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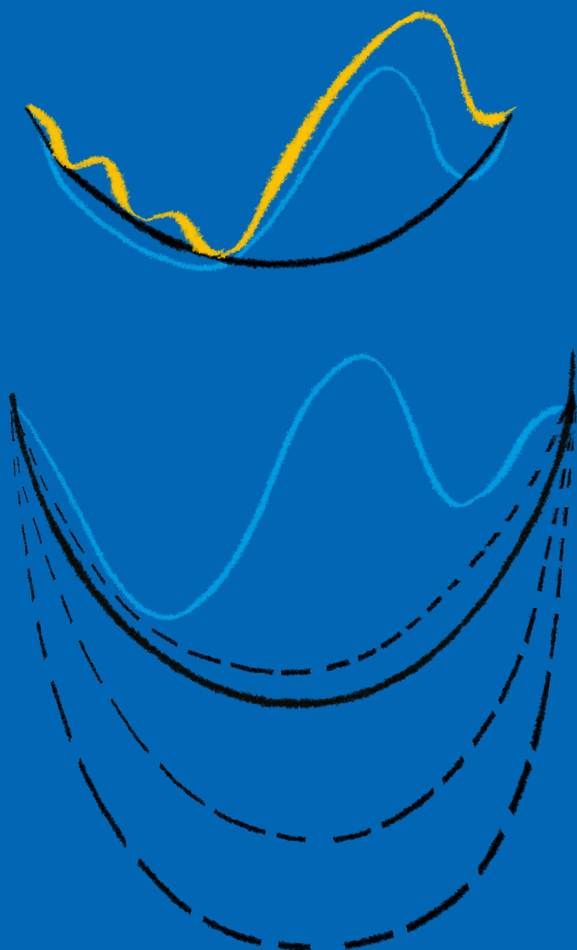

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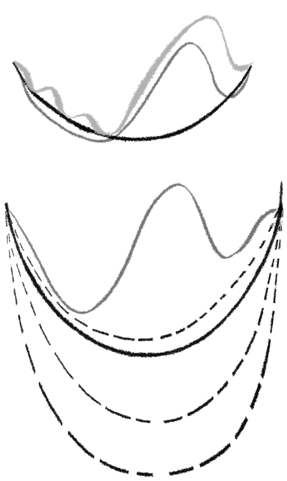

The cover illustration is an interpretation by A. B. Araújo of a pair of hand-drawn figures from the notebooks of Luis Ángel Caffarelli, who is interviewed in the present issue.

Photo by Jim Høyer,
University of Copenhagen

The summer is slowly coming to an end and many of us are preparing to return to our teaching duties. I hope many of you found time for a well-deserved summer break, even if summer is often busy with other duties, such as conferences. I attended several conferences in the late spring and summer, sometimes representing the EMS as president, and at other times simply as a mathematician presenting my own research, and occasionally as both president and a mathematician. I enjoy it all. It is exciting to interact with close colleagues and collaborators and to get a chance to discuss research, but also to meet mathematicians from different areas of the field and from all over the world, not only within Europe.

In May I had the great honor of being invited to Oslo to give a speech celebrating this year's Abel Prize Laureate Luis Caffarelli. I also went to both the 29th Nordic Congress in Aalborg, Denmark and to the 10th Congress of Romanian Mathematicians in Pitești, Romania. I am proud that EMS was directly involved in both events, highlighting its efforts to engage with all regions within Europe. The Nordic Congress was organized by the Nordic societies in collaboration with EMS. As part of the Romanian Congress, the EMS organized the first Balkan Mathematical Conference, and we are grateful to the organizers of the Romanian Congress for hosting us.

The Balkan Mathematical Conference (BMC) is a newly-launched EMS regional conference series. It is modeled on the Caucasian Mathematical Conference (CMC) which has now taken place three times, and we are already beginning to plan the next BMC which will be held in 2025. Please note that the call for hosting has gone live with a deadline of November 30th.[1] I really hope the regional conferences will continue to be a successful way for the EMS to broaden and diversify its reach. I want to take this opportunity to sincerely thank the former EMS vice president Betül Tanbay for her relentless efforts to help make all of this possible.

In my last message in the Magazine I enlisted all of you to encourage your colleagues to join the EMS as members. In my introductory speech at the Nordic Congress, I reiterated the same message. I would like now to repeat what I said then to encourage joining us as members of the EMS. People often ask what benefits come with a membership. There are, indeed, some great benefits, such as reduced registration fees for conferences organized by EMS and reduced prices from EMS Press. I am, however, proud that many offers from EMS do not require membership. It is, indeed, one of the most important goals of the EMS to support open science. As an example, the Magazine you are reading right now is open to all, members as well as non-members, and can be read on the EMS website. Many of the journals from EMS Press are open access for all through the fair principle of Subscribe to Open (S2O). The zbMATH Open offers open access for all to the oldest database for mathematicians. This is all to say, perhaps we should not be asking what we get as members, but rather what we support when we become members of the EMS. In my introductory speech at the Nordic Congress, I stressed that most of what EMS offers is open to all. Indeed, it may not be strictly necessary to become a member in order to partake in the many opportunities offered by the EMS, but this is actually exactly why I urge you to join us and help support our mission.

I wish you all a successful new academic year 2023/24.

Jan Philip Solovej
President of the EMS

[1] https://euromathsoc.org/news/two-ems-calls-99

As usually, this issue of the Magazine contains a number of articles in a variety of topics and I would just like to call your attention to two which illustrate a collaborative practice that has existed between the Magazine and several other publications. The issue starts with a translation into English of an article by Nikolay Tzvetkov first published in French in *La Gazette des Mathématiciens*. This is part of a long collaboration between the EMS and the SMF that has allowed the Magazine to regularly publish English translations of articles originally published in French in *La Gazette*. Similar agreements exist with other publications; for instance, articles originally published in the Magazine have been reproduced in, among others, the *International Association of Mathematical Physics Bulletin*, the *Gazeta de Matemática* (in Portuguese translation), and, in Chinese translation, the *Mathematical Advances in Translation*, a publication of the Chinese Academy of Sciences. Another article in this issue that is part of a series shared with a different publication is the traditional interview by Bjørn Dundas and Christian Skau with the Abel prize winner which is published by the Magazine in the September issue and is reprinted about six months later in the *Notices of the AMS*. These collaborations are important to boost the readership and as a service to the mathematical community way beyond Europe and the usual readers of the Magazine. It would be interesting to expand them to other publications in other languages.

Finally, from 1 September two new editors are starting their duties in the editorial board: Marie Elisabeth Rognes, who will reinforce the Features and Discussion team, and Vesna Iršič, who will be responsible for the Young Mathematicians' Column. A short biographical note of each of them can be found at the end of the issue. I am grateful to both for accepting to dedicate part of their time to our Magazine.

Fernando Pestana da Costa
Editor-in-chief

Nonlinear PDE in the presence of singular randomness

English translation of the paper "EDP non linéaires en présence d'aléa singulier"
published in *La Gazette des Mathématiciens*, **160**, pp. 6–14, 2019

Nikolay Tzvetkov

This paper describes results concerning the construction of low-regularity solutions of nonlinear partial differential equations that depend on a random parameter. The motivations for this study are very varied. However, in the end, the results obtained and the methods used are conceptually very similar.

1 Multiple Fourier series and Sobolev spaces on the torus

For $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we set

$$\langle x \rangle := (1 + x_1^2 + \cdots + x_d^2)^{1/2}.$$

Let $\mathbb{T}^d = (\mathbb{R}/(2\pi\mathbb{Z}))^d$ be the torus of dimension d. If $f\colon \mathbb{T}^d \to \mathbb{C}$ is a function of class C^∞, then for all $x \in \mathbb{T}^d$,

$$f(x) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n) e^{in\cdot x},$$

where $\hat{f}(n)$ are the Fourier coefficients of f. For $s \in \mathbb{R}$, the Sobolev norm of f is defined by

$$\|f\|^2_{H^s(\mathbb{T}^d)} = \sum_{n \in \mathbb{Z}^d} \langle n \rangle^{2s} |\hat{f}(n)|^2. \tag{1}$$

For an integer $s \geq 0$, we have the equivalence of norms

$$\|f\|^2_{H^s(\mathbb{T}^d)} \simeq \sum_{|\alpha| \leq s} \|\partial^\alpha f\|^2_{L^2(\mathbb{T}^d)}. \tag{2}$$

In (2), ∂^α represents a partial derivative of order $\leq s$. For $s = 0$, we recover the norm of the Lebesgue space $L^2(\mathbb{T}^d)$.

The Sobolev space $H^s(\mathbb{T}^d)$ is defined as the completion of $C^\infty(\mathbb{T}^d)$ with respect to the norm (1). In contrast to the case $s \geq 0$, for $s < 0$ the elements of $H^s(\mathbb{T}^d)$ are not classical functions on the torus, but can be interpreted as Schwartz distributions. Note that the Sobolev spaces are nested: the larger s is, the more regular the elements of $H^s(\mathbb{T}^d)$ are; the intersection of all $H^s(\mathbb{T}^d)$ is $C^\infty(\mathbb{T}^d)$. On the other hand, the smaller s is, the larger $H^s(\mathbb{T}^d)$ is; the union of all the spaces $H^s(\mathbb{T}^d)$ is the Schwartz space of $(2\pi\mathbb{Z})^d$-periodic distributions on $\mathbb{R}^d$.

2 Probabilistic effects in fine questions of analysis

2.1 An almost sure improvement of the Sobolev embedding

Let $(\Omega, \mathcal{F}, p)$ be a probability space. Recall that a random variable $g\colon \Omega \to \mathbb{R}$ belongs to $\mathcal{N}(0, \sigma^2)$, with $\sigma > 0$, if the image of the measure p under g is

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2\sigma^2}\Big) dx,$$

where dx is the Lebesgue measure on $\mathbb{R}$. The random variable g then follows the centered normal distribution with variance σ. Similarly, a random variable $g\colon \Omega \to \mathbb{C}$ belongs to $\mathcal{N}_{\mathbb{C}}(0, \sigma^2)$, if $g = h + il$ with $h \in \mathcal{N}(0, \sigma^2)$ and $l \in \mathcal{N}(0, \sigma^2)$ independent.

Let $u \in L^2(\mathbb{T}^2)$ be a deterministic function. There exists a sequence $(c_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$ (which consists of the Fourier coefficients of u) such that

$$u(x) = \sum_{n \in \mathbb{Z}} c_n e^{inx}.$$

Now consider a randomized version of u given by the expression

$$u_\omega(x) = \sum_{n \in \mathbb{Z}} c_n g_n(\omega) e^{inx},$$

where $(g_n(\omega))_{n\in\mathbb{Z}}$ is a sequence of independent variables of $\mathcal{N}_{\mathbb{C}}(0, 1)$. The randomization has no effect on the Sobolev regularity of u_ω (see, e.g., [5]). On the other hand, randomization has an important effect on regularity in Lebesgue spaces $L^p(\mathbb{T}^d)$. Using the rotation invariance of $\mathcal{N}_{\mathbb{C}}(0, 1)$, we obtain that $g_n(\omega)e^{inx} \in \mathcal{N}_{\mathbb{C}}(0, 1)$, and then the independence of g_n ensures that for a fixed x

$$u_\omega(x) \in \mathcal{N}_{\mathbb{C}}\Big(0, \sum_{n\in\mathbb{Z}} |c_n|^2\Big).$$

Since Gaussian variables have finite moments of any order, we get

$$u_\omega(x) \in L^p(\Omega \times \mathbb{T}),$$

which implies that $u_\omega(x) \in L^p(\mathbb{T})$ almost surely, a remarkable improvement in the L^p regularity of u_ω compared to that of u. Note that the Sobolev embedding requires that a deterministic function be $H^{1/2}(\mathbb{T})$-regular in order to conclude that it lies in $L^p(\mathbb{T})$ for any $p < +\infty$ (and this restriction on regularity is optimal). In descriptive terms, randomization saves half a derivative compared with the Sobolev embedding. Like the Sobolev embedding, this effect has been known since the beginning of the 20th century, and it may seem surprising that the interaction between these two phenomena has not been studied more in the past.

Finally, thanks to the Khinchin inequality, in the preceding discussion one is allowed to replace Gaussian variables by a more general family of random variables (e.g., Bernoulli variables).

2.2 *Almost sure products in Sobolev spaces of negative index*

Let

$$u_\omega(x) = \sum_{n\in\mathbb{Z}} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx}, \quad \frac{1}{4} < \alpha < \frac{1}{2},$$

be a random series, with g_n as in the preceding section. It is easy to verify that almost surely $u_\omega \in H^\sigma(\mathbb{T})$ for $\sigma < \alpha - \frac{1}{2}$, but almost surely $u_\omega \notin H^{\alpha-\frac{1}{2}}(\mathbb{T})$. In the following, we fix a number σ such that $\sigma < \alpha - \frac{1}{2}$ (it should be assumed that this number is very close to $\alpha - \frac{1}{2}$). The series u_ω is therefore in a Sobolev space of negative index and it is difficult to define an object like $|u_\omega|^2$. After renormalization, it is nevertheless possible to give a meaning to $|u_\omega|^2$, and even to determine its regularity in Sobolev spaces. Let us start by considering the partial sums

$$u_{\omega,N}(x) = \sum_{|n|\le N} \frac{g_n(\omega)}{\langle n\rangle^\alpha} e^{inx},$$

which are C^∞ functions. Now expand $|u_{\omega,N}(x)|^2$ as

$$|u_{\omega,N}(x)|^2 = \sum_{|n|\le N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}} + \sum_{\substack{n_1\neq n_2\\ |n_1|,|n_2|\le N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^\alpha\langle n_2\rangle^\alpha} e^{i(n_1-n_2)x}.$$

The first term of this expansion (the zero-order Fourier coefficient) contains all the singularity, while the second term has a limit almost surely in $H^{2\sigma}(\mathbb{T})$. We then set

$$c_N := \mathbb{E}\Big(\sum_{|n|\le N} \frac{|g_n(\omega)|^2}{\langle n\rangle^{2\alpha}}\Big) = \sum_{|n|\le N} \frac{2}{\langle n\rangle^{2\alpha}} \sim N^{1-2\alpha},$$

and we define the renormalized partial sum as

$$|u_{\omega,N}(x)|^2 - c_N = \sum_{|n|\le N} \frac{|g_n(\omega)|^2-2}{\langle n\rangle^{2\alpha}} + \sum_{\substack{n_1\neq n_2\\ |n_1|,|n_2|\le N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^\alpha\langle n_2\rangle^\alpha} e^{i(n_1-n_2)x}.$$

The independence of the random variables g_n ensures that the zero-order Fourier coefficient is well-defined. More precisely, we obtain

$$\mathbb{E}\Big(\Big|\sum_{|n|\le N} \frac{|g_n(\omega)|^2-2}{\langle n\rangle^{2\alpha}}\Big|^2\Big) = \sum_{|n|\le N} \frac{4}{\langle n\rangle^{4\alpha}},$$

which has a limit as $N \to +\infty$ if $\alpha > 1/4$.

Similarly, the independence implies that the expectation

$$\mathbb{E}\Big(\Big\|\sum_{\substack{n_1\neq n_2\\ |n_1|,|n_2|\le N}} \frac{g_{n_1}(\omega)\overline{g_{n_2}(\omega)}}{\langle n_1\rangle^\alpha\langle n_2\rangle^\alpha} e^{i(n_1-n_2)x}\Big\|^2_{H^{2\sigma}}\Big)$$

is bounded from above by a term of the order of

$$\sum_{n_1,n_2} \frac{\langle n_1-n_2\rangle^{4\sigma}}{\langle n_1\rangle^{2\alpha}\langle n_2\rangle^{2\alpha}}.$$

The latter sum is convergent under the condition $-4\sigma + 4\alpha > 2$, which is equivalent to our assumption $\sigma < \alpha - \frac{1}{2}$. Consequently, the sequence

$$\big(|u_{\omega,N}(x)|^2 - c_N\big)_{N\ge 1} \tag{3}$$

has a limit in $L^2(\Omega; H^{2\sigma}(\mathbb{T}))$. This limit is by definition the renormalization of $|u_\omega|^2$. We can also establish (using more sophisticated arguments) the almost sure convergence in the Sobolev space $H^{2\sigma}(\mathbb{T})$ of the sequence (3). Note that since $\sigma < 0$, the norm in $H^{2\sigma}(\mathbb{T})$ is weaker than the norm in $H^\sigma(\mathbb{T})$ (where the series $u_\omega(x)$ is defined).

To sum up in a very informal way, the squared modulus of an element of H^σ is in $H^{2\sigma}$, after renormalization. This is a remarkable probabilistic effect which lies at the heart of the study of evolutionary PDE, in the presence of randomness, in Sobolev spaces of negative index. We will further develop this topic in the remainder of this text.

3 Solving the nonlinear wave equation with low-regularity initial data

The wave equation is a typical example of a dispersive PDE. Solving dispersive PDE with low-regularity initial data has a long history, dating back to the seminal works of Ginibre and Velo and of Kato. Kenig, Ponce and Vega, Klainerman and Machedon, and most notably Bourgain have developed tools from harmonic analysis allowing to obtain solutions of very low regularity. The question of the optimality of these results then arose. It was Lebeau's work that launched a series of results on the construction of counterexamples showing the optimality of the assumption of regularity in the previous results. It was in this context that the idea of proving a kind of probabilistic well-posedness for regularities where counterexamples were constructed was introduced in [23], and then implemented in [5, 6].

3.1 Solving the linear wave equation with periodic distributions as initial data

Consider the linear wave equation

$$(\partial_t^2 - \Delta)u = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \qquad (4)$$

where $t \in \mathbb{R}$, $x \in \mathbb{T}^3$, $u\colon \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ and Δ is the Laplace operator. It is readily verified that for

$$(u_0, u_1) \in C^\infty(\mathbb{T}^3) \times C^\infty(\mathbb{T}^3)$$

the solution of (4) is given by the map $S(t)$ defined by

$$S(t)(u_0,u_1) = \sum_{n \in \mathbb{Z}^3} \Big(\cos(t|n|)\widehat{u_0}(n) + \frac{\sin(t|n|)}{|n|}\widehat{u_1}(n)\Big)e^{in\cdot x},$$

where $|n| = (n_1^2 + n_2^2 + n_3^2)^{1/2}$. For $n = 0$, the expression $\frac{\sin(t|n|)}{|n|}$ is naturally understood as its limit t.

Since $|\cos(t|n|)| \le 1$ and $|\sin(t|n|)| \le 1$, it follows from the above definition that

$$\|S(t)(u_0,u_1)\|_{H^s} \le C(1 + |t|)(\|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}). \qquad (5)$$

Since the map $S(t)$ is linear, we can define a unique extension of $S(t)$ for

$$(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$$

and solve (4) with initial data in $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ for arbitrary $s \in \mathbb{R}$.

3.2 The nonlinear problem. Resolution by deterministic methods

The previous discussion makes it easy to solve (4) with singular initial data (in Sobolev spaces of arbitrary negative index). The argument is based on the a priori estimate (5) and the linear nature of the map $S(t)$ (or of equation (4)). The situation changes radically if we consider a nonlinear perturbation of (4). In this text, we restrict our attention to the case of a cubic nonlinear interaction. More precisely, we consider the problem

$$(\partial_t^2 - \Delta)u + u^3 = 0, \quad u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x). \quad (6)$$

For this problem, the crucial information (5) and the linear nature of the equation are lost. Nevertheless, equation (6) is of Hamiltonian type. Therefore, formally, the solutions of (6) satisfy the algebraic relation

$$\frac{d}{dt}\int_{\mathbb{T}^3}\Big((\partial_t u(t,x))^2 + |\nabla_x u(t,x)|^2 + \frac{1}{2}u^4(t,x)\Big)dx = 0. \qquad (7)$$

This relation implies that the Sobolev space $H^1(\mathbb{T}^3)$ is one of the natural settings for the study of problem (6). The starting point of this study is given by the following classical result.

Theorem 3.1. *For any pair $(u_0,u_1) \in H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)$ (real valued) there exists a unique global in time solution of (6) in the class*

$$(u, \partial_t u) \in C^0(\mathbb{R}; H^1(\mathbb{T}^3) \times L^2(\mathbb{T}^3)).$$

If, in addition, $(u_0,u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$, for a given $s \ge 1$, then

$$(u, \partial_t u) \in C^0(\mathbb{R}; H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)). \qquad (8)$$

Finally, the dependence on the initial data is continuous.

Using compactness methods (going back to the work of Leray) we can exploit (7) and obtain a much weaker version of Theorem 3.1, without uniqueness and without the propagation of regularity (8). In Theorem 3.1, uniqueness results from the Sobolev embedding $H^1(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3)$. The L^6-norm appears here naturally when we study the L^2-norm of the nonlinear term u^3. As for the propagation of the regularity, it derives from the estimate

$$\|u^3\|_{H^s(\mathbb{T}^3)} \le C\,\|(1-\Delta)^{s/2}u\|_{L^6(\mathbb{T}^3)}\,\|u\|_{L^6(\mathbb{T}^3)}^2. \qquad (9)$$

The details of the proof of (9) can be found in [1], where estimates of the type (9) are called tame. The key point in estimate (9) is that the s derivatives acting on the expression u^3 are redistributed in such a way that, at the end, the right-hand side of (9) depends only linearly on the strong norm (the one that contains derivatives).

In view of the discussion of the linear problem (4), it is now natural to ask whether Theorem 3.1 generalizes to initial data in $H^s \times H^{s-1}$ for a given $s < 1$. As we shall see below, such a generalization is possible for some s, but not for all. By using Strichartz estimates instead of the Sobolev embedding $H^1(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3)$, part of Theorem 3.1 generalizes to

$$(u_0,u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3), \quad s \ge 1/2. \qquad (10)$$

More precisely, the local well-posedness of (6) can be established under the assumption (10). A more detailed description of Strichartz's estimates would go beyond our objectives in this text. We merely point out that Strichartz estimates can be seen as improvements almost everywhere in time of Sobolev embeddings, when, instead of considering an arbitrary function, we consider a function that satisfies a dispersive PDE. We refer to [24] for more details on Strichartz estimates and the generalization of Theorem 3.1 under assumption (10). It can be conjectured that the global in time part of Theorem 3.1 remains true under assumption (10). The most advanced results towards the resolution of this conjecture are in [8, 21].

3.3 The limitations of deterministic methods

The restriction (10) is optimal with respect to the well-posedness in the sense Hadamard of the problem (6) with initial data in $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$. More precisely, we have the following result.

Theorem 3.2. *Let $s \in (0, 1/2)$ and $(u_0, u_1) \in H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$. There exists a sequence*

$$u_N(t, x) \in C^0(\mathbb{R}; C^\infty(\mathbb{T}^3)), \quad N = 1, 2, \ldots$$

such that

$$(\partial_t^2 - \Delta)u_N + u_N^3 = 0,$$

with

$$\lim_{N \to +\infty} \|u_N(0) - u_0, \partial_t u_N(0) - u_1\|_{H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)} = 0$$

but for all $T > 0$

$$\lim_{N \to +\infty} \|u_N(t), \partial_t u_N(t)\|_{L^\infty([0,T]; H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3))} = +\infty.$$

Well-posedness in the sense of Hadamard requires existence, uniqueness and continuous dependence on the initial data. Theorem 3.2 shows that continuous dependence on initial data fails.

The proof of Theorem 3.2 is based on an idea of Lebeau (see for example [15]): if the initial data are localized at high frequency, then for small times a good approximation of the solution of (6) is given by the solution of

$$\partial_t^2 u + u^3 = 0, \quad u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x) \qquad (11)$$

which is obtained from (6) by neglecting the effect of the Laplacian. In other words, under the hypothesis of Theorem 3.2, nonlinear effects dominate in the regime described above. The solution of (11) manifests the phenomenon of amplification described by Theorem 3.2 and this property propagates to the solutions of (6) by a perturbative, highly non-trivial argument. A detailed proof of Theorem 3.2 can be found in [24].

3.4 Resolution by probabilistic methods beyond the limitations of deterministic theory

Despite the result of Theorem 3.2, we can ask whether a form of the well-posedness of (6) remains true for initial data in

$$H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3), \quad s < 1/2. \qquad (12)$$

The answer to this question is positive if we endow the space (12) with a non-degenerate probability measure such that we have existence, uniqueness and (a form of) continuous dependence almost surely with respect to this measure.

We will choose the initial data for (6) from the realizations of the following random series:

$$u_0^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in\cdot x}, \quad u_1^\omega(x) = \sum_{n \in \mathbb{Z}^3} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in\cdot x}. \qquad (13)$$

Here $(g_n)_{n \in \mathbb{Z}^3}$ and $(h_n)_{n \in \mathbb{Z}^3}$ are two families of independent random variables conditioned by $g_n = \overline{g_{-n}}$ and $h_n = \overline{h_{-n}}$, so that u_0^ω and u_1^ω are real valued. Furthermore, it is assumed that for $n \neq 0$, g_n and h_n are complex Gaussians with distribution $\mathcal{N}_\mathbb{C}(0, 1)$, while g_0 and h_0 are real Gaussians with distribution $\mathcal{N}(0, 1)$.

The partial sums associated with (13) are Cauchy sequences in $L^2(\Omega; H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3))$ for all $s < \alpha - \frac{3}{2}$. Therefore, the initial data (13) belong almost surely to $H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ for $s < \alpha - \frac{3}{2}$. Furthermore, the probability of the event

$$(u_0^\omega, u_1^\omega) \in H^{\alpha - \frac{3}{2}}(\mathbb{T}^3) \times H^{\alpha - \frac{5}{2}}(\mathbb{T}^3)$$

is zero. It follows that for $\alpha > 5/2$ we can apply Theorem 3.1 to the data (u_0, u_1) described by (13). For $\alpha > 2$, we can apply refined deterministic results (based on Strichartz estimates). Finally, for $\alpha \in (3/2, 2)$, Theorem 3.2 applies and we get:

Theorem 3.3 (pathological approximations). *Let $\alpha \in (3/2, 2)$ and $0 < s < \alpha - 3/2$. For almost every ω, there exists a sequence*

$$u_N^\omega(t, x) \in C^0(\mathbb{R}; C^\infty(\mathbb{T}^3)), \quad N = 1, 2, \ldots$$

such that

$$(\partial_t^2 - \Delta)u_N^\omega + (u_N^\omega)^3 = 0,$$

with

$$\lim_{N \to +\infty} \|u_N^\omega(0) - u_0^\omega, \partial_t u_N^\omega(0) - u_1^\omega\|_{H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)} = 0,$$

but for all $T > 0$

$$\lim_{N \to +\infty} \|u_N^\omega(t), \partial_t u_N^\omega(t)\|_{L^\infty([0,T]; H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3))} = +\infty.$$

However, the following result also holds.

Theorem 3.4 (probabilistic well-posedness). *Let $\alpha \in (3/2, 2)$ and $0 < s < \alpha - 3/2$. Using Theorem 3.1, define the sequence $(u_N^\omega)_{N \geq 1}$ of solutions of problem (6) with regular initial conditions given by*

$$u_{0,N}^\omega(x) = \sum_{|n| \leq N} \frac{g_n(\omega)}{\langle n \rangle^\alpha} e^{in\cdot x}, \quad u_{1,N}^\omega(x) = \sum_{|n| \leq N} \frac{h_n(\omega)}{\langle n \rangle^{\alpha-1}} e^{in\cdot x}. \quad (14)$$

Then there exists a set $\Sigma \subset \Omega$ of probability 1 such that for every $\omega \in \Sigma$ the sequence $(u_N^\omega)_{N \geq 1}$ converges when $N \to +\infty$ in $C^0(\mathbb{R}; H^s(\mathbb{T}^3))$ to a (unique) limit that satisfies (6) in the sense of distributions.

Theorems 3.3 and 3.4 show that the type of approximation of the initial data is crucial when establishing the probabilistic well-posedness.

Using compactness methods (à la Leray), we can hope to obtain convergence of a subsequence of $(u_N^\omega)_{N \geq 1}$. The convergence of the whole sequence $(u_N^\omega)_{N \geq 1}$ is beyond the reach of weak-solutions techniques. The fact that the whole sequence converges already contains a form of uniqueness. In [6], one can find a form of uniqueness that can be formulated in a suitable functional framework.

In [6], we also obtain a probabilistic continuous dependence on the initial data, the proof of which makes use of conditioned large deviation properties which seem to be of independent interest.

We can prove the result of Theorem 3.4 for more general randomizations than (13). For example, Gaussian variables can be replaced by Bernoulli variables and the deterministic coefficients $\langle n\rangle^{-\alpha}$ by other coefficients with "similar" behaviour for $|n| \gg 1$ (see [6] for more details).

Theorem 3.4 provides a nice dense set Σ of initial data such that for good approximations we get nice global solutions (but for bad approximations we get divergent sequences, as shown by Theorem 3.3!). On the other hand, due to [22], we also have a dense set of bad initial data, even for the natural approximations by Fourier truncation (or convolution).

Theorem 3.5 (pathological initial data). *Let $0 < s < \frac{1}{2}$. Then there is a dense set $S \subset H^s(\mathbb{T}^3) \times H^{s-1}(\mathbb{T}^3)$ such that for every $(f, g) \in S$, the sequence $(u_N)_{N\geq 1}$ of (smooth) solutions of*

$$(\partial_t^2 - \Delta)u + u^3 = 0$$

with data

$$u_0(x) = \sum_{|n|\leq N} \hat{f}(n)e^{in\cdot x}, \qquad u_1(x) = \sum_{|n|\leq N} \hat{g}(n)e^{in\cdot x}$$

does not converge. More precisely, for every $T > 0$,

$$\lim_{N\to\infty} \|u_N(t)\|_{L^\infty([0,T];H^s(\mathbb{T}^3))} = +\infty.$$

One may even prove that the pathological set S contains a dense G_δ set and consequently the good-data set is not generic in the sense of Baire.

3.5 Going even further

For $\alpha < 3/2$, u_0^ω is no longer a classical function. In this case, it can be interpreted as a distribution belonging to a Sobolev space with negative index. We cannot expect a result like that of Theorem 3.4 for $\alpha < 3/2$. A renormalization is necessary, as shown by the following result established in [19].

Theorem 3.6. *Let $\alpha \in (\frac{5}{4}, \frac{3}{2})$ and $s < \alpha - 3/2$. There exists positive constants γ, c, C, T_0 and a divergent sequence $(c_N)_{N\geq 1}$ such that for any $T \in (0, T_0)$, there exists a set Ω_T such that the probability of its complement is smaller than $C\exp(-c/T^\gamma)$ and such that if we denote by $(u_N^\omega)_{N\geq 1}$ the solution of*

$$(\partial_t^2 - \Delta)u_N^\omega - c_N u_N^\omega + (u_N^\omega)^3 = 0$$

with initial data given by (14), then for any $\omega \in \Omega_T$, the sequence $(u_N)_{N\geq 1}$ converges for $N \to +\infty$ in $C^0([-T, T]; H^s(\mathbb{T}^3))$. In particular, for almost every ω there exists $T_\omega > 0$ such that $(u_N^\omega)_{N\geq 1}$ converges in $C^0([-T_\omega, T_\omega]; H^s(\mathbb{T}^3))$.

Theorem 3.6 was a first step in the study of problem (6) in Sobolev spaces of negative index. In the remarkable recent work [4] the result of Theorem 3.6 was extended to the range $\alpha > 1$.

4 Invariant measures for the nonlinear Schrödinger equation

Let us now consider the nonlinear Schrödinger equation, posed on the torus of dimension two:

$$(i\partial_t + \Delta)u - |u|^2u = 0, \quad u(0, x) = u_0(x), \quad x \in \mathbb{T}^2. \tag{15}$$

Here the solution u is complex valued, but the equation is of first order in time. We have the following analogue of Theorem 3.1 in the context of (15).

Theorem 4.1. *For any $u_0 \in H^1(\mathbb{T}^2)$ there exists a unique global solution of (15) in the class $C^0(\mathbb{R}; H^1(\mathbb{T}^2))$. If moreover $u_0 \in H^s(\mathbb{T}^2)$ for some $s \geq 1$, then $u \in C^0(\mathbb{R}; H^s(\mathbb{T}^2))$. The dependence on the initial data is also continuous.*

Equation (15) is again Hamiltonian in nature. This implies that the functional

$$E(u) = \int_{\mathbb{T}^2} \Big(|\nabla_x u(t, x)|^2 + |u(t, x)|^2 + \frac{1}{2}|u(t, x)|^4\Big)dx \tag{16}$$

is preserved by (15). The Gibbs measure associated with (16) is a "renormalization" of the completely formal object

$$\exp(-E(u))du. \tag{17}$$

This renormalization is a classic procedure in quantum field theory, which would be impossible to present in this short text. Let us just say that the measure obtained by this renormalization is absolutely continuous with respect to the Gaussian measure induced by the random series

$$u_0^\omega(x) = \sum_{n\in\mathbb{Z}^2} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x}, \tag{18}$$

where $(g_n)_{n\in\mathbb{Z}^2}$ is a family of independent complex Gaussians with distribution $\mathcal{N}_{\mathbb{C}}(0, 1)$. Once the measure (17) has been rigorously defined, the natural question is whether we can define a dynamics related to (15) that leaves this measure invariant. The answer to this question is given by the work of Bourgain [3]. The difficulty lies in the fact that (18) does not define a classical function. The object defined by (18) almost surely belongs to the Sobolev space $H^s(\mathbb{T}^2)$ for all $s < 0$. Such a regularity implies that Theorem 4.1

cannot be applied in the context of an initial datum given by (18). This regularity is also beyond the reach of the most sophisticated deterministic techniques. Nevertheless, the following statement can be deduced from [3].

Theorem 4.2. *Using Theorem 4.1, define the sequence* $(u_N^\omega)_{N\geq 1}$ *of solutions of* (15) *for the initial conditions of class* C^∞ *given by*

$$u_{0,N}^\omega(x) = \sum_{|n|\leq N} \frac{g_n(\omega)}{\langle n\rangle} e^{in\cdot x}. \tag{19}$$

Then for any $s<0$ *the sequence*

$$\Big(\exp\Big(\frac{it}{2\pi^2}\|u_N^\omega(t)\|_{L^2}^2\Big)u_N^\omega(t)\Big)_{N\geq 1}$$

converges almost surely in $C^0(\mathbb{R};H^s(\mathbb{T}^2))$ *to a limit which satisfies (in the sense of distributions) a renormalized version of problem* (15)*.*

There is a similarity between Theorems 3.4 and 4.2. One notable difference is the need to renormalize the sequence $(u_N^\omega)_{N\geq 1}$ of Theorem 4.2 in order to obtain a limit. This renormalization is linked to the construction of the measure from the formal object (17) mentioned above.

The result of Theorem 4.2 can be formulated in the spirit of Theorem 3.6. More precisely, we can establish the convergence of the solutions of the problem

$$(i\partial_t + \Delta)u_N + c_N u_N - |u_N|^2 u_N = 0$$

with initial data (19), where $(c_N(\omega))_{N\geq 1}$ is a sequence of real numbers almost surely divergent to $+\infty$.

5 Singular stochastic PDEs

The issues considered in the previous sections are very close to the analysis of PDE in the presence of a singular random source (noise). This topic has received a lot of attention in recent years (see, for example, [7, 9–12, 14]). The closest equation to those in the previous sections is the nonlinear heat equation. More precisely, we consider the problem

$$\partial_t u - \Delta u + u^3 = \xi, \quad u(0,x)=0, \quad x\in\mathbb{T}^3. \tag{20}$$

In this equation, ξ is the space-time white noise on $[0,+\infty[\times\mathbb{T}^3$. The unknown u is a real valued function. There are many physical motivations for considering a PDE perturbed by white noise. A serious discussion of these motivations is beyond the scope of this paper.

The source term ξ represents the singular randomness in (20), while in (6) and (15), the initial datum is the source of the singular randomness. A little experience with the analysis of evolutionary PDE is enough to know that the two situations are very similar and even that, in some cases, for reasons of convenience, we can easily transform the problem with initial data into a problem with a source term and zero as initial data.

A representation in the spirit of (13) and (18) of white noise on $[0,+\infty[\times\mathbb{T}^3$ is given by the formula

$$\xi = \sum_{n\in\mathbb{Z}^3} \dot{\beta}_n(t) e^{in\cdot x}, \tag{21}$$

where β_n are independent Brownian motions, conditioned by $\beta_n = \overline{\beta_{-n}}$ (β_0 is real and for $n\neq 0$, β_n is complex valued). The derivative with respect to t of β_n in (21) is in the sense of distributions.

If $\xi\in C^\infty([0,\infty[\times\mathbb{T}^3)$, equation (20) can be solved by deterministic methods. This is the analogue of Theorems 3.1 or 4.1 in the context of (20). For $N\gg 1$, an approximation of ξ given by (21) by smooth functions is defined by

$$\xi_N(t,x) = \rho_N \star \xi,$$

where $\rho_N(t,x) = N^5\rho(N^2t, Nx)$, with ρ a test function with integral 1 on $[0,\infty[\times\mathbb{T}^3$. This is a convolution regularization, very similar to the regularizations used in (14) and (19). The following statement can be deduced from [12, 17].

Theorem 5.1. *There exists a sequence* $(c_N)_{N\geq 1}$ *of positive numbers, divergent when* $N\to\infty$*, such that if we denote by* u_N *the solution of the problem*

$$\partial_t u_N - \Delta u_N - c_N u_N + u_N^3 = \xi_N, \quad u_N(0,x)=0, \quad x\in\mathbb{T}^3,$$

then $(u_N)_{N\geq 1}$ *converges in law when* $N\to\infty$*.*

It is also possible to have almost sure convergence in suitable Hölder spaces. The initial datum $u(0,x)$ can be non-zero: it just has to belong to a well-chosen function space (see [17]).

The complete analogue of (13) and (18) in the context of problem (20) would be the white noise on $\mathbb{T}\times\mathbb{T}^3$ defined by

$$\xi = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}^3} g_{m,n}(\omega) e^{imt} e^{in\cdot x}, \tag{22}$$

where $(g_{m,n})_{(m,n)\in\mathbb{Z}^4}$ is a family of independent standard Gaussian variables conditioned so that ξ is real valued. The result of Theorem 5.1 remains true for a noise ξ of the form (22).

There are other parabolic PDEs for which a result in the spirit of Theorem 5.1 can be obtained, perhaps the most famous example being the KPZ equation (see [11]).

6 Final discussion

The statements of Theorems 3.4, 3.6, 4.2 and 5.1 are similar. Their proofs also follow the same pattern. First, local in time solutions are constructed. Then we use global information, which is either an invariant measure or an energy estimate, to move towards global in time solutions.

To construct local solutions, we look for solutions in the form

$$u = u_1 + u_2,$$

where u_1 contains the singular part of the solution.

By probabilistic arguments, very close to the considerations in Section 2, u_1 and some maps related to it have better properties than those given by deterministic methods. The whole probabilistic machinery is to be found in this part of the analysis. In the proof of Theorem 3.4, we use almost sure improvements of the Sobolev embedding, while in the proofs of Theorems 3.6, 4.2 and 5.1, we construct almost sure products in Sobolev spaces of negative index.

Next, we solve the problem for u_2 using deterministic arguments. Here the nature of the equation becomes even more important. In Theorem 5.1, the basic tool is elliptic regularity, whereas in Theorems 3.4, 3.6 and 4.2 we make crucial use of oscillations in time (captured by Bourgain spaces, for example).

The transition to global in time solutions in Theorem 4.2 uses an invariant measure as a global control over the solutions. In Theorem 3.4 the globalization of solutions is done by an argument based on energy estimates. It is remarkable that, in the context of Theorem 5.1, we can also use these two methods to globalize local solutions: in [13] the globalization is done using a control coming from an invariant measure, whereas the work [17] uses the (much more flexible) method of energy estimates.

Oh's work [18] establishes the analogue of Theorem 3.2 in the context of Theorem 4.2. To the best of my knowledge, no such amplification result for particular approximations is known in the context of Theorem 5.1.

We have already mentioned that, in Theorem 3.4, we allow more general randomizations compared to Theorem 4.2. This has made it possible to consider randomizations for functions of Sobolev spaces on the whole space $\mathbb{R}^d$ and to prove results in the spirit of Theorem 3.4 for problems posed on the whole space (instead of on the torus). For work in this direction, see [2, 16].

Theorem 3.4 allows more general randomizations than Theorem 4.2, but it says nothing about the transport by the flow of the measure defining the initial data set (whereas the proof of Theorem 4.2 tells us that the initial Gaussian measure is quasi-invariant under the flow). We still do not know the nature of the measure transported by the flow in the context of Theorem 3.4 (see [20] for recent progress on this interesting problem).

The list of references below is far from complete. This is a very active field. For a description of other results directly related to what we have just described, we refer the reader to [9, 11, 24].

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Listen2Intuition: A Mathematics & Arts exhibition project

Karin Baur, Klemens Fellner and Tamara Friebel

Having the right intuition and working hard is common to both the arts and the sciences. The high abstraction levels of mathematics, especially, seem to require an extra amount of intuitive guidance.

Listen2Intuition is a Mathematics & Arts exhibition project, which bridges disciplines by associative thinking, and aims to promote multidisciplinarity on an eye-to-eye level. For several years two mathematicians and an artist "listen" to each other across disciplines, generating artworks, compositions as well as new mathematical research.

1 Introduction: Mathematics & Arts

The story of Listen2Intuition originates in 2014 with a call for unconventional research by the University of Graz. The authors together with Gerhard Eckel from the University of Music and Performing Arts Graz were successful in a highly competitive two-stage selection procedure to ensure funding for the project *Mathematics & Arts: Towards a balance between artistic intuition and mathematical complexity*.

Mathematics and arts are linked throughout the development of the human civilisation: The Vedic scriptures, the Greek philosophers, the Renaissance with da Vinci's universality and Palladio's architecture, to mention just a few, are cornerstones of modern cultures. The twentieth century has seen an increasing influence of mathematical structures and the tools of dynamical systems on the arts.

For instance, the architect Le Corbusier developed the *Modulor*, a sequential system of scales constructed from the golden ratio and human body proportions, to design his outstanding buildings. The composer Iannis Xenakis (in his early years a co-worker of Le Corbusier) transferred the interest in mathematical and graphical structures to the field of music [10]. György Ligeti used a more metaphorical, transformative approach of scientific research in his concept of musical "permeability," according to which a musical structure is "permeable" if it allows a free choice of intervals. And the designs of contemporary architects like Zaha Hadid rely in fundamental manner on the capacities of current 3D design software and fabrication techniques, and thus on a wide spectrum of geometrical and numerical routines.

Already these few examples suggest that whenever mathematics is instrumental in a creative process, a crucial component is the individual intuitive understanding of the artist: Le Corbusier wrote full of inspiration about his Modulor and left a legacy of outstanding buildings, such as the Unité d'Habitation in Marseille, yet he seemed not to have convinced a great many other architects to follow his path: Was it Le Corbusier's talent to make the connection successful in practice?

1.1 Towards a balance between artistic intuition and mathematical complexity

Based on such thoughts, our project *Mathematics & Arts: Towards a balance between artistic intuition and mathematical complexity* set the agenda to *research and explore mathematical structures and dynamical systems* and to explore *intuitive interfaces* for non-mathematicians. The aim is to initiate an *eye-to-eye dialogue across disciplines, be it between maths and arts or, say, between algebra and PDEs*.

Here is an example of a project. Originating from current algebraic research in representation theory, Figure 1 plots triangulations of an annulus connecting a given set of corner points on the inner and outer circles. From a mathematical point of view, the three left triangulations are topologically equivalent. The fourth is a borderline case featuring touching edges. The right-most instance exhibits intersecting edges, and so is no longer a triangulation from a mathematical perspective.

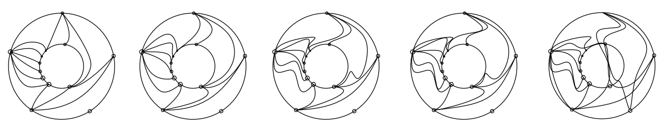
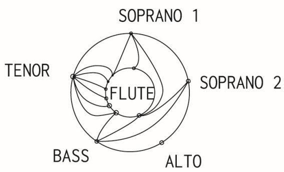
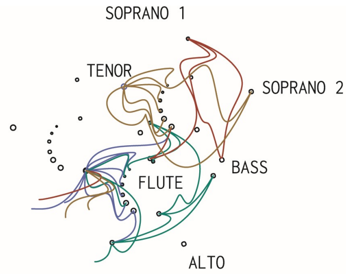


Figure 1. Instances of topological triangulations and beyond.

However, Figure 1 was not created by a mathematician, but by the artist in the team. After having discussed the basic principles of triangulations and their relevance for current research on cluster



Performance Technique: Each voice - Soprano, Soprano 2, Alto, Tenor, Baritone is related in different ways to each other. The singers stand in the following configuration, with the Flute in the centre as left:



Improvisation Schematic Diagram: SSATB & FL

Improvise freely using this diagram to define the points of relation between the players.
Improvisation Duration: 1'00
Score Duration: 6'24

Score

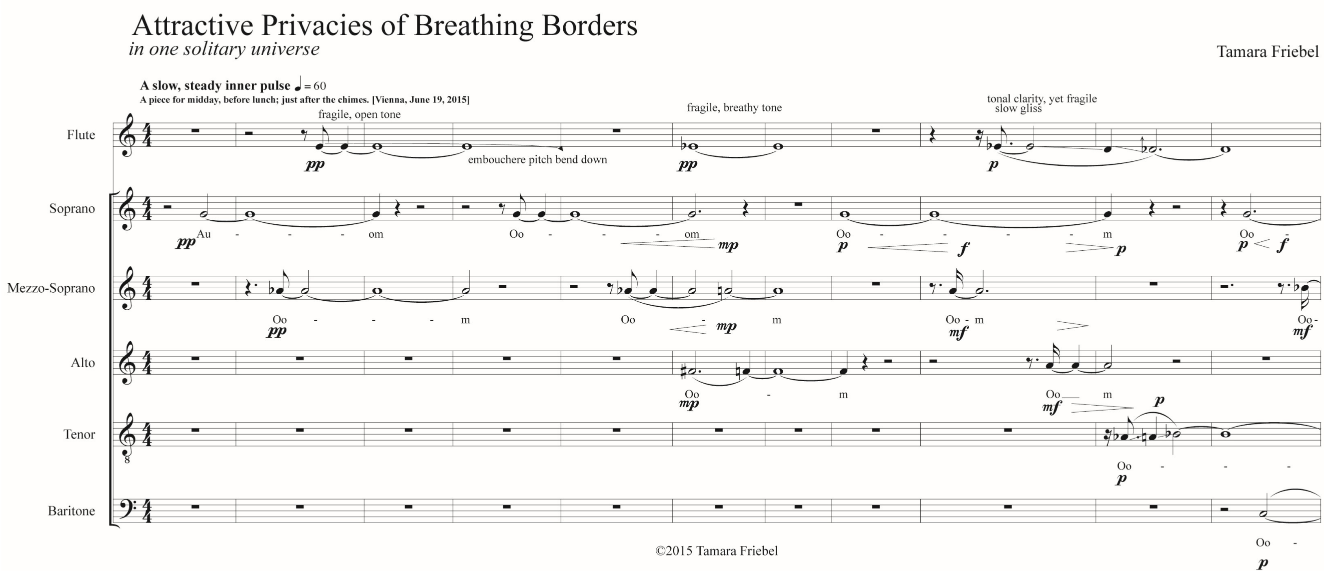


Figure 2. Attractive Privacies of Breathing Borders, in one solitary universe.

algebras, Figure 1 is part of an approbation process guided by years of artistic training. The five instances in Figure 1 represent an aesthetic evolution, which is not afraid of breaking the mathematical rules of triangulations. Instead, there is an aesthetic continuity to the five images.

Figure 1 is intended to serve as an example that it is not necessary (and practically also not possible) for an artist to understand a comprehensive background of mathematical structures. What is necessary in initiating an interdisciplinary process is to *identify representative aspects*, which serve as points of inspiration. In the above example, two aspects were found the most important: First, the geometric possibilities themselves, and second, the topological properties, in particular the flexibility in the shape of edges, which constitutes a metaphor for the elasticity of borders in future artworks.

In the interdisciplinary process of our project, we found it crucial for the artist to decide which aspects play a key role. The decision was made after listening to the explanations of the mathematical background, despite not understanding everything. After having listened, an intuitive assimilation process could start, which allows the participants to transform elements from foreign disciplines into something new within their own discipline. Here, Tamara Friebel continued to explore the geometric repertoire originating from Figure 1, derived from it a graphical concept of the composition *Attractive Privacies of Breathing Borders* before finally producing a score, see Figure 2 and below in Section 3.

2 The set-up of the exhibition project Listen2Intuition

During the work on the project *Mathematics & Arts: Towards a balance between artistic intuition and mathematical complexity*, the question of how to communicate with wider audiences became increasingly important. When talking to friends and colleagues, we realised the challenges faced in explaining our project. Both the theoretical concepts and the practical output of the project were elaborate. It takes time to explain the mathematical background, the intermediate stages of appropriating foreign theories and concepts, to exemplify these processes and, at the end of the day, to discuss the output.

In 2019, the city of Graz initiated the cultural year *Kulturjahr 2020 Graz*.[1] Under the premise *Wie wir leben wollen* (How we want to live), a call was publicised for projects to address questions of contemporary society. The authors of this article submitted (and were granted) a proposal for a pop-up exhibition space, that is, a shop-front meeting place for people to drop in and be guided through our Mathematics & Arts projects. We named this exhibition project *Listen2Intuition*.

2.1 How to address a wider audience?

With neither mathematics nor contemporary music being a natural favourite for a wider audience, one of our challenges was to attract visitors in the first place.

In the planning of Listen2Intuition as a pop-up exhibition space, we were hoping for a good amount of curious people just dropping in by chance. However, even with a popular lunch place located next door, only a few people would just happen to walk in. Not for lack of interest per se. Rather, it seems that everybody has such a busy schedule that before walking into an exhibition by chance, you are already walking somewhere else.

More effective in publicising the exhibition were a well-set-up website[2] and social media activities. What seemed to work best was old fashion invitations and the word of mouth, that Listen2Intuition is something one should visit.

The following two subsections, 2.2 and 2.3, provide some mathematical background relevant to our project.

2.2 Cluster combinatorics and algebraic structures of triangulations

In algebra, a quiver Q is an oriented graph, possibly with loops and multiple edges between vertices. A quiver encodes interactions between mathematical objects (its vertices), while arrows describe the direction of the interactions.

Quivers and their mutations have become very popular in algebra, as they play a key role in the theory of cluster combinatorics, developed first by Fomin and Zelevinsky in, e.g., [6, 7], and on the categorical side, by Buan et al., [3] and by Caldero et al. [4]; see [1] for an overview.

A key source for quivers in this context – quivers without loops and 2-cycles – are triangulations of surfaces. Among them, annuli are of particular interest: The associated quivers are manageable, yet the triangulations provide infinite-type cluster structures which exhibit interesting dynamics.

[1] https://www.kulturjahr2020.at

[2] https://www.listentointuition.com

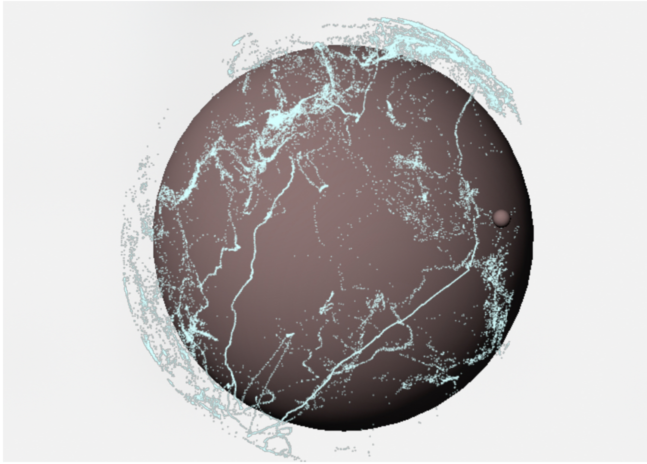


Figure 3. Numerical simulation of a swarm of particles.

2.3 Collective behaviour and swarming patterns

Swarming patterns form due to the collective behaviour of a large number of individuals. Despite all individuals being subject to the same interaction law, collective behaviour, and swarming in particular, can feature highly complicated dynamics due to the amount of crisscross interactions.

Many models of collective behaviour share a common mathematical description in terms of gradient-flow differential equations or discrete stochastic models. Figure 3 shows an example of a large swarm of particles spreading over a sphere. Repulsive–attractive interactions describe individuals that retract when being too close to one another, yet come closer together when being too far away from one another. Examples are provided by the Lennard-Jones potential describing, e.g., crystallisation, or the Morse potential modelling biological swarms. Repulsive–attractive interaction potentials give rise to a stunningly rich variety of patterns, which depend on the details of the interaction potentials in a highly complex (yet not chaotic) manner, see, e.g., [5, 8, 9].

2.4 Growth behaviour of infinite friezes

The interdisciplinary cooperation within our team was not exclusively directed towards connecting mathematics and arts on an eye-to-eye level. In fact, as it happened, we found also connections between the algebra of frieze patterns, in particular properties of so-called growth coefficients of infinite friezes, and ideas from dynamical systems; these were published in [2].

3 Listen2Intuition: An interactive exhibition in four stages

Our exhibition concept for Listen2Intuition divided the experience into four main stages (stations).

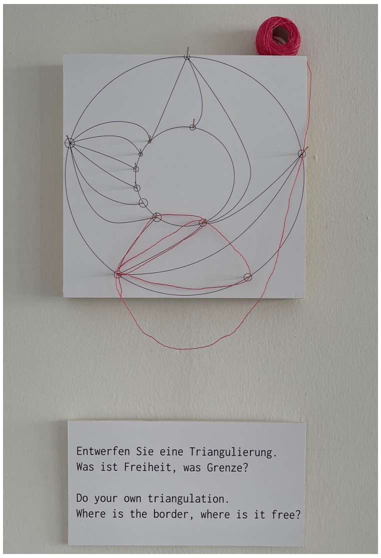


Figure 4. Do-it-yourself triangulation.

Figure 5. Discover your attractive privacies.

The first stage started by presenting the topological triangulations of Figure 1. The visitors were right away invited to invent their own "triangulation" by using a thread that loops around pins as corner points, see Figure 4. We insisted that visitors should feel free to do what they felt like doing as triangulations, and that nothing would be regarded as a mistake. The intention was to be as interactive as possible and to remove any previous-knowledge bias. Even the four-year-old daughter of Fellner and Friebel had great fun making her own triangulations on the pin board.

Next, visitors were presented graphical works, see Figure 5, which were part of the appropriation process of Tamara Friebel leading to the composition *Attractive Privacies of Breathing Borders* for five singers and flute. For example, Figure 2 shows on the top left the triangulation which served as a basic concept of the musicians (five singers and one flute) and how they would relate to one another according to the edges connecting corner points. On the top right, the colourful sketch shows a much more advanced graphical appropriation, which intuitively represents qualities of the later composition *Attractive Privacies*, the final score of which is shown at the bottom of Figure 2.

An important part of the first exhibition stage was also the following anecdote: On one occasion, Tamara Friebel invited professional musicians to improvise the colourful sketch plotted on the top right in Figure 2 as a graphical score. They would only use the syllable OM and not yet know the final score. The musicians were also told the basic concept that the edges of the triangulation represent which musicians could directly communicate with one another and which not. Also, that the flute as the central instrument would have the capacity of linking everybody.

Given only these concepts and the colourful sketch in Figure 2 to be interpreted as a graphical score, the six musicians came up with an improvisation, which sounded relatively similar to the final composition *Attractive Privacies* (of course, without being as refined as a crafted piece). As astounding as this anecdote might sound, it becomes more tangible if one recalls that musicians will always listen to one another even when improvising. They will intuitively form a collective and, when guided by the right input, intuitively appropriate the given material, not unlike how this was done by Friebel in the composition process.

Take-away message. The following observation was a highly interesting aspect of the first stage: We had a total of about 150 guests each making a do-it-yourself triangulation, but no two resembled each other. The combinatorial possibilities are just too many to see repetitions within a sample of 150 tries. The artistic appropriation, however, which is the colourful sketch in Figure 2 together with the rules of interrelating voices, was successful in distilling an essence of the final composition *Attractive Privacies of Breathing Borders*. At least in the sense that with improvisation and composition being similar, one could say that the task became intuitively understandable and essentially reproducible within the paradigm of a musical performance.

The second stage of the exhibition can be seen in Figure 6. It is fair to say that this stage was the one closest related to the name Listen2Intuition.

In an intimate corner with high-quality loudspeakers, a loop of a 1 min 15 sec excerpt of the composition *The Agility* was playing, a duo for violin and guitar written for and first performed at Wien

Figure 6. Listen and see.

Modern.[3] Visitors would experience 40 sec of pure sound with black screen and 35 sec of sound with the video from the live performance. The score of the excerpt could be read along the wall.

The composition *The Agility* brought together the two aforementioned instruments, violin and guitar, whose sounds (despite both being string instruments) do not really mix (which was actually a reason for the composition to be originally commissioned). Visitors were given 3D prints of geometric forms, which were derived from the formal studies on triangulations. Amongst the many formal studies, there was one featuring two subunits, which would not really fit together, yet superpose to a larger entity. These 3D prints were handed to the audience as a metaphor of how a composition was supposed to bring together sounds that do not mix. The audience was then invited to listen to the music. They were asked to use their individual intuition to experience this music, which is actually better than words, to be able to express how two units can be part of an entity while remaining distinct.

By this time in the exhibition experience, the great majority of visitors realised that Listen2Intuition succeeded to draw their mindset into accessing new worlds – worlds that are not imposed by the exhibition's team, but which open up according to the individual personality of the visitors.

Take-away message. The exhibition team felt very positive that Listen2Intuition succeeded to fascinate all (minus one) visitors despite (or rather because of) the quite challenging content, and regardless of any previous knowledge.

The third stage invited visitors to another interactive activity. Figure 7 shows a visitor, who played a zither using ping pong balls. The sound would be recorded using a pick-up microphone, transferred to a laptop, and processed in real time by the music software Max/MSP. A so-called Max patch modulated the sound based on the collective behaviour of a simulated swarm and then played it back to the visitor over the loudspeaker in the middle of the table.

The visitors were given the task to improvise whatever came to mind while observing a "video-score" on the computer screen (again under the premise that they should feel free to do what felt right and nothing was regarded as a mistake).

As part of the third stage, the exhibition team would tell the visitors that by tuning up and down the volume of the loudspeaker with the feedback sound, they would be able to explore the richness of the sounds produced using the rich collective behaviour of a swarm dynamics. From a mathematical point of view, it should be added that there are somehow too many swarming models exhibiting what a human would always intuitively classify as swarming behaviour. In contrast to other areas of applied mathematics, where prototypical equations, like the heat equation, form the basis of a great many of related mathematical models, there exists no single prototypical swarming model.

Take-away message. Stage three of our exhibition wanted to bypass any mechanistic explanation of swarming patterns and

Figure 7. Experience your collective awareness.

[3] Excerpt of Tamara Friebel's *The Agility of Perspective Nearness in a lake of being*, performed by Duo Hedda: https://vimeo.com/482416548 from 41:15 until 42:30.

collective behaviour. We rather wanted to provide an intuitive experience and allow visitors to develop a kind of self-understanding of collective dynamics based on a sound feedback.

The fourth stage invited the visitors into a curtained-off space. There, the composition *Dance Me to My Rebirth*[4] for harpsichord and electronics by Tamara Friebel was played along a video showing swarming patterns. In fact, Figure 3 is a still from that video. Visitors were informed that the task of stage three was also given to the performer Maja Mijatović (yet using the string of the harpsichord instead of a zither, which was the exhibition's practical alternative to a delicate harpsichord). Maja's improvisation was recorded and then used as a layer of electronic music juxtaposed on top of the harpsichord performance of the score of *Dance Me to My Rebirth*.

Hearing the final composition *Dance Me to My Rebirth* for harpsichord and electronics allowed the visitors to relate to their own improvisation experience at stage three. Moreover, being in a darkened space exposed only to video and music gave the opportunity to further process the exhibition experience.

Overall, Listen2Intuition was a great experience for the exhibition team/authors as well as for the visitors. The feedback was so positive that we reopened the exhibition for an additional two weeks later in the year. In total we had about 150 visitors with an age range from 4 to 84 years. On average, every visitor spent almost an hour in the exhibition, which reflects the intensity of the engagement we were so happy to witness. Some visitors did even come back and donate contributions or little artworks to a public board of responses.

3.1 Listen2Intuition will come to Paris in 2024

Listen2Intuition will welcome visitors at the Maison Poincaré in Paris[5] from April to July 2024. Please check the website of the Maison Poincaré for the details to be specified.

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[5] https://maison-des-maths.paris

Karin Baur studied mathematics, philosophy and French literature at the University of Zurich. After completing her PhD at the University of Basel in 2002, she held post-doctoral positions at ETH Zurich, the University of California at San Diego and the University of Leicester. From 2007 to 2011, she was an SNSF professor at ETH Zurich. Since 2011 she was the chair of the Algebra and Number Theory Group at the Institut für Mathematik und Wissenschaftliches Rechnen at the University of Graz. Since October 2018, she is a Zone 2 professor in the School of Mathematics at the University of Leeds. Her research interests are in algebraic, geometric and combinatorial methods in representation theory, including cluster algebras, cluster categories, frieze patterns, surface algebras, triangulations and tilings. She is currently a Wolfson Fellow of the Royal Society and the leader of an EPSRC programme grant in combinatorial representation theory.

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Klemens Fellner studied mathematics and physics at the Technical University of Vienna and bassoon at the Conservatory of Vienna, and finished his PhD in applied mathematics in 2002. His post-doctoral research included long-term visits at the ENS Cachan, Granada, UAB Barcelona, Baker Heart Research Institute Melbourne, before becoming Universitätsassistent at the University of Vienna and senior research assistant at the University of Cambridge. Since 2011 he is professor of mathematics/computational sciences at the Institute of Mathematics and Scientific Computing at the University of Graz. His expertise as applied mathematician ranges from mathematical modelling to the analysis of partial differential equations. His recent research focus is

on (bio)-chemical reaction-diffusion systems and the development of mathematical models for lipolysis. Klemens Fellner is currently dean of the Faculty of Natural Sciences of the University of Graz.

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Tamara Friebel was born in Cohuna, Victoria, in Australia. She currently lives in Austria, having moved there in 2002. She studied architecture at the Royal Melbourne Institute of Technology (RMIT) in Melbourne and at the University of Applied Arts Vienna (masterclass Zaha Hadid), sociology and theology (in universities in Melbourne), and composition at the University of Music and Performing Arts Vienna (masterclass Chaya Czernowin). She completed a PhD in composition at the University of Huddersfield (with Liza Lim) in 2013 with a portfolio of works entitled "Generative Transcriptions, an Opera of the Self." Since 2014 she has been a postdoctoral researcher at the University of Graz as part of the mathematics and art research project "The Collaborative Mind," teaches courses at the University of Music and Performing Arts Graz, and gives diverse lectures in festivals and conferences.

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Abel interview 2023: Luis Ángel Caffarelli

Bjørn Ian Dundas and Christian F. Skau

Bjørn Ian Dundas/Christian F. Skau: *Professor Caffarelli, firstly we want to congratulate you on being awarded the Abel Prize for 2023 for your "seminal contributions to regularity theory for nonlinear partial differential equations, including free boundary problems and the Monge–Ampère equation." You will receive the prize tomorrow from His Majesty the King of Norway.*

We will return to your mathematics, but first it might be a good idea to know something about your background. You were born in Buenos Aires in 1948. How would you describe your childhood?

Luis Ángel Caffarelli: We lived in a nice middle class area of Buenos Aires. At the time when I was a kid I would play soccer a lot with my friends. Another game we enjoyed a lot was to throw a ball, or something else, and see who got it closest to a wall or to a line.

The place where I lived was what I would characterize as an engineering area. My father was a mechanical engineer who worked in the shipping industry, assembling and repairing vessels for navigation in the Río de la Plata bay. When I was 16 years I joined him to assemble a ship engine. This is one of my warmest memories from my adolescence.

[BID/CFS]: *So, your father was an inspiration to you?*

LÁC: Yes, he was. He pushed me, with a lot of love, to do some serious things like engineering or science, or something like that. And I followed his advice. I was admitted to a renowned secondary school, Colegio Nacional de Buenos Aires, run by the University of Buenos Aires. There my scientific interests were channeled by inspiring teachers towards physics and mathematics. I graduated from high school in 1966, and I joined the University of Buenos Aires in March 1967, majoring in physics and mathematics. I received my undergraduate degree in 1970.

As a student I was heavily influenced and inspired by Luis Santaló [1911–2001], Manuel Balanzat [1912–1994] and Carlos Segovia [1937–2007]. Santaló and Balanzat were both Spanish mathematicians who moved to Argentina as a consequence of the Spanish Civil War. Santaló made important contributions to integral geometry and geometric probability, while Balanzat worked in

Abel laureate Luis Caffarelli giving his lecture at the University of Oslo, 2023. © *Ola G. Sæther / The Abel Prize*

functional analysis. They built, jointly with Rey Pastor [1888–1962] and Pi Calleja [1907–1986], a superb undergraduate and graduate mathematics program at the University of Buenos Aires, generating a very strong group in analysis, geometry and algebraic geometry.

The harmonic analyst Segovia was a prominent graduate from Universidad de Buenos Aires who did his PhD at the University of Chicago in 1967 with Alberto Calderón [1920–1988]. While closer to me in age than Santaló and Balanzat, Segovia was always a strong support.

[BID/CFS]: *You then embarked on your graduate studies, receiving your PhD in mathematics in 1972. Your PhD advisor was Calixto Calderón [1939–], the stepbrother of the much older and very famous mathematician Alberto Calderón you mention. Tell us about your graduate work.*

LÁC: Calixto Calderón steered my imagination towards special function theory, which was a vibrant subject associated with finding optimal representations to solutions of partial differential equations (PDEs) by analytic methods consisting of series representations of

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Luis Caffarelli Abel Prize laureate 2023. © *Ola G. Sæther / The Abel Prize*

special polynomial forms. A key point of my thesis involves Abel summability techniques, which I find fascinating considering I am being honored by receiving the Abel Prize! In fact, one of the papers I published with Calixto Calderón in 1974, in the wake of my thesis work, is titled: *"On Abel summability of multiple Jacobi series."*

[BID/CFS]: *May we interject a remark here which, at least tangentially, is related to this? In 2007 you published, together with your former PhD student Luis Silvestre [1977–], the enormously influential paper* "An extension problem related to the fractional Laplacian." *Historically, Abel [1802–1829] was the first to introduce fractional derivation and integration (he effectively used it to solve the generalized isochrone problem – a problem that goes back to Christiaan Huygens [1629–1695]). So broadly speaking, some key notions in your work can be traced back to Abel.*

LÁC: Well, that's why we are here!

Early career and free boundary problems

[BID/CFS]: *After obtaining your PhD at the University of Buenos Aires you went to the University of Minnesota as a post-doc, arriving there in January 1973. What made you choose to go there?*

LÁC: There was a connection established earlier between the mathematics departments at the University of Buenos Aires and the University of Minnesota, so it was quite natural for me to go there as a post-doc student. Also, my PhD advisor Calixto Calderón had arrived there already in 1972, having been offered a permanent position. However, he left the University of Minnesota in the fall of 1974 to take up a tenured position at the University of Illinois at Chicago.

I found my colleagues at the mathematics department at the University of Minnesota to be friendly, extremely generous and dedicated, and they taught me much of what I know. They shared their ideas and gave me guidance as I began my research program.

[BID/CFS]: *Was there some person in particular that was important for your mathematical research while at the University of Minnesota?*

LÁC: Yes, upon arrival I met Hans Lewy [1904–1988]. He was an extraordinary analyst working on nonlinear PDEs and minimal surfaces. I attended a lecture series on harmonic analysis given by Lewy. He gave me two problems, which I succeeded in solving in a few months. Having his support was a game changer for me, influencing strongly my career path. One of the problems Lewy suggested was the *obstacle problem*, which is an example of what is known as a free boundary problem.

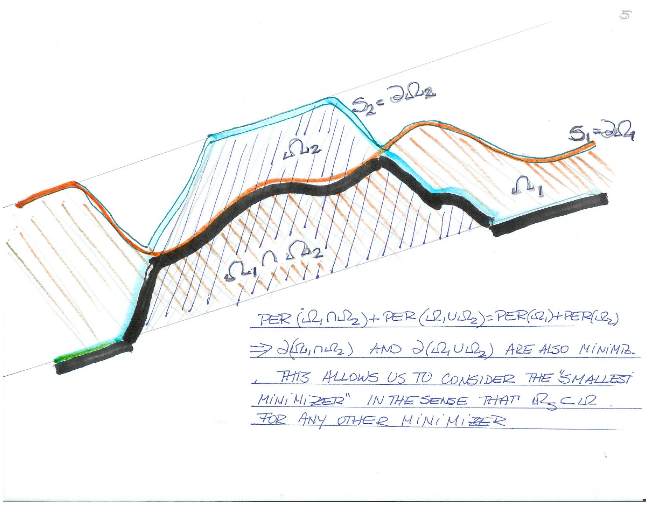
[BID/CFS]: *In 1977 you published a paper titled* "The regularity of free boundaries in higher dimensions" *in the prestigious journal Acta Mathematica. You surprised the mathematical world with this paper, a work whose novelty and brilliance was the basis of your future fame. You were the first mathematician to really understand the free boundary problem in more than one dimension. Furthermore, the methods that you introduced have been extremely powerful and are still being used in many other problems. Could you elaborate on all this?*

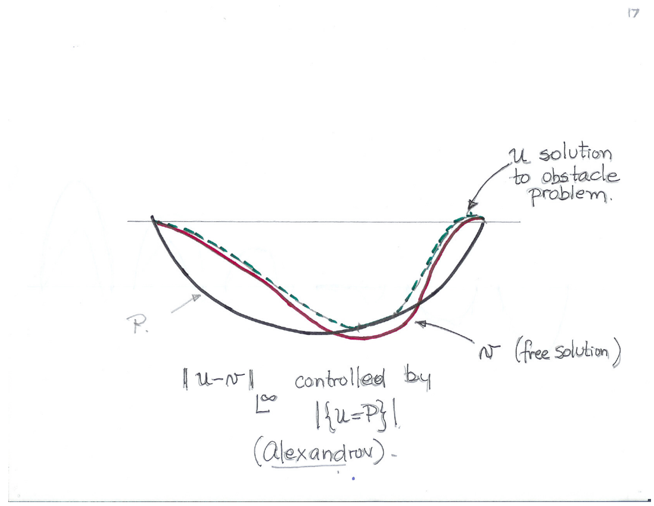
LÁC: Melting ice is an example of a free boundary problem, the free boundary being the surface between ice and water. The surface shifts as ice melts.

Another example of this type of a problem would be a balloon inside a box (or a drop inside a cavity). If the balloon is suspended in the air without constraints, a first approximation to its shape is given by a prescribed mean curvature equation – a mildly nonlinear PDE that we can deduce from the fact that the balloon tries to minimize the energy of the configuration. If constrained to lie inside the box, the surface of the balloon will behave differently depending on whether it presses against the wall or not, giving rise to a strongly nonlinear PDE. The separation curve between the different regions is called the *free boundary*. In this area I have investigated extensively the mathematical problems associated with solid-liquid interfaces, jet and cavitational flows, and gas and liquid flow in porous media.

The Navier–Stokes and Monge–Ampère equations

[BID/CFS]: *Let us put things in perspective. The early 1960s saw great developments in the theory of* linear *PDEs, where a PDE is*





Illustrations created by Luis Ángel Caffarelli (published with his explicit permission)

called linear *if it is linear in the unknown and its derivatives. Many people contributed to this, but the deepest and most significant results were due to Lars Hörmander [1931–2012], according to the citation when he received the Fields Medal in 1962. It is worth noting, though, that some results obtained in the 1950s by Alberto Calderón were very important for this development.*

The upshot of all this is that for linear PDEs there exists a theory. This contrasts with what is the case for nonlinear *PDEs, where it is usually acknowledged that there is no "general" theory, with specialist knowledge being somewhat divided between several essentially distinct subfields. PDEs, and in particular, nonlinear PDEs, are ubiquitous in physics, ranging from gravitation to fluid dynamics. They also are important in mathematics, and have been used to solve problems such as the Poincaré conjecture and the Calabi conjecture.*

Many of the fundamental PDEs in physics, such as the Einstein equations of general relativity and the Navier–Stokes equations, are quasilinear *(meaning that the highest-order derivatives appear only as linear terms, but with coefficients possibly functions of the unknown and lower-order derivatives). On the other hand, the Monge–Ampère equation, which we shall return to and which arises in differential geometry, is* fully nonlinear, *meaning that it possesses nonlinearity in one or more of the highest-order derivatives.*

Among the many open questions are the existence and regularity of solutions to the Navier–Stokes equations, named as one of the Millennium Prize Problems of the Clay Foundation in 2000. Tell us about your involvement with the Navier–Stokes equations.

LÁC: In 1980 Louis Nirenberg [1926–2020], Abel Prize recipient in 2015 (a prize he shared with John Nash, Jr. [1928–2015]) invited me to join the Courant Institute at NYU as a full professor. This experience was a game changer with respect to my academic research. Nirenberg steered my interest towards fluid dynamics and fully nonlinear equations. Walking one day in Chinatown with Nirenberg and Robert Kohn [1953–], we decided to work together on a paper about the Navier–Stokes equations, a set of nonlinear PDEs that models the evolution of viscous incompressible fluid flows (in dimension three).

The result of this "CKN" collaboration was the 1982 paper titled, *"Partial regularity of suitable weak solutions of the Navier–Stokes equations."* We showed that the flow had singularities at most on a set of zero one-dimensional measure (i.e., less than a curve) in space and time. This is a nearly optimal result according to examples given by Vladimir Scheffer [1950–2023].

A more technical way to state the CKN theorem is the following: Let u be a *weak solution* of the Navier–Stokes equations for incompressible fluids, satisfying suitable growth conditions. (To arrive at the idea of a weak solution of a PDE, one integrates the equation against a test function, and then integrates by parts (formally) to apply the derivatives on the test function.) The result says that u is regular away from a closed set whose one-dimensional parabolic Hausdorff measure is zero.

This is the best partial regularity theorem known so far for the Navier–Stokes equations. It appears to be very hard to go further; we need some deep, new ideas.

[BID/CFS]: *Nirenberg has said that you had a "fantastic intuition," which made it hard for collaborators to keep up with you: "He somehow immediately sees things other people don't see, but he has trouble explaining them," Nirenberg said.*

Let us move on to the Monge–Ampère equation, on which you have made seminal contributions, especially on their regularity

His Majesty King Harald V presents the 2023 Abel Prize to Luis Caffarelli.
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properties, i.e., high order differentiability. The Monge–Ampère equation is a fully nonlinear PDE that frequently arises in differential geometry; for example, it is used to construct surfaces of prescribed Gaussian curvature. How did you get started to work on the Monge–Ampère equation and, more generally, on fully nonlinear equations?

LÁC: While at Courant, I listened to a talk of Pierre-Louis Lions [1956–], who left an open question on this problem. I was able to solve it by using ideas from free boundary theory, thus finding a connection between two different parts of PDEs. Then I started to work with Nirenberg and Joel Spruck [1946–], and we published several papers together on the Monge–Ampère equation, including one paper on the complex Monge–Ampère equation (together with Kohn).

These papers, published between 1984 and 1986, were among the first to develop a general theory of second-order elliptic differential equations which are fully nonlinear, with a regularity theory that extends to the boundary. It should be remarked that the paper from 1985, published in Acta Mathematica and titled *"The Dirichlet problem for nonlinear second-order elliptic equations III. Functions of the eigenvalues of the Hessian"* has been particularly influential in the field of geometric analysis, since many geometric PDEs are amenable to its methods.

[BID/CFS]: *Over several decades you wrote many papers, often with co-authors, which have "Monge–Ampère equation" in their titles. It is especially noteworthy that your regularity theorems from the 1990s represented a major breakthrough in the understanding of the Monge–Ampère equation. In particular, according to the Abel Prize citation you "closed the gap in the understanding of singularities by proving that the explicitly known examples of singular solutions are the only ones."*

The fractional Laplacian

Now let's move on to a more recent paper of yours from 2007 (co-authored with Luis Silvestre), a paper we alluded to above, and which is titled "An extension problem related to the fractional Laplacian." *The paper is 15 pages long, but had an enormous impact in many different subfields of PDEs. It is also your most cited paper, with more than 1600 citations on MathSciNet! Tell us about this paper.*

LÁC: Without getting too technical, let me try to explain the main idea, as well as some results, of this paper. The Laplacian is the second-order differential operation $\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$. On $\mathbb{R}^n$ the *fractional* Laplacian $(-\Delta)^s$, for s a real number between 0 and 1, is perhaps easiest understood by means of its Fourier transform, where differentiating corresponds to multiplication with the norm of the variable. So, a fractional Laplacian is mirrored in the Fourier transform by multiplication by the norm of the variable to the appropriate power. Another way to define the fractional Laplacian $(-\Delta)^s$ is via the formula

$$(-\Delta)^s f(x) = C_{n,s} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x-\xi|^{n+2s}} \, d\xi,$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a function and $C_{n,s}$ is a normalizing constant.

The main idea of the paper is proving a result which relates a *nonlocal* problem associated to the fractional power of the Laplacian to a *local* degenerate elliptic problem.

This allows us to prove several regularity results concerning the solutions of problems involving the Laplacian by exploiting purely local techniques. In particular, we prove both a Harnack inequality and a boundary Harnack inequality.

Work style, students and more recent research

[BID/CFS]: *As a mathematician, you are extraordinarily prolific – and extraordinarily sociable. You have published more than 320 papers, you have co-written papers with more than 130 people and advised more than 30 PhD students. Do you have any comments on this?*

LÁC: Through the years, I have had the opportunity to belong to wonderful institutions, starting with the University of Buenos Aires, and then the University of Minnesota, the Courant Institute, NYU, the University of Chicago, the Institute for Advanced Study, Princeton, and for the last 26 years, the University of Texas at Austin.

This gave me the opportunity to befriend and collaborate with extraordinary scientists all over the world. It also led to further opportunities to mentor very talented young people who have invigorated my research with new ideas. I have moved between topics within the wider field of PDEs. There are people who do wonderful things in very concentrated areas. But science is more like a global evolution. It requires the exchange of ideas, looking at things from different angles, slowly improving whatever can be improved.

[BID/CFS]: *Are you more of a problem solver than a theory builder?*

LÁC: Yes, definitely. Broadly speaking, there are two kinds of mathematicians. There are those who develop theories and those who are primarily problem solvers. I belong to the latter group.

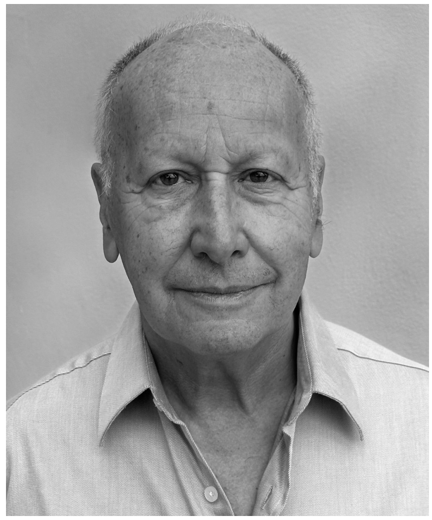
[BID/CFS]: *As you told us, you have spent the last 26 years at the University of Texas at Austin. Your wife Irene Martínez Gamba, who is also a mathematician, is professor of computational engineering and sciences at the same university. You have had more than 20 graduate students while at the University of Texas. Tell us about your research activity there.*

LÁC: My research expanded in an immense way due to the fact that I had graduate students again; many of them I still have long-lasting collaborations with. I am indebted to all of them. Of specific research topics I worked on, I would list three:
(i) Free boundaries in (Lipschitz) surface detection.
(ii) Problems with fractional diffusion; a completely new theory that filled my agenda for many years.
(iii) I kept on working on the Monge–Ampère equation and its regularity theory in fully nonlinear configurations.

I would also mention that I interacted with a group of engineers and natural scientists. I enjoyed enormously having discussions with them, as they somehow relied on some of my ideas.

[BID/CFS]: *We always end these interviews by asking about interests outside of mathematics. Do you have any special interests or hobbies?*

LÁC: I like to cook, but my wife is a better chef than I am. I also like to play some soccer if I have time. I play the piano. I prefer mostly classical music.



Luis Caffarelli, Abel Prize laureate 2023.

[BID/CFS]: *On behalf of the Norwegian Mathematical Society, the European Mathematical Society and the two of us, we would like to thank you for this interesting interview.*

LÁC: Thank you very much.

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Pierced by a sun ray

An obituary of Yuri Ivanovich Manin (1937–2023)

Matilde Marcolli

Ognuno sta solo sul cuor della terra,
trafitto da un raggio di sole:
ed è subito sera.
(Salvatore Quasimodo)[1]

Nostalgia, in Greek, is the pain of returning: the νόστος of the Homeric heroes trying to find their way back in a world forever altered. Return and remembrance are a painful illusion, our longing for a past that can speak to us in a familiar voice.

I promised him I would come in the spring. By the time I was done with my cancer treatment and able to travel again, I was four months too late. I ended up spending a week in his MPI office, looking through the writings he left behind: one week for an entire lifetime. That's when I started writing this piece. On the small board outside the door of his office, along with his photograph and the cover pages of his recent preprints, he had posted a copy of Quasimodo's poem *Ed è subito sera* (Figure 1). Indeed, even after a long life full of so many remarkable achievements, that final nightfall still comes very suddenly and unexpectedly. It still comes much too soon. There was still so much joy of life, of thought, so much energy and creativity left in him, up until the very end of his life.

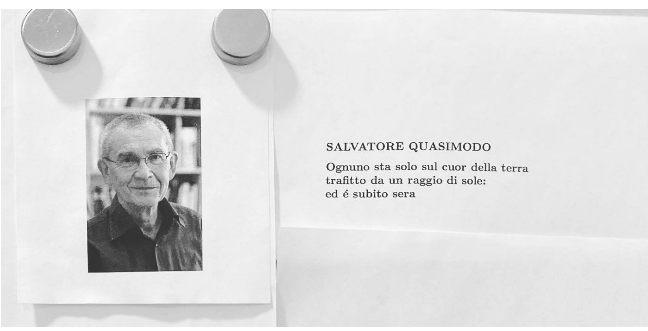

Figure 1. The door of Yuri Manin's office at the Max Planck Institute for Mathematics (MPI) in Bonn

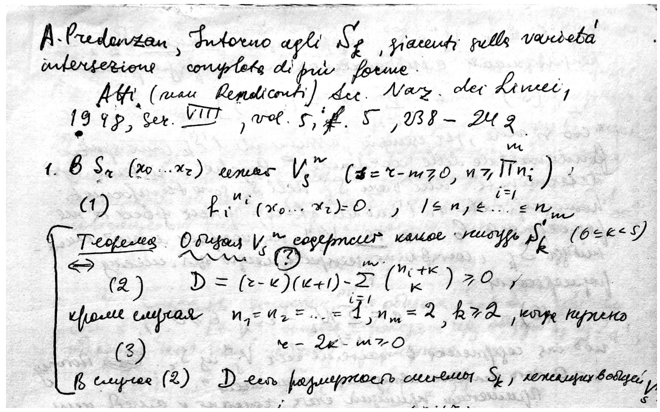

Figure 2. A page from a 1965 notebook.

I used to spend late night hours at the institute, in the eight years when I was his colleague at the MPI. Yuri and Xenia used to tease me when I would walk back to my office, instead of going home, as we were coming back from a concert or a movie and a dinner together. Yes, I used to work at the MPI late at night, but I had never before seen the dawn from the institute windows, like I am doing now every night. I am seeing it from his office windows: a beautiful sight of the first light of day on Bonn's Münsterplatz. I use this time during the night, before anyone else comes in, to reflect on what all this had meant to me over the past twenty years, to read through stacks of his handwritten notes, to look through his books on the shelves – still where I remember them –, to experience the last remnants of his physical presence in the place where he spent the last thirty years of his life.

The oldest handwritten notes I came across in his office are in a small notebook dated 1965. They are notes about results from the Italian school of algebraic geometry: on the first page a commentary on a 1948 paper by Predonzan, *Intorno agli S_k giacenti sulla varietà intersezione completa di più forme* (Figure 2). Just the previous evening I had found among the recent papers left on the desk of his home office a short note expressing his enthusiasm for the classical Italian algebraic geometry (Figure 3).

[1] Everyone stands alone over the earth's core, / pierced by a sun ray: / and immediately it's nightfall.

Two events in the 1960s influenced significantly his view of this field of mathematics: a visit to Pisa, during which he learned Italian, read Dante and an extensive combination of writings by Italian algebraic geometers, and the subsequent visit to Grothendieck at the IHES in Paris, which lead him to write what became the very first published paper on Grothendieck's theory of motives [10]. During the second half of the 20th century the field of algebraic geometry witnessed a sharp divide between the abstract categorical language of the Grothendieck approach, on one side, and the hands-on geometric example-driven approach that can be traced back to the older Italian school. We all know algebraic geometers who fall squarely into one or the other of these two camps, but fewer embrace both views simultaneously, finding ways to combine them with ease in their own work. Yuri always carried with him this simultaneous dual view of algebraic geometry that came to characterize his approach to the field.

By that time, Yuri was already widely famous for his proof, in 1963, of the Mordell Conjecture over function fields [9]. The conjecture establishes a relation between topological and arithmetic properties of curves: if the genus is $g \geq 2$, then the number of rational points is finite. The case over function fields can be restated in terms of a pencil of curves over a projective algebraic surface. The main tool involved in the proof is an abstract formulation of Picard–Fuchs equations in the form of a connection on a bundle whose fibers are de Rham cohomologies, and whose flat sections are the differential equations that govern the structure of period integrals [8]. It was Grothendieck who started calling this construction the *Gauss–Manin connection*. It has since then become a fundamental and widely applied tool in algebraic and arithmetic geometry.

Classical algebraic geometry works by Enriques, Segre, and especially Fano formed the background for his result with Iskovskih [6] on the existence of smooth quartic threefolds that are unirational but not rational, which provided a negative answer to the Lüroth problem in 1971.

The beginning of the 1970s is also the time when he published his first book. *When I felt myself grown up enough to set writing my first book*, he says in the short note left on his home desk, alongside the words *omaggio alla geometria algebrica italiana*, homage to Italian algebraic geometry (Figure 3). The note also contains the Latin translation of Plutarch's πλεῖν ἀνάγκη, ζῆν οὐκ ἀνάγκη (to sail is indispensable, to live is not). Yuri must have likely

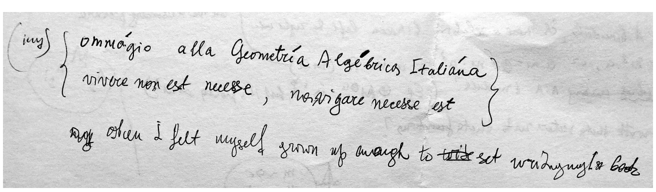

Figure 3. A recent note left on Manin's home desk.

written it just weeks before his death. In Greek ἀνάγκη is necessity by way of fate. Yuri used to say that mathematics chooses us, not the other way around: to him it was fate, destiny, and indispensable vital need.

At the end of the introduction to the book *Cubic forms* [12], Yuri quotes works of Grothendieck, Segre, and Châtelet as main sources of inspiration. The book assembles a significant amount of Yuri's work in algebraic geometry over the preceding years, combined into a beautiful narrative. What happens to the group law of elliptic curves when one moves up to rational points of a cubic surface? The surprising answer is non-associative commutative Moufang loops. The classical geometry of the 27 lines on a cubic surface leads to a modern interpretation in terms of a birational invariant of the surface given by the Galois cohomology group $H^1(k, \mathrm{Pic}(X_{\bar{k}}))$ (that can also be described in terms of Brauer groups), which can be computed from the partition of the 27 lines into Galois orbits. The book is a treasure trove: minimal cubic surfaces and birational transformations, Brauer–Grothendieck group and Brauer equivalence via Azumaya algebras, and other intriguing connections between arithmetic and geometry.

Cubic forms was for a long time the only one of Yuri's books I did not have. It is hard to find, and only last fall I was able to order a good copy from a bookseller. It did not come for a long time, and I assumed it had been lost in the mail. It was delivered to my home in the evening of January 7, the same day when Yuri died. Carl Gustav Jung (another interest Yuri and I shared) called these *meaningful coincidences*: random acausal events that happen to take place at crucial times in our lives and that our psyche therefore invests with meaning.

There are no more lights in the institute windows at night: there used to be plenty, not just mine, but the MPI was a different place at that time. I left in 2008. Yes, I left. It seemed like a good decision at the time – the MPI was changing and my job in California was attractive, but the years kept passing and I never quite felt like I had rooted my life elsewhere. My center of coordinates had stayed in this room, which is now about to disappear. I kept coming back, year after year.

We started working together almost immediately after I joined the MPI faculty in the summer of 2000. Yuri suggested that we revisit his theory of modular symbols [13] from a different perspective, related to noncommutative geometry and the "invisible boundary" of modular curves determined by the irrational points in the boundary of the upper half plane [26]. While the rational points, the cusps, have an algebro-geometric interpretation in terms of degenerations of elliptic curves to the multiplicative group, the irrational points can be regarded as degenerations that no longer exist algebro-geometrically but only as noncommutative tori. Limiting modular symbols exist at such invisible points, determined by an ergodic average of classical modular symbols, and the arithmetic contributions of this part of the geometry can be seen by expressing Mellin transforms of cusp forms in terms of quantities defined entirely on this boundary [26], leading to a more general formalism of modular shadows, Lévy functions, and Lévy–Mellin transforms [27]. The role of noncommutative tori as non-algebro-geometric degenerations of elliptic curves and of the noncommutative boundary of modular curves as their moduli space was the source of Yuri's *real multiplication program* [23], based on the beautiful idea that noncommutative tori with real multiplication (non-trivial Morita self-equivalences) should play the role of the missing geometry behind the explicit class field theory problem for real quadratic fields, like elliptic curves with complex multiplication for the imaginary quadratic case. This promising program still remains unfulfilled.

Modular symbols, periods of modular forms, and *L*-functions formed the central theme of a series of papers Yuri wrote in the 1970s, relating cusp forms and their period integrals and Hecke series. Results include explicit formulae for Hecke eigenvalues, algebraicity results for periods of cusp forms, *p*-adic Hecke series through *p*-adic measures associated to cusp forms and the *p*-adic Mellin transform, an effective algorithm for the computation of the Tate–Shafarevich group of elliptic curves based on Mellin transforms of *L*-functions and modular symbols. In the Helsinki ICM address [15], Yuri summarizes several of these results, presenting as the central theme of number theory the interaction between the two classes of *L*-functions, associated to varieties through cohomologies, and associated to modular and automorphic forms.

There are two extensive sets of handwritten notes I found in his office, one on Lévy functions and what was supposed to be our continuation of [26] and [27], and one on the real multiplication project. Both were likely written in the mid-2000s, probably 2005 or 2006. We had frequent conversations at the time on all of these topics, but during those years I got diverted into other projects (perhaps unwisely handled in retrospect) and much remained undone: here is where the pain of returning becomes especially hard to bear.

But I am here again, night and day, going through these written remnants of Yuri's work: many more sets of carefully organized handwritten notes, references he was reading along with them, drafts of papers, printouts of email exchanges, lecture notes for courses he taught in Moscow, at MIT, at the MPI, at Northwestern. I am reading through all of this, in whatever order the folders come into my hands: everything has already been sorted out, during these months after his death, when I still could not be here.

His work: it makes me think of the ocean, with big waves and deep currents. That's what I see, as I go over folder after folder of manuscripts and notes, those huge waves of work stretching over the years. When I moved to Bonn from Boston in 2000, I arrived in time to catch the last years of the long wave of quantum cohomology and Frobenius manifolds, that started shortly after he had himself moved to the MPI from MIT in 1993, and that dominated his interests for a decade. His work with Kontsevich, starting in 1994 [7], set the foundation for the algebro-geometric form of quantum cohomology and Gromov–Witten invariants, a mathematical construction that originated in the physical theory of quantum strings propagating on a spacetime manifold. This string theory setting leads to a deformation of the usual product on the cohomology of algebraic varieties, where instead of counting the intersection points between cycles, one uses an appropriate counting of algebraic curves connecting them. This algebro-geometric counting requires the geometry of moduli spaces of stable maps and the delicate notion of virtual fundamental classes. The axiomatic treatment of Gromov–Witten invariants and quantum cohomology leads naturally to the study of Frobenius manifolds as the underlying structure (a notion originally introduced by Dubrovin), with diverse applications in singularity theory and integrable systems. A motivic viewpoint on quantum cohomology was developed in his work with Behrend [3], where the moduli spaces of stable maps are shown to determine correspondences between the motive of a variety and the motives of moduli spaces of curves, giving rise to an action of the modular operad on motives. This large body of work on Frobenius manifolds, moduli spaces, and quantum cohomology culminated in his very extensive research monograph on the subject, published in 1999 [22]. My gift for Yuri's 65th birthday was a Frobenius manifolds conference.

The relation between algebraic geometry and physics is the deeper current underlying this long wave of work on quantum cohomology and Frobenius manifold structures. Yuri was the first to clearly demonstrate the importance of algebraic geometry for string theory with the 1986 results (in [18], and with Beilinson in [4]) expressing the Polyakov measure of the bosonic string path integral in terms of moduli spaces of curves, theta-functions, Green functions considered by Faltings in the setting of Arakelov geometry, and holomorphic quadratic differentials. The interaction between algebraic geometry and string theory has deeply transformed, in the decades that followed, both the field of high energy physics and algebraic geometry itself.

At various times, Yuri proposed the idea that not only algebraic geometry, but also arithmetic geometry and number theory should play a fundamental role in physics [20]. One of our first joint papers proposed the use of *p*-adic geometry in the AdS/CFT holographic correspondence of string theory [25], reinterpreting in holographic terms an earlier result of Yuri on the fiber at infinity of Arakelov geometry [21]. This idea of a *p*-adic form of the AdS/CFT correspondence became popular with physicists fifteen years later. I have to admit that Yuri's paper on the fiber at infinity of Arakelov geometry [21] is probably the single paper that was most influential in the development of my own view of mathematics, not only in terms of work directly inspired by it, but more generally in terms of exemplifying what I find beautiful and valuable in mathematical research. Along with the published version of that paper, I kept a copy of the unpublished preprint that preceded it [19], full of his heuristic explanations, drawings, and analogies that formed the background to the final polished result (Figure 4).

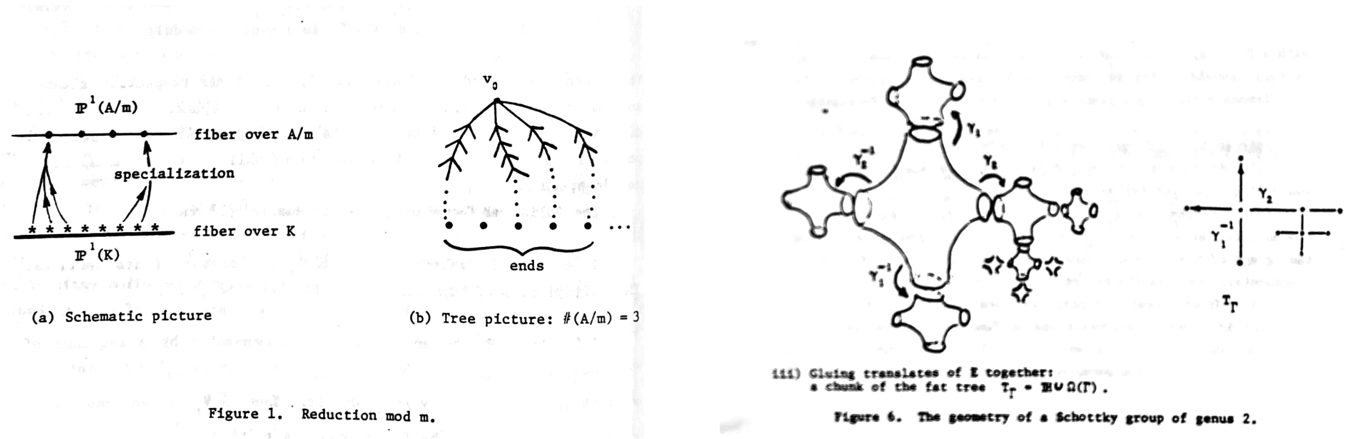

Figure 4. Two drawings from [19].

Within his overall geometrization program, envisioning algebraic geometry as a unifying language in number theory, physics, and information theory, another large wave of work one comes across is the one that happened in the 1980s and early 1990s, encompassing the geometry of Yang–Mills instantons, supergeometry, and symmetries of quantum spaces. The ADHM (Atiyah–Drinfeld–Hitchin–Manin) construction and classification of Yang–Mills instantons on the 4-sphere is certainly the most famous result of this period [1]. Solutions of Yang–Mills equations on 4-dimensional manifolds became, some years later, a major tool in low-dimensional topology, in the form of Donaldson invariants. Yuri's extensive development of an algebro-geometric formulation of supergeometry was motivated by his interest in the physical theories of supergravity and supersymmetric Yang–Mills, for which he gave a precise mathematical formulation. A broad overview of the developments in mathematical physics, around the topics of Yang–Mills instantons, Penrose's twistors, and supergeometry, obtained by him and his students, is presented in his book *Gauge field theory and complex geometry* [17]. Work with Kuperschmidt and Lebedev on integrable systems focused on the hierarchy of higher hydrodynamic equations of Benney type. In this context he also introduced the noncommutative residue of pseudodifferential operators, generalized by his student Wodzicki. After Drinfeld introduced quantum groups, Yuri showed that they can be realized as symmetries of quantum spaces, a natural point of view from the perspective of both physics and noncommutative geometry.

Yuri was the first person who suggested the idea of quantum computing, in his 1980 book *Computable and uncomputable* [14] (Figure 5), a good two years before Feynman made the same suggestion along similar lines of thought.

In the same period he promoted in his seminar the importance of algebro-geometric constructions in the theory of classical error-correcting codes (first used by Goppa in 1981), a topic that his students Tsfasman and Vlăduț then widely developed. He introduced in [16] the asymptotic bound in the geography of error-correcting codes, relating the asymptotic theory of codes to the problem of constructing algebraic curves with sufficiently many algebraic points over finite fields. His impact on the theory of information epitomizes how topics that have by now become huge fields of research had germinated very early inside his mind: nowadays algebro-geometric codes lie at the heart of the edifice of cryptography and quantum information has become an enormous landscape, stretching across mathematics, physics, and computer science.

The themes of computability, of classical and quantum information, and of error-correcting codes resurfaced frequently in his later work, and became a very significant interest in the last years of his life, including a considerable part of our own joint work, for instance [28, 30]. This is another example of what I see as the deep currents in Yuri's work, those long visions that stretched through the decades, acting as a background structure, and emerging periodically to the forefront of his research activities.

Another long-term current in Yuri's work is Diophantine equations and the properties of rational points of varieties. The Hasse principle asserts that the existence of rational points of a variety over a number field can be deduced from the existence of points over all the non-Archimedean and Archimedean completions (that is, one can go from local to global solutions). Yuri first showed that the Brauer–Grothendieck group determines an obstruction

Возможно, для прогресса в понимании таких явлений нам недостает математической теории квантовых автоматов Такие объекты могли бы показать нам математические модели детерминированных процессов с совершенно непривычными свойствами. Одна из причин этого в том, что квантовое пространство состояний обладает гораздо большей емкостью, чем классическое: там, где в классике имеется N дискретных состояний, в квантовой теории, допускающей их суперпозицию, имеется c^N планковских ячеек. При объединении классических систем их числа состояний N_1 и N_2 перемножаются, а в квантовом варианте получается $c^{N_1 N_2}$.

Figure 5. Excerpt from the book [14, p. 15] about quantum computing.

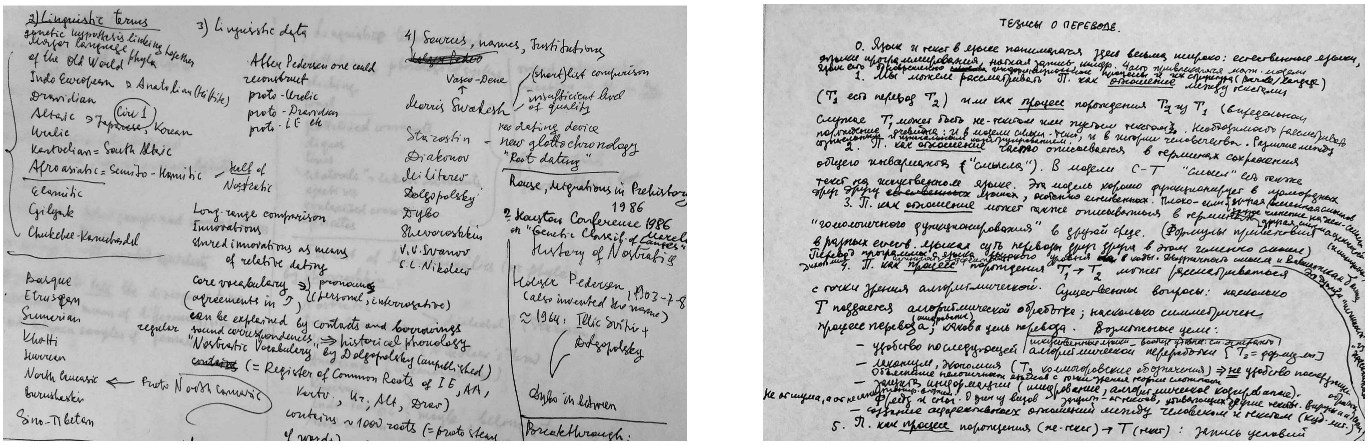

Figure 6. Manuscript fragments about linguistics.

(the Brauer–Manin obstruction) to the Hasse principle [11]. The existence of this general obstruction changed the perspective on Diophantine problems, and the understanding of local-to-global principles. His extensive investigation of rational points of bounded height on cubic surfaces brought him to develop a broad research program relating geometry and topology of varieties to Diophantine properties. Varieties with ample canonical class (intuitively, hyperbolic or negatively curved, like the algebraic curves of genus $g \geq 2$) are expected to have fewer rational points, lying on lower-dimensional submanifolds. For large anticanonical class (elliptic or positively curved, like Fano varieties) one expects many rational points. Work with Batyrev [2] showed two important aspects of the structure of rational points: the presence of accumulating subvarieties and the linear growth of the number of points with bounded anticanonical height on the complement of accumulating subvarieties (Manin's linear growth conjecture). A series of papers in the late 1980s and early 1990s, starting with [5] with Franke and Tschinkel, provided evidence for the explicit form of the asymptotic behavior $cH(\log H)^t$ for the number of points of bounded height for varieties V over number fields, in the ample anticanonical case, with $t = \operatorname{rank} \operatorname{Pic}(V) - 1$. During the last two years of his life, Yuri became interested in the possibility of categorifying the height zeta function, so as to encode, in the form of scissor-congruence type relations, the presence of accumulating subvarieties, and suggested the relevance of homotopy-theoretic methods to this goal [31]. He very much considered this a line of thought that he meant to continue developing.

There are so many other things left in these folders, in his office. It's a beautiful large office, with a lot of light, windows overlooking the central Münsterplatz, a beautiful place from where to watch the ending of a world (Figure 7). All of this will be gone soon, though I am trying to help making it digitally available at some point in the future. Among the things I keep finding: a typewritten manuscript of the Strugatsky brothers (which appears to be unpublished to my inexperienced eyes), writings on a variety of subjects (including the original manuscripts of several essays of [24]), and especially poetry, his own as well as his translations of other poets. Poetry is everywhere: it forms a pervasive ubiquitous subtext both in and around his mathematical thoughts.

I came across some of his old writings in linguistics (Figure 6). I often go by as a linguist these days, but our paths to the subject have been very different and in many ways complementary. He wrote about glottogenesis and protolanguages (like the hypothetical Nostratic) and about psycho-linguistics. He was interested in mathematics as language. I have been interested in language as mathematics, but he often voiced criticism of generative linguistics, as an excess of abstraction away from the more intricate specific

Figure 7. Yuri Manin's office at the Max Plank Institute for Mathematics, Bonn.

functioning of languages. On Christmas day of 2015, as Yuri and I met to work on our own joint paper in linguistics [29], I received an email from Chomsky about my work on syntactic parameters. *Speak of the devil…* – I told Yuri as I showed it to him. It remained an inside joke between us, whenever the topic of linguistics came up in our discussions. In another of those uncanny Jungian coincidences, Noam reappeared in my life on the same day when the MPI held the memorial service for Yuri, in January, to start our currently ongoing linguistic collaboration. Months after Yuri's death, it is still impossible for me to return to thinking about mathematics. He was in every part of mathematics I ever happened to think about (gauge theory, noncommutative geometry, motives, mathematical physics…), and now everything hurts. I always remained an outsider to the world of mathematicians and the one who was able to make me feel at home there is now forever gone. I will return to it one day, probably, but right now I just cannot. By a strange twist of fate, the formal mathematical abstraction of generative linguistics that Yuri used to criticize turned out to be now the only thing that made it possible for me to continue working during the past few months.

I'll take the liberty of ending this piece on a personal note. This is the third time I am asked to write some form of obituary, reminiscence, tribute, commentary on Yuri's death: making a public spectacle of the pain of losing the closest friend I ever had, navigating the acceptable boundary of words. It hurts, every time more. Because I miss him more and more as the months are passing. Because I am here trying to invent implausible motivations for why the show must go on, for why I ought to keep doing the things I used to do, without a main reason for why I did them. I am not requesting that you would all stop asking me: I understand it that, as a long term collaborator and close personal friend, it is my duty to do this kind of writing in his memory. I would just like you to reflect on how duty becomes a proxy for pain when the latter has no proper venue of expression, and on how much we lose as a community if we try to separate our mathematical achievements from the deeper core we stand on, the value of our humanity.

> ἀλλά με κακκῆαι σὺν τεύχεσιν, ἅσσα μοι ἔστιν,
> σῆμά τέ μοι χεῦαι πολιῆς ἐπὶ θινὶ θαλάσσης
> (Homer, Odyssey, XI, 74–75)[2]

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[2] But burn me with my armor and all my belongings, and drop a sign of me on the shore of the grey sea.

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ICERM: 10 years later

Benoît Pausader

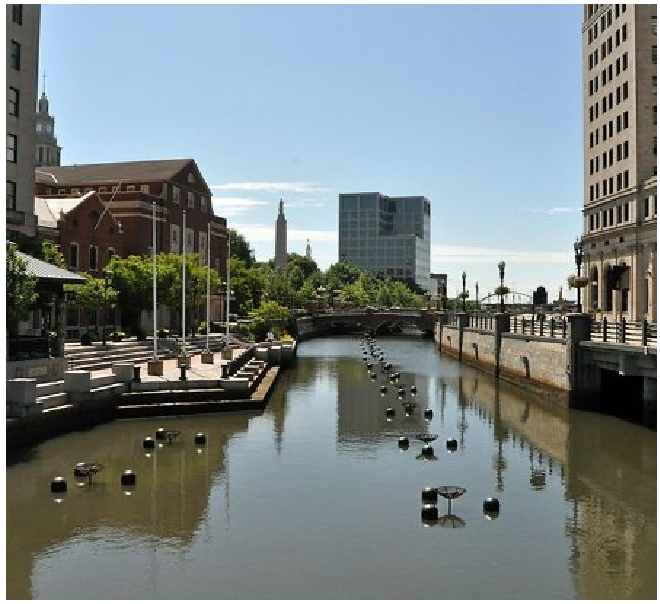
Update on the Institute for Computational and Experimental Research in Mathematics

Ten years ago, in the September 2013 issue of the Newsletter of the EMS, an article presented the newly opened Institute for Computational and Experimental Research in Mathematics (ICERM). This once fledgling center has now grown into a mature institute, and this article aims to reintroduce ICERM to the international mathematical community, provide an update on its achievements over the last decade, and encourage mathematical scientists of all stages to participate in its future.

Since its opening in Fall 2011, ICERM has run 23 semester programs, 74 topical workshops, 36 public lectures, 12 summer training programs for graduate students and multiple other events. Despite the tumultuous last few years that have forced the organizational team to navigate safety concerns, evolutions in the job market, growing awareness of the climate and social impact of research, ICERM has developed a robust operation that supports organizers as they carry out mathematically rigorous, bold, large-scale programs. With guaranteed funding at least until 2025, ICERM encourages all mathematicians to apply to participate in one of the already planned programs, or to propose an ambitious new one.

What is ICERM?

ICERM is an institute with core funding from the National Science Foundation (NSF), located in Providence, Rhode Island. Its principal focus is to bring together pure and applied mathematicians from all specialties and all institutions through a variety of programs. At ICERM, participants come together to collaborate, share, and develop ideas and experiments, and formulate and test conjectures to expand the boundaries of science.



ICERM is located on the top floors of a modern building on the waterfront in downtown Providence.

Among all NSF institutes, ICERM is uniquely focused on experiments in both pure and applied mathematics. Its mission is to "support and broaden the relationship between mathematics and computation: specifically, to expand the use of computational and experimental methods in mathematics, support theoretical advances related to computation, and address problems posed by the existence and use of the computer through mathematical tools, research and innovation." As such it seeks, in particular, to disseminate new computer-based tools and to accompany and facilitate emerging topics in mathematics. Its location on the East coast, integrated into one of the largest scientific eco-systems in the world, allows ICERM to stay at the forefront of recent developments and trends.

ICERM is especially devoted to nurturing young researchers in mathematics. In addition to a few institute postdocs, who stay for a year, each semester program supports a large number of

ICERM's Fall 2019 semester program "Illustrating Mathematics" included an art exhibition for researchers.

6-months-long postdocs and the program gives a chance to meet and interact with many important people in the corresponding field who can become future colleagues, collaborators or employers. Throughout the year, professional development events are organized about all aspects of academic life and work, as well as about life outside of academia.

ICERM's amenities

ICERM occupies about 1,800 square meters on the top two floors of a modern glass building overlooking the waterfront, near downtown Providence, RI. It is centrally located near the train station, restaurants, and the Brown University campus. It is an ideal place for lectures and collaboration, with a large, welcoming communal space and an auditorium that can accommodate 120 people and is fully equipped for online recording and participation. In addition, it contains a conference room, a lecture room and 25 offices for long-term participants in a lower story, with floor to ceiling blackboard-paint on the walls to allow the free flow of ideas and spontaneous discussions and collaborations. For special programs, additional resources can be secured (such as 3D printers for a semester focused on art and mathematics).

The institute provides all long-term visitors with access to a broad range of compilers, numerical libraries, mathematical problem-solving environments, visualization software, statistical packages, and productivity software hosted in a virtual desktop and/or terminal based environment. Additionally, all long-term visitors are given exploratory level access to Brown's high performance computing resources. The goal is to provide participants with a wide variety of tools to experiment with and allow them to sample options that may not be otherwise familiar, in order to maximize potential avenues for research.

In addition, ICERM's IT team works with program organizers to tailor technical offerings to support their vision. This may include leveraging resources at Brown in a customized way, like reserving GPU nodes for computer operations or standing up containerized environments, utilizing cloud-based services like CoCalc and Overleaf, standing up custom systems using Microsoft Azure or Amazon Web Services, or acquiring and supporting physical tools such as 3D printers and VR headsets.

Global pandemics notwithstanding, most programs are in person. However, talks are streamed live on the website, where many recordings of past presentations can also be found.

Programming at ICERM

ICERM hosts a variety of events whose formats vary from one-day meetings to semester-long programs. The following are the principal types of activities, usually involving participants from all over the world.

Semester programs are the backbone of the institute. Programs are selected from applications made by teams of mathematicians from anywhere in the world, and typically organized 2 to 3 years in advance. They bring together about 20 senior participants and a similar number of junior participants who stay in residence for the semester, and about 100 shorter-term visitors, who stay from a few days to a few weeks.

Topical workshops are the shorter, smaller equivalent of the semester programs. They are organized 12 to 18 months in advance and bring together about 30 specialists for a week of conference, collaboration, discussion and/or training.

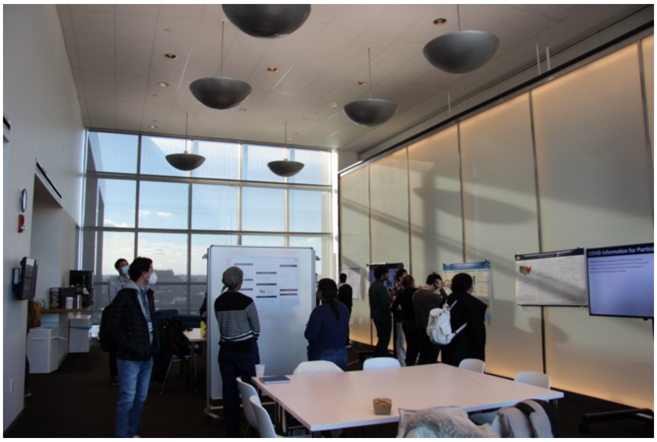
Hot topic workshops are ICERM's rapid-response mechanism to host workshops that can be organized in a matter of weeks to keep up with, and maximize the impact of, a recent breakthrough.

Collaborate@ICERM brings together about 3 to 6 researchers to meet in person to continue an existing collaboration. This is an ideal setting to finish or expand on a collaboration initiated during one of the other ICERM programs.

Summer@ICERM is an eight-week summer program where 2 to 4 faculty organizers lead a team of about 20 undergraduate students on a research project.

Public lectures invite an expert to speak to a large audience on a topic relevant to non-mathematicians to spread awareness of, and interest in, mathematics in the broader public.

ICERM is administratively part of Brown University and ICERM's Director, Brendan Hassett, is a permanent faculty member of Brown.



Early-career researchers gather for a poster session in ICERM's communal space.

From a scientific standpoint, ICERM is largely independent of the university (and the mathematics and applied mathematics departments within) and has a mission to serve the broader community of mathematicians in the US and beyond. The focus on research topics of international significance is guaranteed by Advisory Boards whose members are not affiliated with the university.

ICERM in numbers

As per geography, many (about two-thirds) of ICERM visitors are from America, but the rest come from overseas, principally Europe (more than 20%), and participants and organizers from all over the world are welcome. ICERM is primarily devoted to the whole mathematical community but brings mathematics to the local population through public lectures and the *GirlsGetMath@ICERM* programs.

ICERM can and has partnered with other institutes throughout the world and has helped organize events in India, Japan, Singapore, and South Africa.

The institute actively promotes diversity during all its events. It regularly hosts the Blackwell–Tapia conference (in 2012 and 2018) as well as the Modern Math Workshop (in 2011, 2013 and 2017), and the Conference for African American Researchers in the Mathematical Sciences (2015). The institute is interested in all topics in interaction with mathematics. The themes of past workshops have ranged from "Computational Aspects of the Langlands Program" to "Illustrating Mathematics," from "Computational Biology" to "Knot Theory," and from "Data Science and Social Justice," to "Global Arithmetic Dynamics" and "Computational Aspects of Water Waves." No topic is off the table!

Organizing a semester

Most ICERM activities are relatively easy to organize, thanks to the support of the experienced team of full-time staff. Semester

High school students in ICERM's GirlsGetMath program play cards in the ICERM communal space.

programs are the most demanding for the organizers, but typically follow several steps that we outline below.

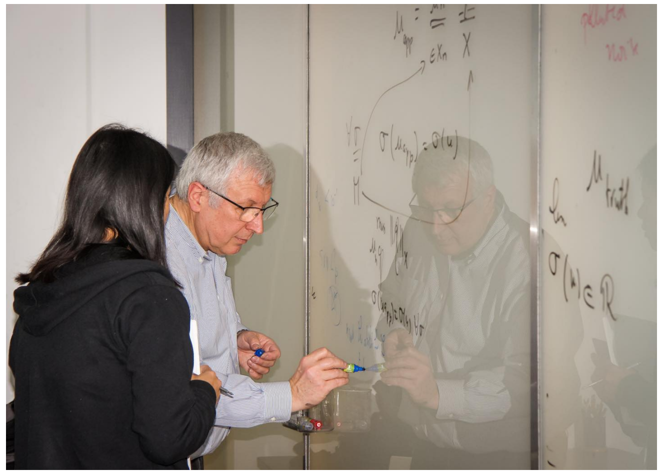
(i) The most important task is to bring together a team of (6–10) organizers around a current mathematical topic. When planning, organizers should consider other institutes' programming schedules, so that as many researchers in the field as possible will be able to attend. Workshops should strike a good balance between theory and application. A diverse group of organizers is encouraged.

(ii) The organizers submit a proposal to the Scientific Advisory Board. Once a program is accepted and there has been an opportunity for feedback, a date will be agreed upon. Programs typically occur two years after acceptance.

(iii) Over the next year, the organizers start to advertise the program and converge on a list of senior participants. The main challenge at this stage is to informally secure the presence of key long-term participants and to make sure that young mathematicians are aware of the application timeline for the postdoctoral positions.

(iv) About a year before the start of the program, invitations are sent to most participants to secure their stay. Accepted postdoc-

Participants attend a workshop in ICERM's Lecture Hall.



Many of ICERM's walls double as workspaces for researchers.

toral fellows are notified in the spring before the program (ICERM postdocs are selected from a pool of applicants on Mathjobs.org).

(v) Around this time the general structure of the program is also finalized. Usually, a program is structured around 3 week-long workshops. The themes and organizing teams of these workshops are decided. In addition, appropriate steps for ensuring a diversity of participants and promoting the development of junior attendees are put in place.

(vi) About 6 months before the beginning of the program, invitations are sent for the workshops, and to short-term participants.

(vii) At the beginning of the semester, the organizers finalize additional events taking place outside of the workshop weeks, e.g., regular seminars throughout the semester and working groups. They also assign mentors to each of the junior participants.

(viii) Throughout the semester, the ICERM staff will organize regular activities and career-building events for the junior participants.

These broad guidelines are flexible and can be adapted depending on the needs of the program. For instance, a program with a significant industry interface where the people concerned may not be able to come to Providence for an extended time can involve *Research Clusters*, which are intensive, several weeks-long events more adapted to the industry timeline.

Conclusions

We hope that this article inspires mathematicians to participate in or organize a program at ICERM. All areas of mathematics, especially nascent and developing mathematical fields, are supported. One of the goals of ICERM is to nurture experiments, help produce conjectures or look for counterexamples and set up standards for explorations of new mathematics. ICERM is the perfect place to explore and create new horizons in mathematics!

Benoît Pausader obtained a PhD in mathematics from the Cergy-Pontoise University in France. After positions at French National Centre for Scientific Research (CNRS), New York University and Princeton University, he is now a professor of pure mathematics at Brown University. He is a co-PI of ICERM, where he organized a semester program on Hamiltonian methods in dispersive and wave equations in the Fall of 2021. His main current interests concern dispersive and kinetic nonlinear partial differential equations.

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How do mathematicians publish? – Some trends

Klaus Hulek and Olaf Teschke

We have already discussed bibliometric measures for the mathematics corpus in this column before. This included the unusual longevity of mathematics citations, effects of delayed publication due to often long and complex refereeing processes, and the specifics of different mathematical areas. It has become clear that purely numerical criteria are often unsuitable to measure mathematical quality or the scientific impact of publications. At the same time, the bibliometric results often depend on mathematical subfields, thus reflecting the structure and different behaviour of mathematical communities. In this column we concentrate on an author-oriented viewpoint. We will derive some quantities which illustrate how the landscape of mathematical publications has changed over the past decades.

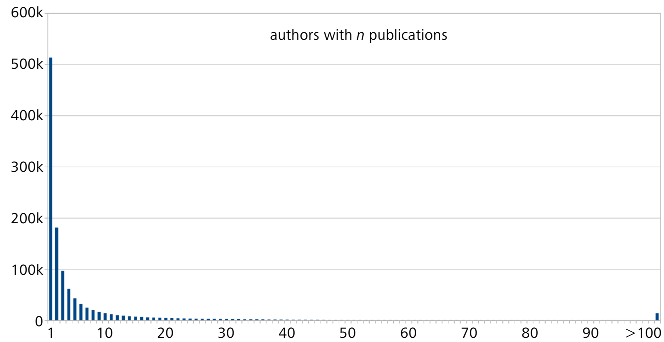

Figure 1. Distribution of publication number for authors in zbMATH Open

1 Introduction

Evaluations and rankings, be it of individuals or institutions, have become part of academic reality. These evaluations range from career-defining assessments of individuals to worldwide university rankings. Although the methodology of many of these evaluations has often been criticised, they remain ubiquitous with extraordinary effects. The effect on individual careers and hence lives can be decisive. On a more global level, these figures not only contribute significantly to the reputation of universities, but also affect the choices of perspective students.

Various parameters are used to evaluate research performance, with bibliometric data playing an important role in (almost) all evaluations. Generating these data, as well as interpreting them, constitutes a major challenge. Therefore, it is important to understand the technical aspects, as well as the different parameters and perspectives, that go into bibliometric data.

In previous articles [1–3], we reported in particular that mathematics citations have an unusually long lifespan compared to other sciences. Taking this into account, together with the often time-consuming refereeing and publication processes, renders useless the measures that count only recent citations (like the traditional impact factor). Also, a strong correlation between the quality of a journal, as assessed by peers, and relative citation counts could not be corroborated. We also see a significant influence of the publication behaviour of mathematicians as compared to other scientists, such as physicists or computer scientists. In fact, publication attitudes also vary significantly, depending on different fields of mathematics.

While our previous analysis was mostly document-based, it is also worthwhile taking a more author-centred point of view when analysing publication behaviour. Such an analysis, however, requires extremely precise authorship data, since otherwise error propagation would disturb any derived quantities, making meaningful conclusions impossible. In this study, we take advantage of the significant progress of the zbMATH Open author disambiguation during the past years. Methods and progress on this matter have been amply described in previous columns [9, 13]. Nevertheless, we would like to mention that currently only roughly 3.5% of authorships are ambiguous (compared to 5% in 2018), despite the growing ratio of authorships involving Chinese names, which cause the most complicated disambiguation tasks. Most large clusters of Chinese names have now been successfully analysed (e.g., 1,529 documents involving the most frequent single name Wang, Wei have been distributed to currently 366 identities). The by now highly efficient author disambiguation will help to eliminate distortions in the subsequent analysis (which will take into account only the 96.5% of unambiguous assignments).

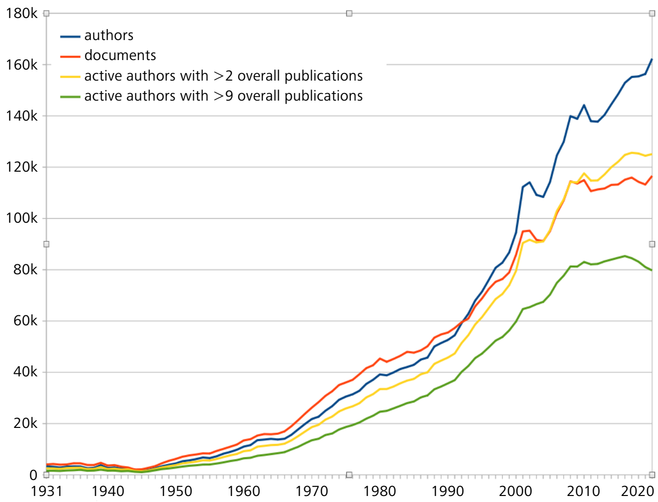

Figure 2. Actively publishing authors per calendar year, in relation to documents

We will first employ the zbMATH Open author database to derive figures on the number of actively publishing mathematicians in a given year. Comparing this with the growth of documents, some effects showing changing publication frequency and collaboration behaviour will become visible. With the assignment of MSC (Mathematical Subject Classification) classes since the 1970s, it is possible to analyse and compare these figures for different mathematical areas. For convenience (and to achieve some historical coherence, avoiding effects from the evolution of MSC) this is done for a set of ten clusters of main MSC classes, which were also employed in previous studies, such as [10, 11].

2 What defines and how large are the mathematics communities?

Zentralblatt für Mathematik und ihre Grenzgebiete (now zbMATH Open, [7]) started in 1931 with the aim of indexing the relevant mathematical research literature in a timely fashion, including related areas of applications. Its indexing and editorial policy differed notably from the earlier *Jahrbuch über die Fortschritte der Mathematik (JFM)*, the main difference being the quicker (though somewhat less systematic) and more international approach [15]. In effect, there are notable differences in the scope of both services, complicating historical comparisons. Since most of the subsequent analysis will involve the Mathematical Subject Classification, which has only been available in a comprehensive form for the data starting from around 1970, we will thus omit the JFM data here.

While the scope of zbMATH has remained largely unchanged over the decades, two adjustments, which are also visible at the level of document counts, should be mentioned: Firstly, the explosion of computer science publications starting at the beginning of the 2000s, required a more precise indexing policy in the MSC area 68. Secondly, after 2010 the number of "mathematical" publications has skyrocketed. This is partly, but not exclusively, due to the growing activity of so-called predatory publishers; see [14] for some discussions. This has led to a stricter indexing policy requiring genuinely new mathematical results.

When one focuses on author counts, instead of publication numbers, one has to keep in mind that the distribution of papers is extremely biased. As shown in Figure 1, about 44% of the authors indexed in zbMATH Open are connected with just one publication. The median author has 2 publications, while the average publication number is about 7.9, with the highest number of publications for a single author being 1769.

There are many reasons why many authors are only connected with one paper. The obvious one is a short career in academia, often just a PhD thesis and one paper derived from this. Other people may have longer careers in research, but may switch to application areas, where they drop out of the scope of zbMATH Open. This makes it harder to define the community of scientists who actively publish mathematics research at a given moment. To make the diagram more meaningful, we add also the figures of mathematicians having more than 2 (the median) overall publications, and more than 9 overall publications, see Figure 2. The last figure excludes, by its very nature, many younger established mathematicians, explaining the decrease of the numbers for recent years.

In spite of the possible methodological issues discussed above, two trends are clearly visible: (1) the number of active authors grows much quicker than the overall number of publications, and (2) the number of established researchers with a larger number of papers grows much slower.

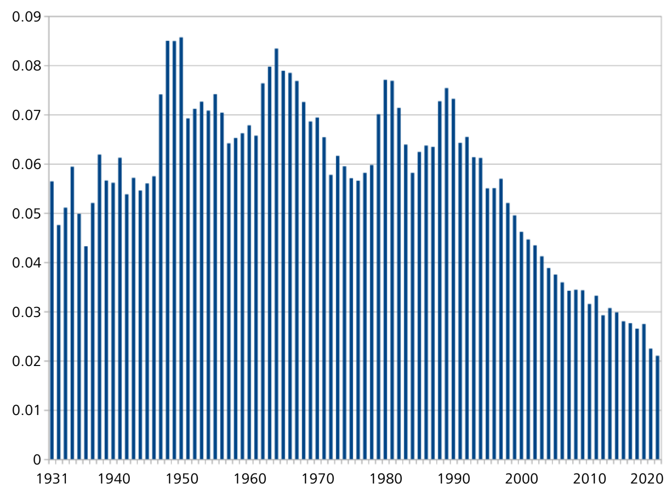

Figure 3. Share of books in mathematical publications

3 Collaboration behaviour and subject-based figures

As the comparison of the document and author numbers in Figure 2 shows, there is a discrepancy between the growth of documents and that of actively publishing mathematicians. Two main effects can conceivably play a role here – the publication frequency and the collaborative behaviour. Due to the large number of authors with very few papers, a detailed analysis of the publication frequency is highly complicated, especially since it then seems appropriate to also involve an analysis of the length of the publications in such a study.

The overall length of publications has actually been decreasing. But this phenomenon is due to the shrinking role of books as shown in Figure 3. Papers in journals have in fact become longer, at least in some areas [5]. Further effects here come from the replacement of printed by fully electronic versions and from different journal policies. Again, this makes a more detailed analysis, which would also need to involve the journal status, as well as the area, quite demanding and thus be beyond the scope of this short note. In other sciences a tendency to split results into smallest publishable units has been reported. At this stage our data do not allow us to draw substantiated conclusions on this for mathematics.

We will, however, see that the changing collaboration behaviour is likely to be a major factor in the increased growth of the number of authors. Historically, mathematical publications were predominantly single-authored. Recently, this has changed significantly, following similar trends in other sciences. Though the overall effect is strongly driven by application areas, the phenomena are visible throughout mathematics. For a subject-specific analysis, we employ the following distribution into mathematical subdomains, as employed in [10, 11]:

- Gen: General Mathematics; History; Foundations. This corresponds to sections 00, 01, 03, 06, 08, and 18 of the Mathematics Subject Classification MSC
- Disc: Discrete Mathematics. Convex Geometry; MSC sections 05, 52
- NTAG: Number Theory. Algebra. Algebraic Geometry. Group theory; MSC sections 11, 12, 13, 14, 15, 16, 17, 19, 20
- Ana: Real and Complex Analysis; MSC sections 26, 28, 30, 31, 32, 33, 40, 41.
- OpTh: Harmonic and Functional Analysis; Operator Theory; MSC sections 42, 43, 44, 46, 47.
- DIEq: Differential and Integral equations; MSC sections 34, 35, 37, 39, 45.
- OptCS: Optimization. Numerical Analysis. Computer Science. Algorithms; MSC sections 49, 65, 68, 90, 93, 94.
- ProbStat: Probability Theory and Statistics. Applications to Economics, Biology and Medicine; MSC sections 60, 62, 91, 92.
- TopGeom: Topology and Geometry; MSC sections 22, 51, 53, 54, 57, 58.
- MaPh: Mathematical Physics; MSC sections 70, 74, 76, 78, 80, 81, 82.

The corresponding diagram of the average number of authors per publication for the calendar years looks as follows:

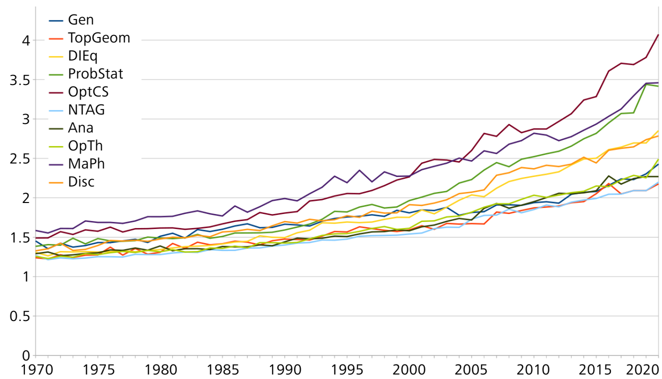

Figure 4. Average number of authors for a paper in clusters of ten mathematical areas

There are significant differences between different clusters. Examples are given by OptCS (where the average now exceeds 4), MaPh, or ProbStat (almost 3.5) and TopGeom or NTAG (about 2.2). In spite of this, however, the overall tendency is clear – collaboration has significantly increased in all fields. With mathematics being a very international enterprise, this seems to hold true globally, although samples indicate that figures may differ geographically; this may be explained both by area correlation or national science policies. However, such an analysis would again exceed the space of this column, and is left for subsequent studies.

Analogously, a breakdown can be made of the actively publishing mathematicians in each field:

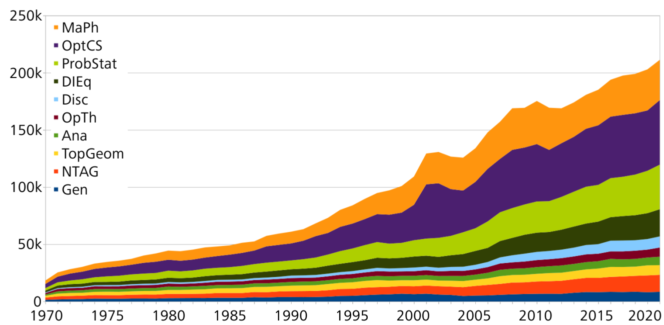

Figure 5. Actively publishing persons in ten clusters of mathematical subjects

There is a small caveat here – actively publishing mathematicians are evaluated separately for each area, so in the cumulative display, people active in several clusters may appear several times

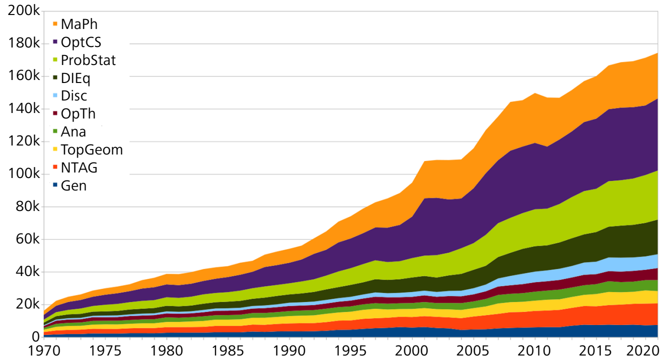

Figure 6. Actively publishing persons with > 2 papers in ten clusters of mathematical subjects

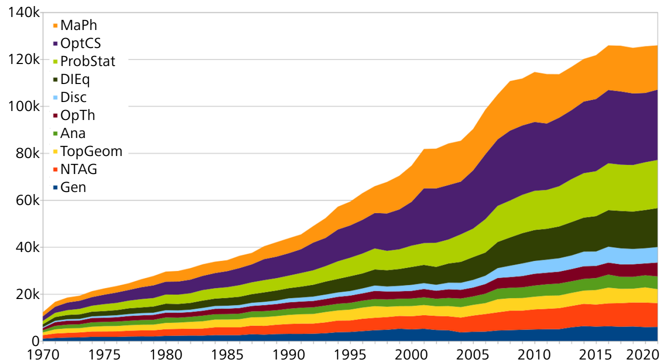

Figure 7. Actively publishing persons with > 9 papers in ten clusters of mathematical subjects

(the comparison with Figure 2 shows that this effect amounts to an about 20% increased height).

The same evaluation can be made for authors with at least 2 and at least 9 publications (see Figures 6 and 7) to obtain an impression on the more stable core of the respective communities. It can be seen that the extreme overall growth in authorships in some areas during the last decades is less extreme when authors with few papers are filtered out. Since the strong growth is concentrated in areas with likewise high collaboration figures, a possible explanation is that the numbers are inflated by many people who appear just a few times as additional coauthors.

Summarizing, we can say that the publication behaviour has clearly changed throughout mathematics towards a more collaborative attitude, but the intensity with which this happens is somewhat different in different areas.

4 Citation and coauthor networks

Another aspect, which is relevant in connection with the observed increased collaboration, is the question as to how citations are distributed within the coauthor network. Although it is for many reasons clear that mathematical achievements cannot be compared on the basis of simple (especially, short-term) citation counts (cf. [1–3]), there is still a prevailing notion that some (possibly vaguely defined) impact is correlated with aggregated citations. For a better understanding of what citations reflect, we would here suggest a first step into an empirical analysis of their distribution in the collaboration network. Although there have been suggestions of a bibliometric index involving collaboration distances [4], it appears that such approaches have never been applied to real-world databases. One reason might be that such an analysis requires very precise authorship data, since otherwise the error propagation would lead to ever more unreliable results as the coauthor distance grows. In bibliometrics, the discussion is mostly restricted to the zero level (i.e., a possible exclusion of self-citations). This is unlikely to provide a comprehensive understanding.

The mathematics collaboration graph has been investigated frequently, especially in [8], based on zbMATH Open data. While the median distance in its large connected component is 5, the situation is different when one looks at the collaboration distance for a citation. Here one would naturally expect shorter collaboration distances. Since higher collaboration distances are linked to a higher error probability, we restrict our discussion to the ranges from 0 (self-citations), 1 (coauthor citations), 2, 3 and more than 3. The distribution shown in the following diagram indicates that these seem indeed the most significant categories.

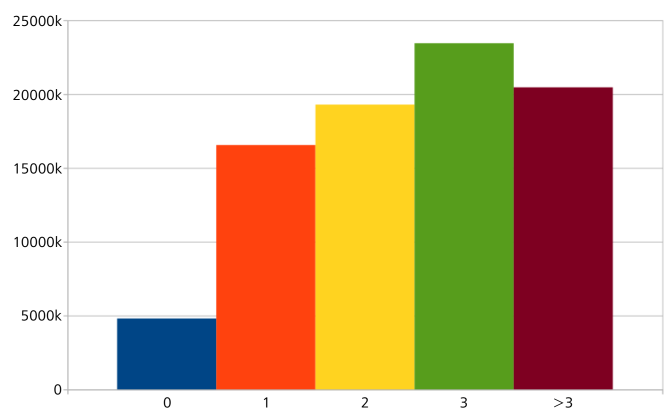

Figure 8. Minimal collaboration distance for citations of zbMATH Open authorships

More precisely, we computed for each authorship in a paper cited in zbMATH Open the minimal collaboration distance to the citing paper (note that due to multiple authorships, the total number is larger than the overall number of matched references in the database). The figures show that both the average and the median collaboration distance are equal to 3. The aggregation for authors, however, seems to indicate that the distribution is somewhat uneven, see Figure 9.

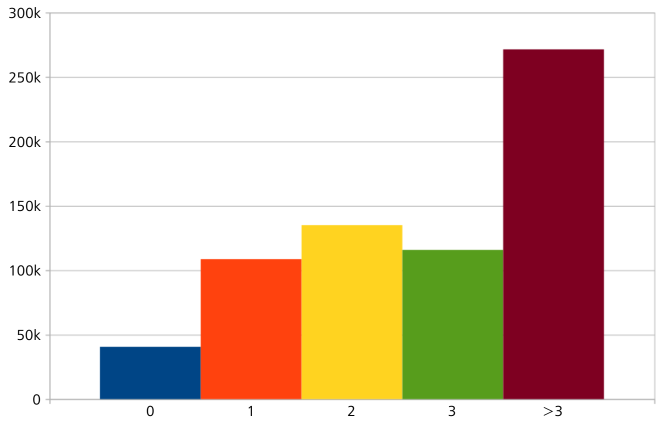

Figure 9. Number of authors in zbMATH Open with median collaboration distance *n* for their citations

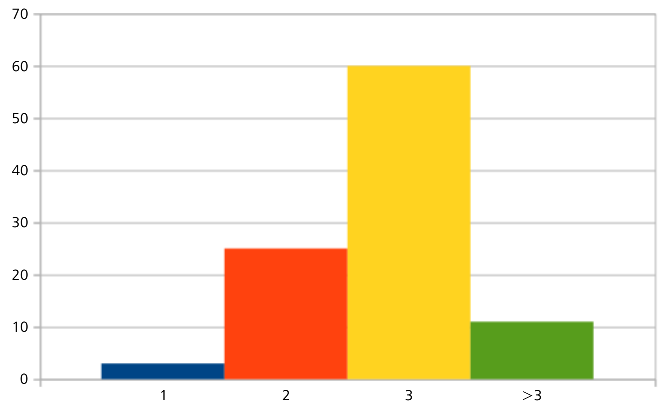

Figure 11. Number of top 100 cited authors in zbMATH Open with median collaboration distance *n* for their citations

Of the 671,513 cited authors evaluated, most (271,435) have median collaboration > 3 distance for their citations, with a second maximum at distance 2. When we restrict this analysis to the top 15,000 cited authors in zbMATH Open (which account for more than half of all citations), the picture is, however, different:

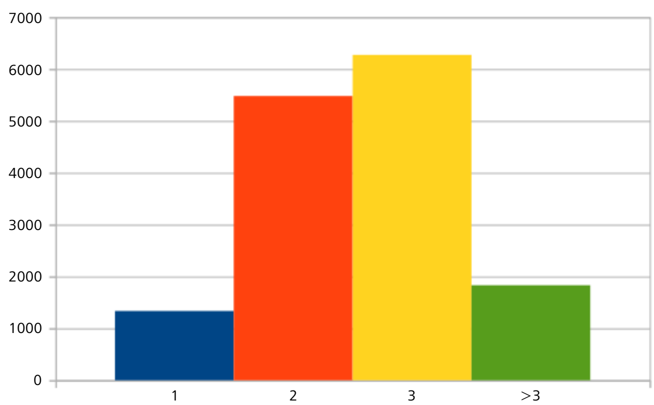

Figure 10. Number of top 15,000 cited authors in zbMATH Open with median collaboration distance *n* for their citations

One sees that the distribution in Figure 9 derives from the large number of rarely cited (and thus presumably also rarely collaborating) authors, which therefore necessarily also have larger collaboration distances. For the 100 authors with most citations in zbMATH Open, the picture is even more clear, see Figure 11.

In the presence of a high number of citations, a median of 3 for the collaboration distance of citations seems indeed to be the default value, which is very much the standard for today's mathematical community. The larger value of 4 occurs almost exclusively for older mathematicians with fewer collaborations (e.g., Kolmogorov, Mac Lane, or Pólya), or in bordering areas for which collaboration paths may exist only outside the database (e.g., Barabási or Hawking). On the other hand, Erdős, who is obviously at a disadvantage due to his huge collaboration network, is almost the only elder famous mathematician with median 2; else, median 2 occurs mostly for younger mathematicians where the citations are more likely to derive from a narrower community. Especially, the rare cases of median 1 (i.e., most citations are self-citations or come from immediate coauthors) indicate almost invariably a very particular citation network.

Finally, we compare the collaboration distance (CD) distribution of zbMATH Open citations for the Fields Medalists (FM) and the highly-cited researchers (HCR) in mathematics in 2022 of the Clarivate[1] database:

CD	0	1	2	3	> 3
FM	7,129	37,576	117,667	193,372	130,562
HCR	29,893	139,980	164,290	175,220	81,515

The huge difference between the distribution in both series is obvious. Although the Clarivate HCR gather a much larger total citation number, only a relative small fraction affects collaboration distances ≥ 2, which usually accounts for most of the citations. By far most of HCR citations derive from the close coauthor network, and the median of 2 differs significantly from the corresponding figure of the most cited authors in zbMATH Open. Even as much as 10% of Clarivate HCR turn out to have an extreme collaboration median of 1 for their zbMATH Open citations, i.e., most of their citations are self- or coauthor citations. The difference of median citation distance for Clarivate HCR in comparison to highest cited zbMATH Open authors may indicate that the Clarivate database contains many more sources that involve large numbers of self- and

[1] Clarivate Highly Cited Researchers in mathematics 2022, https://clarivate.com/highly-cited-researchers/?clv-category=Mathematics (accessed 11 August 2023)

coauthor citations. This adds evidence to the observation in [6] that citations for Clarivate HCR contain a significantly higher number of self-citations. Indeed, the difference exists not just at level zero, it actually becomes even more significant in the full distribution of citations with respect to the collaboration distance.

This indicates that the distribution of citations with respect to the collaboration distance provides a more meaningful impression of the "impact" reflected by citations. However, since it obviously depends heavily on both the age of the author and the size of the areas, it appears not advisable to derive yet another bibliometric measure from it. Rather, the distribution should be taken into account along with other information (such as age or subject specifics), to better understand what is usually hidden in total citation figures.

zbMATH Open bibliographic data used in this column are available under a CC-BY-SA license at the zbMATH Open OAI-PMH API, see [12]. The data used in this analysis reflect the status of the database as of June 30th, 2023.[2]

Acknowledgements. We would like to thank Michael Jost (FIZ Karlsruhe) for implementing the algorithm generating the citation data for various collaboration distances.

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Would it be proper to say that mathematics is blocking students in engineering?

Vladimir Dragović

We discuss the question from the title: Would it be proper to say that mathematics is blocking students in engineering? We consider it from the perspective of the current curricular developments in engineering and related fields and the role of the mathematics component in the education of future engineers and scientists.

Data analytics is entering into all aspects of our lives. This is, in particular, true nowadays for the instructional aspects of higher education. For example, there are new data analytics tools that study the complexity of degree programs and indicate "bottle necks" and "blockers". I have recently encountered a claim, based on the results of such a study, that mathematics is blocking engineering students. In relation to that, I would like to present three analogies which should help us illustrate this issue better and provide a wider and easily understood perspective.

First, I would like to ask the following: Would it be proper to say: "mathematics is blocking students in engineering"?

To understand the question better, we may consider an example that is not directly related to academia. The ongoing Winter Olympics direct our thoughts to ice hockey. Suppose I would like to become an ice hockey player and I do not ice skate. I can try to say:

"Ice skating is blocking me from becoming a hockey player."

But would it be proper? It seems to me that everybody would agree that the answer is no, though the sentence is grammatically correct. At the same time, if a branch has fallen on my driveway due to a recent snowfall, I can easily say:

"The fallen branch is blocking me from driving out."

And now everybody would agree that this last statement makes a perfect sense.

Thus, we have two sentences of a similar syntax structure, but with opposite semantic validity:

"Ice skating is blocking me from becoming a hockey player."

"The fallen branch is blocking me from driving out."

The latter is logical and assumes that I will spend a few hours removing the branch that by accident ended on my driveway and which prevents my driveway from fulfilling its basic purpose: to let me park my car in the garage and take it from there. The former is senseless because no one can play ice hockey without being quite proficient in ice skating.

Developing this statement a bit further in a mental experiment, one can envision an academia for ice hockey without ice skating. There are some obvious advantages to this new form of ice hockey education compared with the traditional one. It is more affordable and more accessible than the pathways that assume ice-skating skills. There are also a variety of jobs which graduates of such an academy can do: they can be very knowledgeable hockey fans, sports journalists, logistic officers in professional hockey clubs, or be active competitors in electronic hockey games. The only thing that these graduates would not be able to do is just to play hockey. So, it may well be that there is a niche for such a new program. And this is all perfectly fine as long as the founders of such an academy clearly state that this is a hockey academy without ice skating. However, if they do not stress that clearly enough, disappointed students and parents can come back to them asking for explanations and compensations, since they were misled into false expectations.

The second analogy is brought by the COVID-19 pandemic, which has been overwhelming for all of us for about two years now. The death toll of the pandemic in the US exceeded 900 000 a few days ago. We all remember the panic that the situation caused one year ago, when the death toll approached 100 000. The death toll per capita in the United States is among the highest in the world, despite the most advanced health institutions, the capacity to create, produce and deliver a significant number of vaccines of the highest quality. We all appreciate the work of the first responders, and no one blames them for this toll. On the contrary, we all try to support them and show our appreciation of their hard work and sacrifice. We all understand that the issues are much deeper and of a systemic nature, above and beyond the reach of the first responders.

In terms of mathematical illiteracy, we are embedded in another pandemic, which has existed much longer. It is not going to go away by itself and for this disease there are no quick and efficient solutions like two bouts of a vaccine. This pandemic is not as lethal, and thus the public awareness of its existence is much lower. The first responders include math departments all around the world. These first responders also deserve appreciation and

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public support. It should be clear that the roots of this pandemic are equally systemic and beyond the reach of the first responders, as in the case of the COVID-19 pandemic. Hurting the mission of the first responders only helps the pandemic to propagate further and faster.

For the third analogy, we can recall that data analytics entered another branch of academia, research, much earlier than the instructional component, a few decades ago. That brought us impact factors, H-indices, and other numerical tools connected with scientific publication. I remember how enthusiastic my colleagues, all excellent mathematicians, were at the early stage of this process. They believed that the objective data would finally bring a well-deserved recognition to them and eventually distinguish them from those who were, in their opinion, not that faithfully devoted to the research ideals.

Some of these other colleagues, however, managed to reorganize and quickly adapt to a new situation. Nowadays we have farms of citations and a variety of tricks and ways to produce good numbers. The issues are so massive that, for example, two committees of the European Mathematical Society are jointly preparing a public awareness campaign about predatory journals. This is all a consequence of a common situation where a measure transforms into a goal. When that happens, we all know that it cannot serve as a measure anymore.

We may assume that there are already predatory degree programs and that soon this may become even more massive and dangerous. We may expect that in the process separating genuine degree programs from the predatory ones, for example, a main criterion for the validity of an engineering program will be the validity and the rigor of its mathematics component. Thus, we are coming to a clear answer to the question posed in the title of this note. Mathematics provides foundational skills and knowledge that engineering students need to grasp in order to successfully embark on their professional careers. Mathematics constitutes an essential part of the engineering education and can't be seen as a road-blocker for students in STEM (science, technology, engineering, and math). Mathematicians need to get out of their shell and reach out to colleagues in other fields, especially the ones for which mathematics serves as foundation. They should also use the data and accumulated experience to improve their mission, their image, and the public awareness of the importance of mathematics, both in research and in teaching. We should be proactive in educating all constituents, stakeholders in higher education, students and parents about the paramount importance of mathematics in contemporary education and society. We should also help our colleagues from neighboring fields not to forget the significance of the mathematics component of their programs for the professional competence and employability of their graduates in the decades to come. This way we will assist our colleagues from other academic areas to avoid traps of rushing into easy, short-term, half-baked interventions that would eventually cause more harm than benefit for their students and their programs in the long run. This is a service to the whole scientific community, and thus to the entire society, that mathematicians should take upon themselves. No one else is going to trace that path for us.

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ICMI column

Anjum Halai

Professional learning networks in mathematics education

Challenges to the quality of basic mathematics education, especially in under-resourced countries, are well documented. It is in response to these challenges that the International Commission of Mathematical Instruction (ICMI) launched its programme of Capacity and Networking Project (CANP). This programme aims to create and sustain effective regional networks in low-income and lower-middle-income countries and develop the educational capacity of those responsible for mathematics teachers. This paper reports on the deliberations among the representatives of CANP, ICMI and the International Mathematical Union in working towards the common goal of making high quality mathematics education available to all.

Introduction

The quality of mathematics teaching and of student achievement in mathematics is a significant and ongoing concern for the communities of mathematics educators, mathematicians, and policy makers in many countries. It is well recognized that the teacher is the most significant factor for quality in the education process. However, there are significant challenges in supplying adequately prepared teachers who are also well supported throughout their career.

In many low-income and lower-middle-income countries (LMIC), teacher education, especially in mathematics, remains weak, compounded by the absence of systemic availability of continuing teacher professional development. In such a context, there is need for viable ways to support mathematics teachers at scale, especially in the remote and hard to reach areas in the LMIC.

Professional Learning Networks offer new spaces in which '[T]eachers may learn and grow as professionals with support from a diverse network of people and resources. With recent advances in technology and widespread access to the Internet, teachers can expand their web of connections beyond their face-to-face networks, seek help and emotional support, and aggregate vast quantities of professional knowledge at any time and from anywhere' [1]. In this spirit the ICMI launched a network for mathematics educators which is described below.

Capacity and Networking Project

The Capacity and Networking Project (CANP) is a flagship programme of the International Commission of Mathematical Instruction (ICMI). It was launched in response to a report by UNESCO [2]. It has two major goals: (a) create and sustain effective regional networks in low-income and lower-middle-income countries of teachers, mathematics educators and mathematicians, also linking these networks to international support networks to enhance mathematics education at all levels; (b) develop the educational capacity of those responsible for mathematics teachers. To date, ICMI has supported five CANPs, each with the common purpose of advancing mathematics education but differing in its approach and methodology. Each CANP has a loosely structured governance mechanism supported by at least two 'representatives' and one member from the ICMI Executive Committee (EC), who serves as a liaison. The CANP representatives are nominated by the mathematics education community in the respective regions to lead and coordinate the activities on the ground and represent CANP in forums such as ICMI. Overall coordination of all the CANPs is provided by one of the two vice presidents of ICMI. Under the current ICMI EC, efforts have been underway to consolidate and enhance the impact of CANP.

On February 15th–16th this year a workshop was conducted with the participation of all five CANPs, along with members of ICMI EC and representatives of the International Mathematical Union (IMU). This meeting was important because all were meeting in-person for the first time since the outbreak of the COVID pandemic. Moreover, several new members had joined as CANP representatives, and it was important for them to meet with the larger group and with the ICMI EC.

Objectives of the workshop included: (i) meet collectively to share progress update and engage in reflection and analysis based on updates; (ii) identify barriers and challenges to sustainability of

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CANPs; (iii) identify support within and externally to address the challenges; (iv) develop guidelines for a plan of action for a way forward.

The workshop was highly interactive. CANP representatives gave presentations and engaged in small and large group discussions. The language of the workshop was English. To enable participation of all, Núria Planas Raig offered facilitation in Spanish and Jean-Luc Dorier facilitated communication in French.

No.	Region	CANP Representatives	EC Liaison
CANP 1	Francophone Sub-Saharan Africa	(i) Adolphe Cossi Adihou (Benin) (ii) Sounkharou Diarra (Dakar)	Jean-Luc Dorier
CANP 2	Central America and the Caribbean	(i) Yuri Morales López (Costa Rica) (ii) Nelly Amatista León de Morales (Venezuela)	Marta Civil
CANP 3	Southeast Asia	(i) Vu Nhu Thu Huong (Vietnam) (ii) Nisakorn Boonsena (Thailand) (iii) Chanika Senawongsa (Thailand) (iv) Pimpaka Intaros (Thailand)	Susanne Prediger
CANP 4	East Africa	(i) Marjorie Sarah Kabuye Batiibwe (Uganda) (ii) Aline Dorimana (Rwanda)	Mercy Kazima
CANP 5	Andean Region and Paraguay	(i) Jorge Daniel Mello Román (Paraguay) (ii) Fredy Yunior Rivadeneira Loor (Ecuador)	Patricio Felmer

The community of mathematics educators across the five CANPs is a significant resource because working at the grassroots level they provide insights into key issues and challenges in promoting mathematics education. The following section summarises some key issues, challenges, and possibilities.

State of mathematics education on the ground: Possibilities and challenges

As the five CANPs mapped out the current state of mathematics education in their respective regions, it was evident that there was considerable practical and research activity in mathematics education at the local levels. These include doctoral programmes in education (mathematics), annual conferences and teacher development programmes. However, very little was known about the quality and impact of these programmes and activities. CANP could support the local efforts to become part of the wider international mathematics education community by helping local activity aspire to international standards of quality, and by supporting international dissemination of findings.

In the regions where CANPs are active, a significant challenge was limited availability and low quality of pre-service mathematics teacher education. In most cases, secondary school teachers were recruited based on their academic qualification, but they lacked pedagogical content knowledge (didactics). In the case of primary school teachers, content knowledge in mathematics was also found to be lacking. In-service and continuing mathematics teacher education was not available systemically. However, there were exceptions, such as in Thailand, when continuous professional development is essential for the promotion of teachers and is frequently conducted.

In the vast and often hard to reach geographical regions of CANP activity, it would be reasonable to create 'regional networks' linking other national bodies across the countries in the region. It was found easier to create regional networks when national mathematics associations or similar bodies were already in place. For example, the presence of the Association of Western and North African Didacticians of Mathematics (ADiMA) provided stability to CANP 1.

Future directions for CANPs

To address the issues noted above, participants of the workshop identified several areas where CANP would focus its efforts.

Mathematics education research

Members agreed that to provide empirical foundation for mathematics education, local research in this field should be strengthened. This could be achieved through enhancing the quality of existing PhD programmes and initiating PhD programmes in mathematics education where none exist.

CANPs could work with the local mathematics education stakeholders, especially the policy makers, to sensitize them to the importance of disciplinary research in mathematics education, as opposed to educational research that is generic. Similarly, CANP could help support specialized mathematics teacher education.

Establish and nurture networks: A three-dimensional strategy

During the deliberations, a three-dimensional framework was presented by Susanne Prediger, one of the participants, as an approach for establishing and nurturing networks of mathematics teachers, mathematics teacher educators, and mathematics education researchers. The three dimensions were personnel, material and systemic. It was suggested that for networks to be sustainable, all three dimensions would need to be strengthened. For example: developing capacity of personnel through workshops, seminars,and conferences; developing materials and resources to sustain relevant activity; and systemic support through changes and intervention in educational policy. The framework was proposed as a guideline, and to raise awareness about the complexity of creating and nurturing professional learning networks.

International community and CANPs

It was noted that a variety of resources and support mechanisms are available to the CANPs from ICMI. Some of these resources can be highly relevant for addressing the apparent disconnect between local and/or regional mathematics education stakeholders and the international community. These include the following:

- ICMI EC members, in particular those in the role of CANP liaison, are available to provide support in the form of online webinars, talks, and other input as appropriate.
- An ICMI newsletter is published four times yearly and includes considerable information about ongoing and forthcoming activity in mathematics education. It could be subscribed at the IMU website.[1]
- AMOR is a series of digital lectures about the work of the ICME awardees, which could be used as a resource for teaching in mathematics education courses.[2]
- The ICMI database project collects information about national curricula, standards, and the structure of education systems.[3]
- International conferences can provide opportunity for networking, for example:
  - ICME-15 in Sydney,[4]
  - upcoming ICMI study 26 on geometry education,[5]
  - ICMI online symposium on socio-ecological perspectives.

[1] https://www.mathunion.org/icmi/publications/icmi-newsletter
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Concluding remarks

Participants deemed the workshop to be a highly successful event. Workshop objectives as described above were fulfilled and provided an opportunity for the more experienced members in CANP to meet with the new entrants. It also provided an opportunity to share and map out the significant work in mathematics education at the grassroots level. Participants of the workshop were able to bond as a team as they took part in cultural activities in Bangkok, alongside the academic work.

Most importantly, the members of ICMI EC and the CANP charted out the issues and challenges they face in supporting mathematics teachers on the ground and deliberated on pathways to support them.

The presence of ICMI President Fredrick Leung, IMU President Hiraku Nakajima, and the IMU Secretary General Christoph Sorger along with other members of the EC showed the great support of the community of mathematicians for the community of mathematics educators, both of whom are working towards the common goal of making high quality mathematics education available to all.

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Solved and unsolved problems

Michael Th. Rassias

The present column is devoted to geometry/topology.

I Six new problems – solutions solicited

Solutions will appear in a subsequent issue.

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Consider the tiling of the plane by regular hexagon tiles, with centers in the lattice L of all $\mathbb{Z}$-linear combinations of the vectors $(1,0)$ and $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$. Glue all but finitely many tiles into position, remove the unglued tiles to form a region, discard some of these tiles, and arrange the remaining n unglued tiles in the region without rotating them, in arbitrary positions such that none of the tiles overlap. Is there a way to slide the unglued tiles within the region, keeping them upright and non-overlapping, so that their centers all end up in L?

Hannah Alpert (Department of Mathematics and Statistics, Auburn University, USA)

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Find two non-homeomorphic topological spaces A and B such that their products with the interval, $A \times [0,1]$ and $B \times [0,1]$, are homeomorphic.

Guillem Cazassus (Mathematical Institute, University of Oxford, UK)

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What is the topology of the space of straight lines in the plane?

Guillem Cazassus (Mathematical Institute, University of Oxford, UK)

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In the standard twin paradox, Greg stays at home whilst John travels across space. John finds, upon returning, that he has aged less than Greg. This is an apparent paradox because of the symmetry in the situation: in John's rest frame, it seems like Greg is doing the moving and so should also be experiencing time dilation. The standard explanation of the paradox is that there is no symmetry: at some point John needs to turn around (accelerate), so, unlike Greg, John's rest frame is not inertial for all times. So let's modify the set-up: suppose that space-time is a cylinder (space is a circle). Now, John eventually comes back to where he started without needing to decelerate or accelerate. In this fleeting moment of return, as the twins pass one another, who has aged more?

Jonny Evans (Department of Mathematics and Statistics, University of Lancaster, UK)

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The *k-dilation* of a piecewise smooth map is the degree to which it stretches k-dimensional area. Formally, for a map $f\colon U \to V$ between subsets $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$, or more generally between Riemannian manifolds,

$$\operatorname{Dil}_k(f) = \sup\{|\Lambda^k Df_x| \mid x \in U\},$$

where $\Lambda^k Df_x\colon \Lambda^k T_x U \to \Lambda^k T_{f(x)} V$ is the induced map on the k-th exterior power and $|\cdot|$ is the operator norm. A map of rank $k-1$ has k-dilation zero, so this can be thought of as a quantitative refinement of rank.

Consider the rectangular prism $R_\varepsilon = [0,1]^2 \times [0,\varepsilon]$.

(1) Let $f\colon R_1 \to R_\varepsilon$ be a map of *relative degree* 1, that is, it restricts to a degree-1 map between the boundaries of the rectangles. Show that the 2-dilation of such a map is bounded below by a $C > 0$ which does not depend on ε.
(2) Now let c_ε be the minimum 2-dilation of a surjective map $f\colon R_1 \to R_\varepsilon$. Construct examples to show that $c_\varepsilon \to 0$ as $\varepsilon \to 0$.

Fedor (Fedya) Manin (Department of Mathematics, University of California, Santa Barbara, USA)

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Given a triangle in the (real or complex) plane, show that there is a natural bijection between the set of smooth conics passing through the vertices and the set of lines avoiding the vertices.

Jack Smith (St John's College, University of Cambridge, UK)

II Open problems

by Dennis Sullivan (Mathematics Department, Stony Brook University; and City University of New York Graduate Center, New York, USA)

A new problem and a new conjecture in four dimensions

Closed oriented two-manifolds were understood in Riemann's time. Klein discovered closed non-orientable two-manifolds in the 1880s. Poincaré discovered that three-manifolds were complicated around 1900. Dimensions four, five and more were then evidently even more mysterious.

Therefore, it came as a surprise in the 1950s that closed manifolds oriented or non-orientable up to cobounding such a manifold of one higher dimension could be completely understood in terms of numerical invariants called Pontryagin numbers (integers) and Stiefel–Whitney numbers (integers modulo two).

Rochlin, mentored by *Pontryagin*, began the pattern by showing in dimension four that the cobordism classes of oriented closed smooth manifolds form an infinite cyclic group. The integer invariant, called the signature, attached to M^4 was computed from the intersection of two-cycles in M^4 as the difference between the number of positive squares and the number of negative squares of the symmetric intersection form. Rochlin proved the formula "the signature equals one-third the first Pontryagin number."

Thom extended this Rochlin pattern to all dimensions using the geometric techniques of Pontryagin and Rochlin plus the algebraic topology techniques of *Serre*, showing that, up to two-torsion, the class of an oriented manifold was determined by the set of Pontryagin numbers, these being the evaluation of products of Pontryagin classes on the fundamental homology class of the oriented manifold. Thom also showed that the non-oriented theory gave a beautiful structure determined by the Stiefel–Whitney numbers.

Hirzebruch, using Thom's work, extended Rochlin's formula for the signature in a rich but explicit fashion to all dimensions; for example, in dimension 8 the signature is one 45th of (seven times the second Pontryagin number minus the evaluation of the first Pontryagin class squared on the fundamental class of the manifold).

Milnor used the seven in that formula to show that the seven-sphere has at least seven different smooth structures. The final answer is 28, where the factor of four is related to the Dirac operator continuation of Rochlin's contribution discussed below. The figure shows one construction of Milnor's generating exotic seven-sphere, which is done by taking the boundary of the eight-manifold obtained by connecting up like party rings tangent disk bundles of the four-sphere as in the E_8 *Dynkin* diagram.

Back to dimension four

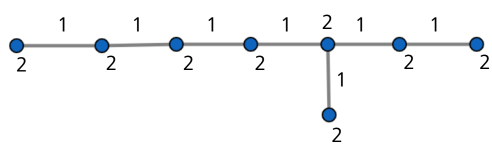
Rochlin's cobordism result depended on showing first that the cobordism group in dimension four was determined by the value of the first Pontryagin class evaluated on the fundamental class of the manifold. Then secondly showing that the signature of any bounding manifold had to be zero. This last proposition is elementary, yet one of the most important facts in manifold topology.

But the most profound point comes now

Rochlin also calculated by a geometric argument à la Pontryagin that if M^4 was almost parallelizable, i.e., parallelizable in the complement of a point, then the first Pontryagin number was actually divisible by 48. Thus the signature of such a closed four-manifold, which Rochlin proved was one third of the first Pontryagin number, had to be divisible by 16. This divisibility by 16 is the celebrated *Rochlin's theorem* about almost parallelizable smooth four-manifolds.

This was at first glance a curious result for the following reason: being almost parallelizable for an oriented closed four-manifold meant exactly that the self-intersection number of any mod two two-cycle was zero mod two, the value mod two being determined by evaluating the second Stiefel–Whitney class on the cycle.

The intersection form for integral cycles up to homology was non-degenerate over the integers by Poincaré duality. Such even-on-the-diagonal unimodular forms inside all symmetric bilinear forms taking integral values were studied in number theory. There it was known these properties meant the signature was divisible by eight and by no more in general. A basic example is the E_8 matrix, where the (inner) products for a special basis are illustrated by the E_8 Dynkin diagram:



Each nodal basis element has self-intersection number two and two nodal basis elements intersect exactly once if and only if there is an edge between them, otherwise the inner product is zero. E_8 is an even unimodular symmetric form of signature eight.

One knows that E_8 generates the indefinite even unimodular forms, in the sense that any such form is a direct sum of E_8's and hyperbolic forms $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Thus Rochlin's theorem shows that half of the elements in the infinite set of even indefinite unimodular forms cannot appear as the intersection form of any smooth closed almost parallelizable four-manifold: namely, those with an odd number of E_8's. An example that does appear is the ubiquitous K3 complex surface whose intersection form is two E_8's and three hyperbolic forms.

This result set the stage for another important development in topology, geometry and analysis.

This relates to definite forms.

In number theory one also knows that there are finitely many unimodular definite symmetric forms of a given rank, the number growing exponentially with the rank.

Donaldson proved that none of those definite forms except the identity form occurs as the intersection form of a smooth four-manifold. This is the first theorem of the unexpected Donaldson theory discovered three decades after Rochlin's theorem.

Freedman at the same time showed remarkably that every unimodular form occurs for closed topological four-manifolds.

Donaldson theory does not prove Rochlin's theorem, because Rochlin's statement involves hyperbolic forms.

In fact, there is an intermediate class of manifolds between smooth and topological where the analysis of Donaldson theory is perfectly valid.

More precisely, there are two such intermediate classes of manifolds, the ones with coordinate charts where the transition mappings are bi-Lipschitz, and the ones where the transition mappings are quasi-conformal.

Let us call these *Sobolev* manifolds.

282. Problem.*

Is Rochlin's theorem true for these Sobolev four-manifolds?

283. Conjecture.*

If Rochlin's theorem is true for Sobolev four-manifolds, then Sobolev four-manifolds are actually smoothable.

Information

Closed topological four-manifolds are almost smoothable, namely, they are smoothable in the complement of a point (see surveys and book by *Frank Quinn*).

Also, except for dimension four, all topological manifolds carry unique Sobolev structures of each type.

The proof makes heavy use of the *Kirby–Edwards* completely elementary and very ingenious construction of paths of homeomorphisms between nearby homeomorphisms in all dimensions (late 1960s).

These paths of homeomorphisms allowed *Siebenmann* in 1969 to construct higher-dimensional manifold counterexamples to the Hauptvermutung soon after he understood the precise role played by Rochlin's theorem about four dimensions in this question.

Operators on Hilbert Space

The signature operator twisted by a vector bundle exists in the Sobolev context. The unbounded version exists in the Lipschitz context. The bounded version, just using the phase of the operator (which contains all of the topological information), exists in the quasi-conformal context. Stiefel–Whitney classes make sense in these settings, so the possibility of constructing Dirac operators also makes sense. This is unknown at present (more below).

Physics

Donaldson theory is part of a larger quantum field theory which has an effective version obtained by integrating out certain variables.

This effective version has expression in terms of Dirac operators which depend on the tangent bundle. One knows that Rochlin's theorem can be deduced in a context using Dirac operators, the Atiyah–Singer index theorem and quaternions (more below).

Physicists believe that Donaldson theory and its effective version *Seiberg–Witten* theory are equivalent. From the perspective of Sobolev manifolds, Rochlin's theorem provides a challenge to and an opportunity for understanding better this belief.

More history

In the middle 1960s this author, as a second year Princeton topology grad student, was following the evidently powerful constructive cobordism techniques of *Browder* and *Novikov* classifying smooth manifolds in a *homotopy type (simply connected) with stable tangent vector bundle specified* plus the covering space method of *Novikov* for showing that the rational Pontryagin classes were homeomorphism invariants. The motivation was to study firstly, *PL-manifolds in a given homotopy type* without PL-stable tangent microbundle specified and secondly, to study *PL-manifolds in a given homeomorphism type* without PL-stable tangent microbundle specified. These formulations, suggested by the influence of *Milnor* and *Steenrod*, had completely calculable outcomes, whereas every other formulation did not have such completely calculable outcomes (simply connected and dimension greater than four).

Given a homotopy equivalence $f\colon L \to M$ one could define in all dimensions numerical obstructions to f being homotopic to a PL-homeomorphism via differences of signatures of V and $f^{-1}V$, where V is a manifold cycle in M and f^{-1} is its transverse preimage in L. These differences were divisible by eight because f is a homotopy equivalence and so pulls back Stiefel–Whitney classes. There were also modulo n versions of this picture where V is a mod n manifold cycle.

The vanishing for a finite generating set of these characteristic invariants of f was necessary for f to be homotopic to a homeomorphism, and further to be homotopic to a PL-homeomorphism if for the mod n characteristic cycles of dimension four the division by 8 was upgraded to a division by 16 using Rochlin's theorem. In higher dimensions than four this vanishing and this refined vanishing were also respectively sufficient in the simply connected case.

(This description for simplicity has absorbed the mod two Arf–Kervaire invariants in dim $4k-2$ [first encountered for $k=1$ by Pontryagin in his misstep of 1942] into the mod two signature invariants in dimension $4k$ by crossing them with $\mathbb{R}P^2$, described in the work with *John Morgan*, Annals of Math., 1974.)

The refined vanishing sufficiency was achieved in 1966 for the PL-homeomorphism case ("On the Hauptvermutung for Manifolds" Bulletin of the AMS, July 1967) and the vanishing sufficiency became valid for the homeomorphism case as a corollary in 1969 of the general topological manifold theory achieved by Kirby–Siebenmann.

The Rochlin refinement by 16 rather than 8 gave an order-two class in the integral fourth cohomology of L canonically defined when f is a homeomorphism. This heretofore unnamed class was dubbed the Rochlin class in the proceedings of the Rochlin centenary conference in St. Petersburg a few years ago.

In the hands of *Kirby* and *Siebenmann*, the entire difference between the PL- and topological manifold categories in higher dimensions could be completely understood by the profound factor of two implied by Rochlin's 16. They proved in 1969 that the homeomorphism f was connected by a path of homeomorphisms to a PL-homeomorphism (higher dimensions and no simply connected hypothesis required) if and only if a *"mod two Rochlin class"* in the degree three cohomology of L with $\mathbb{Z}/2\mathbb{Z}$ coefficients vanished, and all of these classes, referred to as Kirby–Siebenmann classes, are realized by geometric examples.

These two Rochlin classes, the mod two Rochlin type class in degree three of Kirby and Siebenmann obstructing an isotopy of the homeomorphism to a PL-homeomorphism and the integral Rochlin class of order two in degree four obstructing a homotopy of the homeomorphism to a PL-homeomorphism are related by the integral Bockstein operation. The Bockstein operation takes an integral cochain representative of the mod two class and forms 1/2 of its coboundary to obtain an integral cocycle in degree four (so that two times it is obviously a coboundary).

This "Bockstein of the mod two Kirby–Siebenmann class is the order-two integral Rochlin class" discussion is related to the important recent discovery by *Manolescu*, reported at the Rochlin Conference, of the existence of higher-dimensional topological manifolds not homeomorphic to a triangulated compact space.

More information for the Rochlin problem and the Rochlin conjecture

Work of Kirby and Edwards (mentioned above) and work of Kirby depending on that of Novikov was used to show in 1976 that topological manifolds in all dimensions, except for dimension four, could be provided with unique Sobolev structures of either type. This used a substitution of the d-torus used in those works by an almost parallelizable closed hyperbolic d-manifold (D. S. "Hyperbolic Geometry and Homeomorphisms" in the book "Geometric Topology," Academic Press, 1979).

Interestingly, the existence of these almost parallelizable hyperbolic manifolds depends on an argument learned from work of *Deligne* and *Mazur* that the algebraic topology modulo n of a complex algebraic variety can be defined for the algebraic variety reduced mod p for p prime and not dividing n, and not involved awkwardly in the defining equations of the variety.

After the opposite results of Donaldson and Freedman in 1982 it was natural to ask about their results for the intermediate class of Sobolev four-manifolds. The answer was: Donaldson theory works for both classes of Sobolev four-manifolds (S. Donaldson and D. S. "Quasiconformal 4-Manifolds," Acta Mathematica, 1989).

In studying Rochlin's theorem in the Sobolev context, it is useful to know that the index theorem holds there (*N. Teleman*) and that there are local representatives for the Pontryagin classes defined using the bounded phase of the signature operator in *Alain Connes'* perspective of non-commutative geometry (A. Connes, N. Teleman and D. S. "Quasiconformal Mappings, Operators on Hilbert Space and Local Formulae for Characteristic Classes," Topology, 1994).

Considerations related to the construction of Dirac operators and the context of smooth versus Sobolev manifolds plus a smoothability and a Dirac operator conjecture are discussed in D. S. "On the Foundation of Geometry, Analysis and the Differentiable Structure for Manifolds" in the book "Low Dimensional Topology," World Scientific, 1999.

III Solutions

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Consider two positive integers $n \geq 1$ and $a \geq 2$ such that

$$a^{2n} + a^n + 1$$

is a prime. Prove that n is a power of 3.

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Proof by the proposers
Proof 1. Write

$$a^{2n}+a^n+1=\frac{a^{3n}-1}{a^n-1}.$$

If our number is a prime, then all factors of $a^{3n}-1$ must be factors of a^n-1 and our number. Firstly, we will prove that $3\mid n$. Suppose this is not true. Then $a^{3n}-1$ is divisible by a^3-1, and from $\gcd(3,n)=1$ it follows that

$$\gcd(a^3-1,a^n-1)=a-1,$$

hence $a^{3n}-1$ is divisible by $B=a^2+a+1$. Now we have $\gcd(B, a^n-1)=1$ and

$$\gcd(B,a^{2n}+a^n+1)=1$$

(because $a^{2n}+a^n+1$ is a prime). This is a contradiction, which implies that $3\mid n$.

Assuming $n=3k$ and $b=a^3$, we have

$$a^{2n}+a^n+1=a^{6k}+a^{3k}+1=b^{2k}+b+1.$$

Using the above argument we obtain $3\mid k$, and the conclusion follows.

Proof 2. We start by presenting the following classical result.

Lemma 1. *Let $p\neq 3$ be a prime. Then the polynomial X^2+X+1 divides $X^{2p}+X^p+1$ in $\mathbb{Z}[X]$.*

Proof. Let ε be a non-trivial third root of unity. Then, since p and $2p$ have distinct residues modulo 3, it is readily seen that

$$\varepsilon^{2p}+\varepsilon^p+1=\varepsilon^2+\varepsilon+1=0,$$

so ε is a root of $X^{2p}+X^p+1$. However, we know that X^2+X+1 is the minimal (hence irreducible) polynomial of ε over $\mathbb{Q}$, therefore

$$(X^2+X+1)\mid(X^{2p}+X^p+1)$$

in $\mathbb{Q}[X]$. As the polynomials are monic with integer coefficients, it follows that the divisibility holds over $\mathbb{Z}[X]$.

Returning to our problem, suppose that n has a prime factor $p\neq 3$. Write $n=pm$. Then, by Lemma 1,

$$(a^{2m}+a^m+1)\mid(a^{2n}+a^n+1).$$

Since $a\geq 2$, we have that $1<a^{2m}+a^m+1<a^{2n}+a^n+1$, therefore $a^{2n}+a^n+1$ is not prime.

We proved that if $a^{2n}+a^n+1$ is prime, then $n=3^k$.

Remark. A few computational experiments with MAGMA suggest the following conjecture:

Conjecture. For every positive integer $a\geq 2$ the numbers $a^{2\cdot 3^n}+a^{3^n}+1$, $n=0,1,\ldots$, are square-free.

(i) The value $a=2$ gives rise to the sequence $D_n=4^{3^n}+2^{3^n}+1$, $n=0,1,2,\ldots$. We have:

(1) $D_0=4+2+1=7$, is a prime;
(2) $D_1=4^3+2^3+1=73$, is a prime;
(3) $D_2=4^{3^2}+2^{3^2}+1=262657$, is a prime;
(4) $D_3=4^{3^3}+2^{3^3}+1=18014398643699713=2593\times 71119\times 97685839$, it has three distinct prime factors;
(5) $D_4=4^{3^4}+2^{3^4}+1$ is square-free, it has 16 distinct prime factors, the smallest being 487;
(6) D_5 is square-free, but not prime;
(7) D_6, D_7, D_8, D_9 and D_{10} are not primes, but we do not know if they are square free.

(ii) The value $a=3$ gives rise to the sequence $E_n=9^{3^n}+3^{3^n}+1$, $n=0,1,2,\ldots$. We have:

(1) E_0 and E_1 are primes;
(2) $E_2=109\times 433\times 8209$;
(3) $E_3=3889\times 1190701\times 12557612956332313$;
(4) $E_4=70957\times 6627097$
$\times\, 218354731624484548192202389 21$
$\times\, 191497048350296122990338969888688354 57$;
(5) E_5 is square-free, but not prime;
(6) E_6, E_7, E_8, E_9, E_{10} are not primes, but we do not know if they are square-free.

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The Collatz map is defined as follows:

$$\mathrm{Col}(n):=\begin{cases} n/2 & \text{if } n \text{ is even},\\ 3n+1 & \text{if } n \text{ is odd}.\end{cases}$$

Let

$$t_{m,x}:=\min(n>0:\mathrm{Col}^m(n)\geq x).$$

That is, $t_{m,x}$ is the smallest integer such that, if we apply the Collatz map m times, the result is larger than x.

(a) Find $t_{3,1000}$ and $t_{4,1000}$.
(b) Show that, for x large enough (larger than (say) 1000), we have

$$t_{4,x}\equiv 3 \bmod 4 \quad\text{or}\quad t_{4,x}\equiv 6 \bmod 8.$$

(c) In general, for m odd and x large enough, there exists a constant $X_{m,x}$ such that $t_{m,x}$ is the smallest $n>X_{m,x}$ such that $n\equiv c_m \bmod M_m$. Find M_m and relate c_m to c_{m-1}.

Christopher Lutsko (Department of Mathematics, Rutgers University, Piscataway, USA)

Solution by the proposer

(a) Note that, if n is odd, then $3n+1$ is necessarily even. Thus, after applying $3n+1$ we need to apply $n/2$. Therefore, the map $\mathrm{Col}^2(n)$ is bounded by $3n/2+1/2$. Similarly, $\mathrm{Col}^3(n)$ is upper bounded by approximately $9n/2+5/2$. Thus,

$$n > 2000/9 - 5/9 > 221.$$

Moreover, for $\mathrm{Col}^3(n)$ to be as large as possible, both n and $\mathrm{Col}^2(n)$ must be odd. Therefore, we want n odd and

$$3n+1 \equiv 2 \bmod 4$$

or, equivalently,

$$n \equiv 3 \bmod 4.$$

The smallest n larger than 221 which is congruent to 3 mod 4 is 223.

Similarly, $\mathrm{Col}^4(n)$ is bounded by $9n/4+5/4$. Thus

$$n > 4000/9 - 5/9 > 443.$$

Moreover, to ensure we apply the map $x \mapsto 3x+1$ twice, there are two possibilities: If n is odd, then we want

$$3n+1 \equiv 2 \bmod 4,$$

which implies

$$n \equiv 3 \bmod 4.$$

If n is even, then we want

$$3n/2+1 \equiv 2 \bmod 4,$$

which is equivalent to

$$n \equiv 6 \bmod 8.$$

The smallest such number is 446.

(b) The general formula follows from the same line of reasoning.

(c) The values $X_{m,x}$ can be slightly tricky, because of the $+1$ in the definition of the Collatz map; in general,

$$X_{m,x} = \left\lfloor \frac{2^{m-1/2}x}{3^{m+1/2}} \right\rfloor \quad \text{or} \quad X_{m,x} = \left\lfloor \frac{2^{m-1/2}x}{3^{m+1/2}} \right\rfloor + 1.$$

For $m=5$, the same line of reasoning yields $n \equiv 7 \bmod 8$; for $m=7$ the solution requires $n \equiv \bar{3}(2\cdot 7-1) \bmod 16$, where $\bar{3}$ is the inverse of 3 modulo 16 (i.e., 11).

In general, for m odd, we have $M_m = 2^{m+1/2}$ and

$$c_m \equiv \bar{3}(2c_{m-1}-1) \bmod M_m,$$

where $\bar{3}$ is the inverse of 3 modulo M_m.

A similar expression can be derived for m even, however, it is more complicated since n could be either even or odd.

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The light-bulb problem: Alice and Bob are in jail for trying to divide by 0. The jailer proposes the following game to decide their freedom: Alice will be shown an $n \times n$ grid of light bulbs. The jailer will point to a light bulb of his choice and Alice will decide whether it should be on or off. Then the jailer will point to another bulb of his choice and Alice will decide on/off. This continues until the very last bulb, when the jailer will decide whether this bulb is on or off. So the jailer controls the order of the selection, and the state of the final bulb. Alice is now removed from the room, and Bob is brought in. Bob's goal is to choose n bulbs such that his selection includes the final bulb (the one determined by the jailer).

Is there a strategy that Alice and Bob can use to guarantee success? What if Bob does not know the orientation in which Alice saw the board (i.e., what if Bob does not know which are the rows and which are the columns)?

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Solution by the proposer

The strategy is as follows: Alice will choose 'off' for each light bulb in a row, until the last bulb in each row which she will choose to be 'on.' Now if the jailer chooses the final bulb to be 'off,' then that row will be the only row with only 'off' light bulbs. If the jailer chooses that the final bulb should be 'on,' then there will be n 'on' light bulbs. Therefore, Bob's strategy is, if there is a row which is entirely 'off,' then he chooses that row as his n choices. If each row has one 'on' light bulb, then he chooses all 'on' light bulbs.

That strategy works because we have partitioned the $n \times n$ grid into n rows of size n. If Bob does not know the orientation of the board when Alice completed it, then the problem is trickier.

If $n=2$, the same strategy works with the diagonals instead of the rows (since the diagonals are rotationally invariant). If $n=m^2$, then the same strategy works, since we can divide the board into n squares of size $m \times m$s, and use those instead of the rows. If $n \neq m^2$, I do not have a rotationally invariant solution. I conjecture that there is no winning strategy.

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Let p and q be coprime integers greater than or equal to 2. Let $\mathrm{inv}_q(p)$ and $\mathrm{inv}_p(q)$ denote the modular inverse of $p \bmod q$ and $q \bmod p$, respectively. That is, $\mathrm{inv}_q(p)p \equiv 1 \bmod q$ and $\mathrm{inv}_p(q)q \equiv 1 \bmod p$.

(a) Show that

$$\mathrm{inv}_p(q) \le \frac{p}{2} \quad \text{if and only if} \quad \mathrm{inv}_q(p) > \frac{q}{2}.$$

(b) Show by providing an example that, if $1 \le u < v$ are coprime integers and $\alpha := u/v$, then the statement

$$\mathrm{inv}_p(q) \le \alpha p \quad \text{if and only if} \quad \mathrm{inv}_q(p) > (1-\alpha)q \tag{1}$$

is not necessarily true.

(c) What additional assumption should p and/or q satisfy so that the equivalence (1) holds?

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Solution by the proposer

(a) By coprimality, there are integers a and b such that $aq + bp = 1$, where $a = \mathrm{inv}_p(q) + tp$ and $b = \mathrm{inv}_q(p) + sq$ for some integers s and t. Hence, we have

$$\mathrm{inv}_p(q)q + \mathrm{inv}_q(p)p - pq \equiv 1 \mod pq.$$

On the other hand, the left-hand side of the above relation lies in the interval $(-pq, pq)$. Consequently,

$$\mathrm{inv}_p(q)q + \mathrm{inv}_q(p)p = 1 + pq. \tag{2}$$

Therefore,

$$\mathrm{inv}_p(q) \le \frac{p}{2} \quad \text{if and only if} \quad \mathrm{inv}_q(p) \ge \frac{q}{2} + \frac{1}{p}.$$

However, the right-hand side of the above statement is equivalent to saying that $\mathrm{inv}_q(p) > q/2$.

(b) From relation (2) we deduce for a (rational) number $\alpha \in (0,1)$ that

$$\mathrm{inv}_p(q) \le \alpha p \quad \text{if and only if} \quad \mathrm{inv}_q(p) \ge (1-\alpha)q + \frac{1}{p} \tag{3}$$

and

$$\mathrm{inv}_q(p) > (1-\alpha)q \quad \text{if and only if} \quad \mathrm{inv}_p(q) < \alpha p + \frac{1}{q}. \tag{4}$$

The first relation shows that

$$\mathrm{inv}_p(q) \le \alpha p \quad \text{implies} \quad \mathrm{inv}_q(p) > (1-\alpha)q.$$

However, it is clear from the second relation that the converse is not necessarily true. Indeed, choose, for example, $\alpha = 3/7$, $p = 2$ and $q = 5$. Then $\mathrm{inv}_5(2) = 3 > (1-3/7)5$ but $\mathrm{inv}_2(5) = 1 > (3/7)2$.

Generally, with no additional assumptions, it may happen that $\alpha p + 1/q$ is not an integer and $\mathrm{inv}_p(q) = \lfloor \alpha p + 1/q \rfloor > \alpha p$. Here $\lfloor x \rfloor$ denotes the largest integer which is less than or equal to the real number x. In particular, for rational α, the inequality $\lfloor \alpha p + 1/q \rfloor > \alpha p$ is equivalent to the inequality $\{\alpha p\} + 1/q > 1$, where $\{x\} := x - \lfloor x \rfloor$ denotes the fractional part of a positive number x.

(c) In order to prevent the above scenario from happening, we only need to add the assumption $q \ge v$, v being the denominator of α. Then, in view of relation (4), it suffices to show that

$$\mathrm{inv}_p(q) < \alpha p + \frac{1}{q} \quad \text{implies} \quad \mathrm{inv}_p(q) \le \alpha p.$$

Indeed, we readily see that

$$\mathrm{inv}_p(q) < \lfloor \alpha p \rfloor + \{\alpha p\} + \frac{1}{q} \le \lfloor \alpha p \rfloor + \frac{v-1}{v} + \frac{1}{v} = \lfloor \alpha p \rfloor + 1.$$

Hence, $\mathrm{inv}_p(q) \le \lfloor \alpha p \rfloor \le \alpha p$.

We can instead assume that $p \ge v$ and employ relation (3) to prove in a similar fashion the equivalence (1).

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Let $c_n(k)$ denote the Ramanujan sum defined as the sum of kth powers of the primitive nth roots of unity. Show that, for any integer $m \ge 1$,

$$\sum_{[n,k]=m} c_n(k) = \varphi(m),$$

where the sum is over all ordered pairs (n,k) of positive integers n, k such that their lcm is m, and φ is Euler's totient function.

László Tóth (Department of Mathematics, University of Pécs, Hungary)

Proof by the proposer

We use the well-known formula

$$c_n(k) = \sum_{d \mid (n,k)} d\mu(n/d),$$

where (n,k) is the gcd of n and k, and μ is the Möbius function. Let

$$S(m) := \sum_{[n,k]=m} c_n(k).$$

Then for every $m \ge 1$,

$$\begin{aligned}\sum_{d \mid m} S(d) &= \sum_{d \mid m} \sum_{[n,k]=d} c_n(k) = \sum_{[n,k] \mid m} c_n(k) \\ &= \sum_{[n,k] \mid m} \sum_{\delta \mid (n,k)} \delta\mu(n/\delta) = \sum_{n \mid m,\, k \mid m} \sum_{\delta \mid n,\, \delta \mid k} \delta\mu(n/\delta) \\ &= \sum_{\delta a j = \delta b \ell = m} \delta\mu(j) = \sum_{\delta t = m} \delta \Big(\sum_{aj=t} \mu(j)\Big)\Big(\sum_{b\ell = t} 1\Big).\end{aligned}$$

Here

$$\sum_{aj=t} \mu(j) = \begin{cases} 1, & \text{if } t = 1, \\ 0, & \text{if } t > 1, \end{cases}$$

and this gives

$$\sum_{d\,|\,m} S(d) = m.$$

Consequently, $S(m) = \varphi(m)$, by Möbius inversion.

Alternatively, one can show that $S(m)$ is multiplicative in m, and

$$S(p^e) = p^{e-1}(p-1) = \varphi(p^e)$$

for any prime power p^e ($e \geq 1$).

Remarks

If $F(n,k)$ is an arbitrary function of two variables, then

$$\sum_{[n,k]=m} F(n,k)$$

is called the lcm-convolute of the function F. Another example is

$$c(m) = \sum_{[n,k]=m} (n,k),$$

representing the number of cyclic subgroups of the group $\mathbb{Z}_m \times \mathbb{Z}_m$.

More generally, if $F(n_1,\dots,n_r)$ is a function of $r \geq 2$ variables, then the lcm-convolute of F is

$$S_F(m) = \sum_{[n_1,\dots,n_r]=m} F(n_1,\dots,n_r).$$

It can be shown that if F is multiplicative as a function of r variables, then $S_F(m)$ is multiplicative in m. See [1, Section 6] for some more details.

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Show that, for every integer $n \geq 1$, we have the polynomial identity

$$\prod_{\substack{k=1\\(k,n)=1}}^{n} (x^{(k-1,n)} - 1) = \prod_{d\,|\,n} \Phi_d(x)^{\varphi(n)/\varphi(d)},$$

where $\Phi_d(x)$ are the cyclotomic polynomials and φ denotes Euler's totient function.

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Proof by the proposer

More generally, let $f\colon \mathbb{N} \to \mathbb{C}$ be an arbitrary arithmetic function. We show that, for any $n \geq 1$,

$$M_f(n) := \sum_{\substack{k=1\\(k,n)=1}}^{n} f((k-1,n)) = \varphi(n) \sum_{d\,|\,n} \frac{(\mu * f)(d)}{\varphi(d)}, \qquad (1)$$

where μ is the Möbius function and $*$ denotes the Dirichlet convolution of arithmetic functions.

By taking (formally) $f(n) := \log(x^n - 1)$ and using the well-known identity

$$x^n - 1 = \prod_{d\,|\,n} \Phi_d(x),$$

we deduce that

$$f(n) = \log(x^n - 1) = \sum_{d\,|\,n} \log \Phi_d(x),$$

that is, by Möbius inversion,

$$(\mu * f)(n) = \log \Phi_n(x),$$

and identity (1) gives

$$\sum_{\substack{k=1\\(k,n)=1}}^{n} \log(x^{(k-1,n)} - 1) = \varphi(n) \sum_{d\,|\,n} \frac{\log \Phi_d(x)}{\varphi(d)},$$

which is equivalent to the identity to be proved.

Now to prove the general identity (1) write

$$\begin{aligned} M_f(n) &= \sum_{k=1}^{n} f((k-1,n)) \sum_{d\,|\,(k,n)} \mu(d) \\ &= \sum_{d\,|\,n} \mu(d) \sum_{\substack{k=1\\d\,|\,k}}^{n} f((k-1,n)). \end{aligned}$$

By using that $f(n) = \sum_{d\,|\,n} (\mu * f)(d)$ ($n \geq 1$), we deduce that

$$\begin{aligned} A &:= \sum_{\substack{k=1\\d\,|\,k}}^{n} f((k-1,n)) = \sum_{j=1}^{n/d} f((jd-1,n)) \\ &= \sum_{j=1}^{n/d} \sum_{e\,|\,jd-1,\, e\,|\,n} (\mu * f)(e) = \sum_{e\,|\,n} (\mu * f)(e) \sum_{\substack{j=1\\jd \equiv 1 \,(\mathrm{mod}\, e)}}^{n/d} 1, \end{aligned}$$

where the inner sum is $n/(de)$ if $(d,e) = 1$ and 0 otherwise. This gives

$$A = \sum_{\substack{e\,|\,n\\(e,d)=1}} (\mu * f)(e) \cdot \frac{n}{de} = \frac{n}{d} \sum_{\substack{e\,|\,n\\(e,d)=1}} \frac{(\mu * f)(e)}{e}.$$

Thus,

$$M_f(n) = \sum_{d\,|\,n} \mu(d) \frac{n}{d} \sum_{\substack{e\,|\,n\\(e,d)=1}} \frac{(\mu * f)(e)}{e} = n \sum_{e\,|\,n} \frac{(\mu * f)(e)}{e} \sum_{\substack{d\,|\,n\\(d,e)=1}} \frac{\mu(d)}{d},$$

with

$$\begin{aligned} \sum_{\substack{d\,|\,n\\(d,e)=1}} \frac{\mu(d)}{d} &= \prod_{\substack{p\,|\,n\\p\,\nmid\,e}} \Big(1 - \frac{1}{p}\Big) \\ &= \prod_{p\,|\,n} \Big(1 - \frac{1}{p}\Big) \prod_{p\,|\,e} \Big(1 - \frac{1}{p}\Big)^{-1} = \frac{\varphi(n)}{n} \cdot \frac{e}{\varphi(e)}. \end{aligned}$$

Consequently,

$$M_f(n) = \varphi(n) \sum_{e \mid n} \frac{(\mu * f)(e)}{\varphi(e)},$$

which is identity (1).

Remarks

If $f(n) = n$ ($n \geq 1$), then (1) reduces to Menon's identity

$$\sum_{\substack{k=1 \\ (k,n)=1}}^{n} (k-1, n) = \varphi(n)\tau(n),$$

where $\tau(n) = \sum_{d \mid n} 1$. See [2] for references, other generalizations and analogs of these arithmetic identities.

References

[1] L. Tóth, Multiplicative arithmetic functions of several variables: a survey. In *Mathematics without boundaries* (Th. M. Rassias and P. M. Pardalos, eds.), Springer, New York, 483–514 (2014)

[2] L. Tóth, Proofs, generalizations and analogs of Menon's identity: a survey. (2021), arXiv:2110.07271

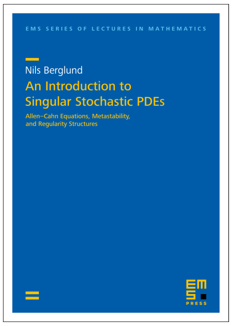
We wait to receive your solutions to the proposed problems and ideas on the open problems. Send your solutions to Michael Th. Rassias by email to mthrassias@yahoo.com.

We also solicit your new problems with their solutions for the next "Solved and unsolved problems" column, which will be devoted to probability theory.

Book reviews

An Introduction to Singular Stochastic PDEs: Allen–Cahn Equations, Metastability, and Regularity Structures
by Nils Berglund

Reviewed by Martin Hairer



The past decade has seen fast paced progress in our understanding of stochastic partial differential equations (SPDEs), especially of the so-called *singular SPDEs*, and this nice little book provides a gentle introduction to the subject. The author wisely eschews the construction of a general theory and instead chooses to focus on the example of the stochastic Allen–Cahn equation, which allows to showcase increasing levels of complexity by varying the dimension of the underlying space.

The deterministic Allen–Cahn equation is the model for phase separation given by

$$\partial_t u = \Delta u + u - u^3, \tag{AC}$$

where u is a real-valued function of time and of d-dimensional space. It clearly admits $u = \pm 1$ as stable stationary states (assuming the spatial variable takes values in a domain without boundaries, like $\mathbf{R}^d$ or the torus $\mathbf{T}^d$, or that the equation is endowed with Neumann boundary conditions) and $u = 0$ as an unstable state.[1] The main subject of study of the book under review is then the behaviour of (AC) under the addition of random noise. More precisely, writing ξ for *space-time white noise*, namely a centred Gaussian random distribution with covariance formally given by $\mathbf{E}\,\xi(s,x)\xi(t,y) = \delta(t-s)\delta(x-y)$, where δ denotes the Dirac distribution, one considers the model

$$\partial_t u = \Delta u + u - u^3 + \sqrt{2T}\xi. \tag{SAC}$$

[1] Depending on the size of the domain, the dynamics can admit further non-trivial saddle points, but the author mostly assumes that the domain is small enough so that this doesn't happen.

Here, the parameter $T \geq 0$ is interpreted as the "temperature" of the system, which is justified in view of formula (BG) below.

The author then studies two types of questions. First, there are "local" questions around the existence and uniqueness of solutions. In the present case however, there is actually an even more basic question that arises, namely, what does (SAC) actually mean? The operation $u \mapsto u^3$ plainly makes sense if u is a (random) function but, since ξ is only a distribution, it is a priori not clear whether (SAC) admits function-valued solutions. In fact, it turns out that this is the case if and only if $d < 2$, so that, in higher dimensions, there is a non-trivial question as to how to even interpret (SAC). The second type of questions studied in this book are "global" questions regarding our solutions. This includes of course the question of global well-posedness, but also the question of the description of the invariant measure for the Markov process generated by (SAC).

Another global question that is being systematically addressed is that of the metastability of the ± 1 steady states. For this, one considers (SAC) at low temperature, namely with T very small. In this case, if one starts with the initial condition $u_0 = 1$, say, then one would expect the solution to remain within a small neighbourhood of 1 for a very long duration. A natural question then is how long it typically takes for the noise to kick the solution over to a neighbourhood of the other stable steady state -1. This question is being tackled using potential-theoretic methods and the book also serves as a nice introduction to this subject.

Regarding the structure of the book, it proceeds by increasing dimension of the underlying physical space, which neatly corresponds to an increase in sophistication of the methods required. Chapter 2 actually starts with "dimension 0", namely the case where the "space" is a finite set Λ of points and the linear operator Δ is a finite-difference operator. In this case, the local questions mentioned above are trivial and one focuses on the global questions. One of the main features of (AC) is that it is a gradient flow for the energy functional

$$V(u) = \int \Bigl(\frac{|\nabla u|^2}{2} + \frac{u^4}{4} - \frac{u^2}{2}\Bigr)\,dx, \tag{V}$$

which yields sufficient control on (SAC) to get global solutions. (In the discrete, "zero-dimensional" case, the integral is with respect to the counting measure on Λ and the gradient is a finite difference.) One furthermore shows that the Boltzmann–Gibbs measure

$$\mu_T(du) = Z^{-1} \exp(-V(u)/T)\, du, \tag{BG}$$

where Z is a normalisation constant and du denotes the Lebesgue measure on $\mathbf{R}^\Lambda$, is invariant for the dynamics. The *plat de résistance* of this chapter is a sketch of the proof of the Eyring–Kramers law: provided that 0 is the only saddle point for V, the expected time to go from $+1$ to -1 is asymptotically as $T \to 0$ of order

$$\frac{2\pi}{|\mu_0|} \sqrt{\left| \frac{\det \operatorname{Hess} V(0)}{\det \operatorname{Hess} V(1)} \right|} \exp\big((V(0)-V(1))/T\big)\big(1+\mathcal{O}(T)\big), \tag{EK}$$

with μ_0 the lowest eigenvalue of the Hessian $\operatorname{Hess} V(0)$.

Chapter 3 proceeds to the continuum one-dimensional case. In this case, while (V) still has an obvious meaning, interpreting (BG) and (EK) is a bit more tricky. In the case of the Boltzmann–Gibbs measure, the problem is that there is no Lebesgue measure in infinite dimensions, while the problem with (EK) is that $\operatorname{Hess} V$ is of the form "Laplacian plus constant", so that it is an unbounded operator. Both of these difficulties can be resolved in relatively straightforward ways, in particular the ratio of determinants in (EK) is nothing but the Fredholm determinant $\det(1 - 3(2-\Delta)^{-1})$, but this gives the author a good opportunity to introduce some of the basic concepts in the study of stochastic PDEs, including a solution theory for (SAC), Schauder theory, the description of space-time white noise, etc.

This lays a good foundation on which to build the study the two-dimensional case in Chapter 4. It is in this case that, for the first time, the word "singular" appearing in the title of the book takes its meaning. Indeed, considering solutions to the *linear* stochastic heat equation

$$\partial_t v = \Delta v + \sqrt{2T}\xi,$$

one already finds that these are no longer function-valued in dimension two, but instead do at best take values in some Besov spaces with strictly negative regularity index. As a consequence, it is unclear a priori what "being a solution to (SAC)" actually means in this case. The author gives a short introduction to Wick calculus, which permits to give a meaning to "renormalised" powers $v^{\diamond p}$ of v by means of a suitable approximation procedure. For example, one has $v^{\diamond 2} = \lim_{\varepsilon\to 0}(v_\varepsilon^2 - C_\varepsilon)$, where v_ε is some smooth approximation to v and C_ε is a suitable chosen (and typically diverging as $\varepsilon \to 0$) sequence of constants. It is then natural to *define* solutions to (SAC) by setting $u = v + w$ and looking for w solving

$$\partial_t w = v + w - v^{\diamond 3} - 3v^{\diamond 2} w - 3vw^2 - w^3. \tag{$\star$}$$

It turns out that this not only provides a well-defined solution theory, but u can be approximated by solutions to a version of (SAC) with smoothened noise, provided that the nonlinearity $-u^3$ is replaced by $3C_\varepsilon u - u^3$. A very interesting consequence discussed in Section 4.6 is that the effect of renormalisation is to turn the Fredholm determinant appearing in the Eyring–Kramers formula, which is no longer well-defined since $(2-\Delta)^{-1}$ is no longer trace class, into the well-defined Carleman–Fredholm determinant $\det_2$.

Chapter 5 finally deals with the three-dimensional case. There, while it is still possible to define $v^{\diamond 2}$ and $v^{\diamond 3}$ as random distributions, the equation ($\star$) for the remainder term is itself ill-posed. Dealing with this problem was one of the original motivations for the development of the theory of regularity structures. Building on the concepts introduced in the previous parts, the main goal of this last chapter is to provide an introduction to the various aspects of this theory (reconstruction theorem, lift of various operations, renormalisation, etc.) in the context of the problem of building a robust solution theory for (SAC). Note that in this case, while a Freidlin–Wentzell type large deviations result is still available and is briefly discussed in Section 5.7, the interpretation and justification of the Eyring–Kramers formula is still an open problem, to the best of the reviewer's knowledge.

As the reader may have come to suspect by now, a complete mathematical treatise of all the aspects mentioned here would take much more space than the roughly 200 pages of this short book. Instead, the style chosen by the author is to provide details for some of the simpler proofs and only rough sketches of the main steps for many of the more advanced statements. This strikes a nice balance between self-contained proofs and references to more advanced material and makes the book a must read for anyone with a graduate-level background in probability and analysis who is interested in a quick introduction to the modern tools used in the analysis of singular SPDEs.

Nils Berglund, *An Introduction to Singular Stochastic PDEs: Allen–Cahn Equations, Metastability, and Regularity Structures.* EMS Press, 2022, 230 pages, Softcover ISBN 978-3-98547-014-3, eBook ISBN 978-3-98547-514-8.

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Les Mathématiques comme Métaphore: Essais choisis
by Yuri Manin

Reviewed by Ulf Persson

At the ICM 2006 in Madrid I attended a lecture by Manin speaking about the different uses of mathematics, as models, theories, and metaphors. Of all the lectures I attended at that congress, this was the one that stuck out to me. It was obviously not a technical talk, but a philosophical one in the best sense of the term, namely, fuelled not by professional pedantry, but by a deep personal curiosity expressed in a very original and captivating way. A year later, a collection of Manin's essays had been translated into English and handsomely published by the AMS under the title of 'Mathematics as Metaphor.' I got the book, read it with delight, as I had read previous books by him as a young man, and in fact I wrote a review of it which was published in 2010 in the EMS Newsletter – incidentally, a fact I had already forgotten this spring. However, I was alerted to it and learned that I had at its end expressed my regret that not more of his essays were available to readers not knowing Russian. Now my wish has been granted. That a wider collection had recently been published, I actually found out from Manin himself in what would turn out to be my last communication with him. I immediately got the book, published by the small French firm, Les Belles Letters, and thus containing French translations of his texts. The AMS version was about 200 pages, while this edition runs well over 500 pages, so one surmises that it is a very significant extension. On the other hand, a mere page count is a bit misleading because the pages of the first edition are larger than those in the latter one and also the font size employed is somewhat smaller. I estimate that the American edition sports about 3500 characters a page, and the French edition about 2000 characters, but still we are talking about a significant extension. My first intention was to single out what was new in the extended edition and concentrate on that, but I have decided to abandon that and treat it as a whole, fully independent of my first review.

We are talking about essays, not scientific articles, and there is of course a significant difference between the two. An essay is, like the terminology indicates, an attempt. Namely, an attempt to come to grips with a subject in a non-technical way using a meta-perspective. You should not write a scientific article if you are not an expert, but anyone is welcome to write an essay on any subject that occurs to them (they need not be published). In fact, any such attempt reminds me of the American diplomat George Kennan, who during his career wrote many dispatches from his various postings with scant hope that they would ever be read, but justifying his activity by claiming that he wrote in order to discover what he thought. This points to a crucial aspect of essay writing, namely, exploration. Karl Popper did not, unlike his colleagues in the Vienna Circle (disparaged by posterity as positivists) reject metaphysics, instead he was thinking of it as proto-science, potentially developing into one.

As indicated, most essays in general may be ignored (which does not necessarily mean that writing them is a useless activity); what makes Manin's essays worth pondering is the originality of his mind and imagination, the precision of his formulations, all supported by his wide culture, and the boundless curiosity which made this culture possible. Essays should be classified as literature, and thus subjected to the demanding criteria such writing invites. Imagination requires obstacles to be circumvented in order to be properly stimulated; this is why, according to Hilbert, mathematics requires more imagination than poetry, or, as claimed by the biographer Peter Acroyd, the writing of a biography requires more imagination than the writing of a novel. But in this general frame there are different kinds of imaginations, the iron-clad laws of logic typically lead to frustration, while writing essays and fiction leaves you more liberty. Arguments need not to be watertight as long as they are exciting, and inconvenient facts can be ignored or simply made up, as typically in fiction; what matters are the ideas, which need not be technically developed. Thus, I cannot resist speculating that the writing of essays (and poetry?) gave Manin a relief from the rigors of mathematical work, but this does not necessarily mean that it should be thought of as a mere diversion – on the contrary, it was an essential component of his mathematical work, without which the latter may not have been possible. His essays are also more accessible to readers, provided they have the required temperament, than his purely mathematical work, although the charm of the latter derives much from being presented in an essayistic spirit (this is why the above-mentioned books made such an impression on my young mind).

The point of an essay is not only to profit the writer but also to inform and inspire the reader, this is why it is very hard for me not to elaborate on Manin's essays, and to just present sober resumes; but then again, they are published and available for everyone to read and engage with in their own ways, so I hope that my taking of liberties can be excused as a kind of homage.

First, what is the nature of mathematics? This is a question that cannot be treated mathematically, but nevertheless must at least to some degree engage every serious mathematician, and even influence the way and why they persist in their obsessions. Manin himself is puzzled why mathematics engages him so much, yet without this potential skepticism in any way dampening his enthusiasm for the subject. Now there is a vulgar idea of mathematics, prevalent not so much among the general public as among philosophers and physicists and other concerned academics. Mathematics is, according to this view, seen as a game; you set up some axioms as rules and then apply logic to it and grind away. From this it does not take much to conclude that mathematics is just a matter of symbolic manipulation, and although its concepts do not have any real meaning (like vertices in a graph), it can still amazingly serve as a useful language and even tool in the study of the real world. The idea that mathematics is applied logic goes at least as far back as Frege and was further developed by his successors Russell and Wittgenstein. On the other hand, the American philosopher Charles Peirce claimed that the integers were more basic than logic, and that mathematicians had no need to study logic, they were anyway able to instinctively draw the necessary conclusions, on which mathematics rests and develops. The emphasis on logic has led to the dictum that mathematics is but a sequence of tautologies, which has been taken to heart by many. Any idea that has spread successfully must have some truth to it, so it is admittedly true that a large part of a mathematician's everyday work may amount to a ceaseless manipulation of symbols. Manin cites, not without approval, the claim by Schopenhauer that when computation begins, thought ends. Mathematics is indeed a very special activity which delights in reasoning using long deductive chains and thereby coming up with true facts in a systematic way. We recall Leibniz's exhortation, stop arguing let us calculate, hoping there would be a verbal calculus which would resolve human problems as neatly as celestial ones (for which calculus was once invented). Manin insists that the logical straight-jacket that mathematics is forced into is necessary – without it, it would degenerate, as anything to remain solid has to be contained. It is the possibility of falsification, that allows things to grow purposefully by pruning off false leads. It is also this that leads to the frustrations of mathematicians, by the presence of what which cannot be willed away. But for the serious mathematicians there is also something else to mathematics without which they would never pursue it. Mathematics involves more than a random walk in a logical configuration space. It requires thinking in a natural language, a thinking that is not in the nature of a computation in some generalized sense, but is meta-thinking whose mission is not to produce new facts, but to distinguish between the interesting and the fruitful, of coming up with new ideas and strategies. Without this meta-thinking mathematics would be a sterile subject indeed. In fact, what the serious mathematician aims for is the elusive goal of understanding, of seeing different pieces coming together, something which cannot be conveyed by mere mathematical formulations, just as little as ideas can be precisely formalized and expressed, at best only conveyed obliquely, and in this elusive vagueness lies their power. One important difference between a natural language and a formal artificial one is that the latter is precise, while the former is vague; as a result, the latter can be treated as a mathematical object. Being vague, natural languages have a recourse to forming metaphors, which, I never tire of pointing out, should never be taken literally, as they then become merely silly; while metaphors in formal languages have no choice but be taken literally. In a natural language nothing stops you from imagining the set of all sets (or the wish to have all ones wishes granted), but in a formal, strict logical setting one is forced to make explicit the different notions of 'set' involved and be forced to adopt a new word for one of them, such as 'class.' The Russell paradox does not affect natural languages, as they thrive on contradictions – in fact, languages evolved socially, meaning in particular that expressing truth is not necessarily the main purpose, rather deception; which incidentally ties up with Manin's fascination with the 'Trickster.' Thus, the metaphorical idea of the diagonal argument when applied 'literally' (in the sense of rigidly logical) has interesting consequences. At the heart of Gödel's argument, as Manin points out, is this partial embedding of the meta-language into a formal one on which it comments. Incidentally, there is much hype connected with Gödel's theorem and Manin's excellent presentation of it has as a purpose to demystify it. As he notes, the theorem has had marginal influence on mathematics as practiced.

What is mathematical intuition? Mathematical and logical concepts are anchored in a physical and hence tangible reality in the human mind. Numbers are in particular associated with the counting of physical objects, such as buttons and shells. One may talk about small numbers such as billions and trillions when they can so be concretely represented; but with the advent of the positional system of representing numbers one was able and hence seduced to write down huge numbers with millions of digits, numbers that in no way can be represented by the counting of physical objects of any kind, only of imagined objects of the mind, such as all possible books in Borges' celebrated story. Let us call such numbers, numbers of the second kind, which for all practical matters can serve as (countable) infinities. Then of course there are numbers of the third kind, represented by those which need a number of second kind to count their digits, and we can proceed inductively, and the whole thing carries an uncanny analogue of Cantor's hierarchy of infinities, except there are of course no precise boundaries between them, but the idea remains (one could of course impose precise demarcations, but that would be artificial and pointless). We are in the realm of natural language after all, where precision is not required. Of course, they are all finite, but even finite numbers can be unbelievably large and induce a sense of vertigo our usual congress with infinity does not involve. What is easier than suggesting infinity by a sequence of dots $1, 2, 3, \dots$ (you get the idea), but to really feel it, your imagination must be suitably stimulated by tangible intuition.

It is tempting to insert a slight digression here, touching upon Manin's interest in Kolmogorov complexity. It is trivial to write down numbers of any kind by using specialized notation (or more generalized inductively-defined functions), but the generic number of, say, the third kind cannot be physically represented in, say, decimal form, which is the type of form that in general is the most efficient. So in what sense can we get our hands on them? How many '7's are there in the decimal representation of a number of type $7^{7^{\cdot^{\cdot^{7}}}}$? Can any solution to this problem be feasibly described in any other sense than by the question itself? Maybe an interesting example of a totally uninteresting question.

Metaphors are important for human thinking, and Manin brings up the notion of the Turing Machine and the influence it had on logic. Yes, machines are tangible objects of the imagination and embody themselves logic in a palpable way – after all, their parts are connected in long chains of causes and effects, like the deductive chains in logical reasoning. Classically, they were represented by the sophisticated machinery of a clockwork; nowadays, we have the computer, although its machinery is not so much exhibited in its hardware, of which most users are blissfully ignorant, but in its software when the old tinkering with cogwheels has been replaced by letting the fingers dance on the keyboard instead, through writing computer codes. As David Mumford has pointed out, a mathematical proof and a computer program have much in common. Indeed, the word 'mechanical' is what we use in describing a mindless manipulation of objects subjected to inexorable laws outside our control.

Set theory was created by Cantor by taking infinities very literally as objects to be mathematically handled (but one may argue that infinite convergent sums actually involve a literal, not only potential infinity, and go back to antiquity – just think of Zeno). The uncountability of the reals is something most of us encounter in our teens, and it is usually considered as something rather metaphysical, apart from mainstream mathematics. However, without the negative aspect of the uncountability of the reals modern measure theory with its countable additivity would be impossible. For it to work, the setting has to be uncountable, and that uncountability could indeed be seen as the metaphysical setting of all those manipulations. It stands to reason that such a theory would have been developed sooner or later and then the uncountability of the reals would have been staring in our faces. Cantor's hierarchy of infinities met a lot of resistance when it appeared, Manin reminds us, and also a lot of skepticism as it was developed. As it is based on human mathematical intuition involving the manipulation of physical objects, which has no longer any relevance, that ordinary expectations would come to grief is not surprising. What could be more natural than picking one object each from a collection of non-empty sets, but the Axiom of Choice has very counterintuitive consequences when applied in, say, an uncountable context, giving rise to the Banach–Tarski paradox, or the well-ordering of the reals. The fate of the continuum hypothesis is a case in point, the physical intuition was that here it was, a subset of the real line just in front of our eyes, it had to be true or not. But it turned out to be a question of mere convention, what rules are allowed or not in forming subsets. Thus, it degenerated to a formal game having no relation whatsoever to our conception of physical reality. The very notion of mathematical Platonism seemed to founder when exploring the transfinite world, where we seem at liberty to bend the rules at our discretion. 'What did the paradoxes and problems of set theory have to do with the solidity of a bridge?' – Ulam rhetorically asked, as reported by Rota. Our sense of the solidity of mathematics seems to be connected to tangible models, such as physical space to classical Euclidean geometry. The real line has for us an almost physical existence. But when it comes to models for set theory, the very notion of a set as a mental construct seems inextricable from a verbal description; but there is only a countable infinitude of such, and hence the existence of countable models even for uncountable sets (where there are two notions of cardinality, one extrinsic, and one intrinsic). Naively we think of all subsets existing of, say, the reals, but from a strict logical and formal point of view, only those which in principle can be described. This threatens, as noted, to indeed reduce mathematics to a game whose objects mean nothing (just like the chess pieces on a board). On the other hand, a piece of mathematics considered as a game has nevertheless some content as a game, and we can ask questions about it, such as its consistency, which we feel is a definite yes or no question, not contingent upon some axioms we introduce in the meta-game of investigation. According to Manin, it is as if we feel that the game itself, defined by its axiomatic rules, is a physical object, and systematically drawing all the conclusions is a physical activity anchored in the real world, no matter how unfeasible in practice; just as concluding that a Diophantine equation must have a solution or not by making an almost physical thought experiment of an infinite search. Manin's attitude to set theory is pragmatic, as that of most mathematicians. He does not seem engaged in the classical controversies and refers to intuitionists and constructivists as somewhat neurotic. Set theory for Manin, like for most mathematicians, provides a convenient language of mathematics, as famously exemplified by Bourbaki. On a more existential level, Manin's attitude to mathematical Platonism is ambivalent; he has described it as psychologically inescapable and intellectually indefensible. What is really meant by that can only be speculated upon. He stresses that his physically tangible intuition, especially when confirmed by mathematical applications to physics as a scientific discipline, makes him inclined to Platonism, an attitude made even more inescapable from his own experience as a mathematician, in particular when studying number theory; but as strong as those convictions may be, they are ultimately based on subjective experience. Of course intellectually Platonism is not amenable to any formal proof, as little as proofs of the existence of God pursued by the scholastics (the concerns of whom seem uncannily similar to those of set-theorists). But as Pascal famously noted 'Le cœur a ses raisons que la raison ne connaît point.'

I would like to conclude this mathematical section with a nice toy example of Manin. Consider a finite set X of m elements. The power set $P(X)$ is naturally an m-dimensional vector space over the field $\mathbb{Z}_2$ with $\varnothing$ corresponding to the 0. Its algebra of functions is given by the Boolean polynomials $\mathbb{Z}_2[x_1, x_2, \ldots, x_m](x_1^2 + x_1, x_2^2 + x_2, \ldots, x_m^2 + x_m)$, thus any such polynomials can be written as a sum of monomials which are naturally identified by the elements (vectors) $x \in P(X)$, where, say, $(1, 1, 0, 1)$ is identified with $x_1x_2x_4$ and 0 with the trivial (constant) monomial 1. Thus, given x we have $x(y) = 1$ iff $x \subset y$. The polynomials (P) are thus tautologically paired with the subsets S of $P(X)$ by $P = \sum_{x \in S} x$. But there is also another way of associating a subset to a polynomial, namely, to associate its zeroes. The fact is that every subset is given by the zeroes of a unique polynomial, so in particular 1 is the only polynomial with an empty set of zeroes. To see this, we have to introduce $I = X$ (and note that $x + I$ is the complement of x and the zeroes of $1 + P$ make up the complement of the zeroes of the polynomial of P). Consider now the polynomial

$$x(u)(x+I)(u+I) = \prod_{i \in x} x_i \prod_{j \notin x} (1 + x_j).$$

We have $x(u)(x+I)(u+I) = 1$ iff $u = x$; thus, for any set S the polynomial

$$\prod_{x \in S} (1 + x(u)(x+I)(u+I)) \quad \Big(= \prod_{x \in S} \Big(\prod_{i \in x} x_i \prod_{j \notin x} (1 + x_j) \Big) \Big)$$

vanishes exactly on S (if $S = \emptyset$, then of course the polynomial is 1). What is the point of this formal almost tautological game? Manin brings it up as a finite version of the Axiom of Choice: given a set of polynomials how do we pick an element in each of the sets they define, or show that the polynomial is 1? Given the polynomial in canonical form (or any random form), this is not so easy in general: do we have to check all the elements of the vector space? This also leads to a particular instance the P/NP problem, an instance which, according to Manin, is intractable at the time.

Now I have not touched upon the section of mathematics and physics, which is greatly expanded, nor upon the essays on general topics from linguistics, Jungian psychology (of which Manin was charmed with many references in his works), art and poetry. Had I done so, the review would have been far too long, not only too long as it already is. Having thus failed to do full justice to the book, I hope that I have at least inspired a few readers to consult the master himself.

Yuri Manin, *Les Mathématiques comme Métaphore: Essais choisis.* Les Belles Lettres, 2021, 600 pages, Softcover ISBN 978-2-251-45172-5.

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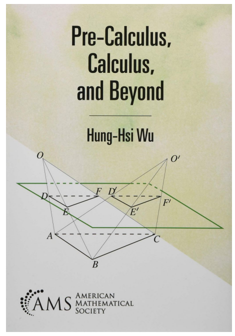
Pre-Calculus, Calculus, and Beyond by Hung-Hsi Wu

Reviewed by António de Bivar Weinholtz



This is the sixth and final book of a series covering the K-12 curriculum, as an instrument for the mathematical education of school teachers. It is the third and final volume of the series dedicated to high-school teachers. Unlike the two previous such volumes, which included topics that had already been treated in the series (to ensure that high-school teachers could have at their disposal a set of self-contained instruments for their mathematical education, expressly written for them, thus not neglecting the pre-requisites to what they have to teach), this final book is composed of entirely new topics.

The first chapter is dedicated to trigonometry and the definition of trigonometric functions with domain $\mathbb{R}$. It starts with the basic definitions, the general notion of extension of a function, then applied to extending trigonometric functions, with the use of the unit circle, to the interval $[-360, 360]$ and finally to $\mathbb{R}$. The laws of sines and cosines, as well as other basic trigonometric identities, such as the addition formulas, are proven in this general setting. It proceeds to the definitions of radian and the new trigonometric functions obtained by switching from degrees to radians, and to the definition of polar coordinates. Finally, trigonometric functions are put to use in the geometric interpretation of complex numbers and the derivation of the De Moivre and Euler formulas, with the exponential notation; applications are given to the study of n-th roots of unity, to a formulation of basic isometries in terms of complex numbers, and to the study of graphs of quadratic functions, with the use of rotations to eliminate the mixed term in general quadratic equations in two variables. The chapter concludes with the introduction of inverse trigonometric functions and a final section where the author analyzes the importance of these functions in the study of general periodic functions, which play a fundamental role in the physics of many phenomena, through Fourier series. More advanced treatments of trigonometric and general exponential functions are given a brief overview, which provides adequate complementary useful knowledge to the readers.

The following chapter proceeds with a rigorous treatment of real numbers. Thus it becomes finally possible to justify what had previously been called "FASM" (the "fundamental assumption of school mathematics") and enabled students to use real numbers, without betraying the basic principles of mathematical studies, from the moment it becomes mandatory for the development of their mathematical instruction, but before it is possible to include in the curriculum a rigorous treatment of the real line, due to the inner complexity of the subject. After an algebraic reformulation of the theory of rational numbers, the introduction of an extra axiom finally leads to the fundamental distinction of the sets of real numbers and rational numbers, that can then be identified with a dense subset of the real line. The concept of limit of a sequence of real numbers is defined and its basic properties are then presented, proved and applied to the rigorous treatment of some of the concepts and properties that had been previously accepted with the use of FASM, namely the existence and basic properties of positive n-th roots of positive real numbers and the fundamental theorem of similarity, followed by a whole chapter dedicated to a full study of the decimal expansion of a number, including repeating and non-repeating decimals, and using the concept and basic properties of infinite series.

A new chapter follows where the delicate concepts of length and area are treated as rigorously as possible at this stage, based on a list of fundamental principles for geometric measurements that are accepted as a guide to the foundation of those concepts, but the inherent difficulties of these topics are explained. In this framework, the author introduces the concept of rectifiable curve and identifies the problems one faces when trying to obtain a rigorous argument that leads to the formula for the circumference of a circle, postponing the final solution to the end of the volume, where a more advanced treatment is given of trigonometric functions, that is put to this use.

Some basic formulas for the area of elementary figures are revisited in this more general setting and obtained using the assumptions of this chapter; a famous proof of the Pythagorean theorem using the concept of area is finally given a proper formulation, whereas it is very often presented to students without the due care to observe that it depends on rather subtle and nontrivial concepts and properties of area and without some apparent geometrical properties being adequately proven. As the author explains in one of his illuminating pedagogical comments, this is another example of how misleading some rather common incoherences in the teaching of school mathematics can be.

After length and area, it is time for the introduction of some comments on three-dimensional geometry and the concept of volume. By the formulation of some elementary principles that, at this stage, have to be accepted without further foundation, the author proceeds to the proof of some basic facts on perpendicularity and parallelism of lines and planes in three-space and to the analysis of Cavalieri's principle, which leads to the formula for the volume of a sphere.

The two final chapters are dedicated to an introduction of derivatives and integrals of real-valued functions of one variable and their basic properties, and applications to trigonometric functions and to new formulations of the logarithmic and exponential functions; they start with the notions of limit of a function in a point and of continuity.

As in the previous volume, this one also contains a very helpful Appendix with a list of assumptions, definitions, theorems and lemmas from the companion volumes. I strongly recommend reading first the review of the first volume (António de Bivar Weinholtz, Book review, "Understanding numbers in elementary school mathematics" by Hung-Hsi Wu, Eur. Math. Soc. Mag. **122** (2021), pp. 66–67). There, one can find the reasons why I deem this set of books a milestone in the struggle for a sound mathematical education of youths. I shall not repeat here all the historical and scientific arguments that sustain this claim, but I have to restate, regarding this final volume, that although it is written for high-school teachers, as an instrument for their mathematical education (both during pre-service years and for their professional development), and to provide a resource for authors of textbooks, the set of its potential readers should not be restricted to those for which it was primarily intended; it should include anyone with the basic ability to appreciate the beauty of the use of human reasoning in our quest to understand the world and the capacity and will to make the necessary efforts, which are required here as for any worthwhile enterprise. Of course, as the content and presentation of the three last volumes of the series is of a more advanced nature, a wider mathematical background is required. This volume being the last of the series, we are now able to fully appreciate the magnitude of the enterprise undertaken by Prof. Wu and how it is indisputable, as I wrote before, that with this set of books at hand there is no excuse left for school (including high-school) teachers, textbook authors and government officials to persist in the unfortunate practice of trying to serve to school students mathematics in a way that is in fact unlearnable…

Like the previous two volumes, this one is punctuated with pedagogical comments that give extremely useful advice regarding what content details should be used in classrooms and which are essentially meant to teachers; mathematical comments are also added to the main text, in order to extend the views of the reader whenever it helps to clarify the subject in question. To the readers interested in the full scope of the pedagogical comments of this volume I also recommend the lecture of my preceding review (António de Bivar Weinholtz, Book review, "Teaching school mathematics: Algebra" by Hung-Hsi Wu, Eur. Math. Soc. Mag. **125** (2022), pp. 50–52), where a detailed description is made of what the author considers to be the main characteristics of mathematics and how they have been neglected in schools for such a long period of time and replaced by what he calls "Textbook School Mathematics" (TSM); the same concern is present in all the topics treated in the present volume.

As it is almost inevitable in any printed book, there are some minor misprints that can be easily detected and corrected by the reader. I just point out some details in formal definitions that deserve some attention.

The definition of the i-th term of a sequence (p. 118) as the value assigned by the function (that the sequence is, by definition) to i is commonly found in these same terms in many mathematical texts, but it can lead to some awkward consequences; for instance, it is then not strictly true that "every sequence has infinitely many terms", as the appreciation of this statement, with the above given definition, depends on the number of *values* of the sequence rather than on the fact that its *domain* (the set of natural numbers) is infinite. With this definition, a *constant* sequence would have only one term…. A formal definition that would allow us to state that the number of terms of a sequence is always infinite, one for each whole number, could be to identify the i-th term of the sequence (s_n) with the ordered pair (i, s_i).

In the definition of convergence of regions (p. 230), apart from the stated condition on the boundaries, one needs some extra condition, as, for instance, the coincidence of the approximating regions with the limit region R outside a "vanishing neighborhood" of the boundary of R. This condition is very easily verified in all the cases where Theorem 4.3 (convergence theorem for area) is applied in this book, and also in the graphical examples that are used in the treatment of area; this treatment, of course, has to rely on some intuitive assumptions at this stage.

Also, the definition of the limit in a point x_0 of a real-valued function defined in a subset I of $\mathbb{R}$ (p. 286) adopted in this book is what we can call the "exclusive" limit, inasmuch as, to "test" the limit of the function, one only considers sequences in I with limit x_0 that never assume the value x_0, as opposed to what we can call the "inclusive" limit definition, where we can also consider such sequences that can assume the value x_0; but in the case of this "exclusive" limit, to ensure that the limit is unique, when it exists, one has to assume as well that x_0 is the limit of a sequence in I that never assumes the value x_0 (x_0 is then usually called an *accumulation point* of I). It is not enough to ensure that it is just the limit of a sequence in I (a *limit point*); if x_0 is what is usually called an *isolated point* of I, i.e., if it is a limit point but not an accumulation point of the domain, with this "exclusive" definition of limit, the function would have every number as limit in x_0 (because to contradict this fact one would have to find a sequence in I that has x_0 as its limit, never assuming the value x_0; but this contradicts the definition of an isolated point). So, either one only considers domains with no isolated points, or one has to define this kind of limit only in accumulation points and not in general limit points of the domain, as the uniqueness of the limit is an essential feature of this concept. Strictly speaking, when considering the algebra of limits of functions, like in Lemma 6.2 (p. 290), one also has to be careful to consider only accumulation points of the domain of the functions obtained by performing each algebraic operation in the pair of functions, as it is not mandatory that if a point is an accumulation point of the domain of each function in the pair it will also have this property with respect to the intersection of domains.

Finally, the definition of continuity (p. 289) is not affected by these subtleties, as it is not dependent on the definition of limit of functions (only in the intuitive motivation of this concept a link is established with limits). In fact, with the alternative ("inclusive") definition of the limit of a function, to be continuous in a point of the domain could simply be defined as having a limit in that point; however, if one aimed to use the adopted "exclusive" limit definition of continuity, one would have to treat separately the isolated points of the domain. Nevertheless, this leads to the conclusion that in the proof of Lemma 6.3, on the "algebra of continuity", one cannot fully rely on Lemma 6.2; once again we could be spared all these subtleties either if one excluded domains with isolated points, or if one considered the "inclusive" definition of limit (in this case, however, with some care also with domains in the algebra of limits).

All these details should not, of course, be brought to a high school classroom, although they can be of some use to teachers.

As in the previous volumes of this series, on each topic the author provides the reader with numerous illuminating activities, and an excellent choice of a wide range of exercises.

Hung-Hsi Wu, *Pre-Calculus, Calculus, and Beyond.* American Mathematical Society, 2020, 417 pages, Paperback ISBN 978-1-4704-5677-1, eBook ISBN 978-1-4704-6006-8.

António de Bivar Weinholtz is a retired associate professor of mathematics of the University of Lisbon, Faculty of Science, where he taught from 1975 to 2009. He was a member of the scientific coordination committee of the new curricula of mathematics for all the Portuguese pre-university grades (published between 2012 and 2014 and recently abolished).

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Report from the EMS Executive Committee Meeting, in Copenhagen, 10–11 March 2023

Richard Elwes

The EMS Executive Committee (EC) plus guests met on 10–11 March 2023, in beautiful (if snowy!) Copenhagen, the home town of the new EMS President Jan Philip Solovej, chairing for the first time. This important moment of transition was kindly hosted by the Danish Mathematical Society and the University of Copenhagen, and took place in the current residence of the University's Centre for the Mathematics of Quantum Theory, the research group led by Jan Philip. On Friday evening, the Danish Mathematical Society hosted a fine dinner for the assembled company near the magnificent Christiansborg Palace, the Danish Parliament building.

1 Officers' reports

After a warm welcome from our hosts, the meeting was opened by the President who paid compliments to his predecessor Volker Mehrmann and outgoing members of the EC, before discussing his activities during the first months of his leadership. He has enjoyed meeting and learning from many and varied people involved in the EMS, especially members of the Society's standing committees whose assorted activities are so central to the EMS's work, as well as representatives of the University of Helsinki which generously hosts the EMS's office, and representatives of affiliated organisations including EMS Press and zbMATH Open.

The new EMS Treasurer Samuli Siltanen then reported on the Society's finances, with the most recent budget approved in the 2022 Council Meeting in Bled (Slovenia). There were then discussions of possible improvements to EMS processes, in both financial terms and those of environmental sustainability. The committee agreed to change the EMS Magazine to being primarily online, rather than having printed copies automatically shipped to every member.

The EMS Secretary Jiří Rákosník, and Vice Presidents Jorge Buescu and Beatrice Pelloni reported on their activities (most of which feature elsewhere in this report).

2 Membership and website

The EC was pleased to approve the Department of Mathematics of the University of Trento as a new corporate member, and honoured to waive the membership fees of the Ukrainian Mathematical Society.

The committee approved 325 new individual members, with the President expressing the hope that the rate of growth in individual membership, which has been constant for a while, might be encouraged to increase through some appropriate publicity and additional membership benefits, such as discounts to attendance at major mathematical meetings.

The EMS website is running smoothly following the complete rebuild and relaunch in March 2021. Incremental improvements are ongoing, with some changes in the technical staffing at EMS Press, the previous primary web-developer having now left with the EMS's thanks for his excellent work. The EMS's membership database is currently in the process of being upgraded, a sizeable technical undertaking which both EMS Office and EMS Press staff are working hard on.

3 Scientific and society activities

Two major EMS-supported meetings are coming up in 2023. The inaugural Balkan Mathematical Conference (BMC)[1] will take place on 1 July, hosted by the 10th Congress of Romanian Mathematicians in Pitești, Romania. Steering Committee member (and EMS Ex-Vice President) Betül Tanbay reported on progress towards this important event, including fruitful collaboration with EMS associate member society MASSEE (the Mathematical Society of South Eastern Europe).

The 29th Nordic Congress of Mathematicians[2] will take place on 3–7 July 2023 in Aalborg, Denmark, and the EC heard about progress towards this meeting.

[1] http://imar.ro/~bmc

[2] https://ncm29.math.aau.dk

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The EMS's marquee event is the European Congress of Mathematics (ECM) which happens every four years and is the second largest mathematical meeting in the world. The 9th edition is coming up in July 2024, in Sevilla, Spain.[3] The EC heard from the Chair of the Organizing Committee Juan González-Meneses (in attendance remotely as a guest) and made decisions about appointments to the congress and prize committees. The EC then considered plans for the 10th ECM to be held in 2028. The location will be decided at the 2024 EMS Council meeting, with initial bids invited by June 2023. The best format for large mathematical meetings in the future, given the financial and environmental costs of flying, is an important matter to consider. The EC plans for a panel discussion of this topic at 9ECM.

Vice President Beatrice Pelloni reported on the inaugural meeting of the EMS Young Academy (EMYA),[4] which she attended alongside all 30 members of this new body. The meeting was exciting and constructive and included practical progress towards determining EMYA's structure and bylaws. The EC agreed to offer non-student EMYA members a two-year free EMS membership (students are already entitled to free membership).

The EC agreed to change the name of the open call for large-scale, inclusive and cross-institutional activities from "EMS Large Events" to "EMS Strategic Activities." The call and application form are available on the EMS website.[5]

4 Standing committees and projects

The EC considered the general set-up of the EMS's standing committees, making several proposals to increase communication between the committees, as well as improve coordination, for example in the setting of budgets, and in the release and format of open calls. The committee then considered reports from the committees in turn.

The Committee on Applications and Interdisciplinary Relations (CAIR) brought a proposal for inaugurating a new joint EMS/ECMI Lánczos Prize for Mathematical Software, which the EC was delighted to approve. Reports were then considered from the committees on Education, ERCOM (European Research Centres on Mathematics), Ethics, Publication and Electronic Dissemination, Raising Public Awareness of Mathematics, and Women in Mathematics. The members of these committees all have the Society's gratitude for their work.

Karine Chemla, Chair of the European Solidarity Committee in attendance as a guest, delivered a report on her committee's activities, noting that the number of applications they receive has dropped in recent years. The EC considered steps to improve the visibility and relevance of the calls. It then approved some funding applications, including for a summer school at CIMPA (Centre International de Mathématiques Pures et Appliquées) on "Graph structure and complex network analysis."

The committee then considered its relationship with the European Open Science Cloud (EOSC), of which the EMS is a member, and made plans for how to continue engaging with this body.

5 Publishing and publicity

EMS Publicity Officer Richard Elwes, in attendance remotely as a guest, reported on his activities, including the state of the Society's social media channels. In November 2022 the EMS created a page on the social media platform Mastodon.[6] As of 1 March 2023, the EMS has over 12000 followers on Twitter,[7] over 5000 on Facebook,[8] around 300 on LinkedIn,[9] and 400 on Mastodon. It also has over 500 subscribers on YouTube,[10] a platform the EMS is yet to make the most of.

The quarterly EMS Digest is currently undergoing a redesign and relaunch, following the retirement of long-standing editor Mireille Chaleyat-Maurel. Discussions on the new format and a new editor are ongoing.

Managing Director of EMS Press[11] André Gaul (in attendance remotely as a guest) presented his update. With 21 Subscribe-to-Open (S2O) journals in 2023, EMS Press is currently the second largest S2O-publisher in the world (after Annual Reviews), indicating the success of this approach. He emphasised the great caution that is needed before further journals can be added to the portfolio, for financial and practical reasons.

The Editor-in-Chief of the EMS Magazine, Fernando da Costa (in attendance as a guest) reported that the magazine is performing well, with a good supply of articles, although the balance of content is not always perfect. He hopes for a productive relationship with EMYA (the EMS Young Academy, see above).

The Managing Editor of zbMATH Open,[12] Olaf Teschke (in attendance as a guest) delivered his report, remarking that it is succeeding in the goal of increasing its visibility, and that it successfully passed a mid-term evaluation by the EMS's co-editor FIZ Karlsruhe.

[3] https://www.ecm2024sevilla.com
[4] https://euromathsoc.org/EMYA
[5] https://euromathsoc.org/strategic-activities

[6] https://mathstodon.xyz/@EuroMathSoc
[7] https://www.twitter.com/euromathsoc
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[10] https://www.youtube.com/c/EuropeanMathematicalSociety
[11] https://ems.press
[12] https://zbmath.org

6 Funding, political, and scientific organisations

The EC discussed its relationships with several scientific and prize-giving panels, making nominations where appropriate. The European Research Council (ERC, a major funding body) was again a major topic of discussion, and the EC agreed to set up a working group to further consider strategies to improve mathematical funding via this route.

7 Close

An online meeting of the presidents of EMS member societies was scheduled to take place in May 2023. Later in the year, the autumn Executive Committee meeting is likewise planned to be held online, at a date to be determined. In the meantime, a new schedule of monthly short online EC meetings will continue.

On behalf of the assembled company, Secretary Jiří Rákosník thanked the President (who also served as a local organiser for this meeting), the University of Copenhagen's Centre for the Mathematics of Quantum Theory, and the Danish Mathematical Society for hosting a very successful meeting and for the generous hospitality.

Richard Elwes is the EMS publicity officer and a senior lecturer at University of Leeds (UK). As well as teaching and researching mathematics, he is involved in mathematical outreach and is the author of five popular mathematics books.

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New editors appointed

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Marie E. Rognes is chief research scientist in Numerical Analysis and Scientific Computing at Simula Research Laboratory, Oslo, Norway. She holds an MSc in Applied Mathematics (2005) and PhD in Numerical Analysis (2009, Centre for Mathematics for Applications, University of Oslo). Her webpage is https://www.simula.no/people/meg.

Her scientific work targets ground-breaking basic research with dual impact in mathematics and/or the life sciences and spans numerical analysis and scientific computing with applications in biomechanics, neuroscience, and physiology. Key scientific expertise includes numerical methods for solving partial differential equations, PDE-constrained optimization including data assimilation, compatible discretizations, automated scientific software, uncertainty quantification, multi-domain operators, nonlinear elasticity, viscoelasticity, porous media and incompressible fluid flow, electrophysiology, and computational brain, cardiac and cancer modeling.

She won the 2015 Wilkinson Prize for Numerical Software, the 2018 Royal Norwegian Society of Sciences and Letters Prize for Young Researchers in the Natural Sciences, was a Founding Member of the Young Academy of Norway (2015–2019) and holder of an ERC Starting Grant in Mathematics (2017–2023). She is a member of the Norwegian Academy of Technological Sciences, as well as of the FEniCS Project Steering Council.

Vesna Iršič is an assistant professor at the Faculty of Mathematics and Physics at the University of Ljubljana. She received a PhD in mathematics from the University of Ljubljana in 2021 and has been a postdoctoral fellow at Simon Fraser University in Canada in 2021–2022.

In 2016–2021 she has been the main coordinator of the voluntary tutoring system at the Department of mathematics in Ljubljana. She has been a member of several organizing committees, in particular of the 10th Slovenian Conference on Graph Theory in June 2023 that hosted more than 300 participants. In 2012–2018 she has also been involved in the preparation of high school students for the International Mathematical Olympiads. She is a member of the recently established EMS Young Academy (EMYA).

She received the faculty's *Prešeren Prize* for master's thesis and an Honorable mention for excellent student achievements at the University of Ljubljana. Her main research area is graph theory, in particular domination, metric properties, and games on graphs. She has been involved in the research of domination games, the *Cops and Robber* game, the strong geodetic problem and the investigation of several families related to Fibonacci cubes.

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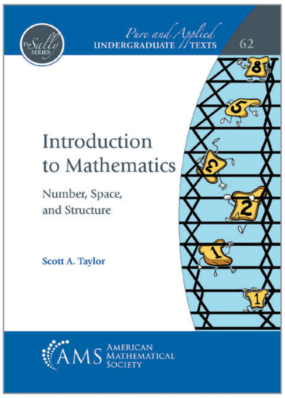

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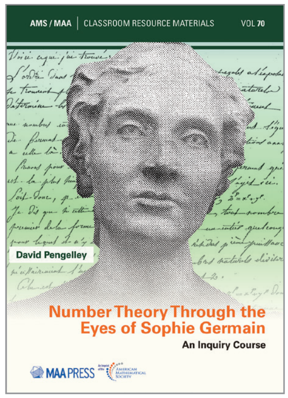

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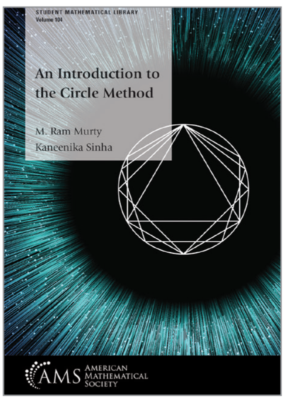

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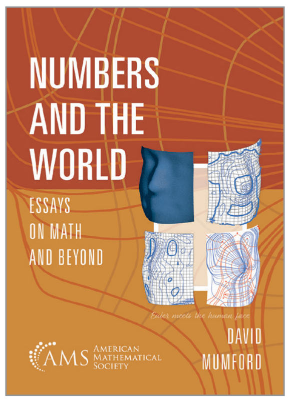

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