me than algebraic geometry. Although, I do use algebraic calculations in positive characteristics when doing computations. For example, when calculating Hodge diamonds using Macaulay2.

UP: Let us go back to your time at UNM and your mathematical work there.

EG: Certainly. Jim Milgram was visiting UNM for a semester and I went to his office to ask him a question. He referred me to another visitor who happened to be in his office. The visitor, whom I did not know, took me down to the library and picked out a book and an article he thought I should read.

UP: … so there was a library …

EG: Wise guy, yes, but few books, yet luckily it did have what we needed that day. We talked for over an hour. Then the visitor told me we should go to the seminar that was about to start. He was the speaker and during his talk he mentioned my work very positively. He said: I am very happy that today I learned some new mathematics. Elizabeth just explained this and that …

UP: That must have been gratifying.

EG: Yes, that was positively surprising, especially since the visitor turned out to be the very famous Dennis Sullivan. So, on the strength of his opinion of what I had done, I was urged to defended my Ph.D. thesis a couple of weeks later, without having time to type it up!

It was amazing, I had been trying to explain what I had done for quite some time, without any success. But, as soon as Dennis found it good, then everyone immediately liked it.

UP: This is how the world works after all. And then?

EG: And then, despite my objections, I was sent to Mexico, where I was supposed to stay for one year.

UP: So things are being done above your head?

EG: Very much so. It turned out impossible for me do any research in Mexico. There were lots of difficulties, for example, I shared the office with a woman who brought her baby to work and the baby would cry very often. This was not the only problem there. There too, I had counter-examples to a seminar talk. The way the speaker and the audience reacted to my counter-examples was unbelievable. The speaker continued on, speaking about a result that had been clearly proved false by my comments. Everything went on as if I did not exist. I tried a second time, but was ignored again. After the talk the speaker told me: 'You are a woman in Mexico, nobody will pay any attention to what you say during my talk.'

I was desperate. That very same day, in the middle of my desperation, I wrote to M. S. Narasimhan, the director of mathematics at the International Centre for Theoretical Physics in Trieste. I explained that I was expected to spend a year in Mexico, but I could not do mathematics in the conditions offered to me. I asked him point blank whether I could go to ICTP, and if so, what was the earliest date I could arrive. He replied back 'Yesterday'. So, right away, I made my way to Trieste. ICTP was another planet, a wonderful place, with a perfect library, and above all with several very wise people who knew lots of algebraic geometry and who were extremely kind and willing to reply patiently to my many questions. In a few months there, I learned far more than in all my previous years of study. That was the actual start of my research career.

UP: You must have had quite a reputation being invited there on the spot.

EG: I never thought about it this way. I was just so relieved that Narasimhan understood immediately that I was in a situation where I needed help. He was an extremely wise person, so a short message from me was enough for him to anticipate my visit by over 6 months.

UP: So once again you going to Trieste really was a turning point in your career?

EG: You could say that it turned me into a mathematician. Being there I got invited to visit Edoardo Ballico, who has been a valuable collaborator ever since. I have learned so much from him. In Trieste I also learned many things by just talking to people over coffee. There I was exposed to lots of new material and I enjoyed listening to various scientists points of view during lectures.

UP: Who were you primarily influenced by?

EG: Narasimhan of course. We all respected and admired his beautiful theorems. His presence attracted many good people, in particular algebraic geometers from India. One of his visitors, Vishwanath, invited me to visit a mathematical institute in India, and that visit turned out to be precious. I not only learned algebraic geometry from Seshadri's group in Chennai, but I also met with strong physicists and learned about instantons and monopoles. Furthermore, I encountered yoga, which has played an essential part in my life ever since. I attribute my good health and physical endurance to yoga. I learned how to avoid minor ailments such as stomach aches which affected me often before. May I insert something?

UP: By all means.

EG: In fact, returning from India I had an unplanned four day stop over in England. So, I contacted Hitchin, and asked if I could come to talk to him a little. He very graciously inserted me into his busy schedule, and as we talked along, he cancelled his following appointments and we ended up talking for eight hours. I learned so much from this meeting. Besides, I never expected that it was even possible to discuss mathematics for so long. I loved it.

The next day he invited me for lunch at his college and we talked a bit more. He had also reserved accommodation for me there for two days and had asked Frances Kirwan to reserve accommodation in her college for me the following two days. I also got to talk with her one afternoon. She was extremely positive and encouraging about my results.

UP: So now being a professional mathematician, how did things play out?

EG: As soon as I finished my Ph.D. I had several job offers in Brazil. My inclination was to go back to Campinas. But, my Peruvian boyfriend, also a mathematician, wanted to live by the beach so we ended up in Recife, northeast Brazil.

Things started out nicely, I saw to it that they hired good mathematicians. As a consequence, the ranking of that mathematics department improved two points in the national classification, which meant a lot.

However, during my third year there, there was a particularly problematic sexist issue.

UP: Would you like to digress on it? Go ahead.

EG: OK. A nice outcome of the various discussions I had with Hartshorne by email was that he invited me to talk at his seminar in Berkeley. Karen Uhlenbeck also invited me to spend a month at UT Austin. There were travel funds available at our mathematics department, but I was turned down based on a ridiculous reasoning. A professor claimed that he was sure Hartshorne could not possibly be interested in the mathematics I was doing, and certainly the reason he was inviting me there was because he wanted to flirt with me. Travel funds were then given to other people who had applied to much weaker universities.

I was told that if Uhlenbeck wanted me to visit her, she should fund my trip herself. She did. I ended up doing the whole trip on funds provided by my hosts, and during it I was recommended to stay in the US and so I did. I spend an entire year at UT Austin, and then got a job in New Mexico.

UP: You mean your alma mater?

EG: No, I went to work at New Mexico State University in Las Cruces, not UNM in Albuquerque.

UP: Maybe we could extend the digression, you promised me to say something about an error in Hartshorne's book.

EG: Yes, this little error was one of the original reasons for the invitation. It is a fun story actually, should I tell it?

UP: By all means, go ahead.

EG: During my Ph.D. I had, as I already told you, two books, Hartshorne and Okonek–Schneider–Spindler. At some point I found an exercise in Hartshorne's book for which my answer did not agree with the expected one at all. I found it unlikely that there would be a mistake in the book, since it is used in so many good schools and was read by so many students over the years. I immediately wrote to Hartshorne, but a week passed without a reply.

Since I could not find anything wrong with my calculations, I asked a couple of mathematicians what answers they got. They said this was too trivial for them to be bothered with.

UP: That figures.

EG: Well … so I thought it was better to ask someone even stronger and I emailed Richard Borcherds. I knew he had a website where he posted solutions to problems in Hartshorne's book. He replied nicely, but said that he could only solve problems in Chapter 1, not in Chapter 3. That was quite fun, to get such a reply from a Fields medallist.

UP: It is getting better and better, please proceed.

EG: So, I decided to take a poll, and wrote to a dozen people. I obtained eight different solutions, and four 'too trivial to be bothered with' replies. One answer, by Ian Morrison, was the same as mine. Then, I realised how a slight change of the problem would yield the desired solution. So, I wrote to Hartshorne again, and he replied in a very amusing way. He wrote: Dear Elizabeth, you get a golden star for finding an error in my book, it has been eight years since the last error was found and I was convinced it was entirely correct by now. So, I replied: Gold! My favourite, I must go collect it. This is how the idea of my visit to Berkeley started.

UP: So on the strength of this you got invited?

EG: Yes, so it was. During the talk I gave at Berkeley, there was this mathematician who interrupted me often and disagreed with several of my statements.

By the end of the talk his observations had generated lots of questions, and together with Hartshorne we stayed another hour after I finished the talk continuing the discussion.

UP: What had you talked about?

EG: I spoke about local characteristic classes of sheaves. At some point Hartshorne asked me: What reference did you use to compute this? So, I told him: I can recommend a very good book, and handed him my copy of his own book. It was very amusing. I had done everything with the Theorem on Formal Functions, directly from his description of it, but somehow it seemed that the explicit calculations in coordinate charts were not so familiar to them. At the end of the discussion, the other guy finally agreed with me. He said: I take it back, I was wrong, you are right. It was incredible. I did not know it was possible for a man to admit being wrong.

UP: This would only happen in mathematics I would say when the love and respect of truth trumps individual ego.

EG: The whole discussion was very interesting and I learned a lot. Then, the guy was leaving, and I called him back and said: Hey, what is your name? I do not know you. So, he shook my hand and said: Okonek. At which point my face must have turned green. I do not think I would have been nearly as bold to disagree with him strongly, had I known who he was. It was a wonderful experience.

UP: What a coincidence.

EG: It was a great privilege to witness such a coincidence. You see, I was a young mathematician who knew very little, discussing with people who knew far more than me. Later, when I proved new theorems, Okonek invited me to speak in Switzerland a few times.

UP: Very well, and returning to your job situation. You followed the advice to stay in the US and got a position at New Mexico State University in Las Cruces.

EG: I really enjoyed it there, I like New Mexico. It is extremely interesting to teach students who work at the national laboratories: Sandia, White Sands and Los Alamos. New Mexico is a very pleasant place to live.

During my fifth year working in Las Cruces, I gave a series of lectures in Münster (Germany), where I had collaborators both in the mathematics and in the physics departments. While in Münster, I got a call from the University of Edinburgh, an invitation to go for a job interview. It came as a complete surprise.

UP: And you got the position?

EG: Yes, I was very happy working at the university of Edinburgh.

UP: Did you have much contact with Atiyah?

EG: Of course, Sir Michael was always extremely positive about my work, and talking to him was always fun. There are lots of fun stories about my interactions with him.

UP: Go ahead. What about his suing you?

EG: It was a joke of course, and we had a laugh, but people around were all quite confused. I pretended to be very scared and said: Sir Michael, please do not sue me.

UP: Tongue in cheek I presume.

EG: Clearly, but in retrospect it would have been great if he had actually sued me.

Elizabeth Gasparim and Sir Michael Atiyah, © A. Ranicki

UP: What on Earth made you leave Edinburgh?

EG: I loved working in Edinburgh (except for the weather of course). There, I had friendly colleagues doing beautiful research. Scotland has a lot of very creative people. I was proud of being part of the University of Edinburgh.

UP: I dare say.

EG: But then some professors from Brazil came to ask me to work in Campinas, offering me full professorship. I did have a permanent position, but a lectureship.

UP: So what?

EG: Of course just the formal change of academic standing would not have made me move. But, according to them, people were only hiring internally, and the quality of geometry was going down year after year because nothing actually new was being accomplished. They insisted that I was in a unique position to help with the variety of themes in the geometry group, they said they needed my help, and asked me to propose a 5 year plan to bring to Unicamp research themes in geometry that had never been done there before.

UP: You were invited to be the big fish.

EG: I did not think of it in that way, I wanted to help. They claimed to need me not just for bringing new research in geometry, but also to stop the predictability of mathematics obtained by people hiring 'subsets of themselves'.

UP: Quite a bait, and you swallowed it?

EG: Yes I did. That seemed to me a worthwhile cause, so I wrote a long-term original research plan, a 10 year one, on themes that had never been studied in Brazil. It was a mathematically productive time. Luiz San Martin gave me his book on Lie algebras, and taught me how to use Lie theory to solve several of the geometry questions I had proposed.

UP: So happy ever after?

EG: Not exactly. By the written contract, after one year in Campinas, I would become a permanent professor, provided some mild conditions were satisfied. I easily satisfied all the requirements, but three years later, there was still no sign of them fulfilling the contract. Then they finally opened a competition, but not in geometry, in a completely general area, so that many people signed up for it. But the real trouble, as they explained was that, 'assuming I won the competition', for one year I would remain without any salary while the process of hiring went through (scary, my mother depended on the money I sent her monthly).

UP: So if the fish is big, the pond is small.

EG: I certainly had not expected it, that the written contract would simply be useless. Well, to be fair, the people who sabotaged the hiring were not the same ones who came to Edinburgh to ask me to come to work there.

UP: So you went to Chile.

EG: Yes, I got a position in Chile. Chile is a comfortable place to work, as it is safer, one can walk around without worries of being robbed. It has a stable economy. Chileans drink tea in the afternoon, like the English and unlike the other South American countries, made of coffee drinkers.

UP: Now let us change track. This interview is after all part of a series of Women in Mathematics. So you are a woman in mathematics. What are the disadvantages and, not to forget, the advantages?

EG: There are difficulties in dealing with sexist people in more than one way. I have already mentioned a few cases.

UP: I am all ears.

EG: Several times I encountered sexual harassment in the form of awful statements of the type 'I will approve your grant request only if you have sex with me'. There are also many cases when sexist men presume that people can not possibly value my research, which is somehow even worse.

UP: Being more insidious?

EG: For example, once I had a wonderful meeting with Edward Witten at the IAS Princeton. It was a great experience. I got to talk to him about some mathematical physics construction that I had been trying to explain to lots of people, yet I could not get anyone to understand me. Witten immediately understood what I said and gave me interesting suggestions. I felt like the Little Prince when finally someone understood his drawing was not a hat, but a snake eating an elephant. I left the meeting very happy and when someone asked me how the meeting went, I replied it was great and that Witten seemed to like the calculations I showed him. Then one guy nearby immediately said: 'Certainly he was just being polite because he found you pretty, he can not possibly be interested in what you are doing'.

UP: That must have been frustrating.

EG: Yes. If I prove a theorem, that does not depend on whether I am female or on how I look.

UP: How is your experience with events designed for women in mathematics?

EG: As far as participation in conferences or gathering for women in mathematics, my experience varies a lot.

On one hand, I have participated in a couple of truly wonderful gatherings of women. Once Karen Uhlenbeck asked me to coordinate the seminar talks for a Women in Mathematics events at the IAS Princeton. I invited strong young female mathematicians and it was a productive event.

At New Mexico State University I got to be supervisor for the female students of the college of Arts and Sciences as part of a mentoring program which was funded by NSF and coordinated by Lisa Frehill, and this was truly a program that helped women to advance in research.

However, in Latin America, I have more than once been declined participation in conferences for women, in various countries, even when I had my own funds to attend. Among the reasons to reject me, the most outrageous one was: 'After you put so much silicone in your body, then you cannot expect to be respected by other women mathematicians'. But, I do not have any silicone implants, nothing of me is made of plastic, this is just how I am made, yet I am being punished for not fitting their ideal of how women mathematicians should look like.

So, it varies a lot. Of course, the difficulties that I have faced are nothing compared to what happens in some countries where women are not allowed to go to school.

I have found it easier to be a female mathematician in Europe. Of course, in Europe there are such brilliant role models.

UP: As who?

EG: Think of the amazing and unique accomplishment of two Nobel prizes in science obtained by Marie Curie. This settles any doubt about a woman being able to do science.

UP: Actually, originally only her husband was considered for the prize, but then the legendary Swedish mathematician Mittag-Leffler allegedly made a suggestion, the rest is history.

EG: My best personal experiences so far as being respected as a scientist regardless of gender have been in India. In every experience I had in discussing with mathematicians and physicists there, the interest for the scientific discussion was clearly stronger than any possible judgment of my gender, or personality, or the colour of my dress.

UP: Really?

EG: My dresses are much too often a topic of discussion.

UP: Go on!

EG: They explained to me that the culture in India is such that people respect and value the work of scientists and find that the variety among scientists is a key point for obtaining the most creative work and for producing better science. Variety of people implies variety of ideas, implies variety of scientific results. This incidentally is why I find it so pleasant and interesting to arrange international gatherings, getting in contact with all kinds of people of different cultural backgrounds.

Maybe I should emphasise that even if I loved being in India and had very positive experiences there, I would not be so presumptuous to claim that India is best as to gender equality. I am only reporting on my own experiences.

UP: Fair enough. This is what we after all are looking for in an interview. And now to the advantages, are there any?

EG: There are less advantages. A woman must do at least twice as much as a man to obtain the same level of recognition. However, meeting women in leading roles, who are able to advance the cause of women as scientists, such as Karen Uhlenbeck and Lisa Frehill is great and I admire their strength. We can look at the teams formed by strong women scientists who support each other as having been positively motivated by the opportunity created by unfair treatment.

UP: This sounds like there are none. What about being surrounded by so many clever men?

EG: (laughing) Funny. Such an idea has never occurred to me. Doing mathematics one does get to meet clever people, men as well as women, and it is a privilege. But then, for example, in the school of economics one also meets with clever people who in addition do not dress like zombies.

Ah … that reminds me of how I found the Geometry and Topology session of the International Congress of Mathematicians in Copenhagen this summer, where I was so very happy to present a short communication. I knew at which metro station to arrive, but the university was enormous and I had no idea in which direction to go. I asked several people, but nobody seemed to know how to get to the mathematics building. Lots of people arrived at the same metro station and walked towards the university campus. So, I decided to follow those men who had the ugliest beards. A winning algorithm. Indeed I arrived at the mathematics building (laughing). The glamorous lives of mathematicians!

UP: We are running out of time, you must be getting tired.

EG: I could continue, but I do have another meeting soon.

Elizabeth Gasparim in Miami, © Nuno Cardoso

UP: We have to think about the readers. I would have liked to ask you about philosophy and mathematics, but if we take a more personal approach?

EG: Approach to what? Mathematics, or life in general.

UP: Why not both?

EG: Then I recommend: Be happy! There is an entire school of spiritual/philosophical approach to life, including science and creativity in general, that sometimes is summarised as: effortless manifestation. There is no reason whatsoever why mathematics should make you miserable, as most people seem to think. I love to swim, to walk, to dance, and when doing so, ideas come effortlessly. It is actually rather easy and pleasant, the life of a mathematician, when you think about it.

UP: And with this pleasant exhortation it might be a good way of rounding off. But before that I cannot pass up the opportunity to refer to your swimming. I have heard that you swim almost at the Olympic level.

EG: I did not learn to swim until I was in my mid-teens, at an age when many already got their medals. I wanted to swim earlier, but there was no nearby swimming pool.

UP: So you were over the hill before you even got to climb it. But you competed? How did that start?

EG: At the university, when I started maths, instead of going to classes, I would spend all the afternoon swimming. Then in Recife, I used to train swimming with the undergraduates, mostly guys, doing about as well as they were, not thinking much about it. Then the coach bribed me to participate in a competition by offering me a good lunch. When I won by quite a wide margin, I was very surprised. I had not realised that by training equally with the young guys I would be well prepared for a Master swim competition. At the end of the race, I looked back: Where are all my competitors?

UP: Maybe, you took a wrong turn.

EG: They were still quite far behind (laughing). I have not participated in the Olympics, but I have won medals in Brazil, in the US, and qualified for the world Master competition in the 200 butterfly, though I did not go. I very much hope to have the opportunity to train again with a team and a coach. I must add though that I am not a competitive person. I swim for fun, just as I do mathematics for fun. I did not train very hard, I prepared myself for competitions by doing about half the training the others did, plus some yoga. It was effective. My colleagues called me the 'effortless butterflier', so it seems that my reputation as a swimmer is better than the one as a mathematician (laughing).

UP: I am very much impressed. I would love to ask you more, it seems that we have material for another interview for another magazine; but enough is enough. Thank you very much for having engaged in this interview.

EG: It has been a pleasure. Thank you so much for interviewing me and for paying attention to my story.

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A richer gathering: On the history of the Nordic Congress of Mathematicians

Laura E. Turner

In this text, we consider the origins and evolution of the Nordic Congress of Mathematicians (NCM)*, held once every four years in Sweden, Finland, Denmark, Iceland, or Norway.*

1 Introduction

The 29th Nordic Congress of Mathematicians (NCM) will be held in Aalborg, Denmark, in July 2023, under the auspices of the Danish, Finnish, Icelandic, Norwegian, and Swedish national mathematical associations and in collaboration with the European Mathematical Society. This particular meeting also coincides with the sesquicentennial of the Danish Mathematical Society. Typically held once every four years, these meetings draw together scholars based not only in these nations, but also well beyond.

But for the scheduling and primary constituents, the modern incarnation of the congress only hints at the spirit the early gatherings were intended to embody. Yet the NCM was the product of a specific time and place. In the political tension following the secession of Norway from Sweden in 1905, the *Scandinavian* Congress of Mathematicians (SCM), as the NCM was formerly known, was originally framed as the extension of a "brotherly hand", one intended to help shape a shared Scandinavian identity for mutual cultural benefit through fruitful scientific cooperation [17].

2 Origin

Although the political map of Scandinavia[1] was relatively stable during the second half of the 19th century when compared to the rest of Europe, political changes early in the 20th century resulted in a reconfiguration of relations between Denmark, Finland, Norway, and Sweden. Of particular importance in the present context was the dissolution of the union between Norway and Sweden in 1905. Unsurprisingly, one outcome of this development was political tension, which temporarily disrupted scientific and cultural exchanges between the two nations. It was in this connection that Swedish mathematician Gösta Mittag-Leffler (1846–1927) conceived of the idea for a pan-Scandinavian mathematical meeting.

By then, Mittag-Leffler was established and well connected within the international mathematical research community. Upon completing his doctorate in Uppsala in 1872, he had studied in Paris and Berlin and built on Karl Weierstrass' work by proving the so-called Mittag-Leffler Theorem of complex analysis, the focus of his research activity between 1876 and 1884. He also founded *Acta Mathematica*, known as the first international journal of mathematics, in 1882, and served as its editor-in-chief until his death. He had a vast network of scientific correspondents and worked actively not only to shape the development of mathematics in his own country but also to shift the image of Sweden from that of a peripheral player to one that was accepted as serious and important among the major mathematical powers. The SCM was also relevant in connection with these latter aims [16, 17].

Mittag-Leffler evidently broached the idea for the SCM to some of his Scandinavian colleagues during the 1908 International Congress of Mathematicians (ICM) in Rome [8, 14]. His subsequent outgoing correspondence references various aspects of the planning, and although his influence is readily apparent in the organizational process, he was not engaged in this work alone. When official invitations were extended in Spring 1909, the signatories hailed not only from his own institution, *Stockholms Högskola*, but also from the two national universities in Uppsala and Lund. The four-day gathering would take place in Stockholm that September.

As for his intentions, they were first and foremost mathematical. "The intention is to *work* at the meeting," Mittag-Leffler stressed in a letter to Finnish mathematician Ernst Lindelöf (1870–1946), "so that we all have the greatest possible scientific benefit from it". Costs should be kept low, and the customary festivities reduced to a minimum.[2] Beyond this, Mittag-Leffler had a vision for the SCM that harkened back to the cultural and political currents of "Scandinavianism" embraced by many Danish and Swedish intellectuals

[1] This controversial term usually refers to Denmark, Norway, and Sweden. "Nordic" also includes Finland, Iceland, the Faroe Islands, Greenland, and Åland. Here, "Scandinavia" will refer to Denmark, Finland, Norway, and Sweden, though sometimes "Nordic" is used interchangeably.

[2] See Mittag-Leffler's letter to E. Lindelöf of 24 March, 1909.

in the mid-19th century, in part a reaction to perceived pressures from powerful nations to the east, south, and west. He advocated a collaboration between Denmark, Finland, Norway, and Sweden in matters of defense, diplomacy, and economic policy [16] and hoped that through mathematical exchange, the SCM could spark a new sort of Scandinavianism [17], one that might strengthen the four nations culturally. As he phrased it at the 1909 meeting,

> In the herb garden of mathematical knowledge grow plants of a most varied kind, and it is not completely the same type of harvest that the [different] colleges of the north usually reap. How much richer our gathering therefore becomes when it includes mathematicians from all the north. [17]

Moreover, according to the report of Danish participant Carl Christian Hansen (1876–1935) published in the Danish newspaper *Politiken*, Mittag-Leffler viewed such collaboration as protective. Denmark, Norway, and Sweden, in particular, had a common historical development and cultural and linguistic connections. As he argued at the final banquet, "individually we are too small to benefit from standing on our own and our culture is too precious not to be carefully guarded" [17].

2.1 *Reception and participation*

The 1909 SCM drew roughly 130 registered participants and boasted 35 speakers from all four Scandinavian nations. Lectures were either general, lasting 45 minutes and treating important progress in various areas of the mathematical sciences, or special, not exceeding 20 minutes. Those giving the latter were instructed to bring necessary formulas and diagrams drawn clearly on cardboard to avoid the loss of time in writing demonstrations on the blackboard. Topics ranged from "real singularities in the three-body problem" (K. Sundman, Helsingfors – now Helsinki) to "convergence of series of orthogonal functions" (J. Mollerup, Copenhagen), to "new arithmetic properties of algebraic numbers applied to diophantine equalities" (A. Thue, Kristiania – now Oslo), to the "mathematical determination of pension" (K. Wicksell, Lund) [7].

Although the costs of the meeting were to be minimal, the affair was nevertheless festive, with formal and social programs modeled after those of the ICM [17]. The congress received several Royal telegrams, and the Crown Prince of Sweden attended Mittag-Leffler's opening lecture. The first evening featured a welcome party at the Grand Hôtel in Stockholm, and one day later, a "special train" shuttled participants, diplomatic representatives, and Swedish Cabinet Ministers to a dinner at Mittag-Leffler's villa in Djursholm [7]. The farewell banquet was held at the historic Hasselbacken restaurant, where 55 attendees each paid 10 Swedish crowns for a meal including coffee, wine and other beverages, and floral decorations and waitstaff [15]. Only congress delegates were invited to this dinner, but many brought their families to the meeting, combining it with a holiday [17]. This practice continued for decades, as did the inclusion of informal activities, often showcasing cultural, historical, or natural treasures. At the 1984 NCM in Reykjavik, for example, 36 individuals accompanied the 129 registered participants, and time was allocated to letting them become acquainted with one another and with the country and its people through a reception at the Kjarvalsstaðir art museum, an evening concert by Hallgrímskirkja Motet Choir, and a day trip through Upcountry Árnessýsla [13]. Similarly, in 1913, participants saw a National Theater performance of Norwegian playwright Bjørnstjerne Bjørnson's 1885 comedy "Geografi og kjærlighet" ("Geography and love") [14], and in 1922, went on a steamboat excursion to the maritime fortress Suomenlinna and the eastern archipelago of Helsinki [10].

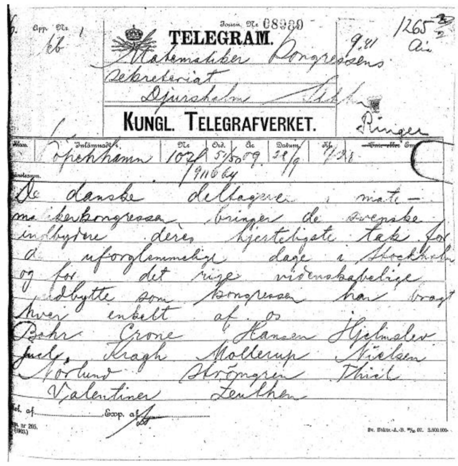
TELEGRAM.
Matematiker Kongressens Sekretariat Djursholm
KUNGL. TELEGRAFVERKET.
De danske deltagere i matematikerkongressen bringer de svenske indbydere deres hjerteligste tak for de uforglemmelige dage i Stockholm og for det rige videnskabelige udbytte som kongressen har bragt hver enkelt af os
Bohr Crone Hansen Hjelmslev Juel Kragh Mollerup Nielsen Nörlund Strömgren Thiel Valentiner Zeuthen

Figure 1. 1909 telegram from Danish participants to Swedish hosts. National Library of Sweden, MS L 233.

That the 1909 meeting was viewed as a success is evidenced by a telegram sent to the Swedish hosts by several Danish participants conveying "their warmest thanks for the unforgettable days in Stockholm and for the rich scientific exchange" that the congress had brought them (Figure 1). Prior to the meeting's end it was decided that the Stockholm gathering was to be the first in a series of congresses. Denmark would host the second in 1911, and in 1913, Norway would host the third. As Mittag-Leffler had proclaimed at the end of his opening speech at the 1909 SCM, "the fruitful cooperation among the culturally connected and equivalent peoples of the North is not the dream of a fool but rather a cornerstone on which the future of the nations is to be built" [17] (Figure 2).

Further attesting to the positive reception of the SCM, the second meeting was supported financially by the Danish State and private individuals, with the Carlsberg Foundation subsidizing the publication of the proceedings. The 1913 meeting and proceedings

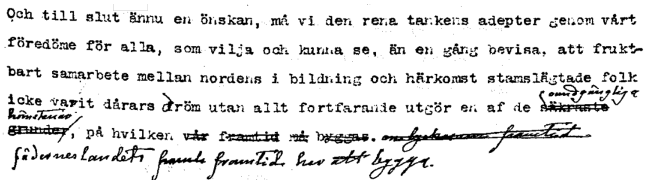
Och till slut ännu en önskan, må vi den rena tankens adepter genom vårt föredöme för alla, som vilja och kunna se, än en gång bevisa, att fruktbart samarbete mellan nordens i bildning och härkomst stamslägtade folk icke varit dårars dröm utan allt fortfarande utgör en af de ~~säkraste grunder~~, på hvilken ~~vår framtid må byggas.~~

Figure 2. Final remarks of Mittag-Leffler's opening speech at the 1909 SCM. National Library of Sweden, MS L 233.

were similarly funded by contributions from the Norwegian government, life insurance companies, and private donations [14]. That said, the perceived value of the SCM to scholars, philanthropists, government officials, and commercial executives did not always translate to the broader public. In [12], Sørensen notes that the opening ceremonies were described in *Politiken* as "extremely subdued", and that

> Only once in the middle of Prof. Zeuthen's lecture came a crash; one of the student marshals couldn't take any more, and just as the renowned old man at the rostrum dwelt on Theodorus of Cyrene and on the long-dead colleague's pursuits of square root extraction, the young man became ill. He fell down with a smack and was lifted up and led out of the hall with one of Scandinavia's most famous mathematicians under each arm. Many eyes followed him with envious expressions.

Sørensen highlights a similar tone in the coverage of the 1913 SCM in Norwegian newspapers, which described the meeting as noteworthy and the lectures as "scientifically interesting" but completely inaccessible to the general public. When Henrik Mohn (1835–1916), a professor of meteorology in Kristiania, spoke to a reporter about the lecture of Gustaf Strömberg (1882–1962) on his analysis of air temperatures in Stockholm, he cautioned that it was impossible to give a popular account. Strömberg's lecture was nevertheless featured in the newspaper *Aftenposten* by a journalist who had

> rarely, in any case, seen more off-putting mathematical equations than those Mr. Strömberg drew on the blackboard […] What were they talking about? Don't ask. Amplitude, lunar period, dynamic effect and an arbitrary coefficient in the 5th or 6th power […]

2.2 Collaboration and Cooperation

Despite its novelty, the SCM was not the first gathering of Scandinavian scholars. A history of scientific collaboration aimed at the exchange of literature and ideas had taken place since 1839, albeit at irregular intervals, via the *Skandinaviske Naturforskermøder* (SN), the meetings of the Scandinavian Association for the Advancement of Science. These meetings saw considerable participation from mathematicians until 1909 [17].[3] That said, no SN meetings were organized between 1898 and 1916; currents of Scandinavianism had suffered a serious setback during the Second Schleswig War, when Denmark received no significant support against Prussia in 1864 in the way of military reinforcements from the United Kingdoms of Sweden and Norway, and again in 1905 as described above.

Regardless of past tensions, the SCM was a shared effort from the outset. For the first 75 years, its location rotated sequentially through Sweden, Finland, Denmark, and Norway. When Iceland became a sponsoring nation, it was added to this list. Although individual congresses were largely determined by local organizers, with local publishers frequently handling the proceedings, the regular rotation of the hosts, the ongoing participation by individuals from all nations involved, and the agreements reached about practical matters like scheduling changes, all attest to the endurance of the cooperative and collaborative aspects of the NCM across time.[4]

3 Evolution

The SCM has changed in several notable ways across the span of its history. From 1984, the first year the meeting was held in Iceland, it has been called the *Nordic Congress of Mathematicians*.[5] From the Middle Ages, Iceland had been ruled by Norway, and then the three crowns of Denmark, Norway, and Sweden under the Kalmar Union. When Sweden seceded from the union in 1523, Iceland became a Danish dependency, and remained so until the 20th century, gaining status as a sovereign state in 1918, and complete independence in 1944.

In contrast to Denmark and Sweden, Finland, and Norway, with universities dating, respectively, back to the 15th, 17th, and 19th centuries, the first university in Iceland was founded in 1911. It offered no studies in any science until the 1940s, when an engineering department was created. A mathematics stream was established at the Reykjavik High School in 1919, but those who wished to study mathematics beyond that level had to travel abroad. One who did so was Ólafur Daníelsson (1877–1957), who left Iceland in 1897 to study in Copenhagen, and in 1909 became the first

[3] *Acta Mathematica*, which had a Scandinavian editorial board, was another cooperative enterprise between the Scandinavian nations. Much has been written about the history of this journal; see, for example, [2].

[4] Further examples of Nordic cooperation in mathematics are the journals *Mathematica Scandinavica*, devoted to research, and *Nordisk Matematisk Tidskrift* (*Normat*), for more elementary topics, both created in 1953 (Normat is no longer in publication). On the histories of these and other Nordic mathematical journals from 1859, see [11].

[5] Somewhat curiously, the congress in 2000 was called the First American Mathematical Society–*Scandinavian* International Mathematics Meeting.

Icelander to earn a doctorate in mathematics. During the 1920s, he published several mathematics textbooks and engaged in research in algebraic geometry. He took part in the 1925 and 1929 SCMs [3]. By 1984, the Icelandic mathematical community had blossomed; 40 of the 129 registered participants were listed as Icelandic, and the Icelandic Mathematical Society, founded in 1947, was instrumental in the organization of the meeting [13].

Figure 3. Participants of the 1934 SCM. Reproduced from [9].

The frequency of the meetings also changed. Although SCMs were held in 1909, 1911, and 1913, the fourth was held in 1916; it was to be merged with the ICM, to be held in Stockholm that year.[6] From 1922, SCMs were held at least three years apart, sometimes reflecting the timing of other international meetings [4], and from 1964 this increased to four years, to place each meeting two years from an ICM [6].[7]

As for the languages of presentations and proceedings, and the body of participants and speakers, during the first few SCMs, talks were given in Danish, Norwegian, and Swedish, with participants from Finland presenting in the latter. This limited participation by mathematicians outside Scandinavia [17]. When Finnish-speaking Finns began attending the SCM in 1922, however, the languages of the academic program shifted primarily to French, German, and English, a decision also intended to make the scientific results more accessible to foreign scholars [10, 17]. Presentations in these languages continued for decades, but ultimately English came to dominate. With time, too, participation from mathematicians at non-Scandinavian institutions also increased, as did the opportunities for participants to speak. That is, while participation was very open in the early years of the SCM, the organizers restricted the opportunity to give a presentation [17]; from 1909 to 1925, only 22 to 34 percent of participants gave lectures. By 1972, however, all participants were offered the opportunity to speak [5], and in 1984 more than half of the participants delivered a longer or shorter talk [13].

In reflecting upon the evolution of the SCM and NCM, one might wonder about the level of involvement by women. In addition to the many wives, fiancées, daughters, and sisters that accompanied men to the SCM, the scientific programs show women participants, even in the early years. Of the 93 who took part in the 1911 meeting, for example, eight were women. They included Thyra Eibe (1866–1955), who had translated Euclid's *Elements* into Danish and was the first woman to earn a degree (*candidata magisterii*) in mathematics at Copenhagen; Inge Lehmann (1888–1993), also Danish, who later became a seismologist and discovered the solid inner core of the Earth; and Louise Petrén (1880–1977) and Elisabeth Stephansen (1872–1961), the first women to earn doctorates in mathematics in Sweden and Norway, respectively. No woman spoke at that meeting; non-male speakers have been relatively few for much of the history of the SCM.

4 Significance and legacy

On the occasion of his 70th birthday in 1916, Mittag-Leffler and his wife, Signe, published his will. In it were plans to create a foundation to support research in pure mathematics in Sweden and the other Nordic countries [15]. At the 25th anniversary of the SCM in 1934, during which the foundation also celebrated its anniversary, participants (Figure 3) were invited to the unveiling of a memorial to Mittag-Leffler at the cemetery in Djursholm. There, in a speech given by Torsten Carleman (1892–1949), then the director of the IML, Mittag-Leffler was described as a "warm friend of understanding and cooperation between the Nordic peoples" [9]. Indeed, while the SCM served Scandinavian mathematicians at personal, local, national, regional, and international levels, fundamentally, it was envisioned and advertised as a means of bridging the post-1905 gap between Sweden and Norway, and of protecting the shared culture and asserting the collective scientific strength of the Scandinavian nations. These aims appear to have been embraced early in the history of the SCM. As Danish mathematician Nils Erik Nørlund (1885–1981) proclaimed at the opening of the 1925 meeting, "When the Nordic countries unite as one, they are not inferior to any country" [17]. These national and regional identities are emphasized in the proceedings of many SCMs, which for decades grouped participants according to nationality.

Hosting the SCM was a measure of belonging; that it became the NCM precisely when Iceland first served as host is suggestive of this theme, as is the assertion of Finnish Chancellor Anders Donner (1854–1938) at the 1922 SCM (when Finland first hosted

[6] The First World War thwarted the plan for the ICM.

[7] Occasionally the timeline has been shifted to mark special events (such as the 100th anniversary of the Institut Mittag-Leffler (IML) in 2016) or out of necessary (the 2020 NCM was postponed due to the COVID-19 pandemic).

29TH NORDIC CONGRESS OF MATHEMATICIANS

The 29th Nordic Congress of Mathematicians, organized in collaboration with the European Mathematical Society, takes place in Aalborg in Northern Jutland, Denmark, in the week July 3–7, 2023, in a hybrid format. The scientific program starts on July 4. It comprises plenary talks, given to a mathematical audience at the beautiful House of Music, and around 30 specialized sessions in a nearby modern university building. Moreover, participants will be able to join in an excursion and in a conference dinner. The organizers are proud to announce that the following mathematicians have agreed to give plenary talks:

Kathryn Hess	EPFL Lausanne, CH
Nina Holden	Courant Institute NY, US
Daniel Král'	Masaryk University, Brno, CZ
Finnur Lárusson	Adelaide University, AU
Jonatan Lennels	KTH Stockholm, SE
Eveliina Peltola	Aalto University, FI, and University of Bonn, DE
Daniel Peralta-Salas	Instituto de Ciencias Matemáticas, Madrid, ES
Nathalie Wahl	University of Copenhagen, DK

For further information about the congress, including fees and registration, onsite and online, please consult the web site https://ncm29.math.aau.dk.

the congress) that Finns placed "special value on not being considered outsiders" within that community [17]. Hosting the meetings was also considered a *right*. When the Danes considered holding the 1922 meeting in Copenhagen, some in Finland expressed their frustration. In a 1921 letter sent to Mittag-Leffler, who was addressed as "Uncle" (an indication of close friendship), Lindelöf wrote: "If we cannot have *this* congress, it would mean that in the future Finnish participation in the [SCM] is all but over" [17].

Participation also forged scientific connections. Norwegian Fields Medalist mathematician Atle Selberg (1917–2007), who spent most of his scientific career at the Institute for Advanced Study in Princeton, would later recall his experiences at the 1938 SCM in Helsingfors, where he gave the first talk of his career. There, he met Lindelöf, Carleman, and Harald Bohr (1887–1951), and was particularly influenced by the lecture of Arne Beurling (1905–1986) on generalized prime numbers, which "made a deep impression" on him [1]. For some, there was a real desire for such contact, as was the case in Finland in 1922. By then, Finland had endured both WWI and civil war, which brought political and economical turmoil and left many unable to afford to travel. According to Lindelöf, the younger Finnish mathematicians, in particular, expected a great outcome from the 1922 SCM, "as most of them have no other possibilities of establishing contact with scholars outside of the country" [17].

The NCM still represents a means of forging and maintaining connections, both within the Nordic countries and now well beyond. This has been especially true since the year 2000, declared "World Mathematical Year" by the International Mathematical Union, when the American Mathematical Society jointly sponsored the 23rd meeting. This practice has now become common, with co-sponsors of the NCM including the French (2005), London and Edinburgh (2009), and European (2013, 2020, and 2023) mathematical societies.

As such, while its core constituents remain the mathematicians of the Nordic nations, over time the NCM has become increasingly international, and its history has been largely forgotten by its participants. When the NCM meets in July 2023, for the seventh time in Denmark and the first time in Aalborg, may its members recall the spirit of friendship embraced by its founders, and the richness of the mathematical harvest that stands to be reaped by all who take part.

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ICM 2022: The first virtual ICM

Background and reflections

Helge Holden

The first ever virtual International Congress of Mathematicians (ICM) took place over 6–14 July 2022. It is without doubt unique in the series of ICMs thus far. I would therefore like to describe what happened, and – with the benefit of several months having passed since the event – add some reflections.

Let me first set out the background to the ICM. The ICM is the flagship event of the IMU, dating back to 1897, which represents a showcase of the best of contemporary mathematics by many of the best mathematicians worldwide. At the opening ceremony, some of the most coveted prizes in mathematics are awarded, most notably the Fields Medals. The format has developed over a long time into a particular structure, with over 200 invited lectures organized across roughly 20 sections. The sections run in parallel. In addition, the program includes about 20 plenary lectures and a number of prize lectures. The invitation to give an ICM lecture is itself considered a distinction. The selection of speakers is of course merit based, but the pool of suitable candidates is relatively broad, meaning that – in addition to the circa 200 happy invitees – there are many more who might expect to be invited but are disappointed in the end. The responsibility for the selection of speakers rests with the Program Committee (PC), where the chair, appointed by the IMU President, is the only publicly known committee member until the opening of the ICM. The composition of the PC is decided by the IMU Executive Committee (IMU EC), and it has become increasingly important that its membership reflects the full diversity of the global mathematical community (gender, geography, area, age, etc.). It is a fact that the format of the ICM has remained rather static over the years. Indeed, it is not easy to identify natural subdisciplines of mathematics. Furthermore, the use of mathematics has spread into most other sciences, and in many cases, in non-trivial ways. In addition, we have the important areas of mathematics education and the history of mathematics. Thus, it is a very difficult task to find the right way to define what should properly be considered as part of an ICM.

Fields Medalist Maryna Viazovska
Photo: Jussi Rekiaro/Unigrafia
Courtesy: International Mathematical Union

To address this problem, the 18th General Assembly (GA) in 2018 introduced the ICM Structure Committee, charged with the task of considering the structure of the ICM, and suggesting natural sections of mathematical subdisciplines and their relative sizes. There is a natural fluctuation in mathematical disciplines – new areas are introduced, other areas suffer a decline in interest, while some areas blossom. A task for the Structure Committee is to assess these developments, and suggest relevant changes. In addition, the Structure Committee can propose other activities as an integral part of the scientific program. The final decision about the structure of the scientific program is then taken by the IMU EC. For ICM 2022, the Structure Committee suggested 19 sections, each with a target number of lectures. In addition, the Structure Committee proposed introducing "Special Plenary Lectures" on assigned topics. The Program Committee made a call for possible topics on *MathOverflow*, which received a massive response. After further discussions, the PC decided on one lecture on formal proof verification, another on gravitational waves, and, finally, one on developments following the proof by Taylor and Wiles of Fermat's Last Theorem. Furthermore, about 20 lectures were designated as special sectional talks, which were very well received by the community. Topics include "Lecture on the Ricci flow after Perelman", "Survey lecture on billiards", and "Survey lecture on motivic cohomology", to mention but a few.

The General Assembly in Helsinki
Photo: Rajupaja Oy | Courtesy: International Mathematical Union

The Program Committee is then left with the task of selecting speakers for the given scientific program. Sectional talks can be shared, and fractions of talks can be assigned to different sections. With the program complete and the speakers selected and invited, mathematicians from all countries around the world can assemble at the chosen venue of the ICM, meeting with friends and colleagues, as well as listening to interesting and stimulating talks during a fully packed nine-day congress. Another nice feature of the ICM is that several countries host receptions in the evenings. However, for reasons well known to all, this year turned out to be different.

The 18th General Assembly in 2018 decided to award ICM 2022 and the 19th General Assembly to Russia with Saint Petersburg as the venue. The concrete planning of the congress started with full speed on the Russian side immediately after this decision. Similarly, the ICM and GA were discussed in depth by the IMU EC at every one of its annual meetings, with further discussions held between meetings as necessary. The collaboration with the Russian organizers went well, and as it should. This happened against a backdrop of the COVID-19 pandemic, which at any given moment threatened to overturn all plans by, e.g., imposing severe travel restrictions or disallowing indoor meetings with large audiences.

However, with the aggressive Russian invasion of Ukraine on 24 February 2022, it was clear the we could not proceed with the GA and the ICM in Saint Petersburg. By coincidence, the IMU EC had its scheduled annual meeting starting 24 February. Following discussions, the IMU EC unanimously issued the following public statements:

- The General Assembly will be organized outside Russia on 3–4 July.
- The ICM will be organized as a fully virtual congress on 6–14 July.
- There will be an IMU Award Ceremony outside Russia on 5 July.
- The ICM will be open to all participants.
- The General Assembly and the ICM will be conducted without any financial contribution from the Russian Government.
- No official or representative of the Russian Government will be part of the organization or activities of the ICM.

At that point, there were no additional financial or human resources available to us, and we only had four months to plan and execute our decisions. For the ICM we contacted several providers of platforms that could host a virtual ICM. Fortunately, the pandemic has resulted in a rapid development of the technology and a surge in the number of high-level providers. After some consultation, we contracted the services of the K.I.T. Group, whose platform would allow for 7,000 participants[1] – twice the number of participants of any previous ICM – and the possibility of Q&A sessions with speakers. All lectures would be recorded and uploaded on the IMU YouTube channel afterwards for posterity. Participation would be free of charge and open to all. We were able to secure generous partial funding from the Klaus Tschira Foundation for the financial outlay of the virtual platform.

We decided to organize a compact and tight ICM, with lectures from morning until evening over 6–14 July according to the CEST

[1] Server capacity scales with the number of participants, and so does the financial cost. While the platform could provide a higher number of participants, we capped the number at 7,000 for budgetary reasons.

Delegates and guests attending the 2022 General Assembly
Photo: International Mathematical Union

time zone, based largely on the schedule from the planned ICM in Saint Petersburg.

Next was the task of finding a host for the in-person GA. We received many generous offers from our member states, and we also solicited a number of offers ourselves. After carefully reviewing several options, we accepted the generous offer from the Council of Finnish Academies to host the GA in Helsinki, Finland. With its historical and important position between East and West, we found Helsinki[2] to be the ideal venue.

Regular invitations to all member states were sent out, and the IMU offered full coverage of accommodation expenses for all delegates. In addition, we offered to cover the travel expenses of one delegate from each member. Thus, the GA represents a substantial burden on the IMU budget. We were uncertain if delegates would be willing to attend the meeting in person – the pandemic having had a strong impact on all of us[3] – but thankfully, a very high number of delegates registered for in-person attendance. During the previous winter, we had thoroughly tested a system for electronic voting that we were prepared to use in case we had to go for a fully virtual GA. While a few delegates could not travel in the end – having tested positive prior to their departure to Helsinki – and a number preferred to attend virtually, at the opening we had around 165 persons on site, with a further 30 participating remotely. The GA went very smoothly, and the electronic voting system worked without a hitch with voting delegates both present in Helsinki and participating remotely.

As usual, many important decisions are taken by the GA – budgets are decided, important elections are carried out, and delegates hear reports about the various activities of the IMU and its commissions and committees. We will not report on the outcome of the elections here, but rather focus on some other important decisions.

The IMU Statutes have essentially remained unchanged since the resurrection of the IMU after World War II. Recently, there has been an increase in the interest in and focus on the freedom of science. The IMU has been a member of the International Science Council (ISC)[4] since its inception, and through that, subscribes to the ISC's mission to "Defend the free and responsible practice of science". In Article 7 of the ISC Statutes, the ISC clearly outlines its aim to support the principle of freedom and responsibility in science. The GA expressed the view that an amendment to the IMU Statutes to incorporate this aim would underscore its importance to the mathematical community. Thus, the GA unanimously added a new Article 3 which reads:

> The Union adheres to the International Science Council's principle of embodying the free and responsible practice of science, freedom of movement, association, expression and communication for scientists, as well as equitable opportunities for access to science, its production and benefits,

[2] Helsinki is of course also historically relevant to the IMU, having previously hosted the ICM in 1978 and housed the IMU Archive until it was moved to its present location at the IMU Secretariat in Berlin in 2011.

[3] In many countries the pandemic was still rather prevalent, not to mention the increased risk of infection through travel and attendance at large-scale meetings. On top of that, Europe was experiencing significant problems with air traffic at the time.

[4] Including its predecessors, the International Council of Scientific Unions (ICSU) and the International Research Council (IRC).

> access to data, information and research material; and actively upholds this principle, by opposing any discrimination on the basis of such factors as ethnic origin, religion, citizenship, language, political or other opinion, gender, gender identity and sexual orientation, disability or age.[5]

The GA also passed a resolution expressing support for all mathematicians affected by the war in Ukraine, and in particular the IMU calls upon its members and other scientific organizations to do everything they can to assist Ukrainian colleagues in these difficult times.

Given the dire situation in Ukraine, the GA was asked to waive Ukraine's membership dues for two years by the Ukraine delegation. The GA resolved to waive the dues in case these were not settled by a third party. In the end, many member states generously offered to support Ukraine directly by paying the relevant dues. Moreover, since several members encounter temporary problems in paying their IMU membership dues, the GA decided in addition to create a more universal way to assist countries in dire situations. The GA decided to introduce the IMU Reserve Fund, with the task of assisting member states experiencing a temporary adverse situation outside their control that makes it impossible for them to cover their IMU dues. The reasons can be varied, e.g., deterioration of the financial situation in a country, hyperinflation, collapse of government, various natural catastrophes, and civil unrest or military conflict. In addition, some countries are subject to international sanctions that prohibit international money transfers. The IMU EC was charged with the task of writing precise regulations, and these have now been distributed to IMU members.

The IMU received one bid for the ICM in 2026 from the US delegation. The GA enthusiastically supported the bid and the next ICM will be held over

> 22–29 July 2026 in Philadelphia, USA

preceded by the 20th General Assembly on

> 19–20 July 2026 in New York City, USA.

On 5 July, the day between the end of the GA and the opening of the virtual ICM, the IMU decided to host the first ever IMU Award Ceremony. In the magnificent Aula of Aalto University, Mr Sauli Niinistö, the President of Finland, welcomed an enthusiastic and fully-packed audience. Here the four Fields Medalists, Hugo Duminil-Copin, June Huh, James Maynard, and Maryna Viazovska, received their gold medals. Appropriately, the first IMU Abacus Medal, partly sponsored by the University of Helsinki, was presented to Mark Braverman. The Leelavati Prize was awarded to Nikolai Andreev. In addition, the Gauss Prize and the Chern Medal were awarded to Elliott H. Lieb and Barry Mazur, respectively, both of whom participated remotely. In addition to the prize ceremony itself, we watched superb short videos on the laureates, and a separate laudatio was given for each recipient on their excellent accomplishment.

On the first day of the ICM on 6 July, proceedings were opened by IMU President Carlos E. Kenig in the old lecture hall of Aalto University. Following this, each of the Fields Medalists and the IMU Abacus Medalists delivered their live prize lectures. The whole event was streamed live and freely accessible worldwide. After the first day, the ICM switched to a fully virtual mode. Every speaker had been given the option of either delivering a live talk (with a recorded back-up), or providing a prerecorded lecture. In many ways, we were utilizing the know-how garnered during the pandemic. Indeed, we have all learned a lot from two years of remote lecturing! Overall, our experience from attending many lectures is that speakers often put in extra effort to give a livelier talk in front of an audience, so this was a welcome element wherever possible, and there were in fact several such efforts coordinated and facilitated by the ICM Satellite Coordination Group.

However, in the last few days before the virtual ICM launched, the K.I.T. Group encountered serious technical problems, which necessitated the restructuring of the format for the virtual ICM at very short notice. This was an exceptionally stressful period for all involved. To cut a long story short, we ended up with a simplified platform that posted all talks on the IMU YouTube channel but eliminated the possibility of a Q&A with the lecturers. This was of course disappointing, but the upside was that, in this format, no registration was necessary, and thus the ICM was truly open to all.

Circumstances outside our control forced us to organize the ICM as a fully virtual event. On the positive side, this gave the IMU the chance to test a dramatic change in the format of an ICM. Some elements of the virtual ICM represented a big step forward – to

The 2022 Fields Medalists (left to right) Maryna Viazovska, James Maynard, June Huh, and Hugo Duminil-Copin
Photo: Jussi Rekiaro/Unigrafia
Courtesy: International Mathematical Union

[5] Other unions have similarly adopted such a text, see, e.g., the International Union of Psychological Science, from whose statutes the text for the IMU's new Article 3 is taken.

The 2022 IMU Award Ceremony in Helsinki
Photo: Jussi Rekiaro/Unigrafia | Courtesy: International Mathematical Union

make all lectures freely available to all shortly after the lectures were given, was clearly exceptionally well received by the community – and more than 500,000 have watched the videos to date. It is clear that this is of lasting value and will be an integral part of future ICMs. Unfortunately, as explained, we were not able to test the possibility of having a remote Q&A with the speakers for technical reasons. At a regular ICM, there are normally few questions, but the virtual format may well invite more questions and direct participation. Finally, regarding participants, for financial reasons we had capped the number at 7,000, but registration was free of charge. The latter was decided for two reasons, one being that truly anyone could participate in this way, and the other being the fact that collecting a small amount from thousands of participants worldwide would be expensive and difficult. However, making the event free of charge does have its drawbacks – signing up thereby has no consequences, and one might feel less committed to participating. With a cap on participants, this could easily mean sacrificing valuable places that could otherwise be made available to those interested in participating.

But one aspect of the Congress that no virtual ICM can replace is the human one – the happy encounter with old friends and colleagues, and the possibility of sharing a dinner and exchanging views on the latest developments in mathematics. The importance and value of this cannot be overestimated. Yet the question of how to best reconcile this in the future with the natural temptation to watch a lecture from your favorite spot at home – especially given the hassles and environmental concerns around modern travel – remains unanswered. Perhaps a happy middle ground can be found? We shall see!

Acknowledgements. The author is very grateful to the IMU Manager Scott Jung for extensive help and improvements in this text. The above recollections are mine and do not necessarily represent those of the IMU.

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ERME column

regularly presented by Jason Cooper and Frode Rønning

In this issue, with a contribution by
Bożena Maj-Tatsis and Esther Levenson

CERME Thematic Working Groups

We continue the initiative of introducing the CERME working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for people working in pure and applied mathematics. Our aim is to enrich the ERME community with new participants, who may benefit from hearing about research methods and findings and contribute to future CERMEs.

Introducing CERME Thematic Working Group 13 – Early Years Mathematics

Group leaders: Bozena Maj-Tatsis, Esther Levenson, Martin Carlsen, Andrea Maffia, and Marianna Tzekaki,

Introduction

Thematic Working Group 13 was established in 2009 at CERME6 and focuses on the mathematics learning of 2–8-year-olds. Learning may take place in (pre-)kindergarten, primary school, and other less formal settings. Therefore, the group focuses on how mathematics is approached implicitly and explicitly by children, teachers, and other responsible adults. A wide range of topics is covered in TWG13, spanning different areas of mathematics and different theoretical and methodological approaches. This wide span is a characteristic feature of TWG13, and one that has been documented at other conferences as well (e.g., the POEM[1] conferences, see [6]). See also [8]. In the next section we present some of the theoretical underpinnings of the papers that have been accepted for presentation in TWG13 over the years.

[1] Perspective On Early Mathematics Learning (see https://www.gu.se/en/education-communication-learning/welcome-to-poem)

Theoretical considerations

One might ask why it is important to investigate the mathematics learning of such young children. The studies within TWG13, together with the accompanying literature, have provided some answers to this question, which can be summarised in three main points. Firstly, mathematics is explicitly or implicitly included in most kindergarten curricula. Thus, in order to plan high-quality mathematical experiences for young children, there is a need to investigate the learning situations that are being provided for children, as well as how to improve and plan for mathematical learning situations. Secondly, mathematics is present in many aspects of children's out-of-school experiences. Therefore, it is worth investigating whether informal learning takes place in such situations. Thirdly, there are connections between the mathematics encountered in pre-school and the mathematics encountered in primary school. Therefore, early mathematical experiences become important for a child's the later mathematics achievement.

The above have led to a change in the focus of early-years mathematics research during the last decades – which in turn is reflected by various national and international programs focusing on early years mathematics. Clements and Sarama [7] express this switch quite eloquently:

> … researchers have changed from a position that young children have little or no knowledge of or capacity to learning mathematics … to theories that posit competencies that are either innate or develop in the first years of life [7, p. 462].

At the same time, it is noteworthy that "not all countries have a mandatory or even recommended curriculum for this age, nor do all have compulsory or financially supported education for young children" [8, p. 107].

Research interests discussed within TWG13

The studies discussed within TWG13 may be divided into three major themes: children's conceptions and mastery of early mathematical skills; providers of early mathematics education; and the settings of early mathematics learning.

Regarding children's mathematical conceptions and skills, it is important to note that early-years mathematics does not does not reduce counting, calculations, shapes and measures. Mathematical processes such as problem solving, reasoning [9], early generalisations and abstraction [12, 14], are present and may be nurtured among young children. Additionally, children's intuitions about mathematical concepts may provide a solid base for the later development of these concepts throughout education. Readers of this magazine may also find it interesting to know that even formal aspects of mathematics, such as definitions, may be discussed and developed in first grade [2].

The first providers of early mathematics education are parents, most of whom are not themselves mathematicians or mathematics teachers. An important question that arises is to what extent can or should parents be involved in their children's mathematical development. This question gives rise to several research questions such as, do parents believe that young children should learn mathematics? Do parents, or other responsible adults have the knowledge to promote children's early numeracy? How may parents and other responsible adults provide opportunities for young children to engage in mathematics at an early age [10]? In school, it is the teacher who plans and implements mathematical activities, as well as evaluates students' knowledge. However, most preschool, kindergarten, and first- and second-grade teachers are not specialists in mathematics. Thus, an important area of research for TWG13 is the study of teachers' knowledge, beliefs, and self-efficacy for teaching mathematics to young children [11]. This knowledge includes knowledge of content (e.g., knowing the definition of a triangle), knowledge of tasks (e.g., what types of activities will promote children's knowledge of triangles), and knowledge of children (e.g., how might a child describe a triangle). Beliefs also impact on teaching [3]. Teachers with a problem-solving view on mathematics may highlight different counting strategies, while teachers with an instrumentalist view may encourage fixed algorithms. Moreover, teachers who themselves suffer from mathematics anxiety may favour resources with immediate use, e.g., a textbook or a worksheet.

The settings and contexts in which children learn vary greatly. Within a school setting, teachers may set up inquiry-based contexts, where children are encouraged to explore, discuss, and justify their thinking to others. Other settings enable children to learn mathematics through bodily movement. For example, whole-body movement can promote children's use of a counting-on strategy [5]. Finger movement on a tablet was found to develop multiplicative awareness [1]. Within TWG13, there has always been much discussion regarding formal versus informal settings, including play contexts. Play is considered vital in mathematics in the early years and can be structured to promote learning [13]. This is especially true for toddlers, ages 2 to 3 [4].

Concluding remarks

As we have mentioned, there are significant differences between the organisation of pre-school education across Europe, complicating the comparison of research conducted in different countries. During CERME12, we discussed how cultural differences need not be a barrier. Instead, we can consider culture as an enriching element in our research and widen our perspectives of early-years mathematics research, for example, by comparing early mathematics learning in out-of-school contexts. Another promising direction for future study concerns affective factors in learning and teaching early years mathematics.

In conclusion, we acknowledge that there exists a plethora of theories related to early mathematics education. As discussed in CERME12, theories cannot be static; they need to be continuously developed and adapted, e.g., by empirical studies [10]. When analysing preschool activities, it is difficult to investigate separately the children, the adults, and the specific activity. The situation under investigation is complex and there is a need to grasp this complexity. Future studies may consider networking theories. Additionally, qualitative and quantitative methodologies could be used complementarily, in order to reach well-verified theories. We invite mathematicians and mathematics educators interested in promoting a mathematical disposition from a young age, to join us at the upcoming CERME13.

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Solved and unsolved problems

Michael Th. Rassias

The present column is devoted to Number Theory.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

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Consider two positive integers $n \geq 1$ and $a \geq 2$ such that

$$a^{2n} + a^n + 1$$

is a prime. Prove that n is a power of 3.

Dorin Andrica and George Cătălin Țurcaş
(Babeş-Bolyai University, Cluj-Napoca, Romania)

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The Collatz map is defined as follows:

$$\mathrm{Col}(n) := \begin{cases} n/2 & \text{if } n \text{ is even,} \\ 3n+1 & \text{if } n \text{ is odd.} \end{cases}$$

Let

$$t_{m,x} := \min(n > 0 : \mathrm{Col}^m(n) \geq x).$$

That is, $t_{m,x}$ is the smallest integer such that, if we apply the Collatz map m times, the result is larger than x.

(a) Find $t_{3,1000}$ and $t_{4,1000}$.

(b) Show that, for x large enough (larger than (say) 1000), we have

$$t_{4,x} \equiv 3 \bmod 4 \quad \text{or} \quad t_{4,x} \equiv 6 \bmod 8.$$

(c) In general, for m odd and x large enough, there exists a constant $X_{m,x}$ such that $t_{m,x}$ is the smallest $n > X_{m,x}$ such that $n \equiv c_m \bmod M_m$. Find M_m and relate c_m to c_{m-1}.

Christopher Lutsko (Department of Mathematics,
Rutgers University, Piscataway, USA)

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The light-bulb problem: Alice and Bob are in jail for trying to divide by 0. The jailer proposes the following game to decide their freedom: Alice will be shown an $n \times n$ grid of light bulbs. The jailer will point to a light bulb of his choice and Alice will decide whether it should be on or off. Then the jailer will point to another bulb of his choice and Alice will decide on/off. This continues until the very last bulb, when the jailer will decide whether this bulb is on or off. So the jailer controls the order of the selection, and the state of the final bulb. Alice is now removed from the room, and Bob is brought in. Bob's goal is to choose n bulbs such that his selection includes the final bulb (the one determined by the jailer).

Is there a strategy that Alice and Bob can use to guarantee success? What if Bob does not know the orientation in which Alice saw the board (i.e., what if Bob does not know which are the rows and which are the columns)?

Christopher Lutsko (Department of Mathematics,
Rutgers University, Piscataway, USA)

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Let p and q be coprime integers greater than or equal to 2. Let $\mathrm{inv}_q(p)$ and $\mathrm{inv}_p(q)$ denote the modular inverse of p mod q and q mod p, respectively. That is, $\mathrm{inv}_q(p)p \equiv 1 \bmod q$ and $\mathrm{inv}_p(q)q \equiv 1 \bmod p$.

(a) Show that

$$\mathrm{inv}_p(q) \leq \frac{p}{2} \quad \text{if and only if} \quad \mathrm{inv}_q(p) > \frac{q}{2}.$$

(b) Show by providing an example that, if $1 \leq u < v$ are coprime integers and $\alpha := u/v$, then the statement

$$\mathrm{inv}_p(q) \leq \alpha p \quad \text{if and only if} \quad \mathrm{inv}_q(p) > (1-\alpha)q \qquad (1)$$

is not necessarily true.

(c) What additional assumption should p and/or q satisfy so that the equivalence (1) holds?

Athanasios Sourmelidis (Institut für Analysis und
Zahlentheorie, Technische Universität Graz, Austria)

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Let $c_n(k)$ denote the Ramanujan sum defined as the sum of kth powers of the primitive nth roots of unity. Show that, for any integer $m \geq 1$,

$$\sum_{[n,k]=m} c_n(k) = \varphi(m),$$

where the sum is over all ordered pairs (n,k) of positive integers n, k such that their lcm is m, and φ is Euler's totient function.

László Tóth (University of Pécs, Hungary)

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Show that, for every integer $n \geq 1$, we have the polynomial identity

$$\prod_{\substack{k=1 \\ (k,n)=1}}^{n} (x^{(k-1,n)} - 1) = \prod_{d \mid n} \Phi_d(x)^{\varphi(n)/\varphi(d)},$$

where $\Phi_d(x)$ are the cyclotomic polynomials and φ denotes Euler's totient function.

László Tóth (Department of Mathematics, University of Pécs, Hungary)

II Open problem

275*. Chowla's conjecture and its relatives

by Terence Tao (UCLA, Department of Mathematics, Los Angeles, USA)

Let $\lambda \colon \mathbb{N} \to \{-1, +1\}$ denote the Liouville function. In [2], Chowla conjectured that

$$\sum_{n \leq x} \lambda(n+h_1) \cdots \lambda(n+h_k) = o(x) \tag{1}$$

as $x \to \infty$, for any distinct natural numbers $h_1, \dots, h_k$ (in fact, Chowla made the more general conjecture that

$$\sum_{n \leq x} \lambda(P(n)) = o(x)$$

whenever P is a square-free polynomial mapping from $\mathbb{N}$ to $\mathbb{N}$). Chowla's conjecture was extended to other bounded multiplicative functions by Elliott [3] (see also a technical correction to the conjecture in [8]).

One can view (1) as a less difficult cousin of the notorious *Hardy–Littlewood prime tuples conjecture* [4], which conjectures an asymptotic of the form

$$\sum_{n \leq x} \Lambda(n+h_1) \cdots \Lambda(n+h_k) = \mathfrak{S}x + o(x) \tag{2}$$

where the *singular series* $\mathfrak{S}$ is an explicit product over primes of factors involving the numbers $h_1, \dots, h_k$. For $k = 1$, both conjectures follow readily from the prime number theorem, but they remain open for higher k. However, the analogue of (1) (and (2) for $k \leq 2$) were recently established in certain function fields [11], and are also known to hold in the presence of a Siegel zero [1, 5, 6, 15]. The conjecture (1) is also known if one performs enough averaging in the $h_1, \dots, h_k$ variables [8].

The logarithmically averaged version

$$\sum_{n \leq x} \frac{\lambda(n+h_1) \cdots \lambda(n+h_k)}{n} = o(\log x) \tag{3}$$

turns out to be more tractable, as it can be analyzed by the "entropy decrement method" [12], which has been successfully used to establish the conjecture for $k = 2$ (see [12], building upon the breakthrough work [7]) and for odd k (see [14, 16]). The conjecture (3) for arbitrary k is also known to be equivalent [13] to the (logarithmically averaged) Sarnak conjecture [10], which asserts that

$$\sum_{n \leq x} \frac{\lambda(n) F(T^n x_0)}{n} = o(\log x)$$

whenever $T \colon X \to X$ is a compact dynamical system of zero entropy, x_0 is a point in X, and $F \colon X \to \mathbb{C}$ is continuous. Many special cases of this conjecture are known, unfortunately too many to list here.

The conjecture (3) would also follow from a *higher-order local Fourier uniformity conjecture* [12], which is somewhat complicated to state in full generality here; however, the first unsolved special case of this conjecture asserts that

$$\int_X^{2X} \sup_{\alpha \in \mathbb{R}/\mathbb{Z}} \Big| \sum_{x \leq n \leq x+H} \lambda(n) e(\alpha n) \Big| \, dx = o(XH) \quad \text{as } X \to \infty$$

whenever $1 \leq H = H(X) \leq X$ is such that $H(X) \to \infty$ as $X \to \infty$, where $e(\theta) \coloneqq e^{2\pi i \theta}$. This is currently only established in the regime $H \geq X^{\varepsilon}$ for a fixed $\varepsilon > 0$ (see [9]).

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III Solutions

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Let $f\colon [0,\infty) \to \mathbb{R}$ be a C^1-differentiable and convex function with $f(0)=0$.

(i) Prove that, for every $x \in [0,\infty)$, the following inequality holds:

$$\int_0^x f(t)\,dt \le \frac{x^2}{2} f'(x).$$

(ii) Determine all functions f for which we have equality.

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Solution by the proposers

(i) We have

$$\int_0^x f(t)\,dt = \int_0^x \big(f(t)-f(0)\big)\,dt.$$

By the Mean Value Theorem,

$$f(t)-f(0) = tf'(c_t)$$

for some $c_t \in (0,t) \subset [0,x]$. Since f is convex, it follows that f' is increasing; hence $f'(c_t) \le f'(x)$. Therefore, $f(t) = tf'(c_t) \le tf'(x)$, and we obtain

$$\int_0^x f(t)\,dt \le \int_0^x tf'(x)\,dt = f'(x)\int_0^x t\,dt = \frac{x^2}{2} f'(x).$$

(ii) We have to find all the solutions to the equation

$$\int_0^x f(t)\,dt = \frac{x^2}{2} f'(x).$$

Denoting

$$g(x) = \int_0^x f(t)\,dt, \quad x \in [0,\infty),$$

the above equation is equivalent to the second-order differential equation

$$g(x) = \frac{x^2}{2} g''(x), \quad x \in [0,\infty).$$

Note that if g is a solution, then

$$g(x) + xg'(x) = \frac{x^2}{2} g''(x) + xg'(x),$$

whence

$$(xg(x))' = \Big(\frac{x^2}{2} g'(x)\Big)'.$$

It follows that

$$\Big(xg(x) - \frac{x^2}{2} g'(x)\Big)' = 0,$$

and so

$$\frac{x^2}{2} g'(x) - xg(x) = C_1, \quad x \in [0,\infty),$$

where C_1 is an arbitrary constant. This last equation is equivalent to

$$\frac{x^2 g'(x) - 2xg(x)}{x^4} = \frac{2C_1}{x^4}, \quad x \in (0,\infty),$$

or

$$\Big(\frac{g(x)}{x^2}\Big)' = \frac{2C_1}{x^4}, \quad x \in (0,\infty).$$

Consequently,

$$\frac{g(x)}{x^2} = \frac{-2C_1}{3x^3} + C_2, \quad x \in (0,\infty),$$

and we get

$$g(x) = \frac{-2C_1}{3x} + C_2 x^2, \quad x \in (0,\infty).$$

On the other hand, g is continuous and $g(0)=0$. This implies that $C_1 = 0$ and $g(x) = C_2 x^2$. We conclude that the sought-for functions are necessarily of the form $f(x) = g'(x) = 2C_2 x$, i.e., $f(x) = Ax$, where A is an arbitrary real constant.

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Let $y(x)$ be the unknown function of the following fractional-order derivative Cauchy problem:

$$\begin{cases} D^{\alpha} y = f(x,y), & 0 < \alpha < 1, \\ y(0) = y^{*}. \end{cases} \tag{1}$$

Find the solution of this problem by solving an equivalent first-order ordinary Cauchy problem, with a solution independent of the kernel of the fractional operator.

Carlo Cattani (Engineering School, DEIM,
University "La Tuscia", Viterbo, Italy)

Solution by the proposer

Before we give the solution of (1), let us make some preliminary remarks about the most popular definitions of fractional derivatives.

The Riemann–Liouville integral of fractional order $\nu \geq 0$ of a function $f(x)$ is defined as

$$(J^{\nu} f)(t) = \begin{cases} \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^t (t-\tau)^{\nu-1} f(\tau)\, d\tau, & \nu > 0, \\ f(t), & \nu = 0. \end{cases}$$

The corresponding Riemann–Liouville fractional derivative of order $\alpha > 0$ is defined as

$$D^{\alpha}_{\mathrm{RL}} f(t) = \frac{d^n}{dt^n} J^{n-\alpha} f(t), \quad n \in \mathbb{N},\, n-1 < \alpha \leq n. \tag{2}$$

The main problem with this derivative is that it assigns a nonzero value to a constant function. To avoid this issue, people often use the so-called order-α Caputo fractional derivative, defined as

$$D^{\alpha}_{\mathrm{C}} f(x) = \begin{cases} \dfrac{d^n f(x)}{dx^n}, & 0 < \alpha \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^x \frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha-n+1}}\, d\tau, & t > 0, \\ & 0 \leq n-1 < \alpha < n, \end{cases}$$

where n is an integer, $x > 0$, and $f \in C^n$.

Riemann–Liouville (RL) and Caputo (C) derivatives are the most popular and have been used in many applications; nevertheless, they both have some drawbacks. For this reason, many authors have introduced some more flexible fractional operators. The most general fractional derivative with a given kernel $K(x, \alpha)$ is defined as

$$D^{\alpha} f(x) = \begin{cases} \dfrac{d^n f(x)}{dx^n}, & 0 < \alpha \in \mathbb{N}, \\ \displaystyle\int_0^x f^{(n)}(\tau) K(x-\tau, \alpha)\, d\tau, & t > 0, \\ & 0 \leq n-1 < \alpha < n. \end{cases} \tag{3}$$

The kernel should be chosen in a such a way that at least the following two conditions are satisfied:

$$\lim_{\alpha \to 0} K(x-\tau, \alpha) = 1 \quad \text{and} \quad \lim_{\alpha \to 1} K(x-\tau, \alpha) = \delta(x-\tau).$$

Moreover, to ensure that one is dealing with a non-singular kernel, one requires that

$$\lim_{x \to \tau} K(x-\tau, \alpha) \neq 0 \quad \forall \alpha.$$

Although several definitions of fractional derivatives are available, they all depend on the proposed kernel, thus implying a subjective and a priori unjustified choice of the fractional operator each time one studies a fractional differential problem. This issue can be avoided by using the following simple definition, which is based on an intuitive interpolation.

Limiting ourselves to the case $n = 1$, the general structure of the Caputo-type fractional derivative

$$D^{\alpha}_{\mathrm{C}} f(x) = \int_0^x f'(\tau) K(x-\tau, \alpha)\, d\tau, \quad 0 < \alpha < 1,$$

is based on the kernel $K(x-\tau, \alpha)$, which is a positive function that decays at infinity (to ensure convergence), while according to (2), the general structure of the Riemann–Liouville first-order derivative is

$$D^{\alpha}_{\mathrm{RL}} f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x f(\tau)(x-\tau)^{-\alpha}\, d\tau, \quad 0 < \alpha \leq 1.$$

Usually, to find the solution of (1), we should first choose the kernel of the fractional operator and then solve the fractional problem by using a suitable numerical method, which roughly consists in constructing and solving an equivalent algebraic/differential (of integer order) problem. In any case, the solution will depend not only on the independent variable x and the initial condition y^{*}, but also on the kernel and on the fractional-order parameter:

$$y(x, y^{*}, K(x-\tau), \alpha).$$

The dependence on the fractional parameter α is essential in solving fractional-order problems. However, the dependence on the kernel leads to an inessential "struggle" about the best choice of the kernel and about its physical/mathematical meaning – an obviously subjective and non-unique choice. Because of this lack of uniqueness, fractional calculus is missing a strong mathematical motivation. On the other hand, there exist many useful mathematical tools, important for the solution of differential problems, that require making choices, such as wavelets, orthogonal polynomials, integral transforms, and many more. Therefore, one can either ignore the uniqueness problem, or try to defend a specific choice by using some reasoning that may still be not sufficient to convince the mathematics community. In the following, we give a solution which is both independent of the choice of the kernel and can be analytically obtained by reduction to an equivalent ordinary differential problem having the same solution as (1).

We search the solution by assuming that the fractional derivative is obtained by linear interpolation between a function and its first-order derivative; consequently, we do not need the integral definition (3) and the accompanying choice of kernel. This is an acceptable assumption, based on the original simple idea in the fundamentals of fractional calculus that the fractional parameter describes a family of interpolation curves. Thus, we set

$$D^\alpha y = (1-\alpha)(y-y^*) + \alpha\frac{dy}{dx}$$

so that the initial value problem (1) becomes

$$\alpha\frac{dy}{dx} + (1-\alpha)y = (1-\alpha)y^* + f(x,y).$$

Now, for $0<\alpha<1$, we easily obtain the following ordinary differential problem that is equivalent to (1):

$$\begin{cases} \dfrac{dy}{dx} - \Big(1-\dfrac{1}{\alpha}\Big)y = -\Big(1-\dfrac{1}{\alpha}\Big)y^* + \dfrac{1}{\alpha}f(x,y), \\ \qquad\qquad y(0)=y_0. \end{cases} \tag{4}$$

For the moment, we can suppose that the initial conditions of (1) and (4) do not necessarily coincide, i.e., $y^* \neq y_0$. However, some further assumptions can be made when the function $f(x,y)$ is given explicitly. In particular, to achieve a perfect equivalence between (1) and (4), we can set $y^* = (1-\alpha)\tau + \alpha y_0 \ \forall \tau \in \mathbb{R}$, and then let $\alpha \to 1$.

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Let $y(x)$ be the unknown function of the following Bernoulli fractional-order Cauchy problem:

$$\begin{cases} D^\alpha y = g(x)y^\beta, \quad 0<\alpha<1,\ \beta\neq 0,1, \\ y(0)=y^*, \end{cases} \tag{1}$$

where $g(x)$ is a continuous function in the interval $I=[0,\infty)$.

Find the solution of this problem by solving an equivalent first-order ordinary Cauchy problem, with a solution independent of the kernel of the fractional operator.

Carlo Cattani (Engineering School, DEIM,
University "La Tuscia", Viterbo, Italy)

Solution by the proposer

We search the solution by simply assuming that the fractional derivative is a linear interpolation between a function and its first-order derivative so that we do not need the usual integral definition of the fractional operator which requires us to choose the underlying kernel.

Thus, we set

$$D^\alpha y = (1-\alpha)(y-y^*) + \alpha\frac{dy}{dx} \tag{2}$$

so that (1) becomes

$$\alpha\frac{dy}{dx} + (1-\alpha)y = (1-\alpha)y^* + g(x)y^\beta. \tag{3}$$

Then taking $0<\alpha<1$ and using equations (2) and (3), we easily get the following ordinary differential problem equivalent to (1):

$$\begin{cases} \dfrac{dy}{dx} - \Big(1-\dfrac{1}{\alpha}\Big)y = -\Big(1-\dfrac{1}{\alpha}\Big)y^* + \dfrac{1}{\alpha}g(x)y^\beta, \\ \qquad\qquad\qquad\qquad \beta\neq\{0,1\}, \\ \qquad\qquad y(0)=y_0. \end{cases} \tag{4}$$

For the moment, we search a general solution of (1) by assuming that $y^*\neq y_0$. Solving separately the two equations

$$\frac{dy}{dx} - \Big(1-\frac{1}{\alpha}\Big)y = -\Big(1-\frac{1}{\alpha}\Big)y^*,$$
$$\frac{dy}{dx} - \Big(1-\frac{1}{\alpha}\Big)y = \frac{1}{\alpha}g(x)y^\beta, \quad \beta\neq\{0,1\},$$

we obtain the respective solutions

$$y(x) = y^* + k_1 e^{(1-\frac{1}{\alpha})x},$$
$$y(x)^{1-\beta} = e^{(1-\beta)(1-\frac{1}{\alpha})x}\Big[k_2 + \frac{1-\beta}{\alpha}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}\,d\xi\Big],$$
$$\beta\neq\{0,1\}.$$

Consequently, the solution of problem (4), which is also the solution of problem (1), takes the forms listed below.

1. Let $y_0\neq y^*$ and $\beta\neq\{0,1\}$. Then

$$y(x) = y^* + \frac{1}{2}(y_0-y^*)e^{(1-\frac{1}{\alpha})x} + e^{(1-\beta)(1-\frac{1}{\alpha})x}\Big[\frac{1}{2}(y_0-y^*) + \frac{1-\beta}{\alpha}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}\,d\xi\Big].$$

2. Let $y_0 = y^*$ and $\beta\neq\{0,1\}$. Then

$$y(x) = y^* + e^{(1-\beta)(1-\frac{1}{\alpha})x}\Big[\frac{1-\beta}{\alpha}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}\,d\xi\Big].$$

3. Let $y_0=0$, $y^*\neq 0$ and $\beta\neq\{0,1\}$. In this case, note that, in order to solve the given fractional-order Cauchy problem (1) via an equivalent ordinary differential problem, we can simply set $y_0=0$ in (4) and thus obtain the solution

$$y(x) = y^* - \frac{1}{2}y^*e^{(1-\frac{1}{\alpha})x} + e^{(1-\beta)(1-\frac{1}{\alpha})x}\Big[-\frac{1}{2}y^* + \frac{1-\beta}{\alpha}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}\,d\xi\Big].$$

4. Let $y_0=y^*=0$ and $\beta\neq\{0,1\}$. Then

$$y(x) = \frac{1-\beta}{\alpha}e^{(1-\beta)(1-\frac{1}{\alpha})x}\int_0^x g(\xi)e^{\frac{1-\beta}{\alpha}\xi}\,d\xi.$$

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Let g be a real-valued C^2-function defined on $(0,\infty)$, strictly increasing, such that $g(x)>1$ for all $x\in(0,\infty)$ and $g(2)<4$. Consider the boundary value problem

$$y'' = -g(x)y, \quad y(0)=1, \quad y'(0)=0.$$

Prove that the solution y has exactly one zero in $(0, \pi/2)$, i.e., there exists a unique point $x_0 \in (0, \pi/2)$ such that $y(x_0) = 0$, and give a positive lower bound for x_0.

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Solution by the proposer

First suppose that $y(x) > 0$ for $x \in (0, \pi/2)$. The function $z(x) = \cos x$ is the solution to the auxiliary initial value problem

$$z'' = -z, \quad z(0) = 1, \quad z'(0) = 0.$$

Therefore,

$$y''z - yz'' = (1 - g(x))zy.$$

Integrating this equality over the interval $(0, \pi/2)$, we obtain

$$\int_0^{\pi/2} (1 - g(x))zy\,dx = \int_0^{\pi/2} (y''z - yz'')\,dx = (y'z - yz')|_0^{\pi/2} = y(\pi/2),$$

and $y(\pi/2) \geq 0$ by the continuity of y. But

$$(1 - g(x))zy < 0 \quad \text{in } (0, \pi/2),$$

so we reached a contradiction. Thus, y has at least one zero in $(0, \pi/2)$.

Next, observe that $\pi/2 < 2$ and by assumption $g(2) < 4$. Consider the function $w(x) = \cos(2x)$, which is a solution to the second auxiliary initial value problem

$$w'' = -4w, \quad w(0) = 1, \quad w'(0) = 0$$

and satisfies $w(x) > 0$ for $x \in (0, \pi/4)$. Suppose that y has (at least) one zero in $(0, \pi/4)$. Denote the smallest such zero by x_1. Then $y(x)$ is positive for $x \in (0, x_1)$ (recall that $y(0) = 1$) and $y(x_1) = 0$; hence $y'(x_1) \leq 0$. An argument analogous to the one above shows that

$$\int_0^{x_1} (g(x) - 4)wy\,dx = \int_0^{\pi/2} (w''y - wy'')\,dx = (w'y - wy')|_0^{x_1} = -w(x_1)y'(x_1).$$

Note that $-w(x_1)y'(x_1) \geq 0$, but the integrand $(g(x) - 4)wy < 0$, so again we reached a contradiction. Thus, y has no zero in $(0, \pi/4)$, so $\pi/4$ is a positive lower bound for the zeros of y.

Finally, suppose that y has more than one zero in $(\pi/4, \pi/2)$, namely, there exist at least two points $x_2, x_3 \in (\pi/4, \pi/2)$ such that $x_2 < x_3$ and $y(x_2) = y(x_3) = 0$. Take the function

$$v(x) = a\cos(2x + b),$$

where a, b are chosen in such a way that $v(x_2) = 0$ and $v(x)$ is negative for $x \in (x_2, x_3)$; see Figure 1.

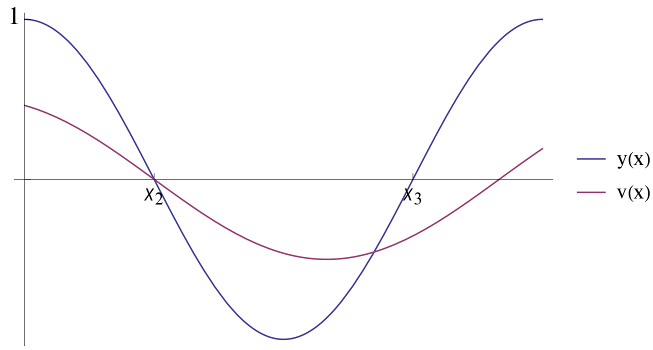

Figure 1. Assuming y has more than one zero in $(\pi/4, \pi/2)$

We have $v'' = -4v$, so $y''v - yv'' = (4 - g(x))yv$, which is positive on the interval (x_2, x_3). Therefore,

$$\int_{x_2}^{x_3} (y''v - yv'')\,dx > 0.$$

On the other hand,

$$\int_{x_2}^{x_3} (y''v - yv'')\,dx = (y'v - yv')|_{x_2}^{x_3} = y'(x_3)v(x_3),$$

and the right-hand side is negative since $y(x_3) = 0$ and $y(x) < 0$ for $x \in (x_2, x_3)$.

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We propose an interesting stochastic-source scattering problem in acoustics. The stochastic nature for such problems forces us to deal with stochastic partial differential equations (SPDEs), rather than partial differential equations (PDEs) which hold for the corresponding deterministic counterparts. In particular, the results of our proposed model will be applied to establish existence and uniqueness for the stochastic solution of a finite element approximation of the stochastic-source Helmholtz equation.

Consider the following approximation problem of a stochastic-source Helmholtz equation:

$$\Delta u + k^2 u = f \quad \text{in } D, \qquad u = 0, \quad x \in \partial D, \tag{1}$$

where $f = \sum_a f_a H_a$ is a generalized stochastic source. For the stochastic problem (1), we use the equations

$$u = \sum_a u_a H_a, \quad f = \sum_a f_a H_a,$$

and we get the collection of deterministic problems

$$\Delta u_a + k^2 u_a = f_a \quad \text{in } D, \qquad u_a = 0, \quad x \in \partial D. \tag{2}$$

Assume that $u_a \in H_0^1(D)$ solves problem (2). Then prove that, for all $v \in H_0^1(D)$, the solution $u_a \in H_0^1(D)$ satisfies

$$-\int_D \nabla u_a \cdot \nabla v\,dx + \int_D k^2 u_a v\,dx = \int_D f_a v\,dx.$$

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Solution by the proposers

We decompose our problem into a hierarchy of deterministic evolution (BVPs), and we give their corresponding variational formulations.

For $|a| = 0$, we get

$$\begin{cases} \Delta u_0 + k^2 u_0 = f_0 & \text{in } D, \\ \qquad\quad u_0 = 0 & \text{on } \partial D. \end{cases} \tag{3}$$

The estimation of a solution of problem (3) is

$$\|u_0\|_{H^1(D)} \le c_0 \|f_0\|_{L^2(D)}.$$

For $|a| = 1$, we get

$$\begin{aligned} \Delta u_1 + k^2 u_1 = f_1 \quad & \text{in } D, \\ u_1 = 0 \quad & \text{on } \partial D. \end{aligned} \tag{4}$$

We take an arbitrary $v \in H_0^1(D)$ and multiply equation (4) by v. Then we get

$$(\Delta u_1)v + k^2 u_1 v = f_1 v \tag{5}$$

and integrate over D. Every term is integrable since $u_a \in H_0^1(D)$, and hence we have $\Delta u_1 \in H_0^1(D)$ and $v \in H_0^1(D)$, so

$$(\Delta u_1)v \in H_0^1(D), \quad k^2 \in L^\infty(D), \quad u_1 \in H_0^1(D), \quad v \in H_0^1(D).$$

Therefore, $k^2 u_1 v \in H_0^1(D)$ and $f_1 \in L^2(D)$, so $f_1 v \in H_0^1(D)$. We obtain

$$\int_D (\Delta u_1) v\,dx + \int_D k^2 u_1 v\,dx = \int_D f_1 v\,dx.$$

We use Green's formula according to which

$$\int_D (\Delta u_1) v\,dx = -\int_D \nabla u_1 \cdot \nabla v\,dx + \int_{\partial D} \gamma_1(u_1)\gamma_0(v)\,d\Gamma$$

since $v \in H_0^1(D)$ is equivalent to $\gamma_0(v) = 0$.

Let $H_0^1(D)$ be a stochastic Hilbert space. If we now assume the bilinear form on $H_0^1(D) \times H_0^1(D)$,

$$a(u_1, v) = \int_D (-\nabla u_1 \cdot \nabla v + k^2 u_1 v)\,dx,$$

and the linear functional on $H_0^1(D)$,

$$\ell(v) = \int_D f_1 v\,dx,$$

then the variational formulation of problem (5) is

$$a(u_1, v) = \ell(v) \quad \forall v \in H_0^1(D). \tag{6}$$

The estimation of a solution of problem (6) is

$$\|u_1\|_{H^1(D)} \le c_1 \|f_1\|_{L^2(D)}.$$

For $|a| = n$, we get

$$\begin{cases} \Delta u_n + k^2 u_n = f_n & \text{in } D, \\ \qquad\quad u_n = 0 & \text{on } \partial D. \end{cases} \tag{7}$$

The estimation of a solution of problem (7) is

$$\|u_n\|_{H^1(D)} \le c_n \|f_n\|_{L^2(D)}.$$

Via the above variational formulations and taking into account $u = \sum_a u_a H_a$, we can prove that the solution u of the stochastic boundary value problem (1) satisfies the following inequality:

$$\|u\|_{H^1(D)} \le c_0 \|f_0\|_{L^2(D)} + c_1 \|f_1\|_{L^2(D)} + \cdots + c_n \|f_n\|_{L^2(D)},$$

where c_i, $i = 0, 1, \ldots, n$, are considered to be in agreement with appropriate built-in weights. The solution u belongs to the space $\{H^1(D), \Omega, F, \mu\}$ which is a stochastic Hilbert space with μ the probability measure defined by $H_a(\omega)$.

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For a Newtonian incompressible fluid, the Navier–Stokes momentum equation, in vector form, reads [4]

$$\begin{aligned} \rho\Big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u}\Big) &= -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}, \\ \mathbf{u} = \mathbf{u}(\mathbf{x}, t), &\quad \mathbf{u} \colon \mathbf{R}^n \times (0, \infty) \to \mathbf{R}^n. \end{aligned} \tag{1}$$

Here, ρ is the fluid density, $\mathbf{u}$ is the velocity vector field, p is the pressure, μ is the viscosity, and $\mathbf{F}$ is an external force field.

(i) Assuming that both the pressure drop ∇p and the external field $\mathbf{F}$ are negligible, it is easy to show that equation (1) reduces to

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u}, \tag{2}$$

and finally to equation (3), where $\nu = \frac{\mu}{\rho}$ is the so-called kinematic viscosity [5].

(ii) Regarding the one-dimensional viscous Burgers equation

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \quad u = u(x, t), \tag{3}$$

prove that an analytical solution can be obtained by means of the Tanh Method [2, 3, 5] as

$$u(x, t) = \lambda\Big[1 - \tanh\Big(\frac{\lambda}{2\nu}(x - \lambda t)\Big)\Big], \quad \lambda > 0.$$

M. A. Xenos and A. C. Felias (Department of Mathematics, University of Ioannina, Greece)

Solution by the proposers
Notice that, for $\nabla p = \mathbf{F} = \mathbf{0}$, equation (1) becomes

$$\rho\Big(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\Big) = \mu\nabla^2\mathbf{u}. \tag{4}$$

Since, for an incompressible fluid, ρ is a nonzero constant, one can divide both sides of equation (4) by ρ and thus obtain equation (2). Now consider the motion of a one-dimensional viscous fluid with fluid velocity u along the x-axis as time passes, $u = u(x,t)$. In this case, equation (2) transforms into equation (3). Introduce the transformation of u given by

$$\begin{cases} u(x,t) = u(\zeta), \\ \qquad \zeta = \mu(x-\lambda t), \quad \mu > 0,\ \lambda \neq 0, \end{cases} \tag{5}$$

with μ representing the wave number and λ the velocity. Then transformation (5) reduces equation (3) to the following ODE for $u(\zeta)$:

$$-\lambda u'(\zeta) + u(\zeta)u'(\zeta) - \nu\mu u''(\zeta) = 0. \tag{6}$$

Integrating equation (6) and taking the integration constant to be zero, we obtain

$$-\lambda u(\zeta) + \frac{1}{2}u^2(\zeta) - \nu\mu u'(\zeta) = 0. \tag{7}$$

The idea behind the Tanh method uses a key property of the functional derivatives all being written in terms of the Tanh function [2, 3]. The following identity is used:

$$\operatorname{sech}^2\zeta = 1 - \tanh^2\zeta, \quad \zeta \in \mathbb{R}. \tag{8}$$

This transforms equation (6) into a polynomial equation for successive powers of the Tanh function. Introducing the new variable

$$y = \tanh\zeta, \tag{9}$$

solution(s) can be sought in the form

$$u(y) = \sum_{n=0}^{N} a_n y^n. \tag{10}$$

Chain differentiation yields

$$\begin{aligned} \frac{d}{d\zeta} &= \frac{d}{dy}\frac{dy}{d\zeta} = \operatorname{sech}^2\zeta\frac{d}{dy} \\ &\underset{(8)}{=} (1-\tanh^2\zeta)\frac{d}{dy} \underset{(9)}{=} (1-y^2)\frac{d}{dy}. \end{aligned} \tag{11}$$

The positive integer value of N is determined after substituting expressions (10) and (11) into equation (7) and balancing the resulting highest-order terms.

Once N is determined, substituting expression (10) in equation (7), one obtains an algebraic system for the coefficients a_n, $n = 0, 1, \dots, N$. Depending on the problem under consideration, μ is either determined or not, while λ is always a function of μ.

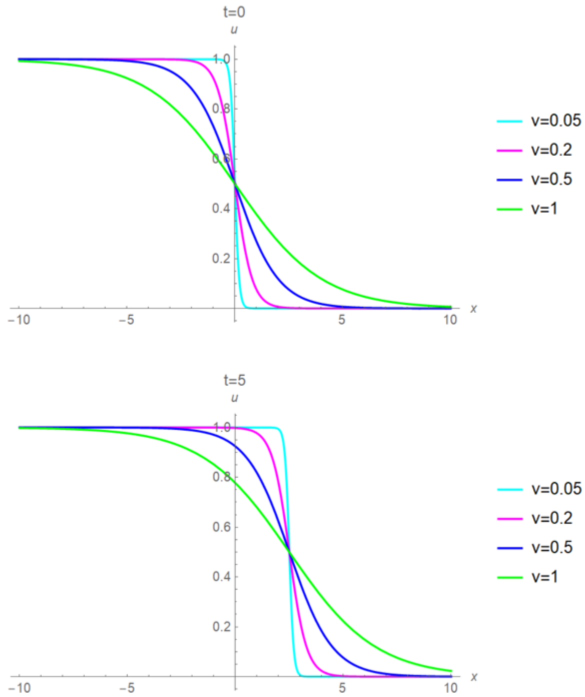

Figure 2. Different right-moving, kink-shaped solutions of the viscous Burgers equation, for $\nu \in \{0.05, 0.2, 0.5, 1\}$ and $\lambda = 0.5$, on the interval $x \in [-10, 10]$ and with $t \in \{0, 5\}$. Small viscosity effects lead to a steeper waveform, whereas larger viscosity effects lead to smoother and wider waveforms.

In the present case, N is found to be equal to 1; hence substituting expression (10) in equation (7) and setting the coefficients of the like powers of y equal to zero leads to the algebraic system

$$\begin{cases} \dfrac{a_1^2}{2} + a_1\nu\mu = 0, \\ a_0a_1 - a_1\lambda = 0, \\ \dfrac{a_0^2}{2} - a_0\lambda - a_1\nu\mu = 0, \end{cases}$$

the solution of which is

$$\begin{cases} \mu = \dfrac{\lambda}{2\nu}, \quad \lambda > 0, \\ a_0 = \lambda, \\ a_1 = -\lambda. \end{cases} \tag{12}$$

Combining (12) and (10) and using expression (9), one obtains

$$u(\zeta) = \lambda(1 - \operatorname{Tanh}\zeta),$$

and finally

$$u(x,t) = \lambda\Big[1 - \operatorname{Tanh}\Big(\frac{\lambda}{2\nu}(x-\lambda t)\Big)\Big].$$

Figure 2 displays right-moving analytical solutions of the viscous Burgers equation for different values of the kinematic viscosity ν.

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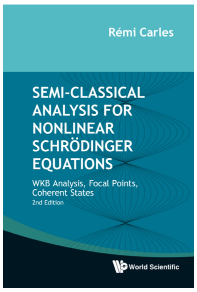
We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions to Michael Th. Rassias by email to mthrassias@yahoo.com.

We also solicit your new problems with their solutions for the next "Solved and unsolved problems" column, which will be devoted to Probability Theory.

Book reviews

Semi-Classical Analysis for Nonlinear Schrödinger Equations: WKB Analysis, Focal Points, Coherent States by Rémi Carles

Reviewed by Jean-Claude Saut



The nonlinear Schrödinger equation (NLS), in its various forms, is undoubtedly one of the most important nonlinear dispersive equations, both for its relevance as an asymptotic model in the so-called modulation regime in many physical contexts (water waves, nonlinear optics, chemistry, Bose condensation, ...) and for its rich mathematical properties.

This book focuses on one aspect of the theory, namely, the semi-classical analysis of the NLS written as

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = Vu^\varepsilon + f(|u^\varepsilon|^2)u^\varepsilon, \qquad (1)$$

where the function $u^\varepsilon = u^\varepsilon(t,x)$, $x \in \mathbb{R}^d$, $t \in \mathbb{R}$, is complex-valued and $\varepsilon \in \,]0,1]$ plays the role of a rescaled Planck constant. The nonlinearity f is local (e.g., power-like), real-valued and smooth, and V is the external potential.

The *semi-classical* analysis concerns the study of solutions to the Cauchy problem for equation (1) as $\varepsilon \to 0$.

The book is organized in three main, essentially independent parts. The first part is concerned with the Wentzel–Kramers–Brillouin (WKB) analysis, that is, for an initial datum of the form

$$u^\varepsilon(0,x) = \varepsilon^\kappa a_0^\varepsilon(x)e^{i\varphi_0(x)/\varepsilon},$$

where the initial amplitude a_0^ε is complex-valued and admits the asymptotic expansion

$$a_0^\varepsilon(x) \sim a(_0(x) + \varepsilon a_1(x) + \varepsilon^2 a_2(x) + \cdots$$

as $\varepsilon \to 0$, one seeks the solution of (1) in the form $u^\varepsilon(t,x) \sim a^\varepsilon(t,x)e^{i\varphi(t,x)/\varepsilon}$, with

$$a^\varepsilon(t,x) \sim a(t,x) + \varepsilon a^{(1)}(t,x) + \varepsilon^2 a^{(2)}(t,x) + \cdots$$

as $\varepsilon \to 0$.

The WKB approach allows to describe approximately solutions in the language of geometric optics. As shown in one chapter, the WKB analysis is also related to some instability phenomena, such as ill-posednes in Sobolev spaces of order less than the critical exponent $s_c = \frac{d}{2} - \frac{1}{\sigma}$, or of negative order, for the "usual" nonlinear equations ($\varepsilon = 1$, $V = 0$ in (1)).

The second part of the book treats the semi-classical limit in the presence of caustics (that is, singularities where the geometric optics picture essentially breaks down), in the special geometric case where the caustic is reduced to a point or several isolated points.

The last part was not present in the first (2008) edition of the book and addresses the nonlinear propagation of coherent states; in a certain sense, it complements the first two parts.

This part deals with the NLS

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \lambda\varepsilon^\alpha|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon,$$

where $\sigma > 0$ and $\lambda \in \mathbb{R}$, and also with the case of Hartree type nonlinearities

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon + \varepsilon^\alpha(K \star |\psi^\varepsilon|^2)\psi^\varepsilon,$$

where either $K(x) = \lambda/|x|^\gamma$, $\gamma > 0$, or K is smooth, real-valued and bounded together with its derivatives.

The notion of coherent states used here is to consider initial data of the form

$$\psi^\varepsilon(0,x) = \frac{1}{\varepsilon^{d/4}}a\Big(\frac{x-q_0}{\sqrt{\varepsilon}}\Big)e^{ip_0\cdot(x-q_0)/\varepsilon}, \quad (q_0,p_0) \in \mathbb{R}^{2d}.$$

Another chapter deals with the linear case

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\Delta\psi^\varepsilon = V(x)\psi^\varepsilon,$$

where V is smooth and at least quadratic. Power-like nonlinearities and the Hartree-type nonlinearity are treated in the next two chapters.

In conclusion, this book presents a welcome and useful overview of an important issue concerning the dynamics of nonlinear

Schrödinger equations. The author has made the necessary efforts to make clear a rather technical topic and to recall various relevant notions, thus making the book essentially self-contained (at a graduate level). He also points out interesting open problems and directions.

Rémi Carles, *Semi-Classical Analysis for Nonlinear Schrödinger Equations: WKB Analysis, Focal Points, Coherent States.* World Scientific Publishing, Second edition, 2020, 368 pages, Hardback ISBN 978-98-1122-790-5, eBook ISBN 978-98-1122-792-9.

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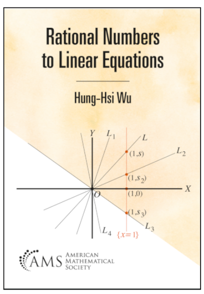
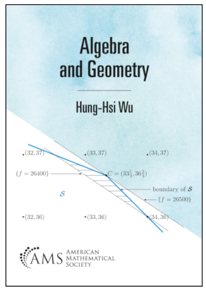
DOI 10.4171/MAG/129

Rational Numbers to Linear Equations and *Algebra and Geometry* by Hung-Hsi Wu

Reviewed by António de Bivar Weinholtz



These are the fourth and fifth books in a series of six covering the K-12 curriculum, as an instrument for the mathematical education of schoolteachers. They follow three volumes entitled "Understanding Numbers in Elementary School Mathematics", "Teaching School Mathematics: Pre-Algebra" and "Teaching School Mathematics: Algebra", and they are the first two of a series of three dedicated to high school teachers. Although all the topics covered in the first of the present books and many of those included in the second one have already been treated in previous volumes, the author wanted to ensure that high school teachers will have at their disposal a set of self-contained instruments for their mathematical education expressly written for them, but not neglecting the mathematical pre-requisites to what they have to teach, which include, of course, the content of middle school mathematics. The first volume starts with a systematic study of fractions, followed by rational numbers, the euclidean algorithm and the fundamental theorem of arithmetic, but with a somewhat briefer approach than in the previous volumes, adequate to high school teachers. It proceeds to the study of geometry, with the introduction of basic isometries, dilation and similarity, giving these subjects a more systematic and advanced treatment than the one adopted in the second volume of this series of six. This study of geometry is continued in the second volume considered in this review, as we shall presently see. The final chapter of the first volume deals with linear equations and their graphs, including the case of two variables, the setting up of coordinate systems, lines and their slopes, and simultaneous linear equations; it starts with general considerations on the use of symbols and symbolic expressions.



The second volume starts with the study of functions, but with a deeper discussion on the abstract concept of function and on the general purpose of studying functions than what could be achieved in the volumes dedicated to basic and middle school mathematics. Then it proceeds to linear functions of one variable, linear inequalities, and an application to linear programming with an introduction to the general problem of optimization, an appendix on mathematical induction, quadratic functions and equations, polynomial and rational functions, exponential and logarithmic functions, polynomial forms, and complex numbers, with the statement of the fundamental theorem of algebra.

The last three chapters continue the study of high school geometry. After a review of the geometric topics included in the previous volume (mainly the basic assumptions and theorems), the book proceeds with the basic theorems of plane geometry, namely, completing the study of congruence and similarity and the fundamental properties of triangles and circles, including the Euler line and the nine point circle, and some other meaningful properties, some of them proposed as exercises to the reader. This covers and goes beyond what is usually considered the content of middle and high school plane geometry, but presented here as a beautiful piece of mathematical writing with the due balance between precision and intuition that was advocated by the author since his first introduction of geometric topics in the second volume of this series. Another juicy chapter follows, in which we are first brought to understand the main ruler and compass constructions, up to the regular pentagon, and the basic ideas that explain the long story of constructibility, from antiquity to the main negative results of the nineteenth century. The final chapter is dedicated to an analysis of the axiomatic method in geometry, with a discussion that synthetizes the basic principles that presided over the presentation of middle and high school geometry in this series of books and that explains why this is the right opportunity for a more detailed approach to evaluate the choice of the set of basic assumptions that are to be accepted without proof, which is an unescapable feature of any deductive system. The author does not advocate that, until the end of high school, students should be confronted with a presentation of geometry based upon one of the known rigorous axiomatic systems that have been proposed, since the pioneering work of Hilbert and other mathematicians, by the end of the nineteenth century, freed the more than two thousand years old approach of Euclid from its flaws. Nevertheless, in this final chapter, he describes Hilbert's system and gives some examples of basic deductions from the axioms; he also shows the consequences of replacing the euclidean parallel axiom by its negation, thus providing an introduction to neutral and hyperbolic geometry. For a discussion on the possibility of slightly different approaches to the first steps in school geometry that nevertheless should respect the general principles advocated by the author, I redirect the readers to one of my previous reviews, namely, António de Bivar Weinholtz, Book review, "Teaching School Mathematics: Pre-Algebra" by Hung-Hsi Wu. *Eur. Math. Soc. Mag.* **123**, 52–54 (2022). I just add a comment on the choice of Hilbert's system as the only example of an axiomatic of elementary geometry that is presented here. While this is certainly the most widely known and the pioneer in modern axiom systems that were able to "fix" Euclid's work, I always like to give some credit to Tarski's system, which in his most recent form (attained only in the 1960s and developed in a single monograph published in German in 1983 and, to my knowledge, nowhere else) deserves, in my opinion, much more attention than it has obtained until now. Not only Tarski's system is of a much more synthetic nature than Hilbert's (only "points" as primitive objects, "betweenness" and "equidistance" as primitive relations and, quite surprisingly, a list of only 10 axioms and an axiom scheme, using logic operators, including identity, which, all taken together, are in a precise sense "shorter" than only one, the "longest", of Hilbert's axioms), but its axioms all admit a very intuitive interpretation. I believe it could be explored—not, of course, to be taught, as such, for middle and high school students, but perhaps to inspire some alternative details in a treatment of geometry of an elementary nature.

The second volume also contains a very helpful appendix with a list of assumptions, definitions, theorems, and lemmas from the first of these two volumes.

For a general appreciation of this set of books, I strongly recommend reading first another one of my previous reviews, namely, António de Bivar Weinholtz, Book review, "Understanding Numbers in Elementary School Mathematics" by Hung-Hsi Wu. *Eur. Math. Soc. Mag.* **122**, 66–67 (2021). Therein one can find the reasons why I deem it a milestone in the struggle for a sound mathematical education of youths. I shall not repeat here all the historical and scientific arguments that sustain this claim, but, as I did previously, I have to state that, while the volumes reviewed here are written for high school teachers, as an instrument for their mathematical education (both during pre-service years and for their professional development), and to provide a resource for authors of textbooks, the set of its potential readers should not be restricted to those for which it was primarily intended; I once again claim that it should include anyone with the basic ability to appreciate the beauty of the use of human reasoning in our quest to understand the world around us and the capacity and will to make the necessary efforts required here as for any enterprise that is really worthwhile. Of course, as the content and presentation of the three last volumes is of a more advanced nature, it requires from the reader a wider set of pre-requisites.

Both volumes are punctuated with pedagogical comments that give extremely useful advice regarding what content details should be used in classrooms and which are essentially meant to teachers; mathematical comments are also added to the main text, in order to extend the views of the reader whenever it helps to clarify the subject in question. People interested in the full scope of the pedagogical comments of these two volumes are referred also to my book review of "Teaching School Mathematics: Algebra" by Hung-Hsi Wu. *Eur. Math. Soc. Mag.* **125**, 50–52 (2022), where a detailed description is made of what the author considers to be the main characteristics of mathematics and how they have been neglected in schools for such a long period of time and replaced by what he calls "Textbook School Mathematics" (TSM); the same concern is present in all the topics treated in the two volumes reviewed here.

As it is almost inevitable in any printed book, there are some minor misprints that can be easily detected and corrected by the reader. I just point out a detail in the formal definition of a function $f\colon D \to T$: the author only refers that it is defined as a set of ordered pairs (d, r) with $d \in D$ and $r \in T$ with the additional property that if both (d, r) and (d, r') are elements in this set, then $r = r'$. As the author does not advocate that such a formal definition should be taught to high school students, it is of no practical importance for classroom use to discuss if the formal definition should be completed with some other requirements, if one wants it to reproduce all the features that one usually associates with functions with a specific domain and "target set". Nevertheless, it is perhaps relevant to note that, for the function to have D as its domain, one should add that, for each $d \in D$, there is an $r \in T$ such that (d, r) is in the set. Moreover, if one wants to define "surjectivity" as an intrinsic property of a function or, in general, distinguish the set T from the "image set" of the function, then one has, in one way or another, "to add" the set T to the definition, distinguishing the function itself from its "graph"; for instance, this can be done by defining the function as an ordered triple where the first and second elements are the sets D, T and the third (the "graph of the function") is a set of ordered pairs with the two properties stated above. One could replace the triple by a pair, eliminating the set D in the definition and reducing the property of the graph to the one stated by the author, as one could then just add that D is the set of the first coordinates of the pairs in the graph, but the set T cannot be defined just by examining the graph. All these "subtleties" should not, of course, be brought to a high school classroom, although they can be of some use to teachers.

Also with respect to high school math, I have to state that, with this set of books at hand, there is no excuse left for schoolteachers, textbook authors and government officials to persist in the unfortunate practice of trying to serve to students this fundamental part of school mathematics in a way that is in fact unlearnable …

As in the previous volumes of this series, on each topic, the author provides the reader with numerous illuminating activities and an excellent choice of a wide range of exercises.

Hung-Hsi Wu, *Rational Numbers to Linear Equations.* American Mathematical Society, 2020, 402 pages, Paperback ISBN 978-1-4704-5675-7, eBook ISBN 978-1-4704-6004-4.

Hung-Hsi Wu, *Algebra and Geometry.* American Mathematical Society, 2020, 375 pages, Paperback ISBN 978-1-4704-5676-4, eBook ISBN 978-1-4704-6005-1.

António de Bivar Weinholtz is a retired associate professor of mathematics of the University of Lisbon Faculty of Science, where he taught from 1975 to 2009. He was a member of the scientific coordination committee of the new curricula of mathematics for all the Portuguese pre-university grades (published between 2012 and 2014 and recently abolished).

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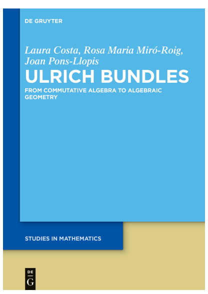
DOI 10.4171/MAG/126

Ulrich Bundles: From Commutative Algebra to Algebraic Geometry by Laura Costa, Rosa María Miró-Roig and Joan Pons-Llopis

Reviewed by Flaminio Flamini



Vector bundles on algebraic varieties constitute an important and ubiquitous topic in mathematics, not only because of their many applications to other disciplines, e.g., commutative algebra, mathematical physics, complex analysis, just to mention a few, but mainly because they offer a fundamental point of view for a thorough understanding of the geometry and topology of algebraic varieties. It is in this context that the book under review, written by internationally recognized experts on vector bundles and their moduli spaces, stands as a fundamental handbook focusing on particular classes of vector bundles, the so-called *arithmetically Cohen–Macaulay* (aCM for short) and more specifically *Ulrich bundles*, which recently have been the object of an intense and fruitful research activity. In the category of all vector bundles with which a given projective variety X can be endowed, a particularly important class is that of aCM bundles, which are basically those bundles with a large number of vanishing cohomology groups; among them, the subfamily of bundles having further the largest allowed number of linearly independent global sections are Ulrich vector bundles on X, and these constitute the main topic of the text.

The study of these bundles has had strong input from commutative algebra, beginning with the work of B. Ulrich in 1984, in connection with *maximally graded Cohen–Macaulay* modules with respect to the associated coordinate ring of X. The attention of algebraic geometers was later drawn by a celebrated paper of D. Eisenbud, F. O. Schreyer and J. Weyman in 2003 where, among other things, the Chow form of X is computed using Ulrich bundles under the assumption that X supports bundles of this type.

Besides these facts, several basic questions on Ulrich vector bundles enter the game (cf. *Eisenbud–Schreyer's conjecture* in Question 3.1.2 of the book): Is it true that any projective variety X supports an Ulrich vector bundle? If so, what is the minimum possible rank of an Ulrich bundle on X (the so-called *Ulrich complexity of* X)? Further basic questions are: If X supports Ulrich bundles of a given rank and of some other extra discrete data, i.e., Chern classes, do they fit in a *moduli space*? If so, what are the local and global properties of such moduli spaces?

From the list of previous questions, it is then clear why Ulrich's bundle theory has become a very active research area, with a constant offer of new results, but also of many still open questions. In approaching this theory, a variety of techniques have been used from several authors, depending on either the rank or the type of variety X on which some of these issues are studied. All these different approaches are scattered throughout an extensive literature making it difficult for an interested mathematician, and in particular a young mathematician who faces these problems for the first time, to keep track of the current state of the art in this topic.

It is with this in mind that the text under review presents itself as a fundamental tool: it provides a detailed introduction to the subject, attempting to emphasize the main reasons for the interest in these bundles, the wide range of areas that are involved in the development of their study, as well as the central questions that remain open in this area; it also gives an as comprehensive and organic as possible collection of all the major techniques used, scattered throughout the literature.

The detailed plan of the book is the following. To make the text as self-contained as possible, Chapter 1 is used to fix notation and introduce general objects and results in vector bundle theory that are used throughout the book. Chapter 2 deals with aCM bundles. First, fundamental preliminaries about these bundles are introduced and the important concepts of varieties of *finite*, *tame*, or *wild representation type* are given (cf. Definition 2.1.14); then the chapter focuses on classification of aCM bundles on relevant projective varieties: projective spaces and Horrock's theorem, smooth hyperquadrics and Knörrer's theorem, Grassmann varieties and, at last, low-rank aCM bundles on some smooth hypersurfaces. Chapter 3 focuses on the main object of the text: Ulrich bundles on a projective variety X. The chapter starts with a beautiful historical background on the subject, then provides the different definitions of Ulrich bundles scattered in the literature. The authors discuss first examples and basic properties of such bundles. Finally, to exhibit preliminary examples towards the Eisenbud–Schreyer conjecture, they review the classification of Ulrich bundles on smooth curves C of degree d and genus g in projective spaces, highlighting the different behavior of families of high-rank, indecomposable Ulrich bundles on C, according to the genus g being either 0, or 1 or, greater than or equal to 2. The rest of the book gives an outlook on the state of the art of the aforementioned questions, stressing the wide range of techniques involved in answering them. Precisely, Chapter 4 deals with the case of complete intersection varieties, in particular hypersurfaces, connections with representation theory and Cayley–Chow forms. Chapter 5 is devoted to the study of Ulrich bundles on smooth projective surfaces in the setting of the Enriques–Kodaira classification. Chapter 6 treats Ulrich bundles of any rank r at least 2 on a smooth complete intersection of two hyperquadrics in the 5-dimensional projective space, as well as low-rank Ulrich bundles on smooth Fano threefolds of index 2 (or even, Del Pezzo threefolds), or on smooth prime Fano threefolds. Chapter 7 discusses some relevant higher-dimensional cases, namely Segre, Grassmann and flag varieties.

Each chapter is endowed with a short final section of further references, historical remarks, and lists of suggested additional reading. Moreover, to make the text as self-contained as possible, the book concludes with an appendix, where important results concerning derived categories and Fourier–Mukai transforms sometimes used in the text are collected, and with an extensive and detailed bibliography.

To sum-up, the book provides a thorough, self-contained and well-organized introduction to a variety of fundamental concepts, techniques and examples concerning Ulrich vector bundles. The clarity of presentation, together with the detailed description of several key examples, makes the book suitable and versatile not only for established researchers with a particular interest in the topic, but also for young researchers, as a motivation to learn these basic techniques in a practical way. For all these reasons, I consider the book a valuable, highly recommended addition to the literature.

Laura Costa, Rosa María Miró-Roig and Joan Pons-Llopis, *Ulrich Bundles: From Commutative Algebra to Algebraic Geometry.* De Gruyter, 2021, 282 pages, Hardback ISBN 978-3-11-064540-8.

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Flaminio Flamini is full professor in geometry at the University of Rome "Tor Vergata", where he moved after a short period at the University of Illinois at Chicago, as visiting scholar, and about 5 years at the University of L'Aquila, as assistant professor. He graduated at the University of Rome "La Sapienza" and got his PhD in mathematics from the Consortium of Universities of Rome "La Sapienza" and "Roma Tre", under the supervision of Edoardo Sernesi. His research interests focus on algebraic geometry, in particular on curves, surfaces and their moduli, as well as on Brill–Noether theory, vector bundles and Hilbert schemes. Additional passions are certainly teaching young people and fun activities with his family (although the daughters are much more technologically and athletically smarter than dad).

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DOI 10.4171/MAG/122

Report from the Executive Committee meeting in Lisbon, 28–29 October 2022

Richard Elwes

Editorial note: Until March 2020, reports of EMS Executive Committee (EC) meetings were regularly published in the EMS Newsletter (as it was then). During the subsequent lockdowns this custom lapsed, as the EC moved to more frequent, shorter, ad hoc meetings online. The EC has decided to revive the practice of publishing meeting reports in the Magazine, for the two major EC meetings per year (though not for shorter ad hoc meetings).

Incoming EMS President Jan Philip Solovej (left) and outgoing President Volker Mehrmann (right)

The EMS Executive Committee (EC) met 28–29 October at University of Lisbon (ULisboa), on the kind invitation of the Portuguese Mathematical Society (SPM). The assembled company was firstly warmly welcomed by Luís Carriço, director of the Faculty of Sciences at ULisboa, and by Fernando da Costa, former president of SPM (also in attendance, of course, as editor-in-chief of this Magazine). On Friday evening, the group was well-entertained at Restaurante Baia do Peixe on Lisbon city centre's beautiful waterfront, at a meal kindly hosted by the current SPM president José Carlos Santos.

This EC meeting marked a moment of transition for the society, as the last to be chaired by Volker Mehrmann, whose term as president ends in December 2022, along with that of Vice President Betül Tanbuy, and Treasurer Mats Gyllenberg. Incoming president Jan Philip Solovej attended as a guest, as did incoming Treasurer Samuli Siltanen, and incoming EC member Victoria Gould. (Current Vice President Jorge Buescu's term continues until 2024, and current EC member Beatrice Pelloni begins a new term as new Vice President in January 2023.)

With greetings and housekeeping finished, business began with reports from the EMS's officers, starting with the chair Volker Mehrmann. Unfortunately, in recent months there has been severe turmoil within the society's office, in part due to the absence of a central EMS team member. This caused correspondence to fall badly behind. The President reiterated the profound apologies he had made to several people who had been inconvenienced. The EC deeply regretted this situation and made decisions on increasing investment in the EMS office, to clear the remaining backlog and ensure such problems never recur.

The President updated the committee on his many recent activities, notably around two major new EMS initiatives: the EMS Young Academy (EMYA) and EMS Topical Activity Groups (TAGs), both of which feature later in this report. The President has also represented the EMS at several recent meetings and events, including the opening of the EU-MATHS-IN Opendesk[1] on 20 September 2022. This new initiative aims to connect companies with industrial mathematicians and compile success stories.

Mats Gyllenberg then delivered his final report as EMS Treasurer after 10 years' efficient management of the society's accounts. He leaves the EMS in robust financial health, and the committee agreed on the need to increase spending on scientific activities.

The EMS Secretary Jiří Rákosník delivered his report, noting that his work has also been impeded by the problems in the EMS office. Nevertheless, he has made good use of the new EMS website and related infrastructure to assist EC decision-making on nominations for posts and other important society matters.

Outgoing EMS Vice President Betül Tanbay delivered her final report sketching her hopes and concerns for the society's future. Having attended the general assembly of the International Math-

[1] https://opendesk.eu-maths-in.eu/

EMS Executive Committee plus guests

ematical Union (IMU), she hopes that the relationship between the two bodies can be deepened in future. She also discussed her recent activities, including regarding EMS participation in the European Open Science Cloud (EOSC). She continues working as member of the General Board of the International Day of Mathematics (IDM, 14 March), whose theme for 2023 is "Mathematics for Everyone".

Ongoing Vice President Jorge Buescu delivered the final of the officers' reports. Having also attended the IMU General Assembly, he supported Betül Tanbay's position. He also represented the EMS at the AMS-EMS-SMF Meeting at Grenoble (18–22 July 2022) which he reported as being a very well-organised and high-level event.

Membership and website

The EC was pleased to approve the application for institutional membership from the SwissMAP Research Station in Les Diablerets, Switzerland. The list of new applications for individual membership would be approved subsequently electronically.

Jorge Buescu summarised the current situation with the EMS webpage, which has reached an approximately steady state, after a total rebuild and relaunch in March 2021. All the main functions (news, jobs, applications, conferences) are now working, with the automated systems providing efficient support for the EMS Office. Nevertheless, some further finessing of the web interface is possible and work on such improvements will continue.

Scientific and society activities

Betül Tanbay reported that preparations for the first EMS-supported Balkan Mathematical Conference (BMC[2]) is going well. The meeting will be held at University of Pitești (Romania) on 1 July 2023.

The most recent and third Caucasian Mathematics Conference was held in 2019, in Rostov-on-Don (Russia). Unfortunately, given the current situation in the region, it was decided to postpone the next meeting until 2025. The committee agreed that this important and successful cycle of meetings should not be forgotten in future.

The 29th Nordic Congress of Mathematicians[3], with the EMS, will be held at Aalborg University (Denmark) 3–7 July 2023.

The 9th European Congress of Mathematics (ECM[4]), the EMS's flagship conference, will be held in Sevilla (Spain) 15–19 July 2024. Preparations are going well. The EC discussed the schedule for planning the subsequent 10th ECM in 2028, with some debate around the appropriate format for large-scale mathematical events in the current world. The bidding process for the conference will get underway shortly and will welcome proposals for innovations to the format.

Good progress has been made on setting up the inaugural EMS Young Academy (EMYA[5]). The new body is due to meet for the first time at the beginning of 2023. The EMYA will comprise 30 young

[2] http://imar.ro/~bmc/
[3] https://ncm29.math.aau.dk/
[4] https://www.ecm2024sevilla.com/
[5] https://euromathsoc.org/EMYA

mathematicians (from 3rd year of PhD study up to 5 years post PhD, at point of nomination) to be selected from a pool nominated by EMS member societies and institutions.

Incoming Vice President Beatrice Pelloni will lead the formation of new evaluation committee for EMS Topical Activity Groups (TAGs[6]). These new bodies should be inclusive and diverse, particularly regarding geographical region and gender and can apply for EMS funding via the Meetings Committee. They will be invited to nominate candidates for EMS committees and speakers for conferences, and to plan and organise applications to European funding programs.

The EC agreed that in the years between biennial Council meetings, the annual meeting of Presidents of EMS members societies will take place online rather than in person.

Following recommendations from the EMS Meetings Committee, the EC agreed to provide financial support to several other scientific meetings, summer schools, and events.

Standing committees and projects

The EC considered each of the EMS's standing committees in turn, nominating new members to replace those who have reached the end of their terms, each of whom leaves with the EMS's sincere thanks. Four committee chairs will cease their roles at the end of 2022: Sophie Dabo of the Committee for Developing Countries (CDC), Stanislaw Janeczko of the European Solidarity Committee, Roberto Natalini of the Committee for Raising Public Awareness of Mathematics (RPA), and Alessandra Celletti of the Committee for Women in Mathematics (WiM). The EC expressed its great gratitude to each of them for all their excellent work. From January 2023 Bengt-Ove Turesson will take over as Chair for the CDC, Karine Chemla for the European Solidarity Committee, Katie Chicot for the RPA Committee, and Mikaela Iacobelli for WiM.

Two vice-chairs of the Education Committee, Gregory Makrides and Tinne Hoff Kjeldsen, joined the meeting remotely as guests to discuss the mission and objectives of that committee. Reports from all ten of the EMS's standing committees were considered: the Committee for Applications and Interdisciplinary Relations (CAIR), CDC, Education, ERCOM (Scientific Directors of European Research Centres in the Mathematical Sciences), Ethics, European Solidarity, Meetings, Publishing and Electronic Dissemination (PED), RPA, and WiM. These bodies are responsible for much of the EMS's work, and the society is grateful for the hard work of all the committee members.

The EC discussed the European Digital Mathematical Library (EuDML), whose General Assembly was held recently. The President reminded the committee of the EuDML's importance as a long-term project.

The committee considered the European Open Science Cloud (EOSC). The President will take part in the EOSC Symposium in Prague in November 2022, where he will deliver a presentation on how mathematics should fit into this project.

Publishing and publicity

EMS Publicity Officer Richard Elwes, in attendance as a guest, delivered a report on his activities, including running the society's social media channels. As of October 2022, the EMS has over 11,000 followers on Twitter[7], over 5,000 on Facebook[8], and over 100 on the newly added LinkedIn page[9].

Managing director André Gaul of EMS Press attended the meeting remotely as a guest and presented his report. The Subscribe to Open (S2O) model is working well; 21 journals are now using the S2O model for 2023 (compared to 17 in 2022). The decision on whether these journals flip to Open Access for 2023 will be made by end of January and will depend on subscription numbers. EMS Press currently publishes around 10 books per year. Proceedings of the ICM (around 6,000 pages) and proceedings of the ECM are in preparation. The latest addition the portfolio is Memoirs of the EMS (MEMS), which is proving a very successful project. EMS Press also plays a critical role in providing IT support to the EMS. The EC expressed its ongoing gratitude for this.

The editor-in-chief of the EMS Magazine Fernando da Costa, in attendance as a guest (and local organiser), reported on the Magazine. There is now a queue of contributions for future issues, a situation the editorial team had been working towards. Once the EMYA is set up, a renewed Young Mathematicians Column will be launched.

The committee also discussed the EMS Digest. Unfortunately, due the problems in the office, the September 2022 issue was never emailed out, although it was released online. With long-standing editor Mireille Chaleyat-Maurel stepping down at the end of this 2022, a new editor will be needed. She leaves this role with the sincere thanks of the EMS.

Funding, political, and scientific organisations

The EC discussed EMS nominations for several scientific panels and awards. It also considered the Coalition for Advancing Research Assessment (CoARA[10]), a global coalition of organisations interested in research funding, who have drafted an agreement establishing

[6] https://euromathsoc.org/EMS-TAGs

[7] https://twitter.com/@euromathsoc

[8] https://www.facebook.com/EuroMathSoc/

[9] https://www.linkedin.com/company/european-mathematical-society

[10] https://coara.eu/

Outgoing EMS Officers (left to right): Treasurer Mats Gyllenberg, Vice President Betül Tanbuy and President Volker Mehrmann.

a common direction for research assessment reform. The EC agreed that the EMS should sign this agreement.

The committee discussed developments at the European Research Council (ERC), where mathematics is struggling to win funding. The EC agreed that the EMS should continue liaising with the ERC and should continue to encourage more and stronger applications.

Close

The meeting closed with further expressions of thanks: firstly, to ULisboa and SPM and their local representatives for hosting this meeting in such fine and hospitable style. Outgoing President Volker Mehrmann, Vice President Betül Tanbuy, and Treasurer Mats Gyllenberg were presented with gifts to express the society's deep appreciation for their dedication to the European mathematical community over the last four years. The incoming President Jan Philip Solovej reiterated these words of gratitude, observing that recent years have been uniquely challenging for several reasons (notably COVID and a major cyberattack on the EMS webpage), and that in these circumstances the outgoing officers' list of achievements (e.g., securing the EMS's financial position, setting up the new publishing house EMS Press, inaugurating EMYA) is a remarkable testament to their vision and commitment to the cause of European mathematics.

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Richard Elwes is the EMS publicity officer and a senior lecturer at University of Leeds (UK). As well as teaching and researching mathematics, he is involved in mathematical outreach and is the author of five popular mathematics books.

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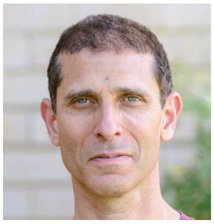
New editors appointed



Jason Cooper is a researcher in the Department of Science Teaching at the Weizmann Institute of Science, Israel. He holds an MSc in mathematics (Hebrew University of Jerusalem), and a PhD in mathematics education (Weizmann Institute of Science, 2016).

One focus of his research is on roles and affordances of university-level mathematics for the development of teachers' mathematical knowledge for teaching at school levels. This line of research brings together mathematicians and secondary-school mathematics teachers to discuss pedagogical and didactic dilemmas, and in so doing, to gain a deeper understanding of secondary-level mathematics and its teaching. Another line of his research is on introducing problem-solving activities into the routine of lower-secondary mathematics lessons.

Jason has been responsible for the *ERME column* (European Society for Research in Mathematics Education) in the EMS magazine since 2017.

Youcef Mammeri is, since 2022, a full Professor of Applied Mathematics in the Camille Jordan Institute, Jean Monnet University, France. He received his PhD in mathematics from University of Lille in 2008, and his accreditation to supervise research (HDR) from University of Picardie in 2015.

His research interests include modeling in life sciences, analysis of nonlinear PDEs and scientific computing, with applications to ecology, medicine, and biochemistry. More particularly, his work deals with the analysis of nonlinear dispersive PDE and advection-reaction-diffusion equations.

He is a member of the French Mathematical Society (SMF), and the French Applied and Industrial Mathematical Society (SMAI). He was the recipient of several grants and was PI for projects funded by ANR, CNRS and Campus France.

His webpage is https://ymammeri.perso.math.cnrs.fr/

European Mathematical Society

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* These discounts extend to members of national societies that are members of the EMS or with whom the EMS has a reciprocity agreement.

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- 25 € for persons belonging to a corporate EMS member society (full members and associate members)
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- A particular reduced fee of 5 € can be applied for by mathematicians who reside in a developing country (the list is specified by the EMS CDC).
- Anyone who is a student at the time of becoming an individual EMS member, whether PhD or in a more junior category, shall enjoy a three-year introductory period with membership fees waived.
- Lifetime membership for the members over 60 years old.
- Option to join the EMS as reviewer of zbMATH Open.