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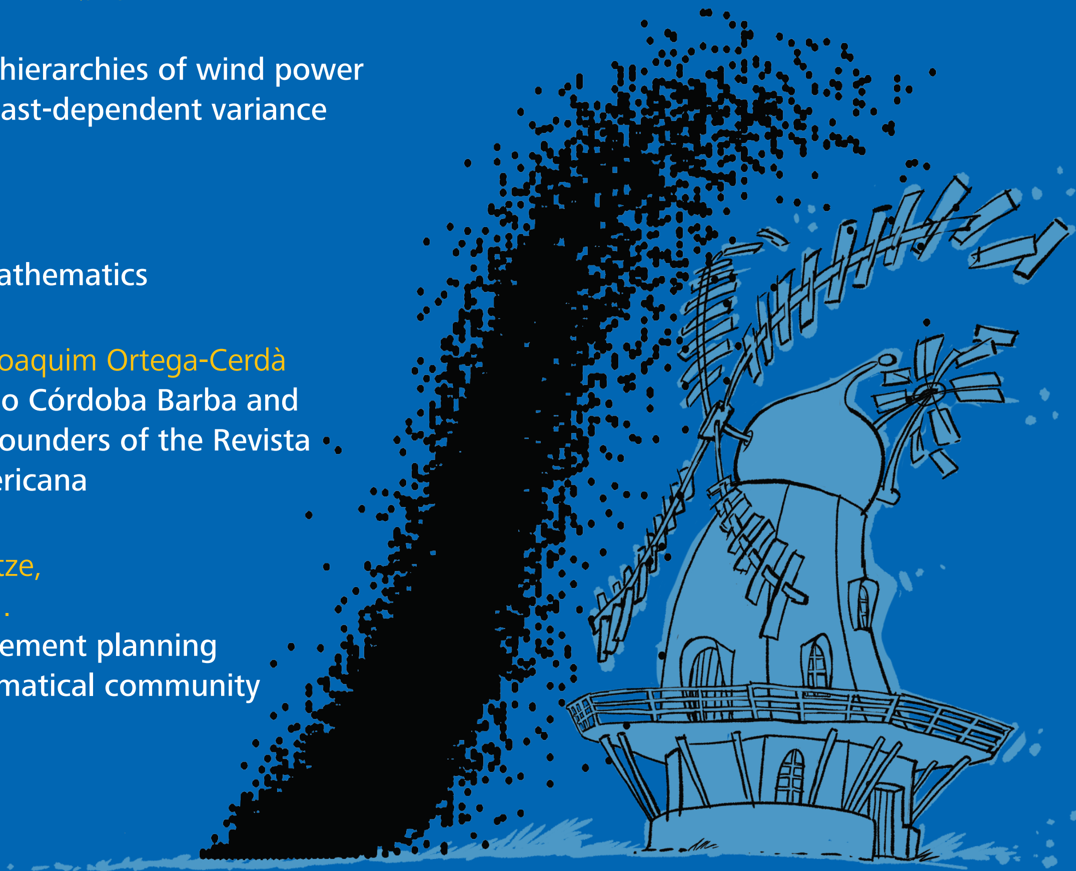
Mikkel L. Sørensen, Jan K. Møller and Henrik Madsen
Reconciling temporal hierarchies of wind power production with forecast-dependent variance structures

François Lê
Words, history and mathematics

Isabel Fernández and Joaquim Ortega-Cerdà
Interview with Antonio Córdoba Barba and José Luis Fernández, founders of the Revista Matemática Iberoamericana

Tobias Boege, René Fritze, Christiane Görgen et al.
Research-data management planning in the German mathematical community



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The cover drawing by A. B. Araújo references the plot of forecasted wind speed from the article by Sørensen et al. in the present issue. The drawing is inspired by traditional Danish windmills such as the Sønderho Mølle in Fanø, Denmark.

A message from the president

Photo by Jim Høyer, University of Copenhagen

I am close to concluding my first year as president of the EMS. It has been a very busy and exciting time. Before my time as president I had not worked very closely with the EMS and as I have already mentioned in a previous message, I spent this year learning the inner workings of the Society. Perhaps the most important thing that I take away from this year, and what I see as the main strength of the Society, is the large and wide-ranging number of mathematicians throughout Europe who work very hard and enthusiastically alongside and for the Society. Indeed, there is a lot of work to be done. I never miss a chance to highlight the importance of our publishing house, the EMS Press, which is by far the main activity of the Society. We, however, do much more than this. To mention some of our core responsibilities: we organize conferences, support young mathematicians, e.g., through our young academy EMYA, engage with funding agencies like the ERC, promote mathematical outreach activities, support mathematicians from less developed regions in and outside of Europe, and in general work to increase the diversity and inclusivity of the mathematical community. To get a sense of the activities supported by the EMS, I encourage you to look at our web page and, in particular, to note the new list of calls for applications and nominations that we have gathered.[1] We also have a very strong social media presence. If you do not already, please consider following us on LinkedIn, Mastodon, or X. We have Richard Elwes, our publicity officer, to thank for our strong and engaged social media presence. Since you are reading this message, you are of course already aware of our Magazine, and I hope you continue to enjoy it.

I wanted to make sure to mention all of this because I believe our level of involvement in the mathematical community has reached a point that requires the EMS to make a real investment in improving and professionalizing its infrastructure and its tools for community building. This will be one of our main goals in the coming year. With the help of Elvira Hyvönen from our Helsinki office and the EMS Press staff, we have already worked on a new members' database which will be launching very soon.

Among the activities I highlighted above I did not mention the important work done at zbMATH Open, which is our openly available mathematical database edited by EMS in collaboration with the Heidelberg Academy of Sciences and Humanities, and FIZ Karlsruhe – Leibniz Institute for Information Infrastructure. I left it out above only because I wanted to take the opportunity now to thank the zbMATH Open editor-in-chief since 2016, Prof. Klaus Hulek, who will be stepping down by the end of this year. Prof. Hulek has worked very hard for zbMATH Open and has been a driving force in making the database successfully open and available to all. I want at the same time to welcome our new editor-in-chief, Prof. Christian Bär, who will take over in 2024. I am looking forward to working with Prof. Bär in securing the continued success of zbMATH Open.

I am afraid I will have to end this message on a somber note. We have all witnessed the eruption of violence near the boundaries of Europe, first in Ukraine and now lately in the Middle East. A lot of innocent people have become victims of this violence. This has inevitably affected many of our colleagues and their families. I do hope for better times for them and for all of us in the near future.

Jan Philip Solovej
President of the EMS

[1] https://euromathsoc.org/calls

Reconciling temporal hierarchies of wind power production with forecast-dependent variance structures

Mikkel L. Sørensen, Jan K. Møller and Henrik Madsen

For electricity grid operators, the planning of grid operations depends on having accurate models for forecasting wind power production on a number of different time resolutions. Together, these time resolutions create what has become known as a temporal hierarchy. Previous studies have considered methods of reconciling forecast hierarchies inspired by least squares methodologies that produce coherent and more accurate forecasts. In this study, we highlight some challenges in the established approach when applied to wind power production, and consider methods which more appropriately take into account the full conditional probability densities. We suggest methods using maximum likelihood techniques to estimate the prediction variance ahead of time. Using base forecasts from a commercial forecast provider together with simpler forecasting models, we test the modified approach against the established reconciliation approach on data from Danish wind farms. The results show significant improvements in accuracy when compared to both the state-of-the-art commercial forecasts and the simpler models.

Introduction

As part of the ongoing green transition, the share of renewable energy being incorporated into the electricity grid is increasing. With this comes the challenge of renewable energy sources, such as wind power, being stochastic in nature, i.e., power production depends on, e.g., the wind speed. Thus, in order to plan and control the electricity grid, transmission system operators (TSOs) require reliable and accurate forecasts of power production. With targets such as the European Union having 70% of power production stemming from renewable energy sources by 2050,[1] this challenge is only becoming greater. Hence, the importance of accurate and reliable wind power forecasts is more crucial now than ever before.

The main goal in this respect is thus to reduce the uncertainty of the forecasted power production as much as possible. One of the more recent developments within forecasting, which has been shown to reduce the forecast uncertainty, is the method of hierarchical forecast reconciliation. The method in its current form was suggested by [7], with further developments by [8, 14, 17]. Hierarchical forecast reconciliation utilizes the hierarchical structures that are often naturally present in data by ensuring that forecasts on different resolutions are coherent. Coherency refers to the fact that the layers should add up throughout the hierarchy in a natural way. An example of this comes from the study on Australian domestic tourism by [1], where forecasts are produced at the national scale, state scale, zone scale, and regional scale. These forecasts form a hierarchy, where one would expect that all the regional forecasts within a zone add up to the zone forecast, all the zone forecasts within a state add up to the state forecast, etc. When forecasts add up in this manner, the forecasts are said to be coherent. However, when the forecasts on each layer are produced independently of each other this will not be the case. The purpose of hierarchical forecast reconciliation is thus to produce a set of reconciled forecasts that have the coherency property based on a set of incoherent base forecasts.

Hierarchies can be either spatial, as was the case for the example on Australian domestic tourism, temporal, as has been explored by [2] and [14], or spatio-temporal, e.g., [9]. In a spatial hierarchy one might examine regional forecasts, see, e.g., [7]. In a temporal hierarchy, which will be the case for this study, the forecasts should add up based on the timescale such that, e.g., the three monthly forecasts within a quarter add up to quarterly forecasts, and the four quarterly forecasts add up to yearly forecasts.

Forecast reconciliation of temporal hierarchies has shown great promise in reducing the uncertainty of the base forecasts and producing more accurate forecasts at all layers [14]. However, the majority of studies on temporal hierarchies only show such improvements using simple models to produce the base forecasts, such as the ARIMA and exponential smoothing models by [2]. These models also often require constant data availability, which under real-time operational conditions cannot always be guaranteed. The established reconciliation approach also assumes that the covariance structure between the forecasts is constant in time,

[1] IRENA, Global energy transformation: A roadmap to 2050; https://www.irena.org/publications/2019/Apr/Global-energy-transformation-A-roadmap-to-2050-2019Edition.

EMS MAGAZINE 130 (2023) — DOI 10.4171/MAG/172

which for many applications may not be the case. For wind power production it is well-known that the forecast uncertainty is highly dependent on, e.g., the wind speed [13]. Furthermore, wind power production is naturally bounded between zero and the installed capacity, which poses a challenge when reconciling, since the established method of reconciling assumes unbounded data. These challenges call for more refined methods to generate reconciled forecasts.

In the literature, forecast reconciliation has been applied previously to cases of both wind power forecasting and solar power forecasting, which both face similar issues. The paper [16] provides a brief overview of the literature on this. In [6] the authors test some of the current methods for forecast reconciliation in a spatial hierarchy with a similar dataset to our case study and are able to achieve significant improvements in accuracy. They do not, however, account for the issue of the non-constant variance structure. The need for varying and mean-dependent variance when reconciling forecasts has also been highlighted by [3] for the case of heat load forecasts, although the approach therein mostly addresses slower seasonal variations. Additionally, [18] demonstrates a method of changing the forecast reconciliation methodology to account for non-negativity.

The purpose of the present study is thus:

- To use real measurements and weather forecasts to examine whether improvements to commercial state-of-the-art wind power production forecasts can be achieved through forecast reconciliation.
- To propose adjustments to the established reconciliation approach (MinT-Shrink by [17]), which addresses the level and time-dependent covariance structure in the wind power framework.

This study is structured such that Section 1 introduces the concept of forecast reconciliation. Section 2 presents the wind power production data used in the study. Section 3 then examines how base forecasts are produced, and introduces the proposed adjustments to the reconciliation process. Section 5 shows the results, and lastly Section 6 concludes on the findings and gives some further perspectives.

1 Forecast reconciliation

For the temporal hierarchy in this study, prediction horizons up to 24 hours will be examined. So, to introduce the concepts of reconciling forecasts based on temporal hierarchies, a simple example hierarchy is constructed within the 24-hour prediction frame with 6-hour, 12-hour and 24-hour forecast resolutions. This example could, e.g., cover the total wind power production in the stated intervals. The hierarchical structure of this example is illustrated in Figure 1.

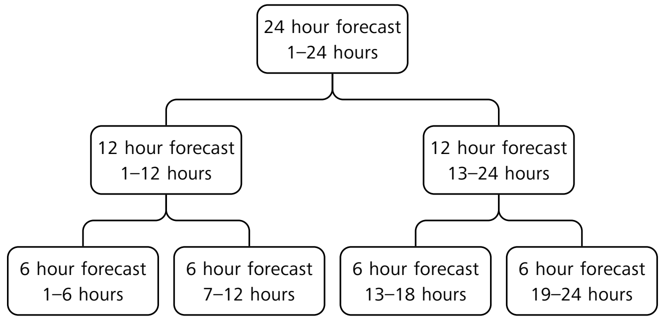

Figure 1. Temporal hierarchy with three levels covering a 24-hour period.

As illustrated by the figure, for such a hierarchy of forecasts to be coherent, the bottom-level 6-hour forecasts should add up pairwise to the second level 12-hour forecast, and the two 12-hour forecasts should add up to the highest level daily forecast. This summation property of the hierarchy is described by a summation matrix $\mathbf{S} \in \mathbb{R}^{n \times m}$, where n is the number of individual base forecasts (in this case 7) and m is the number of forecasts on the lowest resolution (in this case 4). For the hierarchy in Figure 1, the summation matrix is given by:

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The rows of the summation matrix define how the four bottom-level forecasts add up to each level of the hierarchy, so, e.g., the top-level forecast should be equal to the sum of all the four bottom-level forecasts. This also implies that the base forecasts ($\hat{y} \in \mathbb{R}^n$) are ordered from top to bottom, i.e.

$$\hat{y} = \begin{bmatrix} \hat{y}^{24h}_{24h} \\ \hat{y}^{12h}_{24h} \\ \hat{y}^{12h}_{12h} \\ \hat{y}^{6h}_{24h} \\ \hat{y}^{6h}_{18h} \\ \hat{y}^{6h}_{12h} \\ \hat{y}^{6h}_{6h} \end{bmatrix}. \tag{1}$$

For clarity, the notation in this study uses superscripts for naming or other distinctions and subscripts for indices, so, e.g., $\hat{y}^{6h}_{18h}$ is the forecast on the 6-hour resolution 13 to 18 hours ahead.

With the summation matrix and the base forecasts as defined above, the most commonly used method of forecast reconciliation, the MinT-Shrink [17], can be introduced.

The method is based on a generalized least squares approach, which minimizes the coherency errors, i.e., the difference between the reconciled forecast ($\tilde{y} \in \mathbb{R}^n$) and the base forecast ($\hat{y}$). This implies solving the following optimization problem

$$\min_{\tilde{y}} \; (\tilde{y} - \hat{y})^T \mathbf{\Sigma} (\tilde{y} - \hat{y}) \qquad \text{subject to} \;\; \tilde{y} = \mathbf{SG}\tilde{y},$$

where $\mathbf{\Sigma} \in \mathbb{R}^{n \times n}$ is the covariance matrix of the coherency errors $\tilde{y} - \hat{y}$, and $\mathbf{G} = [\mathbf{0}_{n \times (n-m)} \quad \mathbf{I}_m]$. If $\mathbf{\Sigma}$ is assumed to be known, the problem has a closed form solution, given by

$$\tilde{y} = \mathbf{S}(\mathbf{S}^T \mathbf{\Sigma}^{-1} \mathbf{S})^{-1} \mathbf{S}^T \mathbf{\Sigma}^{-1} \hat{y}. \tag{2}$$

In [17] the authors showed that $\mathbf{\Sigma}$ is not identifiable; instead, they propose the trace minimization approach (MinT), which uses the covariance matrix of the base forecast errors, i.e.,

$$\mathbf{\Sigma} = V[\varepsilon],$$

where $\varepsilon = y - \hat{y}$ with $y \in \mathbb{R}^n$ being the observations arranged similarly to the base forecasts in equation (1). This approach has shown great promise in increasing forecast accuracy across all levels of the hierarchy. MinT-Shrink, also proposed by [17], and extended to temporal hierarchies by [14], is an extension of the MinT approach which applies a shrinkage estimator to the correlation.

The method estimates the covariance $\mathbf{\Sigma}^{\text{shrink}}$ from the correlation matrix of the base forecast errors $\mathbf{R}$ using the shrinkage parameter λ:

$$\mathbf{R}^{\text{shrink}} = (1 - \lambda)\mathbf{R} + \lambda \mathbf{I}_n, \tag{3}$$

$$\mathbf{\Sigma}^{\text{shrink}} = \mathbf{\Lambda}^{\frac{1}{2}} \mathbf{R}^{\text{shrink}} \mathbf{\Lambda}^{\frac{1}{2}}, \tag{4}$$

where $\mathbf{\Lambda}$ is the so-called hierarchical variance, which is a diagonal matrix consisting of the empirical variances of the base forecast errors at each level. So, for the example hierarchy in Figure 1, the hierarchical variance is

$$\mathbf{\Lambda} = \begin{bmatrix} (\sigma^{24h}_{24h})^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (\sigma^{12h}_{24h})^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & (\sigma^{12h}_{12h})^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (\sigma^{6h}_{24h})^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & (\sigma^{6h}_{18h})^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & (\sigma^{6h}_{12h})^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & (\sigma^{6h}_{6h})^2 \end{bmatrix}.$$

It is usually assumed that both $\mathbf{\Lambda}$ and $\mathbf{R}$ are either constant or slowly varying (see, e.g., [3]) in time. However, as mentioned in the introduction, this might not be a reasonable assumption for wind power forecasts, since the uncertainty is directly coupled to the wind power generation through the power curve, as discussed by, e.g., [10]. A similar relation between the uncertainty and the prediction is probably also needed in other cases, like for solar power forecasting.

2 Wind power data

The hierarchy that will be explored in this study has eight layers ranging from the 1-hour resolution up to a 24-hour resolution, i.e., the layers will have resolutions 1, 2, 3, 4, 6, 8, 12, and 24 hours. To perform the reconciliation, base forecasts need to be obtained for each temporal resolution in the hierarchy.

The data that these forecasts should be generated from consists of hourly measurements of wind power production from on-shore wind turbines, as well as numerical weather predictions (NWPs) of wind speed and wind direction. These data are available for 15 areas of the DK1 region in western Denmark (Jutland and Funen), as seen in Figure 2. The areas have very different capacities for power production, ranging from as low as 39.3 MW to 746.7 MW, and each of the areas will be forecasted and reconciled individually using temporal hierarchies. The same region has also been examined by [6] using a spatial hierarchy.

Data were available from the beginning of January 2018 and through December 2019, and hence the data were split such that 2018 data were used for training base forecast models and estimating shrinkage parameters, while 2019 data are used for testing the reconciliation out of sample. Out of sample testing will be based on a rolling window containing 12 months of previous measurements and forecasts. This window rolls forward in time, one day at a time, as the data would become available. Each day the coefficients of the base forecast models are re-estimated, and new base forecasts for the following day are produced and reconciled. This mimics how operational wind power forecasting takes place.

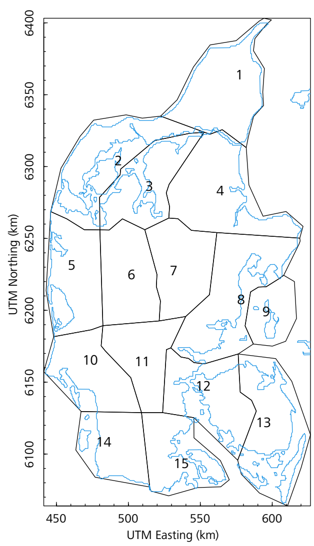

Figure 2. Map of the 15 subareas of the DK1 price area in western Denmark. Coastline in blue.

Area	Rated power [MW]
1	428.1
2	400.9
3	253.3
4	360.8
5	746.7
6	328.0
7	86.9
8	100.4
9	39.3
10	201.6
11	219.7
12	97.3
13	194.4
14	230.2
15	75.6

Table 1. Rated power production for each of the 15 subareas in the DK1 price area given in MW.

From each of the areas depicted, hourly measurements of wind power production, as well as forecasts of wind speed and direction, are available. The power production data is courtesy of Energinet, the TSO that operates the Danish high-voltage transmission grids. The weather forecasts were made by the Danish Meteorological Institute (DMI).

Commercial state-of-the-art forecasts of hourly wind power production were available for the 1-hour resolution (courtesy of ENFOR A/S). These were produced using their wind power forecasting tool WindFor,[2] and will constitute the lowest level of the temporal hierarchy.

Table 2 below summarizes which data are available along with the notations in this study.

Variable	Unit	Description
y	kWh	Measurements of power production
$\hat{y}$	kWh	Forecasts of power production
$\widehat{\text{WSpd}}$	m/s	Forecasted wind speed
$\widehat{\text{WDir}}$	degrees	Forecasted wind direction

Table 2. Data descriptions for variables and the used units. These are all on 1-hour time resolution for each of the 15 areas.

The measurements y are settlement measurements, meaning that they are generally available 8–10 days after the measured production has occurred. This fact will have to be accounted for when modelling the power production, since not having real-time data means that some models, such as the classic ARIMA models, are not applicable. The measurements also include periods of missing data. For shorter periods ($\leq$ 6 hours), measurements were linearly interpolated, while longer periods were left missing, meaning that models could not be updated here.

The forecasts in this study are generated every night at midnight (00:00) and cover the following 24-hour period. The forecasts of wind speed and direction are generated every six hours and are available a few hours later. To generate forecasts at midnight, the NWPs for the following day were taken from the weather forecast generated at 18:00. This also means that the forecast horizon follows the time of day.

The time series in Figure 3 shows some of the volatility in the data, as changes from low production to high production and vice versa can happen within a couple of hours.

The relation between wind speed and power production seen in Figure 3 shows the classic power curve structure, with power production increasing slowly at first until about 4 m/s, then rapidly increasing above the 4 m/s mark, and flattening out when the forecasted wind speed approaches 8 m/s. This power curve structure is also what should be mimicked when modelling the individual layers of the hierarchy.

Figure 3 also illustrates how the variance in power production depends on the wind speed, and hence on the predicted wind power generation. This means that the wind power production cannot be assumed to be Gaussian, since the variance would then be independent of the power production.

Lastly, it is worth noting that the power production does not reach the rated production in the available data. This is likely due to the fact that the turbines within the areas are positioned differently in the terrain, making it very rare for all of them to generate according to their rated power simultaneously.

2.1 *Aggregation*

In order to construct forecasts for the other levels of the hierarchy, the data had to be aggregated up to the 24-hour resolution. Thus, datasets of power production and weather predictions need to be constructed for the 2, 3, 4, 6, 8, 12, and 24-hour resolutions. This is done by aggregating the 1-hour measurements and NWPs. Power production measurements are aggregated by simply adding up measurements, while wind speed and direction forecasts are aggregated by averaging the forecasts. For wind speed, this is a simple arithmetic average, while for the direction, the average has to be computed using the circular mean. This implies converting the angular directions to points on the unit circle, taking the arithmetic mean of the points, with the resulting angle being the circular mean.

At the 1-hour level, the 12 months of training data contain 8760 measurements of power production and as many forecasted values of wind speed and direction. As a result of the aggregation, the 2-hour level has 4380 data points, the 3-hour level has 2920 data points, and so on, up to the 24-hour level, which has 365 data points. This reduction in data quantity puts a limit on how well the higher levels can be modelled. Consequently, if a larger temporal hierarchy should be attempted, more data would be required.

3 Base forecasts

When modelling the levels above the 1-hour resolution, i.e., 2, 3, 4, 6, 8, 12, and 24 hours, it was decided to use beta regression models. These purely statistical models are chosen mainly to address the challenge of power measurements not being available in real time. However, this is also a well-established method of predicting wind energy, see, e.g., [15]. Additionally, choosing such simple models can also show if improvements to already high-quality commercial forecasts are possible using forecast reconciliation, even without employing state-of-the-art forecasts for the other levels.

The beta regression models used in this study are constructed using the `betareg` package in R [4]. The following brief intro-

[2] https://enfor.dk/services/windfor/

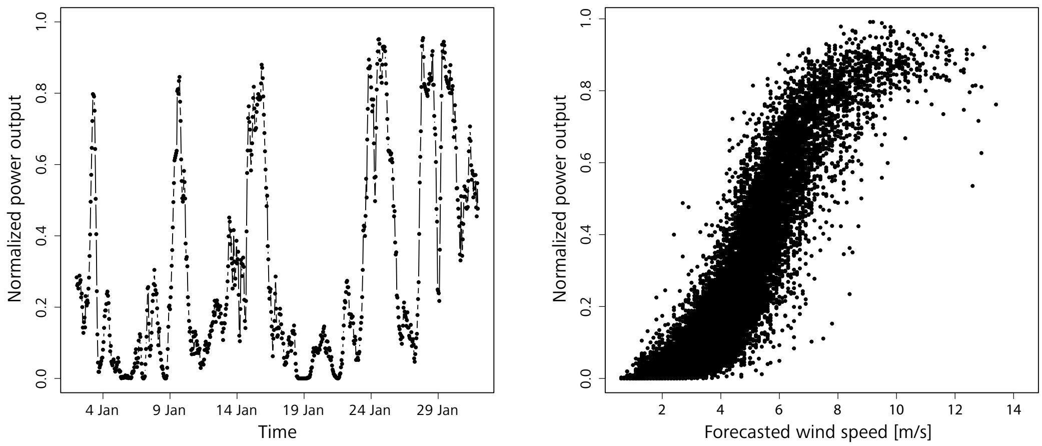

Figure 3. Left: Time series plot of the wind power production in area no. 3 in northern Jutland from January 2018. Right: Relation between forecasted wind speed for all horizons and their corresponding normalized power productions in area no. 3.

duction leans heavily on the documentation for the package, so for a more in-depth walk-through of these methods, see the documentation by [4].

We model the normalized power production as a beta distribution $Y_t \sim \mathcal{B}(\mu_t, \phi_t)$, where $\mu_t \in (0,1)$ is the mean value ($E[Y_t] = \mu_t$) and $\phi_t > 0$ is the precision. Therefore, the variance of the power production at time t can be expressed as

$$V[Y_t] = \frac{\mu_t(1-\mu_t)}{1+\phi_t}. \tag{5}$$

In order to be able to introduce explanatory variables the parameters μ_t and ϕ_t are modelled individually by

$$g_1(\mu_t) = x_t^T \beta, g_2(\phi_t) = z_t^T \gamma.$$

Here β and γ are regression coefficients, and x_t and z_t are regressors; g_1 and g_2 are link functions [11], where g_1 was chosen to be the logit function and g_2 to be the natural logarithm. Given the assumed density, the coefficients β and γ are estimated by maximum likelihood estimation.

When constructing the models for the hierarchy, a method of forward selection was used based on the Wald test with significance level $p = 0.05$. For modelling the mean μ_t, the initial model consisted only of the forecasted wind speed. The two following modelling steps then added the second and third power of the wind speed. The fourth power did not provide a significant improvement and was thus dropped. Next, the contribution of the wind direction was considered by adding the sine and cosine of the wind direction. However, only the cosine was significant, resulting in the following final model for the conditional mean:

$$\begin{aligned} g_1(\mu_{t+h|t}) = \beta_0 + \beta_1 \widehat{\mathrm{WSpd}}_{t+h|t} + \beta_2 \widehat{\mathrm{WSpd}}^2_{t+h|t} \\ + \beta_3 \widehat{\mathrm{WSpd}}^3_{t+h|t} + \beta_4 \cos(\widehat{\mathrm{WDir}}_{t+h|t}), \end{aligned}$$

where $\widehat{\mathrm{WSpd}}_{t+h|t}$ is the forecasted wind speed from the NWP available at time t for horizon h and $\widehat{\mathrm{WDir}}_{t+h|t}$ is similarly the forecasted wind direction. Since the forecasts are produced daily at midnight, t changes in steps of 24 hours, i.e., $t = 0, 24, 48, 72, \ldots$.

When modelling the precision ϕ, a very similar procedure was used. The wind speed terms were significant up to the second order, and only the sine of the wind direction was significant. Additionally, the prediction horizon of the weather forecast was added as a regressor. This was added to address the general tendency that the uncertainty is increasing with the prediction horizon. In the end, the precision is modelled as

$$\begin{aligned} g_2(\phi_{t+h|t}) = \gamma_0 + \gamma_1 \widehat{\mathrm{WSpd}}_{t+h|t} + \gamma_2 \widehat{\mathrm{WSpd}}^2_{t+h|t} \\ + \gamma_3 \sin(\widehat{\mathrm{WDir}}_{t+h|t}) + \gamma_4 h, \end{aligned} \tag{6}$$

where h is the prediction horizon of the weather forecast.

The power curve (Figure 4) resulting from fitting the model to the 2018 training set seems to fit well with what was expected from the data (Figure 3). However, it seems that the wind direction plays a very small role, as the four wind directions (N, E, S, W) are indistinguishable (Figure 4, top).

With forecasted weather variables, the uncertainty of the NWP is translated to power by the non-linear power curve. As nicely illustrated by [10], this explains the form of the standard deviation in Figure 4 (bottom), and this is also in line with what was seen by [12]. Furthermore, the standard deviation clearly depends on the prediction horizon.

Examining the residuals of the fitted models for the area on each resolution, the accuracy can be assessed by the *root-mean-square error* (RMSE), which is computed as

$$\mathrm{RMSE}_{\mathrm{area},l} = \sqrt{\sum_{t=1}^{N_l} \frac{(y_{l,t} - \hat{y}_{l,t})^2}{N_l}},$$

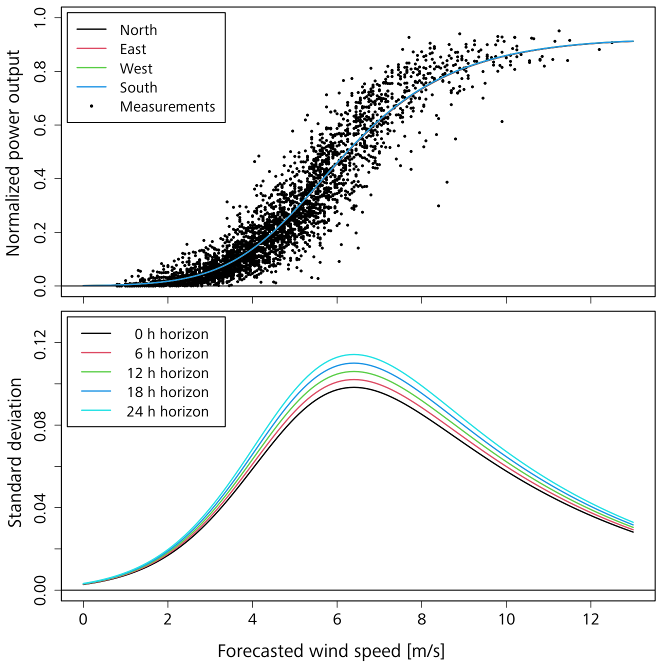

Figure 4. Top: Power curve resulting from fitting the beta regression model to the aggregated 2-hour data for area no. 3. Bottom: Standard deviation of fitted values from beta regression model on aggregated 2-hour data for area no. 3, given the wind direction is fixed in a northerly direction.

where y_t are the wind power measurements at time t, $\hat{y}_t$ are the predicted values at time t, and N_l is the amount of data points at resolution l, i.e. for the 2018 training set $N_{1\text{h}} = 8760$, $N_{2\text{h}} = 4380$ etc.

While the RMSE values as seen in Table 3 increase gradually with the temporal resolution, it is seen that relative to the rated power all forecasts perform well. It should, however, be noted that this is not a fair comparison between the 1-hour forecast and the other resolutions, as the forecasts on the 1-hour resolution were made out of sample in real time, while the forecasts on the other resolutions are in sample.

Model resolution	RMSE[MW]	% of rated power
1 hour	15.50	6.12
2 hours	33.39	6.59
3 hours	48.04	6.32
4 hours	62.26	6.14
6 hours	88.89	5.85
8 hours	112.58	5.55
12 hours	155.79	5.12
24 hours	265.12	4.36

Table 3. Accuracy of the different models for area no. 3 on the 2018 training data. 1-hour predictions were made by a state-of-the-art commercial model, while those for 2 hours and beyond are from the beta regressions.

4 Forecast reconciliation methodology

With the base forecasts that have been generated, the forecasts in the temporal hierarchy can be reconciled. With the MinT-Shrink method, this implies computing the hierarchical variance $\mathbf{\Lambda}$ and the correlation matrix $\mathbf{R}$, inserting them in the equations (3) and (4), and finally computing the reconciled forecast by equation (2).

This on its own is likely to improve the accuracy of the base forecasts, as seen in, e.g., [5] for wind power. However, we propose an alternative which better addresses some of the inherent challenges calling for a functional relationship between the uncertainty (conditional variance) and the predicted wind power generation (conditional mean).

One such challenge is that the boundedness of wind power production should be accounted for, since wind power is inherently bounded both from below at zero and from above at the rated power production. This means that the covariance will depend on the forecasted power production. This dependence is not accounted for in the MinT-Shrink method, where the covariance is assumed to be constant.

To address this challenge, we propose a method for linking the hierarchical variance $\mathbf{\Lambda}$ to the predicted power production.

In each time step t, $\mathbf{\Lambda}_t$ will be set based on a parametric estimate of the prediction uncertainty. In the case of the example hierarchy from Figure 1, at time t, $\mathbf{\Lambda}_t$ becomes

$$\mathbf{\Lambda}_t = \operatorname{diag}\left(\begin{bmatrix} (\hat{\sigma}^{24\text{h}}_{t+24\text{h}})^2 \\ (\hat{\sigma}^{12\text{h}}_{t+24\text{h}})^2 \\ (\hat{\sigma}^{12\text{h}}_{t+12\text{h}})^2 \\ (\hat{\sigma}^{6\text{h}}_{t+24\text{h}})^2 \\ (\hat{\sigma}^{6\text{h}}_{t+18\text{h}})^2 \\ (\hat{\sigma}^{6\text{h}}_{t+12\text{h}})^2 \\ (\hat{\sigma}^{6\text{h}}_{t+6\text{h}})^2 \end{bmatrix}\right).$$

This forecast-dependent hierarchical variance $\mathbf{\Lambda}_t$ then replaces the constant hierarchical variance $\mathbf{\Lambda}$. The shrinkage equations thus become

$$\mathbf{R}^{\text{Var}} = (1-\lambda^{\text{Var}})\mathbf{R} + \lambda^{\text{Var}}\mathbf{I}_n, \qquad \mathbf{\Sigma}^{\text{Var}}_t = \mathbf{\Lambda}_t^{\frac{1}{2}}\mathbf{R}^{\text{Var}}\mathbf{\Lambda}_t^{\frac{1}{2}},$$

which will be denoted MinT-Var.

This approach adapts the variance to the forecast, but not the correlation, which is still estimated from the response residuals. The next step is to allow the correlation structure to change with the forecast. One way of doing this is to compute the correlation from Pearson residuals instead, which are normalized by the forecast-dependent variance.

Additionally, it is well-known from the theory of generalized linear models that Pearson residuals will be closer to normal distributions than the original response residuals, see e.g. [11].

For a set of forecasts at time t, the Pearson residuals may be expressed as

$$\varepsilon_t^p = \mathbf{\Lambda}_t^{-1/2}(y_t - \hat{y}_t) \sim N(0, \mathbf{\Sigma}^p).$$

The second proposal is thus to use both the correlation of the Pearson residuals R^p and a forecast-dependent variance, i.e.

$$\mathbf{R}^{\mathrm{PVar}} = (1 - \lambda^{\mathrm{PVar}})\mathbf{R}^p + \lambda^{\mathrm{PVar}}\mathbf{I}_n, \qquad \mathbf{\Sigma}_t^{\mathrm{PVar}} = \mathbf{\Lambda}_t^{\frac{1}{2}}\mathbf{R}^{\mathrm{PVar}}\mathbf{\Lambda}_t^{\frac{1}{2}},$$

which will be denoted MinT-PVar.

These approaches lead to a flexible and more correct description of the uncertainty than the original method, while also addressing the challenges imposed when reconciling wind power forecasts. The performance of the methods will, however, depend on how well the prediction uncertainty is estimated. This then raises the question of how one should estimate the prediction uncertainty.

In our case, where the beta regressions are used to generate base forecasts, the variance of the predictions is given by equation (5). Conversely, the variance on the 1-hour resolution is not known, since the underlying model is not available. Therefore, for the 1-hour resolution the prediction uncertainty will have to be estimated by another method. Here, a well-known and effective way of doing this is through maximum likelihood estimation, which also allows us to describe how the conditional variance depends on the conditional mean.

Assuming that residuals are normally distributed, a likelihood function can be established for each prediction horizon h. For the set of k observations $\mathcal{Y}_{t+h} = \{y_{t-24k+h}, \dots, y_{t-24+h}\}$ and the corresponding set of forecasts $\hat{\mathcal{Y}}_{t+h} = \{\hat{y}_{t-24k+h}, \dots, \hat{y}_{t-24+h}\}$, the log-likelihood will be computed as

$$\ell(\hat{\sigma}_{t+h}^2; \hat{\mathcal{Y}}_{t+h}, \mathcal{Y}_{t+h}) = \sum_{i=1}^{k} -\frac{k}{2}\log(\hat{\sigma}_{t+h}^2) + \Big(-\frac{1}{2\hat{\sigma}_{t+h}^2}(y_{t-24i+h} - \hat{y}_{t-24i+h})^2\Big).$$

This requires a parametrization of the variances σ_t^2, for which inspiration can be drawn from the beta regression models for the precision as well as the relation between variance and precision given in equation (5). This results in the following parametrization of the variance:

$$\hat{\sigma}_{t+h}^2 = \exp(\alpha_{0,h}) + \frac{\hat{y}_{t+h}(1 - \hat{y}_{t+h})}{1 + \exp(\alpha_{1,h} + \alpha_{2,h}\widehat{\mathrm{WSpd}}_{t+h} + \alpha_{3,h}\sin(\widehat{\mathrm{WDir}}_{t+h}))}.$$

This parametrization represents a slight simplification of the precision model (equation (6)), as the second-order term is omitted. This simplification is justified by examining the *Akaike information criterion* (AIC) of the likelihood functions, with the full precision model and with the simplified model both fitted on the training data. The simpler model had a lower AIC, and hence it was chosen instead of the full precision model. Furthermore, this parametrization has an added constant term determined by $\alpha_{0,h}$. The addition of this term improved the robustness of the optimization and lowered the AIC significantly. Moreover, it is in line with the behaviour of the variance seen in Figure 4 not going fully to zero when there is no wind. Figure 5 shows contour plots of the estimated prediction uncertainty with parameters estimated in sample for area no. 3.

The uncertainty seems to be dependent linearly on the wind speed, while the uncertainty varies sinusoidally with the wind direction, topping at around 270°, corresponding to a westerly wind direction. This is in line with the behaviour observed by, e.g., [13]. The uncertainty related to power output quite clearly follows the polynomial relation from the variance model, with maximum at around $\hat{y}_{t+h} = 0.5$. For the prediction horizon (Figure 5, right column) the picture is less clear. The first few hours have quite low uncertainty, increasing gradually before spiking around the 10-hour horizon, then dropping down again for a few hours before again spiking around the 19–21-hour mark.

To find the coefficients α_h of the model, the log-likelihood is numerically optimized in each time step t for each horizon h using a window of past data. The resulting variance estimate can then be used in the reconciliation of the forecast.

Lastly, to estimate the shrinkage intensities, i.e., λ, λ^{Var}, λ^{PVar} we follow [14] and optimize them by doing a grid search, discretizing the regularization parameter on a grid of $\lambda = 0, 0.01, 0.02, \dots, 1$. The parameters are then chosen based on the greatest in-sample improvements to RMSE for each area. The resulting optimal parameter is then used for the out-of-sample tests.

5 Results

In this section, the out-of-sample performance of the three different methods for reconciling will be examined, those being the MinT-Shrink as proposed by [17], and the two proposed in this study, MinT-Var and MinT-PVar. For testing, a rolling window will move one day at a time. Each day, the coefficients of the base forecast models are re-estimated, and new base forecasts for the following day are produced. Then the variances of the base forecasts are estimated. In the case of the beta regressions, this is done directly using equation (5), and for the commercial forecasts on the 1-hour resolution, maximum likelihood estimation is performed using the windowed data to estimate coefficients. Then the base forecasts are reconciled using each of the three methods. Lastly, the covariances of the residuals are computed, so they can be used for the reconciliation of the following day's data.

To evaluate the forecasts for individual areas, the RRMSE% (*percentage relative root-mean-square error*) is examined as a way

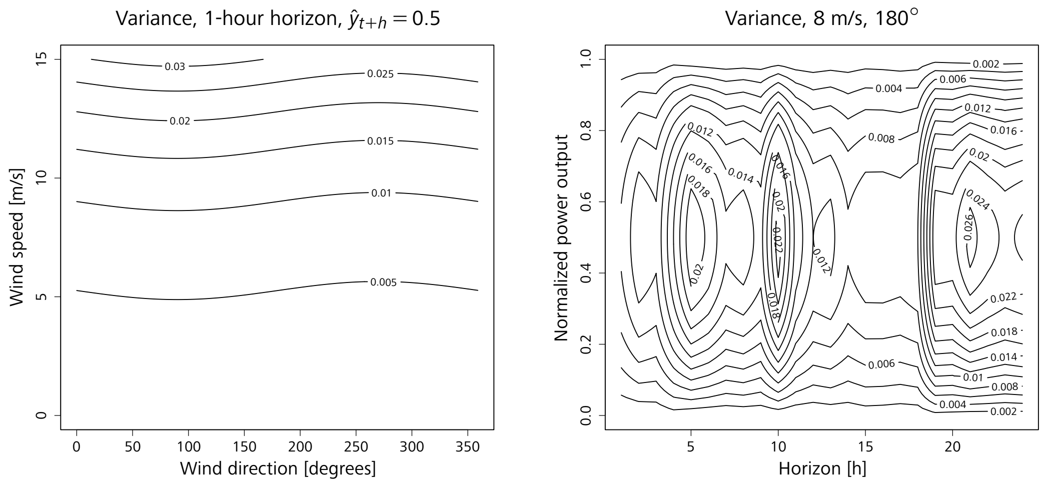

Figure 5. Left: Contour plot of variance for wind speed and direction; 1-hour prediction horizon and normalized power production of 0.5. Right: Contour plot of variance for prediction horizon and power output; 8 m/s wind speed and 180° wind direction.

to evaluate the performance and measure the accuracy of the proposed methods. For each area and for each resolution (l), the RRMSE% is given as

$$\text{RRMSE\%}_{\text{area},l} = 100\%\Big(\frac{\text{RRMSE}_{\text{area},l}}{\text{RRMSE}^{\text{base}}_{\text{area},l}} - 1\Big).$$

When evaluating the forecasts for the entire region as a whole, we use the total RRMSE%, since the areas have very different rated power productions. Hence for each resolution (l)

$$\text{RRMSE\%}^{\text{total}}_{l} = 100\%\Big(\frac{\sum_{\text{area}=1}^{15} \text{RRMSE}_{\text{area},l}}{\sum_{\text{area}=1}^{15} \text{RRMSE}^{\text{base}}_{\text{area},l}} - 1\Big).$$

Similarly, we will examine the performance for the entire region based on forecast horizon. We previously introduced the $\text{RMSE}_{\text{area},l}$ which sums over all prediction horizons. Therefore, we introduce the hourly RMSE as well, which based on the 12 months of results, for a given area, horizon h and resolution l, is given as

$$\text{RRMSE}_{\text{area},l,h} = \sqrt{\sum_{i=1}^{365} \frac{(y_{h-24i} - \hat{y}_{h-24i})^2}{365}}.$$

Again, summing across areas, we get

$$\text{RRMSE\%}^{\text{total}}_{l,h} = 100\%\Big(\frac{\sum_{\text{area}=1}^{15} \text{RRMSE}_{\text{area},l,h}}{\sum_{\text{area}=1}^{15} \text{RRMSE}^{\text{base}}_{\text{area},l,h}} - 1\Big).$$

For all of these scores, if the RMSE of the reconciled forecast is lower than the RMSE of the base forecast, the RRMSE% will be negative, indicating an increase in the accuracy of the forecast. Alternatively, if the accuracy is decreased, the RRMSE% will be positive.

To get a general picture of the performance of the three methods for reconciling wind power forecasts, we examine the $\text{RRMSE\%}^{\text{total}}_{l}$ for all the different temporal resolutions in Table 4.

Table 4 shows that forecasts on all temporal resolutions are improved, with greater improvements in accuracy for the higher temporal resolutions. As expected, for the 2–24-hour resolutions significant improvements in accuracy are observed when reconciled with the commercial state-of-the-art 1-hour forecasts. However, for the 1-hour forecast resolution, improvements are also seen. With the exception of the 24-hour resolution, the two proposed methods both outperform MinT-Shrink on all resolutions, with MinT-PVar showing the best improvements on all resolutions. The difference between MinT-Var and MinT-PVar is quite minor, which tells us that the forecast-dependent variance structure is the main source of the improvements. However, adapting the correlation structure also clearly gives an improvement.

Since the base forecasts on the 1-hour resolution are state-of-the-art and used commercially, improvements here are significantly more valuable for operational purposes than the other resolutions. Therefore, these improvements are examined in further detail in

Temporal resolution [h]	MinT-Shrink	MinT-Var	MinT-PVar
24	−19.34	−19.11	−19.43
12	−17.12	−17.43	−17.74
8	−14.96	−15.13	−15.38
6	−13.71	−13.84	−14.05
4	−13.73	−14.04	−14.26
3	−13.47	−13.85	−13.99
2	−12.87	−13.29	−13.46
1	−1.46	−1.95	−2.08

Table 4. $\text{RRMSE\%}^{\text{total}}_{l}$ of the three methods for each temporal resolution. Greatest reductions are highlighted in bold.

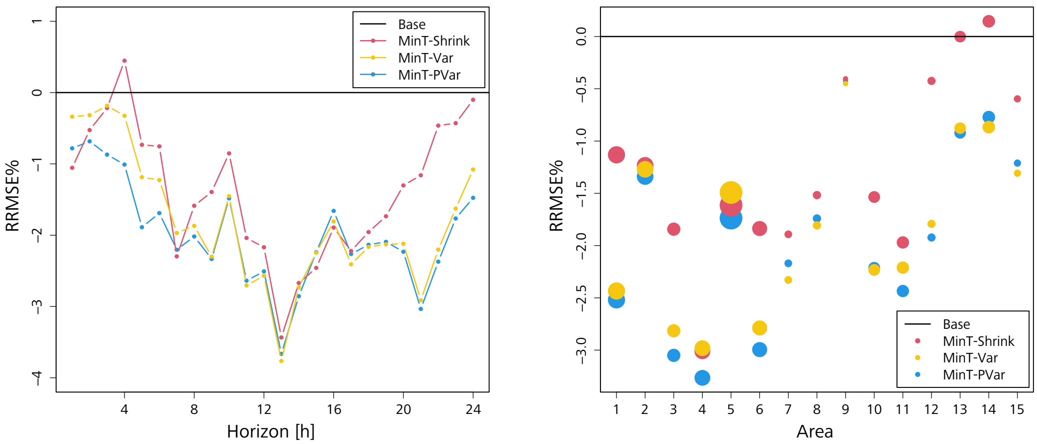

Figure 6. Left: RRMSE%$^{\text{total}}_{l,\,1\text{h}}$ for each of the three methods across the 24-hour prediction period. Right: RRMSE%$_{\text{area},\,1\text{h}}$ for each method on each of the 15 areas. The size of the points has been scaled relative to the rated power production of the area.

the rest of the section. Figure 6 shows how the improvements on the 1-hour resolution are distributed across the 24-hour prediction horizon and across the 15 areas.

The three methods that have been tested are able to improve the accuracy of the base forecast across almost all horizons. Only MinT-Shrink falls below the base forecast in accuracy, and only for a single horizon. There seems to be a pattern to the accuracy improvements for all methods, where the RRMSE drops until the 13-hour horizon and then rises until the end of the 24-hour cycle.

Looking at how the methods perform for each area, MinT-Var and MinT-PVar once again come out as the best methods. In all areas tested, either MinT-Var or MinT-PVar had the best performance. MinT-Var falls behind MinT-Shrink in areas no. 4 and 5, while MinT-PVar is consistently better than MinT-Shrink. This highlights the importance of adapting the correlation to the forecast, as for some areas only adapting the variance is clearly not enough.

Altogether the results show that a reconciliation based on temporal hierarchies can improve even already state-of-the-art commercial wind power forecasts. Although the average improvement of approximately 2% does not seem like much, such improvements are still significant for what are already high-quality forecasts. Further applying the variance estimation techniques proposed in this study results in additional improvements in accuracy, especially when combined with the use of Pearson residuals for correlation estimation.

6 Discussion and conclusion

The investigations performed in this study have shown that, even when using rather simple base forecasts for all the aggregated levels, commercial state-of-the-art base forecasts can be improved using forecast reconciliation based on temporal hierarchies. This finding is very promising, as integrating the method of forecast reconciliation into commercial forecasts can thus help meet the demand for increasingly accurate forecasts. The additional coherency property will also be advantageous for the TSOs, as the planning across multiple time horizons becomes simpler and more coherent.

As discussed throughout the study, reconciling forecasts of wind power production poses some challenges in terms of the non-constant variance. The variance will depend greatly on, e.g., the wind speed. Introducing a method which adapts the variance structure to the weather forecast thus helps in addressing this challenge. This more rigorous approach to reconciliation reflecting the conditional variance dependency on the prediction has resulted in better accuracy of the reconciled forecast.

For cases from other fields where data are close to Gaussian, it is more reasonable to assume that the covariance is constant or at least independent of the mean. Hence, we expect less difference between our proposed method and the MinT-Shrink in these cases.

There is, however, still much that can be done to build upon the results found here, and as such these results can be seen as a step towards an improved methodology for modelling the required covariance structure for reconciling forecasts in cases with non-constant variance. Furthermore, this can help build toward a more general understanding of the uncertainty in forecasts when using hierarchies.

Other authors have also developed methods for handling, e.g., bounded data [18]. It would therefore be interesting to examine how this performs compared to our proposed adjustments, or if these methods could be combined in a meaningful way. Due to the limits imposed by the data in this study, further testing on different data would also be beneficial to more confidently show which method is preferable when reconciling forecasts of wind power production.

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An atlas for all plane curves

Athanasios Sourmelidis and Jörn Steuding

This note deals with an application of Voronin's universality theorem for the Riemann zeta-function ζ. In particular, we show that every plane smooth curve appears, up to a small error, in the curve generated by the values $\zeta(\sigma + it)$ for real t, where $\sigma \in (1/2, 1)$ is fixed. In this sense, the values of the zeta-function on any such vertical line provide an atlas for plane curves.

1 Curves generated by the Riemann zeta-function

Curves appear naturally in life, perhaps not as ideal objects, as Euclid defined a line as "a length without breadth" in his *Elements*, but as orbits of planets, trajectories in physics and technology, or drawings in art. Taking into account their variety, it might be surprising that one can find them all realized in a single curve. Following Tolkien, we may state this also as a "Lord of the Curves" poem:

One Curve to rule them all,
One Curve to find them,
One Curve to bring them all,
And in the plane bind them.

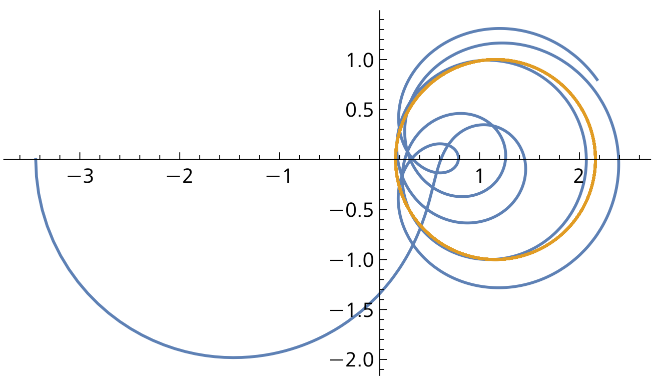

Figure 1. The values of $\zeta(3/4 + it)$ for $0 \le t \le 35$; one can already imagine an approximation of a shifted unit circle (in yellow).

Of course, our statement above needs to be clarified. Here and in the sequel we consider only *finite* and *smooth* curves on the plane, meaning that for each of them there exists a parametrization of the form

$$\gamma\colon [0,1] \to \mathbb{R}^2,\ t \mapsto \gamma(t) \tag{1}$$

such that γ has a non-vanishing first derivative and a continuous second derivative (see [5]); this includes line segments, ellipses, and many more curves that easily come to mind. The single curve that realizes all these smooth curves, however, is an artifact and has to be *infinite* for obvious reasons. In this respect, our theorem below has some implications to our understanding of infinity.

This infinite curve originates from the Riemann zeta-function, defined by

$$\zeta(s) = (2^{1-s} - 1)^{-1} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}, \tag{2}$$

where $s = \sigma + it$ with the imaginary unit $i = \sqrt{-1}$ (in the upper half-plane) is a complex variable with real part $\sigma > 0$. The complex-valued function $\zeta(s)$ plays a central role in analytic number theory and the distribution of prime numbers in particular (see [9]). For our result, we need to allow deviations by a quantity as small as desired. The mathematical language allows for a precise formulation:

Theorem 1.1. *Let $\sigma \in (1/2, 1)$ and $\varepsilon > 0$ be fixed. Then, every plane curve is, up to an error of order ε and affine translation, contained in the graph of the curve $\mathbb{R} \ni t \mapsto \zeta(\sigma + it) \in \mathbb{C}$.*

Here, of course, we regard any curve on the Euclidean plane, via $\mathbb{R}^2 \simeq \mathbb{C}$, also as a curve on the complex plane.

Proof. The proof relies on Voronin's celebrated universality theorem [12] from 1975 which states, roughly speaking, that certain shifts of the zeta-function approximate every zero-free analytic function, defined on a sufficiently small disk – a remarkable approximation property!

For our purpose, we recall the universality theorem [12] in a stronger form: *Suppose that $\mathcal{K}$ is a compact subset of the strip $1/2 < \operatorname{Re} s < 1$ with connected complement, and let $g(s)$*

be a non-vanishing continuous function on $\mathcal{K}$ which is analytic in the interior of $\mathcal{K}$. Then, for every $\varepsilon > 0$, the set of real $\tau > 0$ satisfying

$$\max_{s \in \mathcal{K}} |\zeta(s + i\tau) - g(s)| < \varepsilon \tag{3}$$

has positive lower density (see [11]). The main differences from Voronin's original statement in [12] are the positive lower density of the set of shifts τ (which is already implicit in Voronin's proof, but not in his formulation of the theorem) and the rather general set $\mathcal{K}$, where Voronin considered only disks; this is first apparent in Gonek's PhD thesis [6] and later in Bagchi's PhD thesis [2]. The topological restriction on $\mathcal{K}$ follows from Mergelyan's approximation theorem and its limitations (see [8] and [11, p. 107]). We will also make use of the following observation, due to Andersson [1]: *If $\mathcal{K}$ has empty interior, then the target function g in the universality theorem is allowed to have zeros*.

In order to describe curves, the concept of *curvature* of a smooth curve is essential. We omit the technical definition of this notion, and only mention that the curvature of a curve (with a suitable parametrization (1)) is a real-valued function that measures the deviation of the curve from a straight line. It is a well-known fact that a smooth plane curve is determined by its curvature; this follows from the fundamental theorem of the local theory of curves (see [5]). Let κ be the curvature of a parametrized plane curve (1) with respect to the arc length t (in order to have a unique representation). Define

$$\vartheta(u) = \int_0^u \kappa(t) \mathrm{d}t.$$

Then, a model of the curve with curvature κ on the complex plane $\mathbb{C}$ is given by the parametrization

$$t \mapsto \gamma(t) = \int_0^t \exp\left(i\vartheta(u)\right) \mathrm{d}u,$$

where t ranges through the interval $\mathcal{I} := [0, 1]$. By the universality theorem, more precisely Andersson's observation and (3) with $\mathcal{K} = \{\sigma + it \mid t \in [0, 1]\}$, for every $\varepsilon > 0$, there exists $\tau > 0$ such that

$$\max_{t \in \mathcal{I}} |\zeta(\sigma + it + i\tau) - \gamma(t)| < \varepsilon. \qquad \blacksquare$$

In view of the positive lower density for the set of real shifts $\tau > 0$ that lead to the desired approximation of the target function, it follows that any plane curve appears infinitely often, up to a tiny error bounded by ε, in any curve $\zeta(\sigma + i\mathbb{R})$ with any fixed $\sigma \in (1/2, 1)$ (even with positive lower density). In this sense, *the zeta-function provides a single plane curve that contains all the plane curves with an error too small to be seen with the naked eye!* Note that the Planck length is about $1.6 \cdot 10^{-36}$ meters and, according to quantum mechanics, one cannot *see* anything smaller than this tiny quantity.

Hence, the values of the zeta-function on any vertical line in the right open half of the critical strip provide an atlas for plane curves (where *atlas* should be understood as in geography, rather than as in the mathematical context of manifolds). We note that,

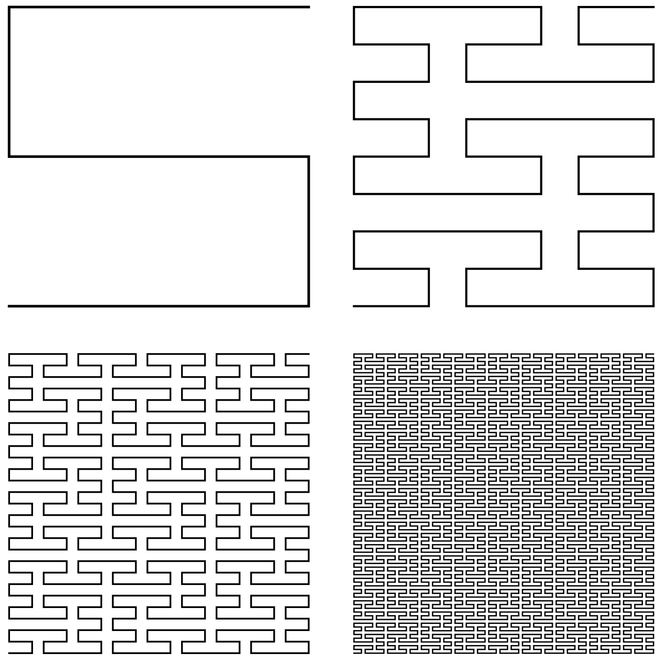

Figure 2. The first four iteration steps for the Peano curve.

in view of the universality theorem, the target function just needs to be continuous if $\mathcal{K}$ has empty interior. This even allows to approximate space-filling curves like the Peano curve, for which a continuous representation (1) exists (see [10] and Figure 2 for the Peano curve as the limit of an iteration). This Peano curve maps the unit interval $[0, 1]$ onto the unit square $[0, 1]^2$. On the contrary, the map $t \mapsto \zeta(\sigma + it)$ is differentiable and, therefore, if t is restricted to a bounded interval, the corresponding curve necessarily has finite length. That nevertheless the approximation of a space-filling curve is possible follows from the inaccuracy hidden behind the epsilon.

Is it possible to extend these results further? To answer this question we recall that, more than a century ago, Bohr (the mathematician Harald, younger brother of the physicist Niels) and Courant [3] proved that $\zeta(\sigma + i\mathbb{R})$ *is dense in $\mathbb{C}$ for every fixed $\sigma \in$* $(1/2, 1]$ (which means that one can find a value $\zeta(\sigma + it)$ for some real t in every neighbourhood of every point of the complex plane). Of course, this result also follows from universality (by choosing a constant target function). For the critical line, however, it is un-known whether $\zeta(1/2 + i\mathbb{R})$ is dense in the complex plane or not. Universality applies neither to the critical line (because of too many zeros of ζ), nor to any vertical line $\sigma + i\mathbb{R}$ with $\sigma > 1$ (because of the absolute convergence of the defining series (2)). These limitations also hold for the approximation of plane curves, which is obvious for $\sigma > 1$ (since then $|\zeta(\sigma + it)| \leq \zeta(\sigma)$); for $\sigma \leq 1/2$, however, this follows from a result of Gonek and Montgomery [7], who showed that the curvature of $t \mapsto \zeta(1/2 + it)$ is negative for $t \geq 3$ and something similar holds for $\sigma < 1/2$ as well. The latter result is conditional subject to the truth of the famous, yet unproven, Riemann Hypothesis that

$$\zeta(\sigma + it) \neq 0 \qquad \text{for} \quad \sigma > 1/2.$$

This open conjecture is one of the seven Millennium Problems in mathematics.

We conclude with a related problem in the universe of numbers. Does every *finite* pattern of digits appear in the *infinite* decimal fraction expansion of the circle number $\pi = 3.14159\,26535\,897\ldots$? There exist real numbers with this property, for example the Champernowne constant $0.12345\,67891\,011\ldots$ (built from the positive integers in ascending order) and it has been proven that *almost all* real numbers have this property (such numbers are called *normal*; see [4]); however, the case of special numbers is difficult and wide open in the case of π.

A more detailed account of our study with additional results in this context will appear elsewhere.

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Interview with Antonio Córdoba Barba and José Luis Fernández, founders of the Revista Matemática Iberoamericana

English translation by Antonio F. Costa González (UNED, Madrid) of the original article in Spanish "Entrevista a Antonio Córdoba Barba y José Luis Fernández, fundadores de la Revista Matemática Iberoamericana" published in volume 24, no. 3, 2021 in *La Gaceta de la RSME*

Isabel Fernández and Joaquim Ortega-Cerdà

Antonio Córdoba Barba and José Luis Fernández created in the 1980s the Revista Matemática Iberoamericana*, a generalist research journal which has become a publication of international prestige. The* Revista*, now part of the EMS Press, has been connected to the RSME (Real Sociedad Matemática Española) for more than a decade. After more than thirty years leading the journal, both have recently left the directorship of the journal.*

Isabel Fernández/Joaquim Ortega-Cerdà: *How came the idea of creating the* Revista Matemática Iberoamericana *(RMI)?*

Antonio Córdoba: At the end of the seventies Pedro Abellanas, director of the Jorge Juan Institute (CSIC, Consejo Superior de Investigaciones Científicas), asked me to prepare a project to relaunch the *Revista Matemática Hispanoamericana*, founded by Rey Pastor in the 1920s, which had been in decline.

My proposal to the Institute and the RSME was to create a journal with an independent editorial board of international character, formed by prestigious mathematicians, and with a budget that would allow to improve its layout.

The proposal was not accepted at that time, but was taken up again years later when José Elguero became President of the CSIC in 1983 and asked me to join his team of scientific advisors. The agenda of the first meeting of this Advisory Council included a decision on whether to maintain or eliminate the *Revista*. I had the opportunity to present my old plans and, to my surprise, not only Elguero, but also the rest of his advisors, offered me their institutional support to carry out the project. And when I was barely thirty years of age and with no publishing experience whatsoever, I found myself embarked on the ambitious task of creating ex novo a mathematical journal of international character and quality. What I did was to contact my mathematical friends (Charles Fefferman, Luis Caffarelli, Yves Meyer and Alberto P. Calderón, among others), who not only did not dissuade me from undertaking such an enormous project, but generously offered me their help, accepting to be part of the Editorial Committee, sending their own papers and asking for contributions. Without this help, RMI would not have been born.

In 1984 we had the first issue ready (in which there were contributions from, among others, two Fields Medal and two Wolf Prize winners). In the meantime, José Elguero had left the Presidency of the CSIC and the new team had some criticisms about the *Revista*, wanting to make changes in the editorial committee, which seemed dangerous to me, since I considered its independence to be very important. I was subject to serious pressure and, eventually, I was summoned to a meeting with both presidents, the president of the CSIC and the president of the RSME, which sounded like an ultimatum to me. But it so happened that at that time Yves Meyer and Charles Fefferman were visiting me, and so I went to the meeting accompanied by them, as well as by Miguel de Guzmán, and carrying with me the printed proofs of the first issue of the RMI. With that escort I only received congratulations from both presidents for the quality of the articles and there were no more inferences. I believe, in retrospect, that it was a key moment for the RMI project.

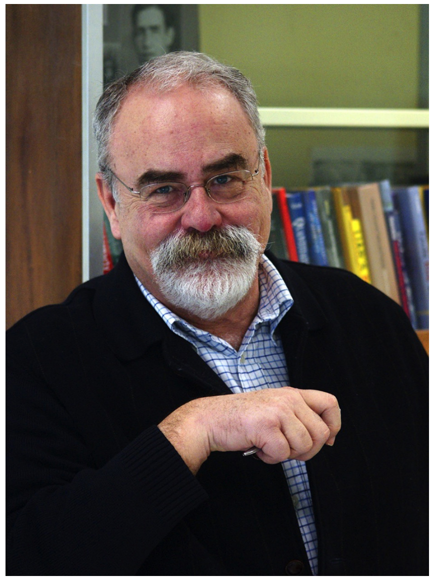
And, fortunately for me, in 1988 I made one of my best decisions in favor of the *Revista*: to invite Josechu for the co-direction.

IF/JOC: *How were the beginnings of the* Revista*?*

José Luis Fernández: Although the journal was formally linked to the CSIC and the RSME, it operated actually quite independently. This independence made it possible to develop the desired editorial line, but the institutional disengagement that it entailed made initial contacts with subscribers, distributors and even potential authors difficult.

Everything was very handmade: the layout was very tedious (the concept of "cut and paste" at that time was literal, with scissors and glue), and mailings took a long time. It was just Antonio and me, and the process was very time consuming.

AC: Perhaps somewhat optimistically, the project considered that, after a period of five years, the *Revista* should be financed by subscriptions (in fact, we achieved this earlier), but at the beginning we depended on CSIC funding and on the calls from the "Comisión Asesora de Ciencia y Tecnología" (public agency of the Spanish government) to which we applied for support. We had



Antonio Córdoba Barba (© Laura Moreno / ICMAT) and José Luis Fernández (© Pedro José Moreno).

problems with the funding received and, after a conflict with the RSME, Pedro Pascual put us in contact with the Ortega y Gasset Foundation, which helped us financially in the early stages. Despite these initial problems, I always believed that, if we managed to consolidate the journal by publishing issues of a high scientific level, the funding problems would be solved and we would not depend on the support of institutions. This objective was finally achieved.

The initial location of the journal was on Serrano Street (Madrid), in the premises of the Spanish Confederation of Mathematical and Statistical Research Centers (of CSIC), which for a brief period replaced the Jorge Juan Institute. Exchanges with other journals were important, because they broadened the visibility of the RMI. Initially, the Mathematics Library of the CSIC benefited from them, but it did not take long for them to express their lack of interest, so they passed to the Mathematics Department of the UAM (Universidad Autónoma de Madrid). The relationship with the EMS led to their drastic reduction and now, in fact, the RMI has practically no exchanges.

IF/JOC: *What was your motivation for creating the journal? What aspects were important to you about how the journal should be?*

JLF: Our desire was to have a journal of a good level in Spain, contributing to the accreditation of our country within the international mathematical panorama, although our initial infrastructure was very precarious. The work of the editors in this aspect was fundamental. Everything worked thanks to contacts. Thanks to Yves Meyer, for example, we obtained a large number of high-level papers from members of the French school in his area, such as Ingrid Daubechies, Guy David (currently editor of the RMI), or Jean-Lin Journé.

AC: Although I have always appreciated the international character of mathematics, I did, like other members of my generation, feel pained by José Echegaray's famous sentence that in mathematical science "there is no name that Spanish lips can pronounce without effort." I believe that having an international journal produced here is a good presentation card for the Spanish mathematical community and benefits us all.

An aspect to which we also gave importance from the beginning was the aesthetics: to have a good typography, good paper, an identity sign… In definitive, to make it attractive so that it would become desirable for a mathematician to have an article in the *Revista*.

IF/JOC: *Who were part of the initial team?*

AC: As mentioned above, the work of the editors was fundamental: Alberto Calderón, who scientifically supported the quality of the project, Charles Fefferman and Yves Meyer, who were two pillars of the editorial committee from the beginning (Fefferman is still with us today, Meyer left the journal seven years ago).

Another fundamental person for the project was Antonio Ros, who has been editor since the beginning and was in charge of covering the area of Geometry. Terry Lyons, who joined the journal later on, also played a decisive role in getting RMI indexed in the Journal Citation Reports (JCR), facilitating motu proprio, together with other members of the editorial board, to be indexed at no cost.

JLF: We also counted on the indispensable collaboration of several people who helped us in the administrative and infrastructure management of the *Revista*, such as Caroline Bintcliffe, Pablo Fernández (who designed the current format of RMI) and Daniel Ortega (UAM), Domingo Pestana (UC3M, Universidad Carlos III de Madrid), and Antonio Gil, an administrative staff member of our department who helped us outside his working hours, even two years after his retirement.

IF/JOC: *When did RMI become part of the EMS Publishing House?*

JLF: It was around 2012. We had previously received offers from Springer and Birkhäuser to publish the *Revista* while maintaining its structure, but we refused them because it was not in our spirit to manage the *Revista* through a commercial publisher.

When the EMS was setting up the Publishing House, Marta Sanz-Solé, president of the EMS at the time, suggested contacting RMI. Manfred Karbe was responsible for the editorial and was very helpful.

As an anecdote, I will tell you that at the first meeting there was a big discussion about the color: whether to keep the initial green, as we wanted, or go to the blue proposed by the EMS. This is the harshest meeting that the journal has ever had in its history (laughs).

AC: But we finally conceded to change the color to the European Community blue, and it was a good decision because in exchange we were able to keep the format of the *Revista*.

JLF: The move to the EMS Publishing House had the great advantage of institutionalizing the *Revista* (at that time, in addition, the relationship with the RSME was made official through its general editor Guillermo Curbera), and we were able to focus solely on editorial work.

IF/JOC: *Although RMI is a generalist journal, it has an important presence of topics related to Analysis; has this been a conscious or unconscious decision?*

AC: We always intended to be universal, but in the beginning we had to rely on our scientific contacts to get good articles, and that ends up imposing a tendency. However, from the beginning we tried to have all areas of mathematics represented, and today the journal publishes on a wide range of topics.

JLF: To some point it is a natural phenomenon that if in the beginning there was a higher proportion of articles on a certain topic, these have attracted other papers on related topics. But we always believed that, for the quality of the *Revista* (perhaps not so much for the impact indices), it was best to have a generalist journal.

IF/JOC: *RMI is currently an established journal in the impact* rankings*; what importance do you give to the impact index?*

JLF: From the beginning it was clear to us that being in the JCR was important for the *Revista*, and we have always aspired for it to be positioned as well as possible.

AC: One must target very high, because if you aim at the middle, you end up going down. If you lower the standards, the quality and prestige of the *Revista* will degenerate, which is so difficult to recover from at a later stage. Our initial strategy was to establish a canon by requiring relevant articles from renowned mathematicians.

JLF: We have always opted for quality over quantity. In particular, in the early days we had to take risks, and we published issues with very few pages.

IF/JOC: *What role do you think journals play right now, or will play in the future, in mathematics research?*

AC: The role of journals has changed a lot. They used to be efficient means of communicating knowledge and research, but they are no longer so efficient because this is now better done by the Internet through web pages and the *arXiv*. The main contribution of good journals is to certify that the published works have been evaluated, are correct and contain relevant results obtained with innovative methods.

This function is very important, and is associated with the quality of the journal and its editors. The work of these is crucial, and it is often difficult to obtain good reports, because their preparation involves considerable work, which is not remunerated. I believe that we mathematicians have to become aware of this task and collaborate generously in it, since it is the best method invented for knowledge to progress and to do so on a solid basis. And that is the role of journals in this era, rather than the transmission of knowledge, because nowadays, due to the *backlog*, articles are published when they have already been circulating for several years.

IF/JOC: *What do you think of the policies of* open access *and the proliferation of so-called "predatory journals"?*

JLF: I think there has been an absolute degeneration of the system. Nowadays it is very easy to set up a journal, and one comes across journals with seemingly serious names that in reality have no refereeing process.

I believe in the philosophy of *open access* and that research should be easily accessible and free of charge to authors. I trust that quality journals are like vinyl records, they never go out of fashion. The role of good quality journals is going to be maintained, quartile algorithms apart, because good articles call for others.

IF/JOC: *Both of you have just left the management of RMI after having been at the head of it since its beginnings. How do you see the present and the future of RMI?*

JLF: Currently, the *Revista* is in very good health. The editorial work is something that takes a lot of dedication, but, fortunately, now the editors can focus on that and not worry about economic and administrative tasks.

AC: I, of course, am very attached to the *Revista*, and I am pleased to see it now directed by mathematicians whose work is so relevant and who are much younger than those who created it. I am sure that they will succeed in maintaining and improving the level of the RMI.

IF/JOC: *We hope so! Thank you both very much. It has been a pleasure talking with you.*

AC/JLF: Many thanks to you.

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Interview with László Lovász

Raffaella Mulas

László Lovász is a Hungarian mathematician and a professor emeritus at the Eötvös Loránd University in Budapest. He was awarded the 1979 SIAM Pólya Prize, the 1982 and the 2012 Fulkerson Prize, the 1999 Wolf Prize, the 1999 Knuth Prize, the 2001 Gödel Prize, the 2006 John von Neumann Theory Prize, the 2007 János Bolyai Creative Prize, the 2008 Széchenyi Prize, the 2010 Kyoto Prize and, most remarkably, the 2021 Abel Prize, which many consider to be the mathematicians' equivalent of the Nobel Prize. He is the former president of the International Mathematical Union, and of the Hungarian Academy of Sciences, in addition to being one of the main collaborators of Paul Erdős.

Raffaella Mulas interviewed him in June 2023, while visiting the Alfréd Rényi Institute of Mathematics in Budapest.

László Lovász in Leipzig, in November 2022.
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Raffaella Mulas: *Thank you so much for taking the time to meet me. It is an incredible honor and a privilege for me to interview one of the mathematicians I admire the most. I would like to start from the beginning. As a teenager, you earned three gold medals at the International Mathematical Olympiad. You also won a Hungarian TV show in which students were placed in glass boxes and asked to solve mathematics problems. Is this true?*

László Lovász: Yes!

RM: *Is this when your passion for mathematics started, and what drove you into mathematics at such a young age?*

LL: Well, it started a little earlier, maybe in the 8th grade, when I joined the math club of my elementary school. I really enjoyed working on the problems that were posed there, and the teacher of the math club, who was also the director of the elementary school, recommended us to subscribe to a Hungarian journal of mathematics for high school students. The journal was established in 1893, and I think it's the oldest one in the world which is still functioning. And that was a great experience! Paul Erdős used to write for the journal as well. He liked to pose some open problems that were easy to formulate but difficult to solve, and he always presented them together with some historical remarks. This was really very inspiring!

RM: *So, you were still in elementary school when you first read something written by Erdős for this journal, right?*

LL: Yes, I think so! One of the first issues that I was looking at had an article of Erdős about combinatorial geometry. Anyway, the teacher of the math club also recommended that I apply for the Fazekas Mihály Gimnázium, a high school that was starting a specialized class for mathematics. The Fazekas Mihály Gimnázium then became quite famous precisely for its mathematics classes, but it also attracted other good students in other areas. For instance, while I was there, in a parallel class there was Éva Kondorosi: She is a biologist and one of the Chief Scientific Advisors of the European Commission. So, it is a very good high school, in general.

There I met several other young people who were recruited for the same class, which turned out to be an excellent community where to learn and do mathematics. The four years I spent there were really fantastic in my life! And since the mathematics class was newly established when I joined, mathematicians from both the university and this institute [Alfréd Rényi Institute of Mathematics] were very interested in what was happening there. They used to give some afternoon classes at our school, and some of us started to regularly visit some of the professors, from whom we got new theory to read or problems to think about. So, many of us started doing some research during high school.

RM: *Wow! Well, this all worked out very well! Let's jump ahead now: Your work spans many areas of mathematics and theoretical computer science. What has been the most exciting research project for you, so far, in your career?*

LL: Oh, I think this has changed over time. Looking back, probably my largest project has been graph limit theory, which started in the early 2000s. When I was at Microsoft, several of us started to work on it, including my wife Kati [Katalin Vesztergombi], Balázs Szegedy, Vera Sós – who passed away a few months ago, very sadly – Jennifer Chayes, Christian Borgs and Lex Schrijver, whom you might have met or will meet in Amsterdam. And others have played some role or contributed at various stages too. I think that this has been my largest project. I have always liked things which connect different areas of mathematics, and the name "graph limits" already indicates that there is a connection going on between graphs and limits.

But I also liked, mostly in the '70s and '80s, when the theory of computing was developing. To me, it was clear that this was a mathematical theory, and that was a very exciting period. I am not sure I contributed so much to that, but I was interested, and I wrote papers. What was exciting as well is that it led to some new graph theory problems. I worked on them also together with Tibor Gallai, who was my mentor during university. There was no official PhD supervisor at the time, but he put in a lot of time and energy to help me get ahead. I remember that he said, "Look at these two problems: the Hamilton cycle problem, and the matching problem. The matching problem has been solved in almost every possible sense, and the Hamilton cycle is very similar. Why is it so difficult, then?" And so, many of us started to think that maybe there was a reason for that. We started to think about it in terms of computational complexity, but we didn't get the right approach there. We then tried to work on Kolmogorov complexity, to see if there is any difference, but that also didn't work out.

Then, in 1972–73, I did what we would now call a postdoc. I went to the United States, to Vanderbilt University, for one year, while my friend Péter Gács, who was also interested in this topic, went to Moscow. There, he worked with Kolmogorov and with Leonid Levin, who was a student of Kolmogorov and who developed the P and NP theory essentially in the same way as (in the West), Stephen Cook and Richard Karp developed it. So, Péter Gács and I both spent a year abroad, and when we met again, we immediately told each other that we could finally see the difference between the matching problem and the Hamilton cycle problem. We were very enthusiastic! And after that, for two weeks we even thought that we could prove that P is not equal to NP. Our proof was nice, but in the end it wasn't right, as it proved something weaker. But anyway, we kept focusing on this area, and we organized a seminar here [at the Alfréd Rényi Institute of Mathematics] and we kept trying to see how computational complexity could be handled mathematically.

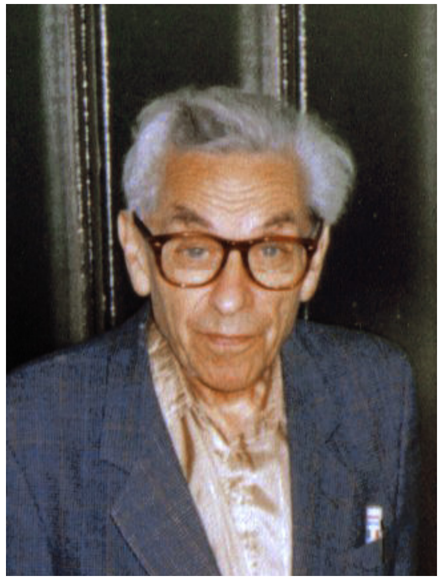
RM: *Amazing! What is your creative process? How do your mathematical ideas take shape?*

LL: Well, I mean, it's of course always a back and forth between trying to solve a problem and then trying to apply the ideas. Maybe one thing which I like probably a little bit more than most of my colleagues is to, sort of, clean up a proof. I don't like to write it down until I get the most essential part of it. And that sometimes is useful because it can lead to a better understanding of the situation. I'll give you one example: I was still in high school, and I was not satisfied with the fact that graph theory was sort of very elementary, and therefore looked down upon by many mathematicians. So, I thought that there should be some kind of algebraic side to it. I reinvented how to multiply two graphs with a new type of strong multiplication, and I thought, "Okay, it's easy to check that this product is commutative and associative; but do we have the cancellation law? Does $A \times C$ being isomorphic to $B \times C$ imply that A is isomorphic to B?" I began to think about it, and eventually, around the end of high school, I came up with a proof. But it was quite complicated, so I wasn't satisfied with it. And I still remember when I realized that, if I don't count subgraphs, but instead I count homomorphisms in the proof, then the claim follows immediately. So, this reinforced my idea that you have to understand what moves the proof, not only to come up with the proof. And I think that is something I like to do all the way. If I prove something, I try to understand what is the best way of looking at it.

RM: *This is great advice! Why is Budapest the hometown of most of the greatest combinatorialists and discrete mathematicians in history, including yourself? Is there something in the water here?*

LL: Haha! Well, there are various explanations. For one thing, which I think is very important, I have to go back a long time. So, Hungary had struck a deal with Austria in 1867 to obtain a certain degree of independence. There was a liberal government in Hungary, and they did many important things, and two of these are relevant. One is that they established general public education for everybody. The other one is that they gave equal rights to Jews. So, there was a large Jewish immigration to Hungary around the end of the 19th century and early 20th century, and the Jewish people sort of created a city life and a scientific life. I'm not saying that Hungary was unprepared: There were already first-class scientists in Hungary, including János Bolyai in mathematics. But anyway, all of a sudden, this mathematical life began to take place, and this is when, for example, this high school mathematics journal which I mentioned was established. The Hungarian National Mathematical Olympiad was also established around the same time, in the 1890s. So, many talented young people were discovered, and this gave an important push.

Now, why discrete mathematics and graph theory? It started with Dénes Kőnig, whose father was also a mathematician; but the Kőnig's Theorem in graph theory is called after the son. Some version of this theorem came out of Frobenius' study of determinants. There is a famous Perron–Frobenius theory about non-negative matrices, and Frobenius was interested in knowing, for a matrix whose entries are variables and some of them are zero, whether the determinant is an irreducible polynomial of the variables. Then, Kőnig wrote a paper where he basically showed that this is all just a combinatorial problem about seeing which variable goes where, and he used bipartite graphs to illustrate the arguments. What's interesting is that he didn't prove "the" Kőnig Theorem (which in this special case amounts to characterizing when the determinant is identically zero), but he just reformulated Frobenius' proof using graphs. Frobenius then wrote another paper in which he did not say very nice things about Kőnig, as he was very much against translating the problem into graph theory; although this is one example where you have to get rid of all the unnecessary signs, sums and everything, and it's all just about the perfect matchings. So anyway, Kőnig got interested in this and then he wrote a textbook in 1937, and he had at least two students, Paul Erdős and my advisor Tibor Gallai. And so, they moved the theory ahead, and many other Hungarian mathematicians got interested in graph theory as a result.



Paul Erdős in Budapest, in 1992.
Wikimedia Creative Commons Licence

RM: *Well, just in case, I will bring a tank of water from Budapest with me back to Amsterdam! Now, you have mentioned Paul Erdős several times already, and you have been one of his main collaborators: How would you describe him?*

LL: He was a very unusual person. Unlike the general picture often painted of him, he was very much concerned about other people. He knew about everybody, what they were doing, and he helped whenever anybody needed either a little money, some recommendation letter, or anything like that. But he didn't care so much for himself.

He couldn't visit Hungary during the Stalinist times, so it was only maybe near the end of the '50s, when he came back to Hungary for the first time after the war or after the Stalinist regime ended. I remember when I was young, maybe a young university student or maybe even a high school student, he was staying in a hotel when visiting Budapest. He was sitting all day in the lobby of the hotel, surrounded by young people, who were between 18 and 35 years old, or something like that, and he was sort of simultaneously working with several people on different problems. "Do you have any idea how to solve this? Oh, I have this additional question, maybe that's easier, or maybe that's also interesting." And sometimes, at lunchtime, he invited whoever was there for lunch at a restaurant. This was very inspiring, and I learned a number of things from him; not only mathematically, of course, but also on the human side.

He always thought that mathematics should be done publicly. He thought that, if you have an idea or have a new result, you shouldn't be afraid of sharing it with other people, because if they contribute to it or carry it on, then it will just be a better result – so you shouldn't try to keep it to yourself! He was always very unselfish, and on at least two occasions, when I was young, he gave me credit which wasn't unjustified, but it was maybe more than I deserved. The first case was when I first met Erdős, and he was already working with Lajos Pósa; you probably know the name. They had almost finished a paper. Pósa was a classmate of mine, and a good friend, and he asked me whether I could prove one of the results in the paper. So, I thought about it, and after a couple of days, I was able to prove their claim. Now, of course, if you know that something is true, then it's much easier to prove it. But anyway, Erdős added a footnote in their paper saying that this result was independently proved by László Lovász, which is not quite true.

The other case was (what is still called) the Lovász Local Lemma, which appeared in a joint paper of ours. Erdős emphasized that this particular lemma was mine, as he realized that it had broader implications compared to the rest of the paper, so he called the lemma after me. But it appeared in a joint paper, so, according to the standard rules, the lemma should be called Erdős–Lovász Lemma, if a name is needed at all. So, he was a very interesting person!

RM: *And from what you are describing, it seems like he was also very generous, both as a person and as a mathematician.*

LL: He was very generous, yes!

RM: *And how old were you when you first met him?*

LL: I was in high school, and I think my friend, Lajos Pósa, introduced us. I don't remember actually where it happened: probably during one visit of Erdős to my school. He used to visit the school about once a year to give a talk for the students there.

RM: *So, he was a very active mentor for young people as well.*

LL: Yes!

RM: *Do you have any other particular memory about Erdős that you would like to share?*

LL: There were several occasions when either we were in the US, or he visited Budapest and stayed with us for a week or two. This was often a little strenuous because he slept very little, and he liked to work for the rest of the day. But of course, we had teaching duties, and kids, and everyday life, and he understood that – but it was clear that he was rather impatient, and he wanted to sit down with us and work as much as possible. He was much older than we were, so it was really a little bit embarrassing to say, "Sorry, we are already tired!"

RM: *Does an Abel Prize laureate ever have difficulties?*

LL: Well, I'm teaching a class, and many more students come than before, because they want to see what an Abel Prize laureate looks like. And of course, just like everyone else, Abel Prize winners also suffer from bureaucratical difficulties: Do these count as difficulties?

RM: *Yes, definitely! And when proving theorems, do you still sometimes feel stuck?*

LL: Of course! In mathematics, 90 percent of the time you are on the wrong path. I mean, you can't see the end of the path, so you have to try! It's also important to be able to turn back.

RM: *You mentioned your wife Katalin Vesztergombi, to whom you dedicate all of your books, and if I'm not wrong you also have many children and grandchildren…*

LL: Yes, we have four children, if this counts as many, and we have seven grandchildren, so far. But our son just got married half a year ago, so we still hope to have even more!

RM: *Besides mathematics, your large graphs, and your (large) family, what makes you happy?*

LL: There is definitely nothing comparable to these! But I like to walk in the nature and just look around, and this is one thing which I try to do regularly with my wife.

RM: *This is nice! What are you looking forward to in the future?*

LL: Well, if you are 75, you have to realize that, mathematically, you cannot really expect to have a career change. But there are some areas that I'm interested in seeing if they lead anywhere, mostly in and around graph limit theory. I am interested in trying to develop limit theory for graphs that are neither very sparse nor very dense, but are sort of in the middle range. There are some papers, but it's still rather far from being able to call it a theory, or something. I've also been thinking about developing some limit theory for matroids, or at least to generalize some of this matroid theory to some kind of continuous rank function, in the spirit of John von Neumann's continuous geometry. So yeah, there are interesting questions in all areas, but at this moment, I am looking at these two areas seriously. Now I also have more time to do mathematics, and there are some very good people with whom I work here, so I really enjoy this.

RM: *Well, I'm looking forward to reading your future results! What advice would you give to young mathematicians?*

LL: During my university years, I have always found it very good to get interested in all areas of mathematics, and not only in my research area. I think that this proved to be very useful in my life, in my research. My advice is to not specialize too early, if possible. But I also understand that our current system is different now, and students have to specialize earlier. Part of this simply comes from the fact that the subjects are growing, so you inevitably end up in one branch which is already difficult enough to learn. For instance, when I was young, graph theory consisted of one or two books, and much of it, if you read it today, would be considered to be very elementary. And other areas are, of course, growing just as fast. So, it's difficult, but I think it's good still, to have some idea of what kind of goals other areas have. What is the main thing that they try to say? What kind of goals do they have? What's the advantage of looking at it in one way and not the other? There are big areas of mathematics where I have very, very little idea about what's going on, but I try to understand a little bit of that, nevertheless.

RM: *I like this advice. My last question is, why do you think it is important to keep studying graphs today?*

LL: Erdős had this idea that, if there are questions, you have to ask them. And, especially in graph theory, he was great at finding good questions – questions that lead to other questions. Eventually, this led to many branches of graph theory that were developed based on his questions and conjectures. Nowadays, the use of large graphs and large networks in various parts of other sciences seems to be inevitable. We saw one example of this with the pandemic: If we have to think about who meets whom, then understanding network properties is crucial for being able to say anything about the spread of a pandemic. Networks are also needed, of course, in brain research and ecological research, for example. So, with or without limits, the study of very large graphs and their properties, the problem of how to model and study them, is very important. And these are all very difficult questions, so I think that the study of large graphs is an exciting new area.

RM: *I completely agree with you, of course. Thank you so much for this very interesting and inspiring interview!*

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I would also like to thank Francesca Arici, Rianne de Heide (again), Conor Finn and my dad, Piergiorgio Mulas, for the support with this article.

Finally, I am very grateful to László Lovász for being so generous with his time, and for his detailed and inspiring answers.

László Lovász and Raffaella Mulas at the Max Planck Institute for Mathematics in the Sciences, in November 2022.
Picture taken by Jürgen Jost

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Words, history and mathematics

François Lê

This article presents some results obtained recently in the history of mathematics using the tools of textometry. The results pertain to two case-studies, devoted to Charles Hermite's style and to the theory of algebraic surfaces at the end of the 19th century, respectively.

It goes without saying that words of natural language have many essential roles in a published mathematical text. They give a title to this text and divide it into sections. They express acknowledgements to people and institutions. They make possible the reference to other texts. They signal the statement of a result and the beginning of its proof. They organise and articulate sequences of computations. Associated with mathematical symbols, diagrams and figures, of which they help fixing the meaning, they allow to express theorems, proofs, motivations, heuristic explanations.

Taking into consideration words of natural language is thus unavoidable if one wishes to understand a given text, be it in a mathematical or a historical perspective. In the latter case, looking at the references and the acknowledgements helps situating a mathematician of the past into various collective frames. Analysing this mathematician's technique may be a way to uncover and grasp phenomena at work at larger scales, such as disciplinary reconfigurations between, say, geometry and algebra. Reading introductions of papers often allows a better appreciation of his or her viewpoint on a topic. And focussing on isolated technical words may also be fruitful, inasmuch as their very use can be the trace of specific traditions: for instance, in the late 19th century, what we call nowadays the genus of algebraic curves was named *Geschlecht*, *genre*, *genere* by German, French and Italian mathematicians, but *deficiency* by the British, and this asymmetry was rooted in different original conceptions of the notion [7].

Investigating specific parts of a text, even reduced to a single word, may thus lead to interesting historical results. But what happens if one tries to deal with the set of all the words that compose a text or a group of texts?

During the past decades, researchers in the history of literature have developed and used computer-aided statistical techniques aimed at handling the whole word mass of large corpora, which allowed them to detect phenomena that were hard to capture with the naked eye: semantic and syntactic peculiarities of authors, assessment of lexical richness, detection of privileged associations between groups of words, thematic classification of texts, etc. Such an approach to textual corpora, where one relies on quantitative computations all the while looking closely at the texts to draw solid conclusions, comes under what is called "textometry," "lexicometry" or "statistical analysis of textual data."[1]

A few years ago, my curiosity led me to try applying the methods of textometry to gain a new view on corpora of old mathematical texts. My hope was that this would be a way to find new (kinds of) results in the history of mathematics or, at least, to confirm intuitions that had been formulated by myself or by others before. In what follows, I present a sample of the results that have been obtained so far. They pertain to two independent situations: the mathematician Charles Hermite on one hand, and the theory of algebraic surfaces in the journal *Mathematische Annalen* between 1869 and 1898 on the other hand.

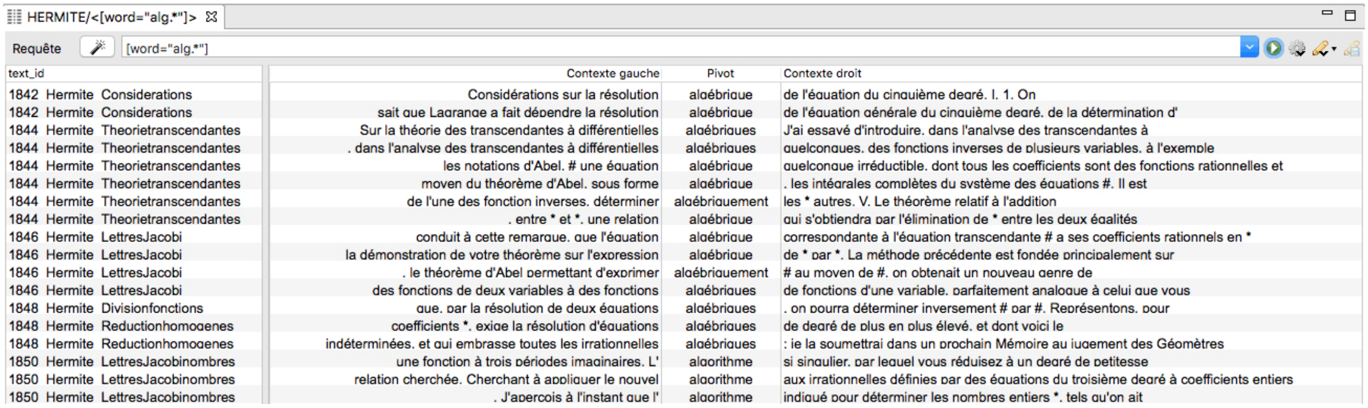
Before delving into these case-studies, let me briefly make some remarks on the general functioning of the chosen textometry software and on the associated terminology.

1 Counting words

The software that has been selected for my investigations is the open-source one TXM [5]. Given a corpus of texts formatted in an appropriate manner, this software begins by listing all the words that constitute these texts. It also attaches to each word a lemma, that is, the entry that would correspond to this word in a dictionary, as well as a grammatical label that indicates the part of speech corresponding to the word. For instance, the word "Theorems" would be associated with the lemma "theorem" and the grammatical label "common noun."

The corpus can then be interrogated according to various requests, the results of which are typically displayed as lists: list of the most frequent lemmas, list of the words containing a given chain

[1] See for instance [10] for an overview on the topic.



HERMITE/<[word="alg.*"]>

Requête [word="alg.*"]

text_id	Contexte gauche	Pivot	Contexte droit
1842 Hermite Considerations	Considérations sur la résolution	algébrique	de l'équation du cinquième degré. I. 1. On
1842 Hermite Considerations	sait que Lagrange a fait dépendre la résolution	algébrique	de l'équation générale du cinquième degré. de la détermination d'
1844 Hermite Theorietranscendantes	Sur la théorie des transcendantes à différentielles	algébriques	J'ai essayé d'introduire. dans l'analyse des transcendantes à
1844 Hermite Theorietranscendantes	. dans l'analyse des transcendantes à différentielles	algébriques	quelconques. des fonctions inverses de plusieurs variables. à l'exemple
1844 Hermite Theorietranscendantes	les notations d'Abel. # une équation	algébrique	quelconque irréductible. dont tous les coefficients sont des fonctions rationnelles et
1844 Hermite Theorietranscendantes	moyen du théorème d'Abel. sous forme	algébrique	. les intégrales complètes du système des équations #. Il est
1844 Hermite Theorietranscendantes	de l'une des fonction inverses. déterminer	algébriquement	les * autres. V. Le théorème relatif à l'addition
1844 Hermite Theorietranscendantes	. entre * et *. une relation	algébrique	qui s'obtiendra par l'élimination de * entre les deux égalités
1846 Hermite LettresJacobi	conduit à cette remarque. que l'équation	algébrique	correspondante à l'équation transcendante # a ses coefficients rationnels en *
1846 Hermite LettresJacobi	la démonstration de votre théorème sur l'expression	algébrique	de * par *. La méthode précédente est fondée principalement sur
1846 Hermite LettresJacobi	. le théorème d'Abel permettant d'exprimer	algébriquement	# au moyen de #. on obtenait un nouveau genre de
1846 Hermite LettresJacobi	des fonctions de deux variables à des fonctions	algébriques	de fonctions d'une variable. parfaitement analogue à celui que vous
1848 Hermite Divisionfonctions	que. par la résolution de deux équations	algébriques	. on pourra déterminer inversement # par #. Représentons. pour
1848 Hermite Reductionhomogenes	coefficients *. exige la résolution d'équations	algébriques	de degré de plus en plus élevé. et dont voici le
1848 Hermite Reductionhomogenes	indéterminées. et qui embrasse toutes les irrationnelles	algébriques	: je la soumettrai dans un prochain Mémoire au jugement des Géomètres
1850 Hermite LettresJacobinombres	une fonction à trois périodes imaginaires. L'	algorithme	si singulier. par lequel vous réduisez à un degré de petitesse
1850 Hermite LettresJacobinombres	relation cherchée. Cherchant à appliquer le nouvel	algorithme	aux irrationnelles définies par des équations du troisième degré à coefficients entiers
1850 Hermite LettresJacobinombres	. J'apercois à l'instant que l'	algorithme	indiqué pour déterminer les nombres entiers *. tels qu'on ait

Figure 1. A screenshot of TXM showing the beginning of the list of all the words beginning with "alg," placed within their textual neighbourhoods.

of characters, list of verbs to the present indicative, etc. It is also possible to get the lists of given words together with their close textual neighbourhoods, and to sort them according to several criteria (see Figure 1). A non-trivial issue for the researcher is then to determine which lists will be significant for a given historical purpose, and to make sense of them. The same remark holds for the other, more complex functions that are provided by the software TXM; I will explain their basic principles when I display their utility in my case-studies.

Before that, the problem of the mathematical formulas must be raised. For some technical and methodological reasons I have not succeeded yet to take formulas into account satisfactorily in the quantitative treatment. Hence a radical operation of razing these formulas has been done (manually) during the initial text formatting: each in-line formula has been replaced by a symbol $\ast$, and each displayed formula has been replaced by a symbol #. This operation thus only allows to keep track of the number of formulas in the statistical counts. That said, since the quantitative results must always be supported by a reading of the original texts, the content of the formulas is taken into account in the interpretative phase. Moreover, as will be seen, it turns out that the mere counting of the symbols $\ast$ and # provides some information on the texts.

Another comment must be made on the technical terminology coming from textometry. Without entering into details, a *word* can be a word of natural language, a punctuation mark, a number written with digits or any symbol such as § or $\ast$. The *frequency* of a word in a corpus designates its absolute number of occurrences. A word of frequency 1 is called a *hapax*. Consider for instance the sentence within the quotation marks: "The continuous function $\ast$ is a uniformly continuous function." It contains 10 words,[2] distributed into 6 hapaxes and 2 words of frequency 2.

Finally, one is often lead to compare several corpuses (or several parts of one corpus). A question, then, is to ascertain if some words are over- or under-represented in one corpus or the other, considering their size difference – the main difficulty is that making a linear adjustment of the numbers is not satisfactory enough because it does not take into account the actual word distribution in the corpus.[3] Based on computations associated with a hypergeometric model of word distribution, the notion of *specificity* allows to answer this question nicely, through the calculation of a *specificity score*. In the case of two corpuses, the over- (resp., under-) representation of a word corresponds to a positive (resp., negative) specificity score; the higher the latter, the stronger the over-representation.

2 Charles Hermite's style

Celebrated by his contemporaries as one of the most influential mathematicians of the 19th century, Charles Hermite (1822–1901) is still renowned nowadays for various mathematical achievements. Beyond the objects and theorems named after him, his 1873 proof of the transcendence of e is probably the most emblematic result that we owe to him. The year 2022 has marked the bicentenary of his birth, and has caused a number of new historical research projects on him.[4]

The question about Hermite that interested me was to describe his style, that is, to account for the impression that one gets when reading his mathematical writings. More precisely, while concentrating on his mathematical publications, the aim was to focus on the

[2] Do not forget the period!

[3] See [10, pp. 122–123] and the reference to Pierre Lafon's works given there.

[4] A forthcoming special issue of *Revue d'histoire des mathématiques* is devoted to this bicentenary. It will include the paper [9], which corresponds to the present section. More generally, for a rich and deep study of numerous aspects of Hermite's work, see the publications of Catherine Goldstein, such as [3, 4].

literary side of his writing, and not on what could be called his mathematical style. In other words, the idea was to ignore the facets of his works related to how given mathematical domains and objects intervened in his proofs, or how epistemic values shaped his practice, for instance. Instead, I wanted to bring to light the mechanisms through which his texts acquire a particular literary taste.

In fact, other historians had already hinted at such a question. In a booklet devoted to several works on irrationality and transcendence, Michel Serfati wrote:

> In the middle of the 18th century, Lambert […] clearly exposes his mathematical intuitions […] and resorts quite frequently to what could be called light metaphors and to active verbal forms, which are the grammatical consequences of an explicit "I." […] Thirty years later, Legendre makes passive forms predominant, forms which are characteristic of the contemporary style […]. In Hermite, a man with a modest personality, it is almost the modern style, synthetic, neutral in form and content, characteristic of the modern exclusion of the author in the mathematical text. [13]

Quite surprisingly, an opposite image of Hermite's style appears in a paper of Catherine Goldstein, in a passage where she comments on the possible youthful sources of Hermite:

> More difficult to pinpoint, but quite characteristic, the flavour of Hermite's mathematical prose itself reminds the reader strongly of these French authors [Lagrange, Legendre, Cauchy, Fourier]. The style is discursive and oriented towards the description of processes. [3]

As will be shown, my analysis tended to confirm (and enrich) Goldstein's assertions rather than Serfati's.[5]

In any case, the stylistic features evoked in these quotations are exactly of the kind that interested me, and that I wanted to quantify precisely using the tools of textometry. The following lines thus aim at showing that Hermite can be seen as a mathematical narrator whose presence in the written texts is made explicit through various markers, such as the use of the first person singular associated with verbs that describe the mathematical action in a lively way, and with the lexical field of the personal views.

To do so, the approach will be comparative. Indeed, and as it is apparent in the previous quotations, the assessment of the particularities of an author is always done (even if implicitly) with regards to a certain point of reference, consituted by another author or by more general norms of writings. In Hermite's case, I decided to

[5] It must be emphasised, though, that the two historians were looking at two distinct pieces of Hermite's publications, which possibly led to different appreciations. My own approach considers at once almost all the published articles of Hermite.

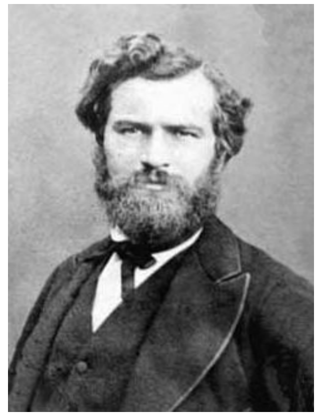



Figure 2. Charles Hermite (on the left) and Camille Jordan (on the right). Left portrait: © Mathematische Gesellschaft in Hamburg

confront him with Camille Jordan (1838–1922), a French mathematician separated from Hermite by about one generation, and who shared with him a number of research topics.

Since the objective is to investigate Hermite's prose in his mathematical publications, our corpus of reference is made of his research papers written in French, as they appear in his *Œuvres complètes*. This represents 186 papers published between 1842 and 1901, and a total of 364,412 words. Jordan's corpus is made in the exact same way and gathers 122 articles published between 1861 and 1920. Although it contains less texts than Hermite's corpus, it has more words, with a total of 591,732 words (see Table 1).

	Texts	Words	Hapaxes
Hermite	186	334,001	2,045
Jordan	122	529,766	2,014

Table 1. Numbers of texts, words and hapaxes in Hermite's and Jordan's corpuses.

2.1 *The personal comments*

One possibility to enter into this mass of words is to examine the hapaxes. These particular words are, indeed, a usual way to grasp the side of an author's style related to the notion of lexical richness: the more hapaxes a corpus contains, the richer it is. From such a numerical point of view, with 2,045 hapaxes for Hermite against 2,014 for Jordan, the former's corpus appears as lexically richer as the latter's, especially given the size difference between them. However, it is first and foremost on the semantic content of the hapaxes that I would like to expand here.[6]

[6] The numerical side of the problem is linked to the question of lexical diversity. This question is investigated in [9], together with that of lexical sophistication.

Let me first recall that, generally speaking, hapaxes fill up to 30%–45% of the number of distinct words of a (French) literary text, depending on the genre and the author, which might seem very high at first sight. In Hermite's case, the hapaxes represent about 32% of the distinct words. Many of them are completely usual words, as is exemplified by the hapaxes *essai* ("attempt"), *rencontrés* ("met," in a masculine plural form) and *Comparaison* ("Comparison").[7] Other hapaxes are technical terms whose unique use reflects marginal mathematical questions within Hermite's works, like in the case of *quadrique*.[8] A perfectly analogous phenomenon is to be seen in Jordan, for whom *torrents* and *cirques* are hapaxes involved in a paper related to questions of mountainous geography.

The other hapaxes reveal two important differences between our authors. The first one is linked to a structural feature of the Hermitian corpus: 70 of its 186 papers are extracts of letters that have been published in research journals at Hermite's time. On the contrary, there exists only one such paper in Jordan's corpus.

Now, a large part of the Hermitian hapaxes is clearly due to this epistolary format. They refer in particular to many variations on the theme of the delay of answer, the associated excuses, the acknowledgements and the sociable chat. Thus, in a letter written to Carl Gustav Jacob Jacobi in 1847 (and published in an 1850 issue of *Journal für die reine und angewandte Mathematik*), Hermite wrote:

> *Près* de deux années se sont *écoulées*, sans que j'aie encore *répondu* à la lettre pleine de bonté que vous m'avez fait l'honneur de m'écrire. *Aujourd'hui* je viens vous *supplier* de me pardonner ma longue *négligence* et vous exprimer toute la joie que j'ai *ressentie* en me *voyant* une place dans le recueil de vos Œuvres. Depuis longtemps éloigné du travail, j'ai été bien *touché* d'un tel témoignage de votre *bienveillance*; permettez-moi, Monsieur, de croire qu'elle ne m'*abandonnera* pas […].[9]

All these hapaxes, which clearly colour Hermite's texts in a characteristic way, have almost no counterpart on Jordan's side because of the quasi non-existence of published letters.

Nevertheless, the texts written by Hermite which are not extracts of letters are also rich in hapaxes. Many of these words have a meliorative function, and are used in passages where Hermite comments on diverse mathematical questions, objects, theorems or works. Examples of such hapaxes include *mystère, paradoxe, prestige*, *lumière* ("light"), *guide*, *inattendu* ("unexpected"), *magnifiques, stérile*, …, or those of the following quotation, where Hermite talks about a certain formula that had been earlier worked on by Leopold Kronecker:

> M. Kronecker, en la donnant comme l'expression analytique d'un de ses théorèmes, avait bien évidemment *pressenti* la signification qu'elle *recevrait* dans la théorie des fonctions elliptiques, et, à cet égard, je ne puis trop *admirer* la *pénétration* dont il a fait *preuve*.[10]

Such comments are much scarcer in Jordan's corpus. In fact, most of the hapaxes which are about the expression of personal viewpoints refer to a specific episode, namely a scientific quarrel with Kronecker that occurred in 1874 [1]. The hapaxes *contradicteur* ("detractor"), *excusable, objective, incontestable* ("indisputable"), *jugé* ("judged"), *complaisance* ("complacency"), … occur in the related publications.

From this point of view, the Hermitian prose appears as more personal than the Jordanian one. By writing to his colleagues and his friends, and by abounding with various personal comments on many issues, Hermite appears very explicitly within his mathematical texts.

Such a picture can also be drawn from the inspection of the common nouns, the adjectives and the adverbs that are specific to Hermite. For instance, *méthode, recherche, facile* ("easy"), *important*, *beau* ("beautiful") and *essentiel* are terms which are abnormally more used by Hermite; they clearly display the semantic field of the expression of the personal viewpoints on diverse aspects of the mathematical work. In this case, too, there are no equivalents of such words in Jordan's work, who thus appears as being more neutral, or less directly committed in his publications.[11]

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[7] Because of the capital C, *Comparaison* is not the same as *comparaison* from the viewpoint of word counting, even though the two words have the same lemma.

[8] Throughout this paper, all the French and German terms that are not translated are supposed to have a transparent meaning in English.

[9] "Almost two years have passed, and I have not yet responded to the letter, filled with kindness, that you made me the honor to write to myself. I am coming today to beg you to forgive my long negligence, and to express all the joy that I have felt by seeing for myself some room in the collection of your works. Having been away from work for a long time, I have been very touched by such a testimony of your benevolence. Allow me, Sir, to believe that it will not abandon me […]." The French words in *italics* are the hapaxes.

[10] "Mr. Kronecker, by giving it as the analytic expression of one of his theorems, had obviously foreseen the meaning that it would receive in the theory of elliptic functions, and, in this respect, I cannot but admire the insight he demonstrated."

[11] The examination of the terms that are specific to Jordan is interesting because many of them are related to the proof by contradiction, with *hypothèse, absurde, inadmissible, contraire*, and the French adverbs of negation *ne*, *pas*. A systematic counting of this kind of reasoning shows that it is used 599 times by Jordan, and only 35 times by Hermite…. This curious feature, however, is not related to the question of style as I wished to understand.

2.2 The mathematical action

To describe how Hermite conducts the mathematical narration, the grammatical categories of the verbs and the personal pronouns are now considered.

First of all, it is telling to look at the list of the verbs which are the most specific to Hermite: *ai, savoir, conduit, tire, faisant, donne, supposant, vais, conclut, trouve, obtient, obtenir, écrire, observe, a, été, remarque, employant, parvenir, trouvera*. For Jordan, the most specific verbs are: *sera, contient, formé, contiendra, pourra, contenu, aura, Soient, contenant, déplace, Supposons, forme, être, existe, serait, déplacent, seront, transforme, succéder*.

From the semantic point of view, Jordan's list contain a few technical verbs (such as *contient*, *contiendra*, *contenu*, *contenant*, which are inflections of the infinitive *contenir*: "to contain") as well as a certain number of stative verbs (such as *sera, serait, seront*, which come from the infinitive *être*: "to be"). On the contrary, one finds on Hermite's side a variety of verbs which evoke the description of processes, with the inflections of *conduire, tirer, trouver, observer, écrire* ("to lead," "to draw," "to find," "to observe," "to write") among others.

Furthermore, as the conjugated forms of these verbs let us guess, the grammatical subjects that are associated with them are not of the same kind: for Jordan, these subjects are very often the mathematical objects themselves – for instance, it is a group that "contains" an element – while the action described by Hermite's verbs is taken care of by a person, real or fictive – it is someone who "draws" a conclusion from a premise.

This asymmetry is confirmed by the inspection of the personal pronouns. Those that are over-represented in Hermite's corpus are related to the different forms of the first person singular *je* ("I"), the semi-impersonal *on* ("one") and the second person plural. The latter is due to the French *vouvoiement*, which appears exclusively in the published letters. It is however remarkable that the over-abundance of the *je* and the *on* holds even after removing these special publications from the comparative counting. Jordan, on his side, favours the employment of the personal pronouns related to *il* and *elle* ("he/it," "she"). The *il*, for instance, almost never refers to a person; its occurrences either designate mathematical objects or are used in impersonal, fixed phrases such as *il y a*, *il faut*, *il existe* ("there is," "one has to," "there exist(s)").

These are the differences that can be observed within the set of all personal pronouns. Considering now the overall numerical distribution of grammatical categories in each corpus, it turns out that Hermite's texts contain significantly more such pronouns than Jordan's ones. This echoes the fact that many verbs used by Hermite cannot have a mathematical object as their subject, and are often associated with the *je* or the *on*. On the contrary, in Jordan, many sentences have a mathematical object as their subject, even if this object is referred to by a single letter, as in: "Thus, G contains n substitutions." This kind of writing thus tends to diminish the number of personal pronouns used by Jordan.

Finally, the two lists of verbs given at the beginning of this subsection reflect the existence of an imbalance between the modes and the tenses in which the verbs are conjugated. Indeed, as is disclosed by a computation of the specificity scores of these modes and tenses, the simple future is clearly over-represented in Jordan's writings, while the present indicative, the infinitives and the participles proliferate on Hermite's side.[12]

The case of the future is particularly interesting. To a great extent, the simple futures used by Jordan are associated with the expression of mathematical facts, as in: *Ce système ne contiendra donc en général qu'une fraction des substitutions du système primitif*.[13] The future is also used by Hermite, but in a different way: the preference goes to combinations of a conjugated form of *aller* ("to go") and an infinitive: *Cette remarque faite, je vais étudier de plus près les quotients …*.[14] These two expressions of the (grammatical) future do not have exactly the same meanings and colours. In particular, the one used mostly by Hermite contributes to animate the mathematical narration with a kind of immediacy in the description of processes.

To finish this discussion on the verbs, let me remark that it is quite characteristic that *Supposons* ("Let us suppose") is specific to Jordan, whereas *supposant* ("supposing") is on Hermite's side: both verbs have obviously the same meaning, but the way they are conjugated implies different turns of phrases and different writing flavours.

Although other grammatical categories, such as the conjunctions or the demonstrative adjectives, could complete this picture, I will not elaborate on them for reasons of space. Still, it is quite amusing to take a look at the list of the beginnings of sentences (made of two words) which are specific to one author or the other (see Table 2). One finds in it several characteristics that echo what has been explained above: on Hermite's side, the involvement of the first person singular, the use of verbs of action, but also the employment of adverbs and other phrases which are yet other testimonies of the liveliness of his prose: *Cela étant*, *De là*, *Voici maintenant* ("This being said," "From this," "Here is now"), etc. More impersonal formulations are to be found in Jordan's list, many elements of which also point to sentences with mathematical objects as their subject.

As the absolute numbers in Table 2 recall, everything that has been stated about the specificities has to be first and foremost interpreted as relative results. In particular, there is no question of asserting that Hermite never uses the simple future, or that he never writes sentences of which the subject is a mathematical object, etc.

[12] Jordan's corpus is also abnormally rich in conditional, subjunctive and, to a lesser extent, imperfect indicative. This imbalance is to be linked to the over-use of proofs by contradiction that we alluded to above.

[13] "In general, this system will thus contain only a fraction of the substitutions of the primitive system."

[14] "This remark being made, I am going to study closer the quotients …."

Sentence beginning	Freq. H.	Freq. J.	Spec. score
Cela étant	195	3	93.5
C'est	234	55	68.0
Or,	283	36	61.0
De là	80	8	31.0
Je me	52	4	21.4
J'observe	39	0	19.7
Effectivement,	43	42	19.1
Je remarque	39	1	18.3
J'ai	53	12	16.1
Ainsi,	62	20	15.8
On trouve	40	5	14.9
Maintenant,	32	1	14.8
Voici maintenant	29	0	14.7
…	…	…	…
D'autre	3	180	−24.9
Soit ∗	68	512	−28.1
Soient ∗	42	409	−29.4
En effet	74	547	−29.4
D'ailleurs	29	385	−33.0
Les substitutions	5	285	−38.9
On aura	33	533	−50.4
Le groupe	2	332	−50.6
Donc ∗	4	417	−61.2
Si ∗	11	813	−115.2

Table 2. The specific beginnings of sentences with their absolute frequencies in each corpus. Negative specificity scores mean that the corresponding phrases are under-represented in Hermite, thus over-represented in Jordan.

The specificities tell us that some features are significantly not equally distributed within the two corpuses.

2.3 Hermite, Jordan … and the others?

A natural question that arises, now, is to determine whether the characteristics of Hermite's style that have been sketched in this section only appeared because the opposite figure was Jordan. Among others, the relative depersonalisation of Jordan's prose might be linked to the fact that he was born 16 years after Hermite. Hence it would be illuminating to confront Hermite with a mathematician who would have been educated and who wrote papers roughly during the same period as Hermite did.

Ideally, this new mathematician should also have devoted approximately the same number of works to the same mathematical domains as Hermite did. Indeed, as my second case-study shows, mathematical domains are not indifferent to the matters of specificities of words and grammatical categories.

3 The theory of algebraic surfaces

Turning now to the theory of algebraic surfaces, the corpus of reference is made of all the papers dealing with this subject, written in German and published in *Mathematische Annalen* between 1869 and 1898.[15] This represents 75 papers and 632,926 words. A notable difference with the Hermite–Jordan case is that the corpus is a collective one: 26 authors are to be counted, the most prolific of whom are Alfred Clebsch (1833–1872) and Rudolf Sturm (1841–1919), with 6 and 7 papers, respectively.

When tackling this corpus on algebraic surfaces, the issue was not to deal with the notion of style. My intention was to see what original information on the corpus could be brought by textometric techniques, by investigating the words of natural language, the punctuation marks and the symbols ∗ and # standing for mathematical formulas.

To get a first view on the corpus, let us consider the common nouns and the proper nouns that appear the most (see Table 3).

Common nouns	Freq.
Fläche	8,041
Punkt	7,876
Kurve	4,934
Ordnung	4,698
Gerade	4,458
Ebene	4,020
Gleichung	2,544
Knotenpunkt	1,398
Kegel	1,308
Schar	1,176
Grad	1,163
Doppelpunkt	113
Abbildung	990
Zahl	971

Proper nouns	Freq.
Clebsch	114
Cayley	66
Cremona	60
Salmon	57
Zeuthen	45
Schläfli	49
Sturm	41
Fiedler	41
Crelle	36
Kummer	27
Lie	26
Borchardt	25
Schubert	24
Noether	21

Table 3. The most frequent lemmas of common nouns and proper nouns in the corpus on algebraic surfaces.

The common nouns are not very surprising: they designate the main objects of the research on algebraic surfaces at the end of the 19th century, and associated objects and notions. For instance, *Fläche*, *Punkt*, *Kurve*, *Gerade* mean "surface," "point," "curve" and "line," while *Ordnung* and *Gleichung* mean "order" (a synonym of degree) and "equation." Two nouns at the end of the list hint at particular topics that we will encounter again later: *Abbildung*, which can be translated as "representation," refers to the question of representing a surface on another one (typically, the plane), which

[15] The results of the present section come from [8].

can be interpreted nowadays as establishing a birational transformation between these surfaces. The word *Zahl* ("number") recalls that many enumerative questions are dealt with in the corpus.

As for the proper nouns, let me just note that the trio composed of Alfred Clebsch, Arthur Cayley and Luigi Cremona dominate the list – they form what Wilhelm Fiedler called "the capital C, which is now […] marching at the head of mathematical Europe in the field of analytic geometry."[16] The presence of George Salmon's name among the most used ones reflects the fact that his famous books on analytic geometry, including the *Treatise on the Analytic Geometry of Three Dimensions* (four editions in 1862, 1865, 1874, 1882), are widely cited at the time.

It would be possible to deepen such little investigations. Rather, I would like to present two other pictures of the corpus, corresponding to two ways of looking at it. The first one consists in confronting the corpus to another one, in the image of what has been done for Hermite. The second one uses techniques of lexical classification of the texts of the corpus.

Surfaces		Invariants	
Co-occurrent	Spec.	Co-occurrent	Spec.
Gleichung	74	Wurzel	46
Form	53	determinierend	36
Wurzel	28	genügen	34
setzen	27	Lösung	24
stellen	24	befriedigen	21
Elimination	23	Auflösung	15
eliminieren	20	Seite	13
Faktor	19	Elimination	11
befriedigen	18	ergeben	9
erhalten	17	links	9
Koordinate	17	bestehen	8
homogen	15	fünft	8
genügen	15	rechts	8

Table 4. The most specific lemmas of the content words which are co-occurrents to the lemma *Gleichung*.

3.1 Surfaces vs. invariants

The corpus of comparison that has been chosen is made of all the papers dealing with invariant theory,[17] written in German and published in *Mathematische Annalen* during the same period as above, between 1869 and 1898. This algebraic corpus gathers 105 articles and 460,327 words. The authors are 39 in number, among whom 8 also contribute to the theory of surfaces.

A function provided by the software TXM computes the *co-occurrents* of given terms, that is, words which appear significantly more than others in all the neighbourhoods of the given ones. For instance, let us consider the co-occurrents of the words having *Gleichung* ("equation") as their lemma. The first results are given in Table 4 where, for the sake of clarity, articles, propositions and other function words have been excluded. The specificity scores that can be seen in the table measure how characteristic the co-occurrence is.

Some of the co-occurrents are in the two lists, such as *Wurzel* ("root"), *befriedigen* ("to satisfy") and *Elimination*. They are part of the standard vocabulary associated with algebraic equations, and their attraction to *Gleichung* is quite natural.

Among the co-occurrents which are proper to one corpus or the other (or have highly different specificity scores), some recall themes or objects that are characteristic of each mathematical domain. It is the case for *homogen* and *Koordinate* for the geometry, and for *Lösung*, *Auflösung* and *fünft* ("solution," "resolution," "fifth") for the invariants, which echoes the publications on invariant theory dealing with the theory of algebraic equations, and especially that of the fifth degree.

The words *Seite*, *links* and *rechts* ("side," "left" and "right"), which appear only on the invariant table, hint at other aspects of the mathematical work. First, an inspection of these terms within the texts that contain them proves that in the corpus of invariant theory, they almost exclusively refer to the (left or right) side of an equation. Their relative absence in the corpus of algebraic surfaces seems to be tied to a particular kind of mathematical practice: in invariant theory, the pieces and sides of equations are frequently observed, transformed and then re-injected in other equations or equated to zero in order to carry on with a proof. They are also more often the objects of some descriptive comments of the mathematicians who study them. On the contrary, such ways of doings are much scarcer in the corpus of algebraic surfaces.

That the equations are more at the core of the mathematical work in invariant theory can also be seen by studying the specific verbs. As Table 5 shows, almost all the verbs that are specific to the corpus of algebraic surfaces are what I call verbs of geometric action, i.e., verbs of which the subjects or the complements are mathematical objects such as points, curves or surfaces, and which describe the behaviour of such objects: *schneiden*, *treffen*, *liegen*, *berühren* ("to cut," "to meet," "to lie," "to touch") are some of them. On the side of invariant theory, the specific verbs refer to another kind of action: for instance, *ersetzen, verschwinden, ausdrücken, berechnen*, ("to replace," "to vanish," "to express," "to compute") clearly refer to the lexical field of the algebraic operations.

[16] Letter from Fiedler to Cremona, dated March 1867. See [6].

[17] In the 19th century, *invariants* are objects associated with n-ary forms. For instance, the discriminant $\Delta = b^2 - ac$ is an invariant of the quadratic form $f = ax^2 + 2bxy + cy^2$: if (x, y) is transformed into (x', y') by an invertible linear transformation of determinant r, and if one writes $f = a'x'^2 + 2b'x'y' + c'y'^2$, then $b'^2 - a'c' = r^k(b^2 - ac)$ for an appropriate integer k.

Surfaces		Invariants	
Verbs	Spec.	Verbs	Spec.
schneiden	273,5	ersetzen	60,7
treffen	179,3	verschwinden	53,9
liegen	176,8	ausdrücken	52,6
berühren	171,0	multiplizieren	48,2
gehen	115,2	setzen	34,0
entsprechen	97,4	bezeichnen	30,5
abbilden	64,4	bedeuten	28,5
legen	29,5	berechnen	22,5
begegnen	28,6	auslassen	20,7
hindurchgehen	26,4	folgen	20,4
zerfallen	25,0	entstehen	20,3

Table 5. The most specific lemmas of verbs.

This picture is made more complete by examining the common nouns that are specific to the corpuses. Contrary to the case of algebraic surfaces, the corpus of invariants contains, apart from specific nouns referring to the objects proper to the domain, several ones such as *Faktor*, *Formel*, *Ausdruck* ("expression"), *Identität* and *Operation*, which are part of the lexicon of the algebraic computations. Moreover, the term *Gestalt* ("shape") is also among the nouns that are specific to the corpus of invariant theory, which underscores the fact that observing the aspect of equations and formulas is an important facet of the research of the time.

To finish with the comparison between surfaces and invariants, let me briefly mention that considerable differences exist in the distribution of grammatical categories among the two corpuses. Indeed, the corpus of algebraic surfaces is marked by a very clear over-abundance of common nouns and articles, as well as substituting relative pronouns, indefinite attributive pronouns and commas; in invariant theory, symbols standing for mathematical formulas are legion, while adverbial relative pronouns and periods are also over-represented.

Such imbalances highlight two modes of writing, which are quite different from one another. The corpus of algebraic surfaces contains markedly more sentences whose subjects are the geometric objects themselves, and are often designated by a word of natural language. These sentences are wide and rich of relative propositions, as testifies the over-abundance of commas and of substituting and attributive relative pronouns.[18] As for invariant theory, sentences are shorter, turned to displayed mathematical formulas and their manipulations: the over-representation of adverbial relative pronouns is explained by a huge number of *wo* ("where"), used in sentences such as: "This leads to: #, where ∗ designates the given invariant."

[18] In German, relative clauses that follow a main clause are preceded by a comma.

These characteristics emerged from a global comparison between the two corpuses. What is striking is that similar differences of writing mathematics can also be found, yet at a more restricted scale, inside the corpus of algebraic surfaces.

3.2 Lexical classes

Another way to investigate the corpus of algebraic surfaces from the textometric point of view, indeed, is to explore the possibilities offered by the functions of lexical classification. Without going into technical details, the idea is just to group into classes the texts that have a similar lexical profile, for instance when taking into account all the common nouns, proper nouns, verbs, adjectives, adverbs and symbols ∗, # that compose them.

The software TXM thus partitions our main corpus into six classes. The aim, then, is to understand if and how such a partition is relevant and significant from a historical perspective. One option is to try to interpret and characterise these classes by combining several viewpoints. Here, only three will be evoked: that of the mere topics of the lexical classes, that of the confrontation with the network analysis of the corpus, and, quite briefly, that of the grammatical imbalances between the classes.

It turns out that the lexical classes coincide to a considerable extent with as so many citation clusters, in the sense that the texts composing each class cite each other a great deal and share some common references, but rarely cite the texts of the other classes (or do so to deprecate them). Such a superposition of the lexical classes and the citation clusters is quite remarkable, considering that each classification is made on the basis of very different criteria: the vocabulary of texts on one hand, the links of citation on the other one. Somehow, this proves that the belonging of a paper to some research dynamics determines its vocabulary, and conversely.

As for the thematics, the six classes can roughly be described as follows – their numbering follows their size, in terms of text numbers. The (very marginal) first one consists of a few papers of Sophus Lie on minimal algebraic surfaces, a topic at the boundary with differential geometry. The second class gathers papers which are devoted to what was called "line geometry" at the time, that is, an approach of space geometry where the basic element is the line (and not the point or, in the dual view, the plane). The third class studies many special surfaces, by tackling a variety of issues such as their singularities, their shape and the making of models of them, typically in plaster (see figure 3).

The fourth class is that of enumerative geometry, the main questions consisting in counting geometrical objects that satisfy given conditions. The fifth one congregates papers falling under what is called the "new synthetic geometry"; in a word, this phrase refers to the avoidance of any recourse to projective coordinates to study algebraic surfaces, and to the need to conceive the latter with "purely geometrical" procedures. The sixth and last class is all about the topic of surface representation, where one tries to find

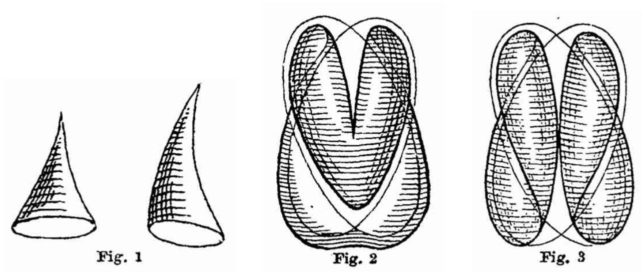

Figure 3. These drawings come from [11], which belongs to the third class. They display the shape of a surface in the neighbourhood of particular singular points.

birational correspondences between a given surface and another, and to deduce from this many results on the first surface.

These themes appear clearly in the lists of the words which are specific to each class. As hinted to above, perhaps more surprisingly, the classes happen to possess grammatical specificities as well. For the sake of brevity, I will focus on the fifth and sixth classes, and provide only a few examples of such specificities.

The fifth class, on the new synthetic geometry, is characterised by a colossal over-representation of in-line mathematical formulas, and a more relative under-representation of common nouns and displayed formulas. This is a trace of a way of writing that is very characteristic of these texts, where very long paragraphs without any displayed formula come after one another interminably, the names of geometrical objects being often followed, or even replaced, by a mathematical symbol (see Figure 4).

Haben wir nun umgekehrt auf einer G^2 drei Gerade g_1, g_2, g_3 und auf einer C^2 drei Gerade c_1, c_2, c_3 so, dass g_1 die c_2 und c_3, g_2 die c_3 und c_1, g_3 die c_1 und c_2 schneidet, so giebt es eine Fläche 2. Grades F^2, in Bezug auf welche g_1 und c_1, g_2 und c_2, g_3 und c_3 reciproke Polaren sind. Ordnen wir nämlich den Punkten (g_1, c_2), (g_1, c_3) die Ebenen $[c_1, g_2]$, $[c_1, g_3]$ und den Punkten (c_1, g_2), (c_1, g_3) die Ebenen $[g_1, c_2]$, $[g_1, c_3]$ als Polaren zu, so bilden alle dadurch bestimmten Flächen 2. Grades ein Büschel (oder eine Schaar) durch ein windschiefes Vierseit, dessen Ecken und Seitenflächen die Doppelelemente der durch jene Zuordnung auf g_1 und c_1 bestimmten Involutionen sind. Dem Punkte (g_2, c_3) entsprechen daher in Bezug auf

Figure 4. Extract of a paper by Friedrich Schur [12], belonging to the fifth class.

As for the sixth class, I will just note that it contains an abnormally high number of common nouns, articles and relative pronouns, compared with the other classes. In other words, although at a smaller scale, the same phenomenon that had been seen in the comparison between algebraic surfaces and invariants is observed here: the class is distinguished by an over-representation

Verbindet man je zwei aus demselben Punkte der Doppelcurve entspringende Punkte ihrer Abbildung durch eine Gerade, so umhüllen diese eine Curve, deren Tangenten den Punkten der Doppelcurve eindeutig entsprechen, und deren Geschlecht daher dem der Doppelcurve selbst gleich ist.

Man sieht daraus, dass die Abbildung der Doppelcurve ausser den oben angeführten Doppelpunkten noch gewisse Specialitäten besitzt, welche sie von anderen Curven mit gleichem Grade und Geschlecht unterscheiden. Sind nämlich die Coordinaten eines Punktes der Doppelcurve durch Ausdrücke der Form

$$\varrho x_i = \varphi_i(\lambda, \mu)$$

gegeben, wo zwischen den Parametern λ, μ eine algebraische Gleichung

$$\psi(\lambda, \mu) = 0$$

stattfindet, so müssen die Coordinaten einer Geraden in der Abbildung, welche die beiden einem Punkte der Doppelcurve entsprechenden Punkte verbindet, sich ähnlich darstellen; die Coordinaten eines Punktes der Abbildung der Doppelcurve müssen also die Form annehmen:

$$\varrho \xi_i = \chi_i(\lambda, \mu) + \vartheta_i(\lambda, \mu) \sqrt{\Omega(\lambda, \mu)},$$

wo χ_i, ϑ_i, Ω rationale Functionen ihrer Argumente sind. Ist also

Figure 5. Extract of a paper by Alfred Clebsch [2], belonging to the sixth class.

of long sentences expressed mainly with words of natural language, while only a few mathematical formulas are present in the corresponding texts (see Figure 5).

It is thus noteworthy that even within a relatively small topic, such as that of algebraic surfaces, such different ways of writing theorems and proofs exist, associated with as so many citation clusters and lexical classes.

4 Methodologies

Would it have been possible to detect this phenomenon, or the phenomena which I showcased throughout these pages, without the textometric tools? Probably yes, but maybe with a more conjectural status: one strength of textometry is that it allows to confirm such intuitions. At the same time, using it invites us to explore texts in an original way and to formulate new kinds of research questions, even on corpuses that have been studied by others before.

History of mathematics, just like other research disciplines, evolves with time. The breadth and the depth of the historical knowledge keeps growing. Its norms of rigour change. And the manner of interrogating the past is subject to development as well: trying new methodologies, comparing them to one another and reflecting on them is part and parcel of the historian's work.

From this point of view, exploiting the tools of textometry to investigate corpuses was also a way for me to investigate the very workability and relevance of using them. As I see it, that some results have been obtained and that new questions arose seem to be a result in itself, and an encouragement to explore this path further.

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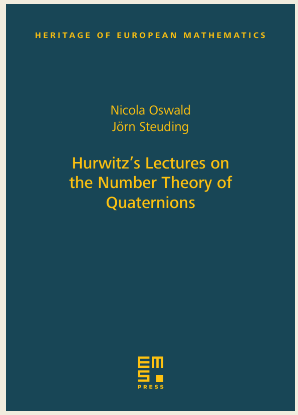

Hurwitz's Lectures on the Number Theory of Quaternions

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Quaternions are non-commutative generalizations of the complex numbers, invented by William Rowan Hamilton in 1843. Their number-theoretical aspects were first investigated by Rudolf Lipschitz in the 1880s, and, in a streamlined form, by Adolf Hurwitz in 1896.

This book contains an English translation of his 1919 textbook on this topic as well as his famous 1-2-3-4 theorem on composition algebras. In addition, the reader can find commentaries that shed historical light on the development of this number theory of quaternions, for example, the classical preparatory works (of Fermat, Euler, Lagrange and Gauss to name but a few), the different notions of quaternion integers in the works of Lipschitz and Hurwitz, analogies to the theory of algebraic numbers, and the further development (including Dickson's work in particular).

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Ancient Indian mathematics needs an honorific place in modern mathematics celebration

Steven G. Krantz and Arni S. R. Srinivasa Rao

The Indian tradition in mathematics is long and glorious. It dates to the earliest times, and indeed many of the Indian discoveries from a period starting 5000 years ago correspond rather naturally to modern mathematical results. Celebration of Indian mathematics needs to consider the personalities among ancient mathematicians who laid a solid foundation for modern thinking. Our main purpose here is, by presenting very briefly some of the main contributions of ancient Indian mathematicians and astronomers, to argue and convince the reader that before the great Ramanujan, there have been thousands of years of rich mathematical discoveries in India and those personalities' work also needs to be honored on Indian Mathematics Day.

The government of India announced in 2012 that every year, Indian mathematician Srinivasa Ramanujan's birthday, December 22, will be celebrated as national mathematics day.[1] The year 2012 was the great Ramanujan's 125th birth anniversary. The government of India released a commemorative stamp on that occasion as well. Brilliant contributions in number theory and combinatorics by Ramanujan are well known [1, 2, 13, 16]. However, deep astronomical and mathematical developments in India are several thousand years older than Ramanujan. In this comment, we try to recollect a few gems of the ancient Indian mathematics and its mathematicians who did fundamental work in number systems, mathematics of astronomy, calculus, etc., over more than 5000 years.

Our main purpose for writing this article is to argue and convince that, while giving Ramanujan's brilliant achievements during the past 125 years their due place, reducing the Mathematics Day in India to the celebration of Ramanujan's birthday (who was born in the 19th century) is somewhat short-sighted. Our goal is to make sure Indian Mathematics Day is seen as a celebration of thousands of years of deep-rooted mathematical thought processes and discoveries since the times of *Shulba sutra*. Moreover, it should also be devoted to celebrating many very strong mathematicians, such as, say, Harish Chandra, who have come since Ramanujan's time.

The origins of the mathematics that emerged in the Indian subcontinent can be seen around the Shulba sutra period, around 1200 BCE to 500 BCE. During this period the numbers up to 10^{12} were counted (in Vedic Sanskrit this number was referred to as *Paradham*). The Vedic period mathematics was confined to the geometry of fire-altars and astronomy, and these concepts were used to perform rituals by the priests. Some of the famous names from that era are Baudhayana, Apastamba, and Katyayana. In Table 1 we describe Sanskrit sounds and their corresponding English numerals. Indian mathematics also introduced the decimal number system that is in use today and the concept of zero as a number. The concepts of sine (written as *jaya* in Sanskrit) and cosine (*cojaya*), negative numbers, arithmetic, and algebra were found in ancient Indian mathematics [6]. The mathematics developed in India was later translated and transmitted to China, East Asia, West Asia, Europe, and

[1] Prime Minister's speech at the 125th Birth Anniversary Celebrations of Ramanujan at Chennai: https://archive.is/20120729041631/http://pmindia.nic.in/speech-details.php?nodeid=1117 (accessed on April 15, 2023).

Ancient Indian number	०	१	२	३	४	५	६	७	८	९
Sound	shunya	ekah	dva	triyoh	caturah	panca	sat	sapta	astah	nava
English number	0	1	2	3	4	5	6	7	8	9
Sound	zero	one	two	three	four	five	six	seven	eight	nine

Table 1. Sanskrit and English numerals 0 to 9 and their sounds.

Saudi Arabia. The classical period of Indian mathematics was often attributed to the interval from 200 CE to 1400 CE, during which works of several well-known mathematicians, like Varahamihira, Aryabhata, Brahmagupta, Bhaskara, and Madhava have been translated into other languages and transmitted outside the sub-continent.

The number systems present since the Vedic days, especially since the Sukla Yajurveda and their Sanskrit sounds, were as follows: 1 (*Eka*), 10 (*Dasa*), 100 (*Sata*), 1000 (*Sahasra*), 10^4 (*Aayuta*), 10^5 (*Laksa* or *Niyuta*), 10^7 (*Koti*), 10^{12} (*Sanku* or *Paraardha*), 10^{17} (*Maha Sanku*), 10^{22} (*Vrnda*), 10^{52} (*Samudra*), 10^{62} (*Maha-ogha*).

In addition to the number systems of ancient India, still today in India are heard the popular *"Vishnu Sahasra Nama Stotra,"* which dates back to the Mahabharata epic. In this, there is a verse that sounds like Sahasra *"Koti Yugadharine Namah."* If we translate this verse, then, as we saw above, *Sahasra* means 1000, and *Koti* means 10^7, so a simple translation of the phrase *Sahasra Koti* could mean 10^{10}. The entire phrase has been interpreted in different ways. We do not list here all possible interpretations and confine ourselves to number systems.

The deep investigations in astronomy and the solar system, geometry, and ground-breaking mathematical calculations by ancient and medieval great scholars in India, for example, Baudhayana, Varahamihira, Aryabhata, Bhaskara I & II, Pingala, Madhava, and many more, are well known (see [5, 7, 10] and [17, p. 423]).

It seems that celebrating national mathematics day in India only as part of Ramanujan's birthday is confining the glory and celebration of Indian mathematics to a little over 100 years of the past. Schools and colleges across India have celebrated Ramanujan's birthday for many decades, but that is different from exclusively limiting national day only to the great Ramanujan.

A good deal can be written on ancient scholar's work from India; material on this can be found, for example in [3, 6, 8, 12, 15]. In this opinion piece, we highlight only a few of them.

Shulba sutras were believed to have started in India around 2000 BCE through verbal usage. Their compilation in Sanskrit started perhaps 1000 years later by Baudhayana then by Manava, Apastamba, Katyayana and consisted of geometric-shaped fire-altars for performing ancient Indian rituals [3, 12]. Some of these sutras also contain the statements of Pythagorean theorems and triples. For example, Apastamba provided the following triples:

$$\left\{ \begin{matrix} (3,4,5), \ (5,12,13), \ (15,8,17), \\ (12,35,37), \ (36,15,39) \end{matrix} \right\}$$

for constructing fire-altars [15].

These sutras can be used to find the approximate value of $\sqrt{2}$ [12, 15], using the expression

$$\sqrt{2} \approx 1 + \frac{1}{3} + \frac{1}{3 \cdot 4} - \frac{1}{3 \cdot 4 \cdot 34} \approx 1.41421568627451.$$

६ (6)	१ (1)	८ (8)
७ (7)	५ (5)	३ (3)
२ (2)	९ (9)	४ (4)

६ $+n$	१ $+n$	८ $+n$
७ $+n$	५ $+n$	३ $+n$
२ $+n$	९ $+n$	४ $+n$

Figure 1. Magic squares (*anka-yantra*) for the Sun (top) and for the other eight planets (bottom) taken from ancient Indian literature.

In the Vedic period astrology (*Jyotisha*) of India, the magic squares (*anka-yantra*) were used to please and worship nine planets of the solar system [9, 18]. Figure 1 is about ancient magic squares for the Sun and the other eight planets in our solar system.

In the 5th century CE, Aryabhata calculated, among many other things, that the moon orbit takes 27.396 days, the value of $\pi = 3.1416$, etc. He is believed to have started the study of properties of sine and cosine in trigonometry.

According to [12], the Leibniz infinite series

$$4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots\right)$$

was known in the works of Indian mathematician Madhava, who lived three centuries before Leibniz.

In the 12th century CE, Bhaskara described in his famous book *Bijaganita* the rules of algebraic operations on positive, negative signs, rules of zero (*shunya*), and infinity (*anantam*). His book also shows how to obtain solutions to intermediate equations of the first degree [14]. Bhaskara's book titled *Siddhanta siromani* provided a detailed account of Indian astronomy and its development. Computations of the planetary movements, shapes of planets, rotation axis, lunar month days, etc., were explained in detail. See Figure 2. Pavuluri Mallana translated Mahavira's *Ganitasarasamgraha* from *Sanskrit* in the 11th century to another ancient indian language, *Telugu*; Joseph [11] thinks that this stood as a role model for other subsequent translations. Bhaskara's *Lilavati Ganitam* was for the first time translated from *Sanskrit* to *Telugu* in the 12th century by Eluganti Peddana [4], and into English first in 1816 by John Taylor, then in 1817 by Henry Thomas Holbrooke, who was considered as the first European Sanskrit scholar.

What we advocate in this piece is for an exposition of deep-rooted mathematical knowledge in India, and not an exhaustive

Bhāskarācārya, b. 1114

Līlāvatī gaṇitamu

Vijayanagaraṃ, 1936

Bhāskarācārya, b. 1114.
Bhāskarācārya viracitamagu vāsanābhāṣya sahita Līlāvatī gaṇitamu / Pīḍamarti Kr̥ṣṇamūrtiśāstri praṇītamagu Āṃdhravyākhyānamu. - 1 kūrpu. - Vijayanagaraṃ : Vidyāmudrākṣaraśāla, 1936.
ii, (1), 2, 2, iii, v, xiii, iii, 546 p. :port. ; 22 cm.
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ప్రస్తుతమాంగ్లభాషయందు అంకగణితము (Arithmetic) అనుశీర్షికక్రింద చర్చింపబడు విషయములన్నియు సంగ్రహముగ లీలావతియందు వర్ణింపబడియున్నవి, అవి క్రమముగ పరికర్మాష్టకము Four rules of Arithmetic, Square, Square root, Cube, (cube root) భిన్నములు (Fractions) వాని ప్రయోగములు (application to problems of daily life) త్రైరాశికము (Rule of Three) క్షేత్రవ్యవహారము(Mensuration). ఇంతియగాక ఉచ్చ బీజగణితమునకుజేరిన శ్రేఢీవ్యవహారము (Progressions) వ్యక్తి భేద, అంకపాశ వ్యవహారములు (Combination and Permutations) కుట్టక వ్యవహారము(Solutions of Indeterminate Equations) కూడ నీగ్రంథమునందు వర్ణింపబడియున్నవి. కుట్టకవ్యవహారసందర్భమున వితతభిన్న(Continued Fractions)ఆసన్నమాన (Convergent)ఆనయనములునుసంగ్రహముగసూచింపబడియున్నవి. శూన్యపరికర్మసిద్ధాంతములుకూడ వర్ణితములై యున్నవి.

Figure 2. Screenshots from the book "Lilavati Ganitamu" written by Bhaskara (also known as Bhaskaracharya) in the year 1114 CE. Top: Bhaskara wrote the book in Sanskrit. These screenshots were taken from its translated version into another ancient Indian language, Telugu, by Pidaparti Krishnamurti Sastri in 1936, published by Maharaja College, Vizianagaram, Andhra Pradesh, India.[2] Bottom: Here the contents of the book are mentioned in Telugu, as well as in English.

account of all possible results and conclusions. Several of the ancient texts in the language *Sanskrit* are either lost or preserved in museums.

Srinivasa Ramanujan's work undoubtedly shines as part of modern Indian mathematics but thousands of years ancient mathematical discoveries, the introduction of various branches of pure and applied mathematics needs a proper representation in any celebration of India's contribution to world mathematics.

We hope that this short list of significant examples will convince the readers as well as the decision makers of the need to incorporate and celebrate all the rich past and contemporary history of Indian mathematics during the Indian Mathematics Day.

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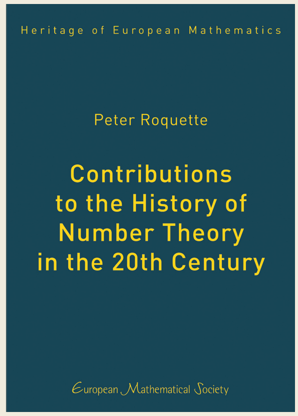

Contributions to the History of Number Theory in the 20th Century

Peter Roquette

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The 20th century was a time of great upheaval and great progress, mathematics not excluded. In order to get the overall picture of trends, developments and results it is illuminating to look at their manifestations locally, in the personal life and work of people living at the time. The university archives of Göttingen harbor a wealth of papers, letters and manuscripts from several generations of mathematicians – documents which tell us the story of the historic developments from a local point of view.

This book offers a number of essays based on documents from Göttingen and elsewhere – essays which are not yet contained in the author's Collected Works. These little pieces, independent from each other, are meant as contributions to the imposing mosaic of history of number theory. They are written for mathematicians but with no special background requirements.

Involved are the names of Abraham Adrian Albert, Cahit Arf, Emil Artin, Richard Brauer, Otto Grün, Helmut Hasse, Klaus Hoechsmann, Robert Langlands, Heinrich-Wolfgang Leopoldt, Emmy Noether, Abraham Robinson, Ernst Steinitz, Hermann Weyl and others.

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Research-data management planning in the German mathematical community

Tobias Boege, René Fritze, Christiane Görgen, Jeroen Hanselman, Dorothea Iglezakis, Lars Kastner, Thomas Koprucki, Tabea H. Krause, Christoph Lehrenfeld, Silvia Polla, Marco Reidelbach, Christian Riedel, Jens Saak, Björn Schembera, Karsten Tabelow, Marcus Weber

In this paper we discuss the notion of research data for the field of mathematics and report on the status quo of research-data management and planning. A number of decentralized approaches are presented and compared to needs and challenges faced in three use cases from different mathematical subdisciplines. We highlight the importance of tailoring research-data management plans to mathematicians' research processes and discuss their usage all along the data life cycle.

1 Introduction

Scientific progress heavily relies on the reusability of previous results. This in turn is closely linked to reliability and reproducibility of research, and to the question whether another researcher would arrive at the same result with the same material. In mathematics proofs, together with references to definitions of mathematical objects and already verified theorems, traditionally contained all the information needed in order to verify results. However, the advent of computers has opened up new resources previously deemed impossible, while increasing the need for well-adapted research-data management (RDM). For example, algorithms are now implemented to arrive at new conclusions. The size of examples has exploded several orders in magnitude. And some proofs have become too complicated for even the brightest minds, such that software is consulted for thorough understanding and verification.[1] Studies [12, 13, 18, 19] from various fields of applied mathematics show that nowadays many results cannot be easily reproduced and hence verified.

As we outline in Section 2, there are research data in all subdisciplines of mathematics that need responsible organization and documentation in order to ensure they are handled according to the FAIR principles [25] for sustainable, reproducible, and reusable research. One way to achieve this is via a tailored research-data management plan (RDMP), describing the data life cycle over the course of a project [17] and providing guidance to fulfill funding requirements.[2] In mathematics, it is particularly important to treat the RDMP as a living document [4] because the mathematical research process is hardly projectable and does usually not follow a standardized collection–analysis–report procedure. In subfields with experience in using such documentation, three-fold reports – at the grant-application stage, as a working document, and as a final report – have proven useful. We discuss this in Sections 3 and 4, spotlighting examples from different subfields, and conclude this article by listing central topics for RDMPs in all areas of mathematics.

2 Mathematical research data

Following [9, p. 130], we define *research data* as all digital and analog objects that are generated or handled in the process of doing research.[3] In mathematics, research data thus include paper publications and proofs therein as well as computational results, code, software, and libraries of classifications of mathematical objects. A non-exhaustive list of possible formats and examples is presented in Table 1 and Section 4. The apparent diversity of mathematical research-data formats is also reflected in other characteristics, such as their storage size, longevity, and state of standardization [3, 8, 10, 22, 24], leading to RDM needs and challenges that are very specific to the discipline of mathematics.

One of the most apparent challenges is the question what metadata are sufficient for reusability. We will answer this question partially for the mathematical subfields presented in Section 4.

[1] See, e.g., the story outlined in https://xenaproject.wordpress.com/2020/12/05/liquid-tensor-experiment, solved with the lean project https://github.com/leanprover-community/lean-liquid.

[2] E.g., at the European level https://ec.europa.eu/info/funding-tenders/opportunities/docs/2021-2027/horizon/guidance/programme-guide_horizon_en.pdf, and at the German national level https://www.dfg.de/download/pdf/foerderung/grundlagen_dfg_foerderung/forschungsdaten/forschungsdaten_checkliste_de.pdf.

[3] This is in line with the notions employed by the DFG https://www.dfg.de/foerderung/grundlagen_rahmenbedingungen/forschungsdaten/index.html, forschungsdaten.info https://www.forschungsdaten.info/themen/informieren-und-planen/was-sind-forschungsdaten, and the MPG https://rdm.mpdl.mpg.de/introduction/research-data-management, e.g.

Research-data type	Examples of data formats
Mathematical documents	PDF, LaTeX, XML, MathML
Literate programming sources	Maple Worksheets, Jupyter/Mathematica/Pluto Notebooks
Domain-specific research software packages and libraries	R for statistics, Octave, NumPy/SciPy or Julia for matrix computations, CPLEX, Gurobi, Mosel and SCIP for integer programming, or DUNE, deal.II and Trilinos for numerical simulation
Computer-algebra systems	SageMath, SINGULAR, Macaulay2, GAP, polymake, Pari/GP, Linbox, OSCAR, and their embedded data collections
Programs and scripts	written in the packages and systems above, in systems not developed within the mathematical community, input data for these systems (algorithmic parameters, meshes, mathematical objects stored in some collection, the definition of a deep neural network as a graph in machine learning)
Experimental and simulation data	usually series of states of representative snapshots of an observed system, discretized fields, more generally very large but structured datasets as simulation output or experimental output (simulation input and validation), stored in established data formats (i.e., HDF5) or in domain-specific formats, e.g., CT scans in neuroscience, material science or hydrology
Formalized mathematics	Coq, HOL, Isabelle, Lean, Mizar, NASA PVS library
Collections of mathematical objects	L-Functions and Modular Forms Database (LMFDB), Online Encyclopedia of Integer Sequences (OEIS), Class Group Database, ATLAS of Finite Group Representations, Manifold Atlas, GAP Small Groups Library
Descriptions of mathematical models in mathematical modeling languages	Modelica for component-oriented modeling of complex systems, Systems Biology Markup Language (SBML) for computational models of biological processes, SPICE for modeling of electronic circuits and devices, and AIMMS or LINGO as a modeling language for integer programming

Table 1. Mathematical research data come in a variety of data formats. Updated table based on [24, pp. 26–27].

However, as the authors of [3] note, 'the meaning and provenance of [mathematical research] data must usually be given in the form of complex mathematical data themselves.' It is thus not surprising that there is no common, standardized metadata format yet. A search in the RDA Metadata Standards Catalog[4] at the time of writing reveals five hits, four from a subfield of statistics and one from economics, none of which could encode information about, say, a computer-algebra experiment. This lack of standardization is in contrast to other disciplines such as the life sciences, where the OBO Foundry[5] hosts more than one hundred interoperable ontologies to describe and link research results, including common naming conventions [1, 20].

Another important aspect of mathematical research data is its particular data life cycle. Again in contrast, for instance, to the life sciences, where older results can be overruled by new evidence, mathematical results that have been proven true remain true indefinitely. Since they cater for other disciplines such as the physical, social, health or life sciences [24, Fig. 1 and discussion], mathematics has a particular responsibility to science to preserve their results in a sustainable manner. We discuss this aspect and how mathematics can be embedded in interdisciplinary research pipelines in more detail in Sections 4.1 and 4.2. Section 4.3 stresses the role thorough documentation plays in this context, using classifications as an example.

3 Status quo of RDMPs in mathematics

In the narrower sense of data (rather than research data), it is a common claim in the community at the time of writing that mathematics rarely produces data[6] and that the few data available need no particular management.[7] This is often based on an interpretation of data being something computational, and mathematics

[4] https://rdamsc.bath.ac.uk/subject/Mathematics%20and%20statistics

[5] https://obofoundry.org/principles/fp-000-summary.html

[6] Usually, only statistics is mentioned as a data-producing subdiscipline, see, e.g., https://wissenschaftliche-integritaet.de/kommentare/software-entwicklung-und-umgang-mit-forschungsdaten-in-der-mathematik.

[7] See, e.g., the unofficial document https://www.math.harvard.edu/media/DataManagement.pdf.

being a discipline which is very much paper rather than computer based. Anecdotal evidence suggests that this view is widely established, that there is little knowledge about general RDM, that existing local facilities are hardly used, and that RDMPs are not a standard tool at any stage of the research process.[8] The proposal [24] has identified the need to build common infrastructures for all subdisciplines of mathematics, and mathematics-specific DFG guidelines for FAIR research data will be developed in the foreseeable future.[9]

Now, the question of what these guidelines should be is not trivial. A large number of questions from a general RDMP catalogue,[10] are irrelevant for a community which produces foremostly theoretical results. For instance, for mathematicians the cost of producing data is rarely relevant – unlike, e.g., in the life sciences where data might have to be collected in the field. In the same vein, ethical or data-protection questions most often do not play a role, save for, for instance, industry collaborations or studies conducted in didactics. Large parts of the community have little training in legal aspects as, for example, formulae cannot be assigned proprietary rights. In order to avoid the impression that thus all general RDMP questions apply only to sciences different than mathematics, it is imperative to design bespoke catalogues of questions. These should (a) use unambiguous language, for instance, using the term 'research data' rather than the more specific 'data' which many mathematicians do not handle in their research, and (b) avoid superfluous topics while at the same time including sufficient detail, for instance, for mathematics' metadata and preservation needs identified in the previous section. Now, rather than endeavoring to find a one-size-fits-all solution, in the subsequent section we identify important RDM questions for a number of use cases which are known to the authors – focusing on metadata, software, data formats and size, versioning, and storage – and provide those with what we consider to be sensible answers. We use the remainder of this section to report on two RDM solutions implemented in DFG-funded Collaborative Research Centers (CRC).

The CRC 1456 'Mathematics of Experiment' includes 17 scientific projects in applied mathematics, computer science, and natural sciences such as biophysics and astronomy, aiming to improve the analysis of experimental data. The research data here are extremely diverse (e.g., mathematical documents, notebooks, programs, simulation data or experimental measurements) and their handling is supported by the CRC's dedicated infrastructure project. In regular RDM meetings, four themes are recurrent. First, reusage scenarios: especially in interdisciplinary research the same datasets may be processed or used by different groups; documentation, curation, and publication should be tailored to those groups' needs. Second, reproducibility, both computationally and practically in data recreation. Third, metadata: finding accurate descriptors to help the user understand cross-scientific research data. And fourth, visibility: receiving recognition for stand-alone research data beyond a journal publication is hard. This last topic is usually not part of a standard set of RDMP questions but aims to provide an incentive to increase the effort in research-data creation, publication, and curation.

The CRC 1294 'Data Assimilation' includes 15 interdisciplinary research projects focusing on the development and integration of algorithms, e.g., in earthquake prediction, medication dosing, or cell-shape dynamics. Researchers are thus confronted both with diverse research data and varying cultural data-handling habits. A central project supports their RDM, and IT infrastructure to facilitate collaborative work and knowledge perpetuation to advance good scientific practice is provided. In particular, the CRC designed an RDMP template in collaboration with the University of Potsdam's research-data group. This covers policies and guidelines, legal and ethical considerations, documentation, and dataset-specific aspects. A vital component of the training is then the classification of the digital objects that are reused and created by the individual researchers. This helps them to develop tailored strategies to improve the quality and reproducibility of published results and to sensitize their research-data handling throughout the data life cycle.

4 Use cases

We now consider four very different mathematical use cases and discuss their particular research-data needs. These use cases have been identified by the different mathematical subfields in [24] as particularly representative for the research community. Central in these expositions for us is to find out how, using RDMPs, we can provide the best, case-specific guidance to make a project reusable.

4.1 Applied and interdisciplinary mathematics

In numerous scientific fields real-world problems are simplified, e.g., to experiments, and subsequently described in abstract ways using mathematical models. If a model is combined with input data, it forms a concrete instance of such a problem. With the help of algorithms, the input data are then transformed into output data. Following validations, the interpretation of outputs provides the solution of the initial problem in a so-called Modeling–Simulation–Optimization workflow [24, p. 77]. For complete RDM,

[8] In fact, RDMPs became compulsory in DFG-funding applications only in March 2022, and there are no statistics available on how many mathematics proposals included such a document. See also https://www.dfg.de/foerderung/info_wissenschaft/2022/info_wissenschaft_22_25/index.html.

[9] https://www.dfg.de/foerderung/info_wissenschaft/2022/info_wissenschaft_22_25/index.html

[10] For instance, the current questionnaire supplied by the DFG-funded research-data management organiser RDMO https://github.com/rdmorganiser/rdmo-catalog/releases/tag/1.1.0-rdmo-1.6.0.

such workflows should be documented in detail as part of an RDMP.

A standard RDMP questionnaire includes some guidance for the documentation of workflows, such as the main research question, involved disciplines, tools, software, technologies, processes, research-data aspects, and reproducibility. Using this as a template, a tailored questionnaire is currently being developed within the framework of MaRDI[11] to document workflows in detail. This is divided into four sections dealing with the problem statement (object of research, data streams), the model (discretization, variables), the process information (process steps, applied methods), and reproducibility. It is aimed at all disciplines and differs only slightly in whether a theoretical or experimental workflow is documented. The central element of the questionnaire is to establish connections between different steps of the research process in order to improve interoperability of research data. The description of an individual process step, for example, requires the assignment of the relevant input and output data, the method and the (software) environment. At the same time, the documentation of the methods, software, input and output data requires persistent identifiers (e.g., Wikidata, swMATH, DOI) in addition to topic-dependent information.

We consider the documentation of a concrete workflow combining archaeology and mathematics as an example. This is based on [11], was created by Margarita Kostre independently afterwards, and described in personal communication as 'very helpful for the reflection of the own work.' The author commented that she will use workflow documentation in the future again, as she believes it facilitates interdisciplinary communication, e.g., about the status of a project, its goal, and data transfer, it provides better clarity in larger collaborations and allows colleagues to enter a project more easily. The aim of this work is to understand the Romanization of Northern Africa using a susceptible infectious epidemic model. On the process level, the workflow starts with data preparation, e.g., collecting, discretizing, and reducing archaeological data. Once a suitable epidemics model is found, the inverse problem is solved to determine contact networks and spreading-rate functions. Subsequent analysis allows the identification of three different possibilities of the Romanization of Northern Africa. The detailed documentation can be found on the MaRDI Portal.[12]

4.2 Scientific computing

While research in pure mathematics strives to determine an ultimate truth, applied or computational mathematics in majority need to deal with approximations to reality: models are usually expressed in terms of real or complex numbers and only finite subsets of these can actually be implemented on computer hardware. Consequently, the result of a computation depends on the format of the finite-precision numbers used and on the specific hardware executing the computations, making a detailed documentation of the computer-based experiment crucial and reusability of code a must-have [5]. Thus, the input data and results of a computer experiment and also the precise implementation (code, software, and hardware) of the algorithms used constitute important research data.

Absence of such details in documentation makes applied mathematics face the same reproducibility issues (e.g., [2]) as other scientific fields. Still mathematical algorithms make up the foundation of many computational experiments, for instance as solvers for linear systems of equations, eigenvalue problems, or optimization problems, and are thus at the heart of science today. This responsibility calls for rigorous RDM and documentation in RDMPs.

The main difficulty in establishing RDMPs in scientific computing seems to be in creating incentives to adhere to common standards. In case of a single multi-author paper within a larger project cluster, there are two levels to this question: the funding context and local RDM. Regarding the first, incentives should clearly address reporting requirements and incorporate rewards for sustainable RDM, rather than merely counting publications and citations, to ensure the cluster can *stand on the shoulders of giants* instead of *building on quicksand*. The beneficiaries here are other researchers in the project and world-wide. Consequently, global RDM needs to answer what is reported where and why. Regarding the local context, for the collaborative work of the authors, incentives are far more evident. Thorough RDM, documented in a living RDMP, not only accelerates the paper writing, it also improves the reusability of information for future endeavors of the individual authors. Questions center around 'When is the code/data provided? Where in the (local) infrastructure is it stored? By whom? Who is processing it next?'

Consequently, RDMPs should be modularized to enable the single modules to change at their appropriate pace. While the global management rules of a project cluster may not change at all, or at best very slowly, the findings in a single work package may alter the RDMP and thus RDM needs high agility to react to changes. For the software pipeline of an example paper that means: a task-based RDMP, updated as the pipeline evolves, needs to fulfill the requirements of [5], while for the project cluster sustainable handover, following (e.g., [6]), needs to be addressed in the overarching RDMP.

4.3 Computer algebra and theoretical statistics

Large parts of the German mathematical community consider themselves as not doing applied work. This includes fields such as geometry, topology, algebra, analysis or number theory, and also mathematical statistics, for instance. However, these researchers increasingly use computers, too, to explore the viability of proof strategies, test their own conjectures or refute established ones.

[11] https://www.mardi4nfdi.de

[12] https://portal.mardi4nfdi.de/wiki/Romanization_spreading_on_historical_interregional_networks_in_Northern_Tunisia

As a consequence, *classifications*, the systematic and complete tabulation of all objects with a given property, grow wildly in size and complexity. They give a complete picture for some aspect of a theory and may be used in many ways, from the search of (counter)examples over building blocks for constructive proofs to benchmark problems.

For instance, the L-Functions and Modular Forms Database (LMFDB) [23] contains over 4.8 TB of data relating objects conjectured to have strong connections by the Langlands program: number fields, elliptic curves, modular forms, L-functions, Galois representations. It includes tens of millions of individual objects and stores the relations between these. Entries contain detailed information on reliability, completeness, and several versions of the code needed to compute them. The database has a public reporting system which allows all users to have visibility of any issues or errors.[13]

Computing mathematical objects for classification can often be algorithmically hard and time-consuming. But once computed, results are final and independent of the software used. With larger computer clusters and better algorithms, it is unreasonable and unsustainable to expect researchers who want to build on existing research to repeat individual computations. This expected reuse increases the need for responsible RDM and triggers challenges which need to be addressed in an RDMP. In particular, four themes are central in this regard. First, how can researchers ensure that their research data are correct and complete? Is the connection of mathematical theory and code sound? Second, how can other researchers access, understand, and reuse the research data? Third, how can one ensure longevity of their research data? And fourth, how can researchers report errors/corrections and upload new versions of research data if necessary?

These questions are neatly addressed in the LMFDB mentioned above. To show how things can go wrong without proper RDM, we discuss a classification of all conditional independence structures on up to four discrete random variables, originally published in a series of papers [14–16]. Of the $2^{24} = 16\,777\,216$ *a priori* possible patterns of how four random variables can influence each other, only 18 478 ($\approx 0.11\%$) are realizable with a probability distribution. Šimeček, the author of [21], digitized this result and then left the field after his PhD in 2007. His research data was deleted in 2021 from his former institute's website – the only public place which ever held the database.[14] It was encoded in a packed binary format which is hard to read, search, and reuse. Some files supporting the correctness of the classification for binary distributions use an unspecified, compiler-specific binary serialization format for floating-point data.[15] The programs used for the creation and inspection of the database were written in a dialect of the Pascal programming language, which has not been maintained since 2006. The sparse documentation is in Czech.

This situation can only be fixed by recreating the database from scratch, including proofs. An RDMP for this project should emphasize the need to list and document each step of redoing the computations, the use of standard data formats with rich metadata for interoperability and searchability of the database, and ensure future reusability of Šimeček's results.

5 Discussion and outlook

The problem of reusability strongly relates to a phenomenon called *dark data*, which 'exists only in the bottom left-hand desk drawer of scientists on some media that are quickly aging' [7]. If research data are not available, they are of course neither traceable nor reusable or FAIR. This phenomenon extends from lost USB sticks and conflicting cloud-based collaboration tools like Dropbox[16]nd Overleaf[17]ithout local backup to papers containing very condensed complicated proofs that can only be taken up in future work if access to handwritten notes of the authors is also possible. A prime example of this is presented in Section 4.3 where unavailable research data is in stark contrast to the everlasting truth of mathematical results. RDMPs are a tool of choice against such issues, serving as a basic measure to organize the full data life cycle.

From the three case studies considered in Section 4, we derive that RDMPs in mathematics in particular (a) stimulate reflection, clarity, and interdisciplinary communication, (b) require flexibility and modularization as living RDMPs, and (c) facilitate the documentation of iterative computational processes by fostering research-data interoperability and reusability.

We further conclude that archiving and preservation is key in any mathematical subdiscipline. As a very first step to improve the status quo, all research results necessary for reusability (data, code, notes, …) should be stored in a sustainable and findable manner, using resources already documented in an RDMP before a project starts. Ideally, in a second step, citable repositories with persistent identifiers for these research data can be chosen, and, in a third step, these can be annotated with interlinked metadata, implemented via knowledge graphs. Because of the diversity of mathematical research data, the choice of metadata should be made carefully with possible reusage scenarios and interest groups in mind, also documented in an RDMP. If code is part of a publication, thoughts should be given to the detail of documentation

[13] Other classification databases targeted at specific audiences are listed at https://mathdb.mathhub.info.

[14] A backup is still available on the Internet Archive at http://web.archive.org/web/20190516145904/http://atrey.karlin.mff.cuni.cz/~simecek/skola/models/.

[15] A set of scripts for reading these files is available at https://github.com/taboege/simecek-tools.

[16] https://www.dropbox.com

[17] https://www.overleaf.com

and again appropriate citable long-term repositories. In addition, an RDMP should be used as a tool to identify legal constraints, like the compatibility of software licenses, before any actual work is conducted.

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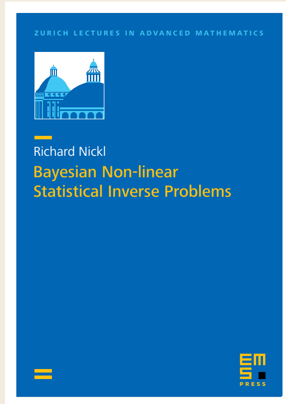

Bayesian Non-linear Statistical Inverse Problems

Richard Nickl
(University of Cambridge)

Zurich Lectures in Advanced Mathematics

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The present book presents a rigorous mathematical analysis of the statistical performance, and algorithmic complexity, of Bayesian methods based on Gaussian process priors in a natural setting of non-linear random design regression.

Due to the non-linearity present in many of these inverse problems, natural least squares functionals are non-convex and the Bayesian paradigm presents an attractive alternative to optimisation-based approaches. This book develops a general theory of Bayesian inference for non-linear forward maps and rigorously considers two PDE model examples arising with Darcy's problem and a Schrödinger equation. The focus is initially on statistical consistency of Gaussian process methods, and then moves on to study local fluctuations and approximations of posterior distributions by Gaussian or log-concave measures whose curvature is described by PDE mapping properties of underlying 'information operators'. Applications to the algorithmic runtime of gradient-based MCMC methods are discussed as well as computation time lower bounds for worst case performance of some algorithms.

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Why should mathematicians ask politicians to avoid the ancient dilemma of pure and applied mathematics?

Mohammad Sal Moslehian

Due to the commercialization of science and technology, there is evidence of politicians paying attention mainly to research with immediate applications and benefits. Furthermore, unrealistic requests are made regarding the direct applicability of results in mathematics, a circumstance that may negatively affect all basic and theoretical studies. This paper not only shows that the distinction between pure and applied mathematics is historically and practically unnecessary and unhelpful, but also emphasizes the romantic idea that pure and applied mathematicians should constitute a community and work together to change politicians' minds.

The dilemma of "pure versus applied mathematics" dates back to ancient Greece, where the first studies the world of ideas and the second investigates the world of the senses.

Pure mathematics can be regarded as the study of abstract concepts independent of any application to the physical world; see [3]. This description goes back to Plato's metaphysical view of mathematics as the study of ideas or eternal unchanging abstract forms; cf. [4]. Although many of these concepts originate from the real world, the applicability of the results is not the primary concern of pure mathematicians. Mathematicians intend to show the truth of mathematical propositions. In fact, inventing or discovering mathematical structures, generalizing notions, solving important mathematical problems, and briefly seeking "beauty" motivates pure mathematicians to deepen, strengthen, and expand existing mathematics. Hardy [9, pp. 84–85] asserted that "a mathematician, like a painter or a poet, is a maker of patterns" and added: "The mathematician's patterns, like the painter's or the poet's, must be beautiful; the ideas, like the colors or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics."

Pythagoras stated that "mathematics is the way to understand the universe." It is mentioned in [10] that "someone who had begun to read geometry with Euclid, when he had learned the first theorem, asked Euclid, 'what shall I get by learning these things?' Euclid called his slave and said, 'Give him a coin since he must profit by what he learns.'" None of the philosophical schools of mathematics – platonism, formalism, logicism, intuitionism – provide a completely convincing and comprehensive explanation of why mathematical structures do describe the real world. Recently, Max Tegmark [19], a mathematician-physicist, offered a different explanation. He believes that the universe itself is an abstract mathematical structure. He introduced the Computable Universe Hypothesis, stating that the mathematical structure that is the external physical reality is defined by computable functions. In addition to these views, the humanism of Reuben Hersh[1] considers mathematics to be part of human culture and history, which originates from the nature of our physiology and physical environment. He believes that mathematical structures adapt to the world around us for the same reason our lungs adapt to the Earth's atmosphere.

Many scientific theories are formulated and expressed using mathematical concepts and symbols. A part of mathematics, named applied mathematics, deals with modeling and simulation of phenomena and related calculations and helps other scientists to better understand nature, and describe and control it. Applied mathematics develops those mathematical methods that are used in various sciences and technologies. The help that applied mathematics provides to other disciplines in order to solve their problems is sometimes so significant that it has given rise to specific branches of mathematics. Examples include data science, mathematical biology, and financial mathematics. Penelope Maddy [14], a contemporary mathematical philosopher, based on historical pieces of evidence in mathematics, says that an applied mathematician invents a model that corresponds to a physical phenomenon that not only is not clearly visible in the phenomenon itself, but is also more complex than it.

Applied mathematics can be regarded as the bridge between pure mathematics and concrete applications of mathematics in the world. It is generally recognized that the practical value of mathematics lies in its applications, but that pure mathematics is indispensable to make such (future) applications possible, and that there are valid intrinsic reasons for doing mathematics, such as curiosity.

[1] https://www.edge.org/conversation/reuben_hersh-what-kind-of-thing-is-a-number

Pure mathematicians provide a solid framework and rigorous scientific basis which enables applied mathematicians to develop efficient methods or invent useful tools to assist physicists, computer scientists, engineers, biologists, medical researchers, economists, etc., in solving real-world problems. Sometimes this may be reversed in, for instance, physics, that is, mathematicians take ideas from physics and incorporate them into their abstract theorems and theories. Archimedes, for example, proved a geometric theorem while inspired by the law of levers in mechanics. Furthermore, some fields such as machine learning help mathematicians better understand the behavior of objects, structures and systems that are too large or too complex, discover patterns, and apply them to formulate conjectures; see [6] for examples in topology and representation theory.

In [7], the world of mathematics is described as a pyramid at the apex of which mathematical applications to other sciences, commerce, and industry are found. In the middle part of the pyramid, applied mathematics together with data science, mathematical biology, financial mathematics, computer science, scientific computation, information theory, etc., shine. The base of the pyramid is pure mathematics, consisting of logic, number theory, algebra, analysis, and geometry. There is, however, no clear boundary between these sections, and in various places, one can observe an entanglement between pure mathematics, applied mathematics, and mathematical applications. It is notable that if the base of the pyramid is not large enough, the pyramid may not be as stable as required. Some of the achievements of pure mathematicians have found practical applications, while others, so far, have not, but also the latter are still needed to maintain and consolidate the pyramid. We cannot predict where, when, and how a specific piece of pure mathematics will become useful for applications. The bottom of the pyramid is made up of topics related to the fundamentals of mathematics, such as mathematical logic and set theory, which are not considered to be applied in the conventional sense, but without which the pyramid of mathematics cannot be properly erected and stabilized.

In addition, mathematics generally improves decision-making methods in students' minds when we teach them problem-solving techniques, modeling the real world, and cognitive skills. Therefore, it plays an important role in shaping what is named "logical thinking" in humans, apart from what we usually call applications.

Mathematics goes freely beyond the bounds of thought, although it always has an eye for the nature of problems in other branches of science. Ideas in pure mathematics are based on a mental interest in problem-solving toward discovering, establishing, and exploring new structures. "There is nothing more practical than a good theory," said Kurt Lewin, a German-American psychologist [13, p. 169].

History of mathematics has shown that what were once considered mental, abstract, and useless results, were often used at other times by other sciences such as physics, chemistry, computer science, and engineering. To prove this claim, we take a brief look at the history of mathematics and mimic some ideas of [1, 2, 11, 16].

- Number theory is one of the purest fields of mathematics and was considered by many to be a mental game. Fermat's Little Theorem (1600 AD) and Euler's Theorem (1800 AD) are together considered the backbone of the RSA algorithm (named after Rivest, Shamir, and Adleman), which is a public-key cryptosystem. RSA is widely used to secure data transmission, internet communications, e-commerce, and blockchains.
- Conic sections (circle, ellipse, parabola, and hyperbola) were introduced by Apollonius of Perga (250 BC). Some people thought about these as a playground for the mind, until Johannes Kepler (17th century) realized their practical application in describing the orbits of planets.
- In the 19th century, Klein, Beltrami, and Poincaré showed that the geometric structures developed by Nikolai Ivanovich Lobachevsky (hyperbolic geometry) and Bernhard Riemann (elliptic geometry) are as logically consistent as Euclidean geometry. It was a revolutionary discovery and a completely pure one, until in the 20th century Albert Einstein utilized Riemann geometry in his theory of general relativity, where the space-time is curved. Non-Euclidean geometries have since then found applications in cosmology to study the structure and evolution of the universe.
- The complex numbers appeared in the 16th century and were so abstract that they were called imaginary numbers. These numbers gradually found applications in mathematics themselves for polynomial factorization, and in signal processing and electrical circuit calculations. Also, the theory of functions of several complex variables developed by Weierstrass and others in the 19th century has found applications in quantum field theory in the second half of the 20th century.
- Probability theory was established by Girolamo Cardano in the 16th century for solving gambling problems and was well developed by Kolmogorov in 1935 and then applied to statistical mechanics. Probability theory and statistics play a key role in current life, in particular in risk assessment, modeling, and reliability.
- The origins of graphs go back to 1732, when Leonhard Euler posed the seven-bridges problem of Königsberg. Recall that this problem asks whether the seven bridges of the city of Königsberg over its river Preger can all be crossed precisely once during a walk through the city that returns to its starting point. The field of graph theory is considered part of pure mathematics but found its first applications more than one hundred years later when in 1847 Kirchhoff studied electrical networks. Other applications in chemistry, computer science, and social sciences appeared subsequently.
- In the early 20th century, Gottlob Frege analyzed the concepts of arithmetic to show why mathematical reasoning is applicable

as a deductive procedure as applied to statements about the world; see [17, Chapter 1].

- Matrix theory was introduced by Cayley in the 19th century with no connection with applications. In 1925, Heisenberg applied matrices to describe his understanding of the atomic structure, and they are now a key tool in all sciences including coding, economics, and wireless communications.
- Linear algebra, developed in the early 20th century, is devoted to studying linear equations and linear mappings in the framework of vector spaces. It is a crucial ingredient in Google's PageRank algorithm and is used in developing artificial intelligence algorithms. In return, some topics of mathematics such as inverse problems have been influenced by artificial intelligence; see [12].
- Partial differential equations were developed in the 18th and early 19th centuries as applied mathematics, but were also pursued by mathematicians of pure temperament. Maxwell, a physicist, introduced a set of coupled partial differential equations, called now the Maxwell equations, and discovered that light is an electromagnetic wave and there must be other such waves with different wavelengths. Later, the radio waves that the theory predicted were found by Hertz.
- The Radon transform is an integral transform that was introduced in 1917 by Johann Radon. About 50 years later, it was used for tomography, a visualization process in which an image is constructed from the projection data connected with cross-sectional scans of a part of the body.
- The Fourier transform is a map of a function space decomposing a function of time or space into functions depending on spatial frequency. Its history goes back to the 19th century when Fourier expanded a function into an infinite series of sines and cosines. The wavelet transform was created by Alfred Haar and Norman Ricker in the first half of the 20th century. The wavelet transform represents a signal in both the time and frequency domains, whilst the Fourier transform represents it only in the frequency domain. The wavelet and Fourier transforms are now used in the design of computer graphics and in medical devices such as MRI machines, or heart, brain and diabetes monitors.
- Group theory was introduced by Lagrange and Galois in their study of symmetry and symmetry transformations. It greatly influenced the development of cryptography, crystallography, and musical set theory.
- The Entscheidungsproblem was a challenge posed in 1928 by David Hilbert, asking whether there is a way to determine the correctness or incorrectness of mathematical statements in a finite number of steps. In the 1930s, Alonzo Church and Alan Turing answered his question in the negative. Turing formulated an abstract machine, called now the Turing machine, which is considered the basis of modern computers.
- Lawrence Klein believes that economics is a mathematical discipline. John Keynes methodologically inspired his revolutionary economic theory from non-Euclidean geometry; cf. [5].
- Classifying and specifying digital images of millions of fingerprints can take up a very large storage space. Wavelet theory makes it possible to compress information quickly, relatively, and simply in such a way that we can compare new individuals in an investigation. In addition, the theory provides faster information retrieval.
- Linear algebra, mathematical analysis, probability, and statistics are used to analyze big data, that is, large or complex data sets. Big data is applied by information specialists, in particular in the modeling of financial markets.
- Biomathematics is the field using mathematical models for understanding phenomena in biology. It uses mainly linear algebra, differential equations, dynamical systems, probability and statistics.

Our daily lives are tied to many advanced technologies such as computers, the internet, and smartphones, while mathematics constitutes their scientific basis. For instance, mathematics is applicable in securing information in financial transactions, removing landmines and tumors, understanding climate change, developing advanced medical devices, redefining architecture, influencing human behavior, improving weather prediction, satellite communications, and searching for a second Earth.[2]

Due to the commercialization of science and technology, most politicians pay attention only to research with immediate applications and direct profit. Such unrealistic requests have negatively affected all basic and theoretical studies and have caused controversy among some mathematicians concerning the importance of both pure and applied mathematics. Free thought is the nature of research at universities, and the demand for immediate application limits it and darkens the horizon for the advancement of science and ultimately technology. Obviously, usefulness is important for research in industrial fields, but in theoretical fields this concept is ambiguous. However, politicians are not usually impressed by the news about settling a conjecture or the eternal beauty of a certain theorem proved by a pure mathematician. And here, applied mathematicians can greatly help the community by convincing them about the importance of math.

In recent years, great resources (grants, projects) are concentrated on applied research; it is not surprising that this happens. In fact, applied mathematics (including statistics and computer sciences), as well as the speed of computer simulations, are now such that problems from other disciplines can be tackled these days that were completely out of reach a few decades ago. However, it is necessary to support basic research. Furthermore, applying scientometric indicators such as the h-index for granting mathematicians causes some unfair decisions, since not only such indicators do not

[2] https://ima.org.uk/case-studies/mathematics-matters

consider the "hardness" of the topics on which mathematicians are working, but also in some countries applied mathematicians usually have scientometric indicators higher than pure mathematicians, and therefore such a comparison is not appropriate. Also, the highly-visible Fields medals are traditionally awarded for research in pure mathematics, which is somehow annoying because it may split the community and drive applied people away from the general community.

In some countries, there is a cultural difference between applied and pure math departments, where each side may view the other with some sort of suspicion, sometimes even disdain.

Indeed, pure mathematicians are criticized by some of their colleagues who work in applied fields, along the following lines:

- The goal of pure mathematicians is to expand the boundaries of knowledge, while science, industry, and society have more serious problems to explore.
- Research in pure mathematics is not directly aimed at solving society's problems.
- Pure research refers to a vague future in which applications may or may not materialize.

In contrast, pure mathematicians mainly argue as follows:

- Looking this deep into the immediate future applications of pure mathematics is not reasonable. Any kind of real innovative application of mathematics is based on basic research in pure mathematics.
- Pure mathematicians establish and develop precise and correct mathematics used in applications.
- Pure mathematicians ensure that mathematics can be reliably and confidently used in science, industry, and society.

The distinction made between pure and applied mathematicians comes back to their different criteria for doing mathematics. For example, an applied mathematician may use quasi-empirical arguments, such as a non-rigorous method to effectively solve an important problem, while a pure mathematician never does that; see [15]. However, computers and automated reasoning may help pure mathematicians formulate and prove their conjectures. For example, the proof of the Four Color Problem by Kenneth Appel and Wolfgang Haken utilized computers to test a large number of cases. Thus pure and applied mathematics can interact with each other.

Practically, any discussion on the distinction between pure and applied mathematics makes a negative impression on the possible collaborations between pure and applied mathematicians aimed at solving deep problems and improving their faculty; see [20]. In the past there was no such distinction – Kepler, Euler, and Gauss were of course both. In mathematics departments of many prestigious universities, research in pure mathematics is considered as important as research in interdisciplinary and emerging disciplines in applied mathematics. Because they look at mathematics as an integrated system: a device in which the existence of each part, whether pure mathematics or applied mathematics, is essential for its proper functioning. Pure and applied mathematicians constitute a community that needs to work together.

We should pay attention to both pure and applied aspects of mathematics and measure their achievements solely based on depth, breadth, and impact. Ian Stewart [18] said that the very distinction between pure and applied mathematics is looking increasingly artificial, dated, and unhelpful. Sometimes at conferences, after a presentation of results in pure mathematics, a person in the audience may ask: "What is the application of these results?" There is no problem in asking this question as long as it is aimed at helping to connect more pure and applied branches of mathematics. But in many cases such a question arises from a misunderstanding of mathematics, its history, and its structure (a continuum), and shows superficial thinking; see [8].

It is better to consider pure and applied mathematics as two sides of the same coin and recognize and cherish all achievements that have rich and important mathematical content. Pure and applied mathematicians should stand together to explain the word of importance and make science effective progress.

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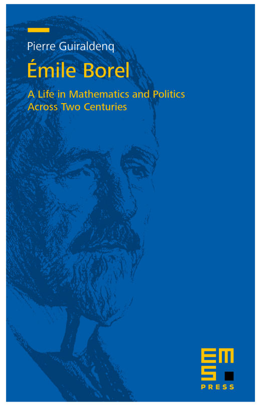

Émile Borel
A Life in Mathematics and Politics Across Two Centuries

Pierre Guiraldenq
(École Centrale de Lyon)

Translated and edited by
Arturo Sangalli

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Émile Borel, one of the early developers of measure theory and probability, was among the first to show the importance of the calculus of probability as a tool for the experimental sciences. A prolific and gifted researcher, his scientific works, so vast in number and scope, earned him international recognition. In addition, at the origin of the foundation of the Institut Henri Poincaré in Paris and longtime its director, he also served as member of the French Parliament, minister of the Navy, president of the League of Nations Union, and president of the French Academy of Sciences.

The book follows Borel, one of France's leading scientific and political figures of the first half of the twentieth century, through the various stages and the most significant events of his life, across two centuries and two wars.

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EMYA: The European Mathematical Society Young Academy

Beatrice Pelloni, Volker Mehrmann, Róisín Neururer and Irene De Blasi

regularly presented by Vesna Iršič

Recently, EMS has initiated the instalment of the EMS Young Academy (European Mathematical Young Academy – EMYA). In this column we present the story behind the establishment of EMYA – presented by Beatrice Pelloni (vice-president of the EMS) and Volker Mehrmann (former president of the EMS) in Section 1, and the current state of EMYA – presented by Róisín Neururer (chair) and Irene De Blasi (co-chair) in Section 2.

1 The instalment of EMYA

Europe has a great tradition in Mathematics. But the mathematical community is very fragmented and the overall communication between the different mathematical communities and societies in Europe is not so well established. Furthermore, in the European Mathematical Society the interaction happens to a large extent on the level of established scientists and functionaries who meet at council or committee meetings. But the shaping of the future development of mathematics in Europe should involve the voice and the ideas of the next generation.

With this in mind, inspired by the successful models in some national societies, in recent years the EMS has discussed how to increase the support for the young generation of mathematicians in Europe. This support must involve their mathematical development and their career perspectives, but in order to shape the future, the young scientists should also participate more actively in the development of the EMS and its planning and decision procedures. They should also give impulses for future directions of research and for procedures to improve the well-established processes in the EMS.

As a consequence of this discussion and several iterations of ideas, the EMS council, at its meeting in Bled, approved the establishment of the EMS Young Academy (EMYA). The EMYA was added as an EMS structural element to the EMS by-laws. It was decided that every year, by the deadline of 31 July, each member society/institute nominates two young mathematicians (at least 3rd year PhD students up to 5 years after PhD) for the Young Academy, respecting gender and other diversity issues. The EMS Executive Committee (EC) forms a selection committee (of 5 members respecting mathematical field and gender diversity as well as regional balance) that selects 30 new members of the Young Academy. The selected ones remain members for a period of 4 years, so that at full scale EMYA will have up to 120 members representing the young generation of European mathematicians, with diversity in gender, geographical origin and mathematical expertise. The EMYA selection committee has a mandate for 2–4 years and the membership can be renewed once.

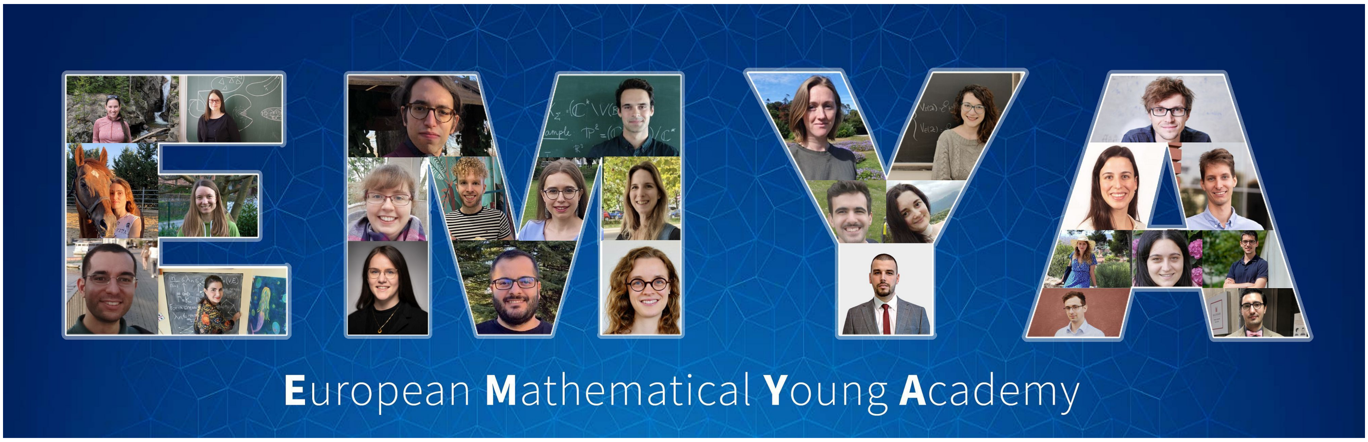
The role of EMYA includes an advisory function, commenting on and proposing procedures for the development of EMS (research plans, workshops, schools, organisation, web presentation, publications, etc.). The EMYA will have a representative on the EMS EC, elected by the EMS council with a 2-year mandate. Meetings of the EMYA will be supported with a reasonable budget by the EMS.

The first cohort of EMYA members was elected at the end of 2022 and the group has established itself at 30 members. There is a 50/50 gender balance in this cohort, with 18 countries represented and a good breadth of mathematical fields, from algebra and analysis to optimal control theory, from statistics to maths education. As their first action, EMYA organised itself, by agreeing by-laws and organisational structure. An EMYA committee composed of 9 members was elected. This committee comprises a chair, a co-chair, a secretary, a vice-secretary, a treasurer, a communications officer, an EMS Magazine column editor, a diversity officer, and an EMS EC representative.

We are looking forward to a fruitful and harmonious collaboration across generations of European mathematical scientists.

2 What is happening with EMYA now?

We are delighted to introduce you to the newly formed EMS Young Academy, or EMYA. It is a wonderful honour and great responsibility for us, as chair and co-chair of the inaugural cohort of 30 young mathematicians, to play a role in establishing the vision and activities of EMYA over the coming years. We hope to be able to represent and advocate for young mathematicians, making proposals and promoting initiatives to support our colleagues across Europe.



A collage of current members of EMYA, created by Dušan Džamić and Hana Turčinová.

The journey to the formation of EMYA began in late 2022, with a call to member societies, after which the inaugural cohort of 30 young mathematicians were selected. Our first meeting was held in March 2023, facilitated by members of the EMS Executive Committee. A working group was established to draft a set of by-laws and outline procedures for the running of EMYA, as well as to assign roles and responsibilities. These by-laws were formally proposed during the 2nd meeting of EMYA in June, where they were unanimously approved, and committee officers were elected. For the full list of EMYA members, as well as the elected committee officers, we invite you to view the EMYA webpage.[1]

At present, the 30 members span 18 different countries and are at various stages of their academic career, from 3rd year PhD students through to those in their 4th year after PhD. Additionally, there are a wide variety of mathematical fields represented among our membership – a brief survey of AMS subject classifications revealed that 28 are represented. We hope the diversity of our membership will only increase as it grows in number. At full scale, EMYA will have 120 members.

EMYA echoes the aims and responsibilities of the EMS, but with a specific focus on young mathematicians. In particular, we aim to give a voice to, and encourage participation of, young mathematicians within the EMS; promote and support the work of young mathematicians across Europe; and propose scientific activities of interest to our community. To achieve these aims, we hope to organise a wide range of initiatives, both scientific and social, which will foster connections among young mathematicians in Europe and provide them with opportunities to develop research and academic skills, as well as create opportunities to share our views on what it means to be a young researcher in our academic system.

Since supporting our young community is the primary aim of EMYA, we would welcome ideas and suggestions from any young mathematician across Europe about how we can support them in their careers and, broadly, their academic life. We strongly encourage you to follow us on our social media channels,[2] to stay up to date with our activities and contribute to our discussions. We also ask our EMS colleagues to share the news of our organisation with their colleagues and PhD students and invite them to contact us, to ensure that our representative role is as effective as possible.

We look forward to seeing how EMYA can benefit from and contribute to the EMS over the coming months and years, hoping that, in the future, it will become a reference point for any young mathematician aiming to build their life in this wonderful community.

[1] http://www.euromathsoc.org/EMYA

[2] https://www.facebook.com/groups/303163669063503, http://www.linkedin.com/in/young-academy-ems-569468299, https://twitter.com/EMSYoungAcademy

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She is well known for her work on the qualitative behaviour of partial differential equations, including equations from mathematical physics, particularly those involving a realistic set-up in bounded domains. She was the Olga Taussky-Todd Lecturer at ICIAM 2011, and the LMS Mary Cartwright Lecturer in 2019. She was elected Fellow of the IMA in 2012, and Fellow of the Royal Society of Edinburgh in 2020. She is currently a vice-president of the European Mathematical Society and a deputy chair of the International Centre for Mathematical Sciences.

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Volker Mehrmann received his diploma in mathematics in 1979, his PhD in 1982, and his habilitation in 1987 from the University of Bielefeld, Germany. He spent research years at Kent State University (1979–1980), at the University of Wisconsin (1984–1985), and at the IBM Research Center in Heidelberg (1988–1989). After spending the years 1990–1992 as a visiting full professor at the RWTH Aachen, he was a full professor at Chemnitz University of Technology from 1993 to 2000. Since then he has been a full professor for numerical mathematics at Technical University of Berlin and was from 2008–2016 the chair of the Research Center Matheon in Berlin.
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ICMI column

Yoshinori Shimizu and Renuka Vithal

Mathematics curriculum reforms around the world: Report on the 24th ICMI Study

Introduction and background

School mathematics curriculum reforms are a widespread and very long-standing practice across the world. Yet it is only in this *twenty-fourth* ICMI Study that this critically important and most impactful area in mathematics teaching and learning has come under scrutiny as an ICMI Study. The last ICMI Study that focused on mathematics curricula was ICMI Study 2 on *School Mathematics in the 1990s* [2].[1] This ICMI Study 2 followed after the publication of a seminal volume five years earlier, *Curriculum Development in Mathematics* [1] that provided an overview of school mathematics curriculum reforms in the preceding decades.

The ICMI Executive Committee announced the launch of *ICMI Study 24: School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities* in Hamburg, Germany, in July 2016 during the 13th International Congress on Mathematics Education (ICME-13). A diverse International Programme Committee (IPC) met the following year in Berlin, in November 2017, to finalize the ICMI Study 24 Discussion Document, which set out the scope of the study and called for papers for the ICMI Study 24 Conference [4]. The Study Conference took place in Tsukuba, Japan, a year later in November 2018, after which the task of compiling the Study Volume began. The work of developing the Study Volume, which in itself was a complex task given the study topic, faced several additional setbacks during the devastating global challenge of COVID-19, but endured and was finally published in June 2023 [5].

ICMI Study 24 demonstrates a diversity of studies and findings from international experience and research that can and do influence the nature of curriculum changes, and the possibilities of educational reform and its implementation: curricular design results; a revised role for components in the teaching of mathematics (e.g., mathematics content, pedagogy, and assessment); the role of technology; and new cognitive, sociocultural and sociopolitical perspectives. The consideration of curriculum reforms from various perspectives and constructs (mathematical literacy or competencies, for instance) raises many issues from scientific, political and cultural points of view, which need to be taken into account by communities of researchers, teachers and policymakers involved.

The scope of the Study, the Discussion Document and the Study Conference

ICMI Study 24 was conceptualized to focus on *School Mathematics Curriculum Reforms: Challenges, Changes and Opportunities*. This topic invoked not only questions about changes in curriculum design but also about the implementation of these changes across an educational system. What functioned (or not) at the time of implementing a curricular change? What are the limitations? How have resources (e.g., textbooks and technology) influenced the reforms and their enactment? How should large-scale teacher preparation be conducted to achieve the reform goals? How do diverse social, economic, cultural and national contexts condition the nature and extent of curricular reforms, especially teacher expectations, attitudes and beliefs, and the social and cultural background of students? How do assessments of students' learning influence curriculum reforms? The study opened opportunities for a synthesis or meta-analysis of different aspects of school mathematics reforms historically, geographically and globally.

The overarching question of this ICMI Study, as captured in the Discussion Document, was to explore what school these reforms have been or are taking place, especially at a meta, macro or system level; and to learn about the many different aspects of mathematics curriculum reforms from past experiences, to specify the current status and issues in reforms worldwide, and to identify directions for the future of school mathematics. The Discussion Document was disseminated in December 2017, inviting participation in the Study Conference, which took place in the Tsukuba International Congress Center from November 25 to 30, 2018. The conference was organized around working groups in the five themes described in the next section. These groups met in parallel during the con-

[1] ICMI Studies may be accessed at https://www.mathunion.org/icmi/digital-library/icmi-studies/icmi-study-volumes.

EMS MAGAZINE 130 (2023) — DOI 10.4171/MAG/170

ference and their work is captured as chapters in the ICMI Study Volume. The Conference Proceedings provided the foundation for discussions during the conference and the production of the Study Volume.

The ICMI Study 24 Conference attracted 96 participants and 68 papers from various countries or regions and from educational systems with different cultural, economic, political, and historical backgrounds. The Study Volume included 71 authors, with 66 participants contributing chapters from the Study Conference. This diversity provided rich discussions on current and future thinking about school mathematics curriculum reforms, cases of reforms, and opportunities to juxtapose different cases highlighting commonalities and differences. The range of countries (approximately 30) represented in the Study Conference translated into a diverse authorship in the Study Volume, drawing on mathematics curriculum reforms across the world, while acknowledging underrepresentation from some world regions. The diversity of the Study Conference and of the Study Volume also brought a variety of positionalities in respect to mathematics curriculum reforms as receivers or drivers of a particular reform and with differing knowledge, skills, expertise and experiences.

The main themes in ICMI Study 24

When developing the Discussion Document for ICMI Study 24, the IPC came to a consensus on five main themes, which provided the basis for the Study Conference Proceedings and for the Study Volume. For each theme, a set of questions was identified which addressed curriculum elements such as content, pedagogy, textbooks, technology, assessment, teacher professional development, curriculum development, design processes, and the role of agents. Contributions were invited to the separate themes and distinguished by the theme foci and questions.

A. Learning from the past: Driving forces and barriers shaping mathematics curriculum reforms

This theme provides historical perspectives on school mathematics curricula, highlighting key issues around past reform movements, thereby ensuring that their lessons and challenges can inform future movements. The first chapters in this theme present four cases of national reforms in the period since the 1960s and then extend the empirical landscape framed by each case. It addresses research questions about the aspects of mathematics teaching, and learning processes certain international reforms attended to, identifies key stakeholders in curriculum reforms, factors that underpinned curriculum reforms and barriers that inhibited reform efforts. Relations between curriculum reforms and cultural values are also analyzed, as is the question of mathematical content and how it is treated and affected by past curriculum reforms. The theme concludes by summarizing the drivers and barriers of school mathematics curriculum reforms.

B. Analyzing school mathematics curriculum reforms for coherence and relevance

This theme on coherence and relevance of school mathematics reforms examines the role and importance of mathematics as a school subject and its relation to other subjects in an educational system. Chapters in this theme begin by examining the notion of coherence in depth, within and between components of curricula, and between the curriculum and the system in which it is enacted. Then the focus moves to the relations between mathematics and other disciplines and explores the role mathematical modeling plays in transdisciplinary approaches to school curricula. The next chapter identifies the increasing range of physical and digital curriculum resources that have been developed to support particular curriculum reforms, the characteristics of such resources, and the constraints that weigh on achieving the goals of coherence and relevance. A chapter that examined theories and methodologies for researching and analyzing mathematics reforms and their limitations is also featured. Some guiding principles deriving from the theme are set out in the concluding chapter.

C. Implementation of reformed mathematics curricula within and across contexts and traditions

The cultural, social, economic, historical and political contexts and positions for the implementation of a school mathematics curriculum are important considerations in reform efforts. Chapters in this theme begin by sharing experiences and examples about the implementation of mathematics curriculum reforms in different countries or regions from a plenary panel, which demonstrate how reforms are diverse, multifactorial, uncertain and require both top-down and bottom-up strategies. This is followed by a chapter that examines factors that intervene within mathematics curriculum reforms and seeks 'processes, models, or best/common practices' that can be relevant for the progress or success of a reform. Next, the initial preparation and professional development of teachers in curriculum implementation, and the interrelation between reform and teachers' actions are analyzed. The concluding chapter proposes several 'laws' from the studies in this theme.

D. Globalization and internationalization, and their impacts on mathematics curriculum reforms

This theme points to factors that advance globalization and internationalization, and influence mathematics curricula through rapid changes in the nature of communication and availability of information. It begins with an exposition on the definition of key concepts. This is followed by a chapter on the emergence of numeracy and mathematical literacy and their relationship with curriculum reform processes. Then the impact of TIMSS and PISA is compared in economically and geographically diverse countries. The inclusion of new areas in recent mathematics curriculum reforms of algorithmic/computational thinking is examined next. The theme concludes by mapping out future visions of the impact of internationalization and globalization on school mathematics curriculum reforms, and offers recommendations for future reforms.

E. Agents and processes of curriculum design, development, and reforms in school mathematics

This final theme acknowledges that curriculum reform processes are as much a political matter as they are educational; and nowadays involve a broad range of stakeholders with vested interests. The first chapter comprises four contributions from prominent leaders of mathematics curriculum reforms in different cultures, countries and contexts. The next chapter proposes a model of curriculum reform as a system of: agents (who is involved); objects (what materials, etc. they are working with); processes (how agents work with objects and other agents); and in terms of arenas (where the reform takes place). This is followed by a chapter on communication and negotiation among stakeholders in different communities of practice. The professional dynamics stemming from the relationship between the stakeholders leading the reform and the stakeholders responsible for translating the official curriculum into the classroom is examined in the next chapter. The concluding chapter addresses implications for active curriculum reform work, challenges for conducting curriculum reform research, and future research directions.

Reflections and commentary on ICMI Study 24

The 24th ICMI Study Volume was released on June 29, 2023, as an open access book, and within a month had garnered more than 100,000 downloads. There are several points of reflection in the Study Volume. The first is raised by Jeremy Kilpatrick, in a chapter that captures a historical perspective and his reflections on the current status and future trends in school mathematics reforms. A second point of reflection is offered by Berinderjeet Kaur in a chapter as a reaction to two contrasting curriculum perspectives, namely, the OECD Learning Compass 2030 framework (Miho Taguma) and the Common Core State Standards in Mathematics in the USA (William McCallum), with reflections based on her experience and involvement in the Singapore school mathematics curriculum reforms. A third point of reflection consists of two commentaries on the volume as a whole, from two leading scholars in mathematics education research with a keen interest in school mathematics curriculum reforms, who did not participate in the ICMI Study 24 Conference. Anjum Halai draws attention to the importance of language in mathematics reforms as an equity issue; and Paola Valero offers a cultural-political reading of mathematics curricula.

The final point of reflection is offered by the editors in the introductory and concluding chapters. In the first chapter, it is acknowledged that school mathematics curriculum reform has been a diverse and widespread practice but an un- or under-explored area of research in mathematics education, and therefore there is not much scholarly work to guide and understand this critical aspect of mathematics education. The final chapter distills key learning points from the themes and chapters that may be of use to school mathematics curriculum reform researchers, practitioners and policymakers. This includes: difficulties in defining school mathematics curriculum reforms; inadequate theories and methodologies for studying mathematics reforms; significant shifts in the content of mathematics curriculum reforms; the crucial role of teachers, teacher education and professional development to make or break a curriculum reform; the growing importance of resources and technology in reform efforts; the alignment of components of a mathematics curriculum reform; and recognition of how these reforms are context bound and have invariant aspects.

This ICMI Study allowed for a more informed and comprehensive analysis of the roles of different actors, of the many aspects influencing and shaping mathematics curriculum reforms and of the possibilities and means to tackle a curricular reform in the current scenario. It is as crucial an issue for the global South and the global North, given the widespread changes taking place in societies, as they confront challenges of growing inequality, unemployment, poverty, mass migration, environmental disasters, various forms of discrimination and conflicts, to name but a few, within which school mathematics reforms must take place. New phenomena such as actions against infectious diseases (e.g., COVID-19), and the massive shift to online teaching and learning create urgent imperatives related to mathematics curricula. ICMI Study 24 demonstrates that further research and publications on mathematics curriculum reforms are needed, notwithstanding recent volumes [3, 6], which point to the potential for evidence-based mathematics curriculum policy generation and implementation. By continuing to study curriculum reforms across diverse contexts, key messages and lessons may be derived to inform, improve and better conduct future mathematics curriculum reforms.

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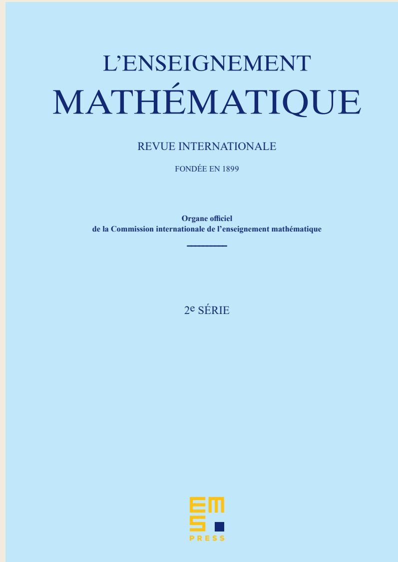

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ERME column

regularly presented by Jason Cooper and Frode Rønning

In this issue, with a contribution by Shai Olsher

CERME Thematic Working Groups

The European Society for Research in Mathematics Education (ERME) holds a bi-yearly conference (CERME), in which research is presented and discussed in Working Groups (TWG). We continue the initiative of introducing the working groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for research mathematicians. Our aim is to extend the ERME community with new participants, who may benefit from hearing about research methods and findings and who may contribute to future CERMEs.

Introducing CERME Thematic Working Group 22 – Curricular Resources and Task Design in Mathematics Education

Group leader: Shai Olsher

Group rationale

Curricular resources and task design carries within its scope a two-fold connected perspective. At the macro level, teachers, lecturers, and students work with mathematics curriculum resources, both digital and traditional, inside and outside the lecture hall or the classroom. Individually or collectively, teachers and instructors select, (re)design, modify, and interact with such resources for lesson preparation, student assessment, and the planning of their courses. These resources (e.g., educative curriculum materials) are the focus of professional development sessions, where mathematics teachers, often with education researchers, design and transform curriculum resources, including blended materials, and in the process develop design capacity and valuable knowledge for teaching. At the micro-level, curriculum resources contain mostly tasks derived from textbooks or other sources. The representation of these tasks in resources, their sequencing, and the lecturers' and teachers' actions during their enactment can limit or broaden the cognitive demand the tasks impose and affect students' views of the subject matter. Thus, they can influence the opportunities afforded to students to make mathematical connections and to develop mathematical concepts, skills, or habits of mind. The literature indicates that tasks play a key role in effective teaching. There has been an upsurge in publications on various aspects of task design (e.g., task features that can help generate certain forms of mathematical activity); methods of task analysis (e.g., analyses of the learning affordances of certain kinds of tasks); and principles for task implementation in conventional and digital learning environments (e.g., factors affecting the fidelity of implementation of tasks in the classroom). Students can also be involved in task design activities to foster their reflections about what they know, understand, and do.

TWG history

The Thematic Working Group 22 – Curricular Resources and Task Design in Mathematics Education was launched at CERME 10 (2017) in recognition of the growing area of research in the field of curriculum resources, their process of design and implementation by various stakeholders, and the specific design of mathematical tasks, by both teachers acting as designers of their own teaching as well as in curricular resources such as textbooks. As presented in the rationale, this field shares approaches, methods, and research topics found in other areas of mathematics education research (e.g., design and use of technological platforms, professional development of teachers), it has its own distinctiveness, and dual focus of the TWG, on both the macro level of curriculum research and micro-processes of task design and implementation research provides also a unique opportunity to connect and discuss the mutual influence of different units of analysis. Historically, only a small number of lecturers involved in tertiary education participated in this focused thematic working group.

TWG topics

The following section provides several exemplary issues which emerged in the TWG "Curricular Resources and Task Design in Mathematics Education" during the 2022 ERME conference. These topics include research on teachers' and students' interactions with curriculum materials, theoretical foundations and methodologies of task analysis, the collaboration between teachers, between teachers and researchers, and students for designing tasks and resources and for analyzing their implementation, and affordances and constraints of digital and conventional tasks and resources.

Concerning students' interactions with curriculum materials, one central question is about the affordances of digital curriculum resources and their influence on the learning process. This question was addressed by several studies using students' reflections about the study content and learning process, which is regarded as an important aspect of learning in K-12, but also in university-level courses. In addition, studies suggested finer definitions and characterizations for given student interactions that could manifest through concrete design features of mathematical tasks such as problem posing and working backward.

Another aspect that was highlighted by studies was the interrelatedness of curriculum resources with the social and cultural context in which they are developed and used. These studies stressed the contextual and cultural influences on teachers' and students' interactions with curriculum resources, raising the issue of how to read and interpret studies about curriculum resources from different social and cultural contexts, and suggesting that results from one context cannot easily be transferred to another. This issue is manifested also in university teaching, which embodies unique contextual influences, but also common phenomena such as the stability and resistance to change among practitioners.

Regarding the analysis of the selection and characterization of tasks by teachers, one assumption was that understanding task perceptions of prospective elementary teachers could help predict their eventual modification and appropriation for classroom use, which can affect teachers' practice. These studies raised the issue of "good" mathematical tasks – good for what purpose? A study of the collaborative task characterization by teachers as suitable for introduction or enrichment and presented several dilemmas. Among the results was the observation that introductory tasks should have an easy entry level and not require pre-knowledge of the upcoming concept, while an enrichment task should require relatively deep conceptual pre-knowledge. Teachers' verbalization of task characteristics was one outcome, but not all tasks met all criteria. A relevant question to university level lecturers is what kind of discourse and analysis is taking place when it comes to tasks in university courses, and do they see a potential benefit in such discussions and studies?

Concerning the use of carefully designed curriculum materials to support the implementation of particular learning goals and enhance mathematical competences, one question is how to design tasks and items so as to provide students with opportunities to make mathematical connections and develop mathematical concepts, skills, and habits of mind. The characteristics of task design can influence the processes that characterize students' interaction with the tasks or items themselves, and among these, studies focused on how the design of specific tasks and resources influences students' reading processes. The interrelation between enhancing efficient reading and enhancing students' comprehension and reasoning, the role played by metacognitive processes in guiding students' reading processes when interacting with tasks with certain characteristics, and the effect of students' age were the focus of different studies. An interesting common facet was methodological: the use of eye-tracking technology to investigate students' reading processes when interacting with the designed tasks and resources.

Several issues related to the design of learning environments, in particular, when the focus of the design is the role of tangible tools or physical objects, considered as products of digital design, or as tools to be combined with digital ones. Studies emphasized the role played by the teacher's orchestration in combining the use of digital and tangible tools, noting that teachers should make the connection between digital and tangible tools clearer if they want students to work effectively with a combination of these tools. In addition, viewing teachers' learning when they make, share, and use manipulatives, studies suggest communities of practice as a framework for understanding teachers' learning of digital fabrication for mathematics education.

Many of the research topics seem a priori relevant for tertiary education, yet it is difficult to even imagine what they would look like there (e.g., what kinds of digital resources are relevant for a proof-oriented mathematics course). While some results that focus on the analysis of digital resources in undergraduate Linear Algebra courses are to be presented at CERME 13, this suggests possible avenues for an emerging field of research.

TWG future

With the growing connectivity of online resources, one future focus of the TWG is expected to be on expanding the populations that interact in new ways with curricular resources (e.g., student's participation in the curricular design process). Yet while other TWG's could share a common interest in curricular design in the form of digital platforms, or task design and implementation as means to focus on teachers' knowledge or professional development, TWG 22 participants have been traditionally focused on a curriculum-centered perspective, which allows gaining insight on the curricular resources and the different agents that interact with them. Curriculum resources are part of every level of mathematics education, from preschool to university courses, and as such TWG 22 warmly welcomes participation and contributions from the wider mathematics community. Specifically, mathematicians who are interested in changing their use of resources and/or researching the effect of such change may find CERME's TWG 22 an interesting venue for learning and collaborating.

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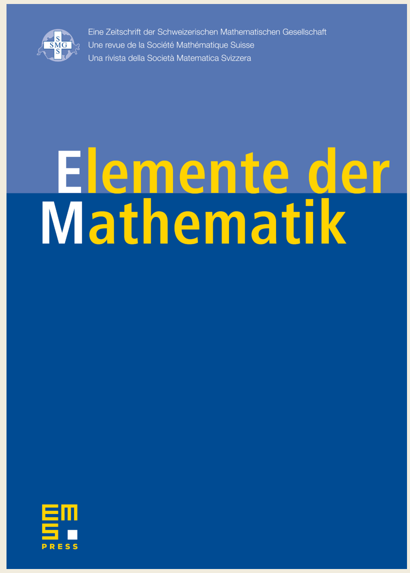

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A REST API for zbMATH Open access

Marcel Fuhrmann and Fabian Müller

1 Introduction

For some time now, zbMATH Open has been offering its services in various formats and through different media. Biweekly printed volumes for long decades in the past, distribution via CD-ROMs and DVDs, and the now standard (and only) access option as an online service, have been incarnations of the service over the years. Typography and design have changed many times in the printed volumes as well as the online version. Furthermore, the online version has offered new ways of distributing the data for specific purposes. The contents of zbMATH Open are available in HTML, BibTeX, and PDF form, as well as via an XML-based OAI-PMH API (more on these three-letter acronyms below). In 2023 a new option was added, a so-called REST API, which makes the data available in a machine-readable fashion, for automated use by researchers. We will outline the features of this new API and describe new use cases made possible by this new way of accessing zbMATH Open.

2 Why do we need APIs?

API is an acronym for "application programming interface." It denotes any part of a software system that makes it accessible through programs and machines, as opposed to human users, and is an important part of technology for software developers. The "interface" component of an API is a system that allows two or more software programs to interact and understand each other without communication problems. One could think of it as a sort of intermediary that helps software programs share information with each other.

Creating an API establishes a common language for wildly divergent systems to communicate to each other. Instead of having to learn each other's specifications, both entities can interact in a common format, which can also be used by any other application or service. The so-called OpenAPI Specification defines a standard, language-agnostic interface to HTTP APIs. This allows both humans and computers to discover and understand the capabilities of the service without access to source code. The user can understand and interact with the remote service with a minimal amount of implementation effort. An OpenAPI definition can then be used by documentation generation tools to display the API, code generation tools to generate servers and clients in various programming languages, testing tools, and many other use cases. This provides advantages for users and data providers alike, granting access in a machine-readable, automatable way for interested partners, e.g., research groups.

3 The OAI-PMH API of zbMATH Open

In 2002, an informal organisation of providers of repository services called the "Open Archives Initiative," or "OAI" for short, created a common protocol that allowed users to automatically harvest metadata from such a service in a unified way using programmatic tools. Unsurprisingly, this was called the "Protocol for Metadata Harvesting," or "PMH." It is supported by numerous providers of scientific or other databases. In mathematics, of course, by far the largest of them is the arXiv. In September 2020, zbMATH Open started to support this protocol as well.[1]

The obvious advantages of using such a common protocol are that it is well established, easy to implement, and has a substantial positive impact on working mathematicians. Aggregators, archives, and bibliometric researchers commonly use the OAI-PMH, and there are ready-made client libraries available in every major programming language. Furthermore, this protocol is well suited for harvesting the entire open collection of zbMATH Open document data.[2] Moreover, it is possible to restrict the harvesting process to well-defined subsets, and to only consume updates since the last download, obliterating the need to download all the data every time.

[1] Available at oai.zbmath.org. For further information, see [2].

[2] The data made available through the OAI-PMH API is provided under a CC-BY-SA 4.0 license, as are all other data offers by zbMATH Open made through APIs. Legal constraints by publishers that are outside the control of zbMATH Open are currently restricting the amount of data that can be distributed through APIs.

4 Why do we need another API?

Although the OAI-PMH API has numerous advantages – not least among them its widespread adoption – it also has various shortcomings:

- The standard format supports only a very limited set of metadata (although additional formats can be defined).
- It is XML-based only.
- It only supports filtering by time and by subset.
- It is only designed to retrieve data, not to update or add any.

Due to this lack of flexibility, we decided to create a new REST API for zbMATH Open, with the OAI-PMH API as a starting point. The acronym "REST" stands for "representational state transfer" and is a style of software architecture that defines how different systems communicate, again improving reusability and interoperability. An API that complies with these REST principles is known as a REST API.

The new zbMATH Open REST API[3] now gives access to a number of additional benefits. First, improving the data set offered by OAI-PMH, a complete set of bibliographic metadata for zbMATH documents is available to any interested party. The complex metadata structure of any document covers a vast variety of information. A quick overview of the available data sets is shown in Figure 1.

Furthermore, the API offers access to information about published authors and their work, their awards, any external IDs, and so on. Other data endpoints cover information about the current version of Mathematics Subject Classification codes for documents (see [1]), metadata information about books and journals, any published document available within the zbMATH Open database, and metadata information about research software. All these available data sets are similar to what is available at the zbMATH Open website. The difference is that the data is machine-readable and therefore can be easily used for any purpose, e.g. any bibliographic research efforts.

Most importantly, it is now possible to search in the REST API in exactly the same way as on the website zbmath.org, including arbitrary combinations of logical operators and filtering by all supported fields (in particular, arbitrary time frames).

The search result is a machine-readable status report in JSON format, containing the requested data, but also including information about the search execution (for example, whether it was successful or not, and if not, why). In the future, other formats will be supported as well (e.g., XML).

All the information is documented in the Swagger UI (see the screenshot of Figure 2),[4] so that, based on the information there, other developers may create their own APIs for their own

Figure 1. Screenshot of data structure of an example result, as seen in the browser.

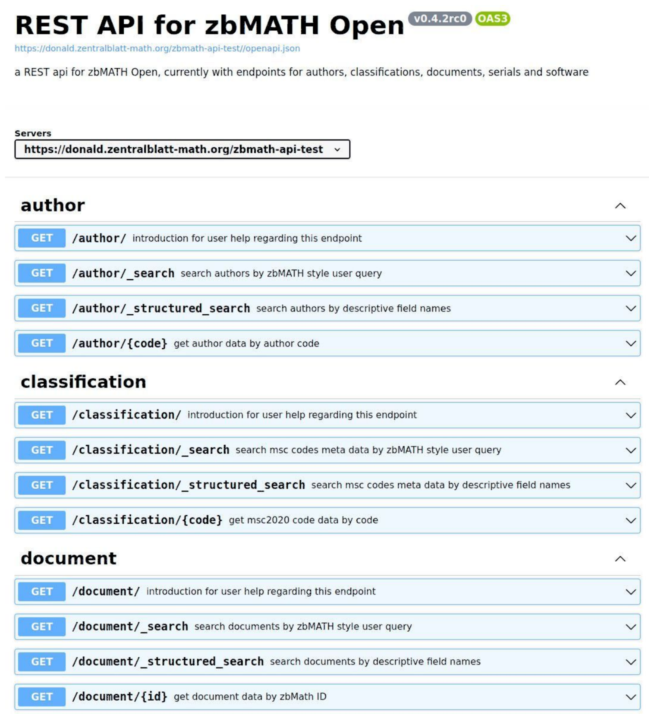

Figure 2. Screenshot of the user interface of zbMATH Open REST API.

[3] Available at https://api.zbmath.org.

[4] For more information, see https://swagger.io/docs/.

research projects. Like in the case of the OAI-PMH API, in some instances, parts of the information (e.g., abstracts or references, based on the agreements with the respective publisher) need to be redacted for legal reasons. The API takes care of that automatically.

5 Outlook: Pushing data via the REST API

Currently, we are working on adding more features to the zbMATH Open API. So far, data can only be retrieved from the database. A next step is to enable users to also add new data in an automated way. We identified a number of use cases for that, which would improve the updating process of the zbMATH Open database and facilitate the incorporation of input from the user community.

The most straightforward use case would be to enable the upload of bibliographic metadata information about documents from publishers directly. Although some larger publishers have their own data delivery processes, for many (especially smaller ones) it can be quite helpful to offer a well-documented way of adding their bibliographical data to zbMATH Open.

Other use cases involve uploading references or external IDs (like DOI or arXiv identifier) of existing documents to enrich the data contained within the zbMATH Open database. Moreover, the upload of, and therefore the access to, full-text documents is planned to be possible in the near future. This requires resolving any licensing issues, in particular the uploader (e.g., the original author) needs to certify that they have the right to upload the full text for distribution by zbMATH Open.

Finally, adding or correcting new information about authors and their connection to existing articles (if this information was missing before) is going to be possible via the zbMATH Open REST API.

6 Final thoughts

To summarize, the new zbMATH Open REST API will be a useful tool for automated access to the data stored within the zbMATH Open database. It will extend the possibilities currently offered by the OAI-PMH API (without superseding it). With the oncoming features of adding and/or updating information to the database in an automatable way, open access for zbMATH Open data will continue to expand.

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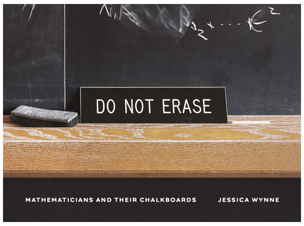
Book review

Do Not Erase: Mathematicians and Their Chalkboards
by Jessica Wynne

Reviewed by John J. Watkins



Chalk – the soft white sedimentary rock many of us use on chalkboards to express our mathematical thoughts and ideas – is the central character in *Do Not Erase: Mathematicians and Their Chalkboards*. This remarkable book exposes many layers, as often happens in geology, each layer a story that is beautifully told by photographer Jessica Wynne.

Wynne tells us in her introduction something about the way her idea for this book came about. Her parents were teachers and on weekends she and her sisters would often play in classrooms that were "hot and musty and smelled of teenagers and chalk." Years later she led a group of photography students on a study abroad trip to India where a school principal took them up to the roof of a schoolhouse. Chalkboards filled with lessons in Hindi ran all along the low wall surrounding the roof. Wynne photographed each chalkboard, beautiful and yet inaccessible to her.

In *Do Not Erase* Wynne similarly displays more than a hundred photographs of chalkboards that express the beauty of mathematics and the creativity of mathematicians from around the world. Each photograph is accompanied by a short essay on the facing page where an eminent mathematician talks about their chalkboard, explaining the images on the board and the collaborative human process that produced them.

My favorite chalkboard in this book is one that, as mathematician Nancy Hingston explains, could be understood by almost anyone, even a creature in another galaxy (see Figure 1). It is simplicity itself. On the left she has drawn a rather ordinary looking doughnut, except that it has two holes instead of just one, and below it the exact same doughnut, but this time with the arch from one hole passing through the other hole so that the resulting doughnut is now a bit tangled. Then on the right side of the chalkboard she has a very long "doughnut" with six holes and next to it an absolutely gorgeous drawing of a mathematically equivalent, incredibly tangled object with the same six holes that immediately reminds one of a wonderful sculpture by Henry Moore.

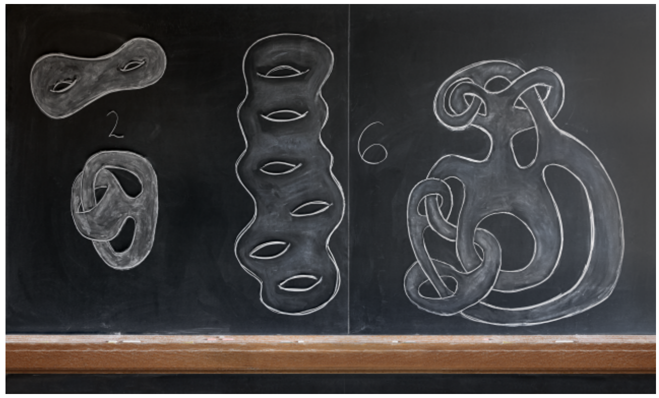


Figure 1. A photo of Nancy Hingston's chalkboard from the book.

My favorite essay in this book is by Fields medalist Terence Tao on the twin prime conjecture: we have known since the time of Euclid that there are infinitely many prime numbers, but we do not yet know whether there are infinitely many *twin primes* such as 17 and 19, or 179 and 181, or 1487 and 1489. Tao uses a wonderful metaphor involving fog and mist, mountain peaks and fertile valleys, to describe the interconnected web of ideas and results that surround this famous unsolved problem, and his chalkboard is filled with formulas, theorems, still more conjectures, and various arrows linking everything together.

Another essay tells us about two mathematicians, who had been working obsessively for a very, very long time on a problem called the Willmore conjecture, when they suddenly cracked the problem during a family Thanksgiving Day celebration as things finally clicked for them. André Neves and Fernando Codá Marques had just found the exact optimal shape for a surface with one hole that minimizes a natural bending energy called the *Willmore energy*. Their optimal surface is lovely and looks like a very fat doughnut with a tiny hole in the middle that you could just barely get your little finger into. After nailing their solution at long last, it looked so natural and simple to them that they could finally just relax and go back and enjoy the rest of Thanksgiving Day with everyone else.

Do Not Erase is a mathematics book that at its core is fundamentally about beauty and creativity. My copy currently resides on a coffee table where it joins several of other art books. It now sits quite comfortably right next to *Monet: A Retrospective* and photographer John Fielder's *Colorado 1870–2000*.

Jessica Wynne, *Do Not Erase: Mathematicians and Their Chalkboards.* Princeton University Press, 2021, 240 pages, Hardcover ISBN 978-0-691-19922-1.

John Watkins is an emeritus professor of mathematics at Colorado College, with wide interests in graph theory, ring theory, and the history of mathematics. John also loves skiing and climbing the mountains of Colorado.

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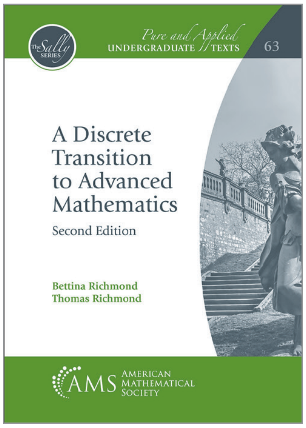

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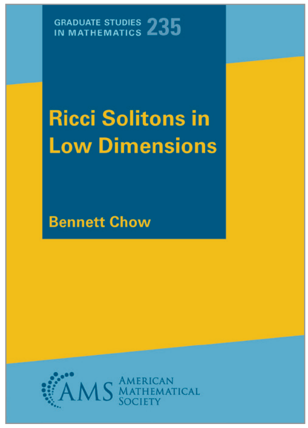

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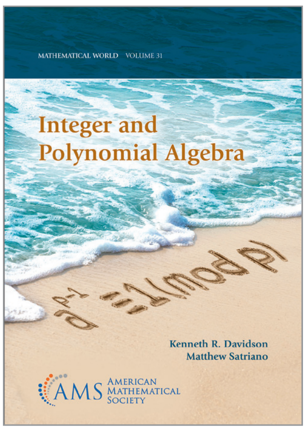

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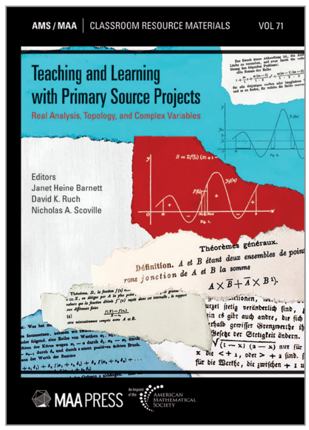

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