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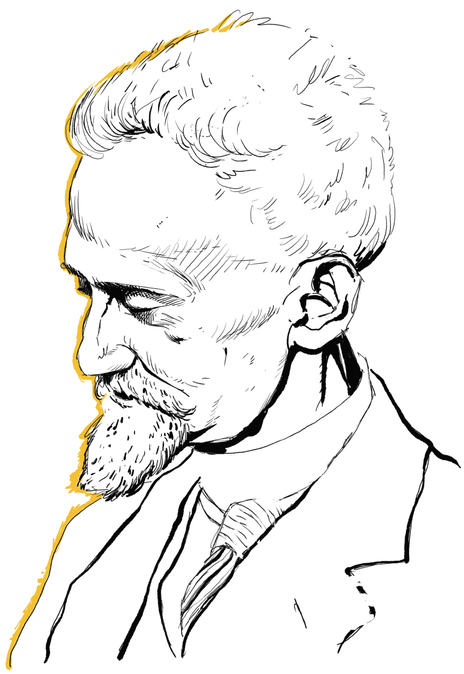

The cover illustration is a portrait of Felix Hausdorff by A. B. Araújo.

Photo by Jim Høyer, University of Copenhagen.

In my last message I emphasized how academic freedom is a core value of the EMS and that we are concerned to be seeing it challenged in a changing world. Indeed, since then, the EMS published a statement in support of academic freedom.[1]

Academic freedom is being challenged in many places, and mathematicians and other scientists are being targeted. The EMS also published a statement expressing concern for our esteemed colleague Professor Mikhail Volkov from Russia who has been forced to retire from his position at Ural Federal University and has been denied the possibility of traveling freely.[2]

Let me now turn to something more positive and highlight some of the new initiatives we have been planning at the EMS. The first is not really new but rather the relaunch of our Strategic Activities that we had to pause for several reasons but are excited to be returning to. We plan to open up for these activities initiative again in the fall. In its new iteration it will be our EMS Topical Activity Groups (TAGs) who will have the opportunity to apply for these Strategic Activities to promote larger scale inclusive events across geographic boundaries and disciplines.

EMS is also continuing its ambition to engage with policymakers and is in the process of forming an external policy group to strengthen this engagement. As I mentioned in an earlier message, we have already formed an EMS–ERC working group that interacts closely with the ERC in support of their goal to widen participation. ERC funding is of course very competitive, but we believe that ERC and, in particular, the mathematics panel PE1 are working hard to support mathematics across Europe and across sub-disciplines. This is not always easy, and we hope together with the EMS–ERC working group to increase the perception of ERC as inclusive and welcoming to all areas of mathematics and all aspects of diversity. We believe there is potential for an increased number of excellent applications to ERC.

Our community engagement manager Enrico Schlitzer is planning a campaign to increase EMS membership, especially among early career researchers, those who do not need to pay for membership the first few years. One particular initiative is to launch masterclasses for early career EMS members on different topics of interest to them. One such topic is the question of maneuvering in the complicated landscape of publishing. While this is particularly aimed at early career researchers, I think this may also prove very useful for those of us who are more experienced. In particular, I often find, talking to colleagues, that there are a lot of misunderstandings about the whole issue of open access. Luckily, there is general support among mathematicians for open access but also a general belief that this is not an important issue worth worrying about because most of us use repositories like the arXiv. I disagree and think we ought to take this issue seriously and concern ourselves with it. Many research funding agencies like, e.g., ERC subscribe to the Plan S open access requirements. Using repositories like ArXiv is Plan S compliant as the green open access route. It, however, requires the refereed author version of the manuscript to be immediately available upon publication and most commercial publishers do, in fact, not allow this. Many of us either ignore this or do not need to care because we live in a rich country (or belong to a rich institution) that has an agreement with commercial publishers. This is certainly not a fair system and does not support "equal access to open access." There are many other issues regarding open access that I will not get into here. My main mission is to encourage you to stay informed.

Let me finally remind you of the upcoming EMS supported first Indo-European Conference in Mathematics to be held in Pune, India, January 12–16, 2026.[3]

I hope to see many of you there.

I wish you all a great summer and a great start to the new academic year.

Jan Philip Solovej
President of the EMS

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[1] https://euromathsoc.org/news/european-mathematical-society-statement-on-academic-freedom-169
[2] https://euromathsoc.org/news/european-mathematical-society-expresses-concern-for-professor-mikhail-volkov-168
[3] https://euromathsoc.org/news/the-first-indo-european-conference-in-mathematics-to-be-held-in-pune-india-(january-12-16-2026)-160

Brief words from the editor-in-chief

Dear readers of the EMS Magazine,

In this issue of the EMS Magazine, you will find another article by a winner of the 2024 EMS Prizes – this time by Adam Kanigowski – as well as a contribution from Reinhard Siegmund-Schultze, recipient of the Otto Neugebauer Prize, awarded for original and influential work in the field of the history of mathematics.

As usual, the Magazine includes a range of articles on diverse topics, along with our regular columns on discussions, societies, education, and book reviews.

In celebration of the *International Women in Mathematics Day* – May 12, we are pleased to feature a special contribution: a piece based on a panel discussion organized by the *Anillo Matemáticas y Género* (Mathematics and Gender) project, funded by the Chilean National Agency for Research and Development. The panel focused on issues of social justice, equity, and inclusion. Rather than presenting the discussion in the form of a traditional article, we chose to convey its spirit through a series of cartoons created by Coni Rojas-Molina.

Finally, you may have noticed that Issue 135 of the EMS Magazine reached your mailbox later than expected. I apologize for this delay and want to assure you that we – both the editorial team and EMS Press staff – are working hard to streamline our processes to ensure timely delivery of future issues.

Donatella Donatelli
Editor-in-chief

Chaotic properties of smooth dynamical systems

Adam Kanigowski

1 Introduction

The fields of dynamical systems and ergodic theory are concerned with evolution of a given system under time. Given an initial condition (or a set of initial conditions) one is interested in its long term behavior. One of the founders of modern dynamical systems is Henri Poincaré. The famous *Poincaré recurrence theorem* [47] states that a measure-preserving dynamical system will return (infinitely often) arbitrarily close to its initial state. Subsequent fundamental achievements came with *ergodic theorems*, proven by John von Neumann in the L^2 setting [41] and by George Birkhoff in a stronger almost everywhere situation [3]. They imply that, under the natural ergodicity assumption on the system, the time average is equal to the space average. In particular, they yield quantitative information on the set of return times in the Poincaré recurrence theorem. These theorems gave first general statements on random behavior in dynamics. This naturally sparked questions on what finer chaotic properties one can (typically) observe in naturally appearing dynamical systems, which became one of the central directions of study in the realm of (smooth) dynamical systems in the second half of the last century. More precisely, people were interested in random behavior of deterministic systems, i.e., in how sensitive can the evolution of a deterministic system be (depending on the initial condition). One of the main motivations was coming from dynamical systems modeling real-life phenomena, such as evolution of weather or motion of planets in the solar system. Most notably, in the 1960s Edward Lorenz [38], a meteorologist and mathematician, was conducting an experiment on predicting the weather in a given place on earth. He came up with a deterministic dynamical system modeling the evolution of weather and noticed that even a tiny perturbation of the initial condition would typically lead to a completely different outcome after a relatively short time. This phenomenon became known as the *butterfly effect* and this led to the discovery of the so-called (chaotic) *Lorenz attractor* (see Figure 1[1]).

At around the same time some fundamental mathematical tools were developed which allowed to rigorously *measure* randomness of deterministic dynamical systems. The entropy of an abstract measure-preserving system was defined by Kolmogorov and Sinai in the late 1950s [35, 56]. Systems with positive entropy would then exhibit strong randomness, at least in 'parts' of the system. Other fundamental objects defined for smooth systems on manifolds are the *Lyapunov exponents*. Positivity of Lyapunov exponents is another manifestation of strongly chaotic properties. There are also other qualitative and quantitative properties that yield a hierarchy of possible chaotic behaviors. By qualitative properties we mean properties that can be defined for an abstract dynamical system. Some main examples are: ergodicity, weak mixing, mixing, absolutely continuous spectrum, positive entropy, K-property and Bernoulli property. Quantitative properties use some additional structure of the system (in most cases smooth or piecewise-smooth structure). Some main examples are: rates of deviation of ergodic averages, rates of weak mixing and mixing (decay of correlations), central limit theorem, large deviations. The last seventy years have seen a spectacular development in studying chaotic properties of

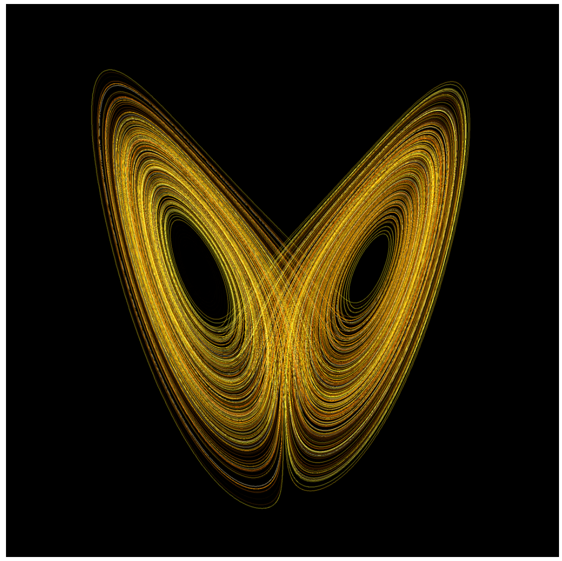

Figure 1. The Lorenz attractor. Nearby initial conditions diverge at exponential speed.

[1] Taken from https://commons.wikimedia.org/wiki/File:Lorenz_system_r28_s10_b2-6666.png.

(smooth) dynamical systems. In the sections below I plan to give a selective account of some important classical results in the theory, highlight some of the more recent developments, and discuss some open problems and future directions.

2 Randomness in deterministic systems

One of the fundamental discoveries in smooth dynamical systems in the last seventy years is the fact that deterministic (smooth) dynamical systems can behave very randomly. In particular, in many aspects such systems can behave like a system generated by a sequence of independent coin tosses. The first class of smooth systems for which such phenomena where observed is that of the *uniformly hyperbolic* systems, or *Anosov* systems [1], i.e., systems for which the tangent space at every point splits into contracting and expanding directions for the derivative of the map f and moreover these directions are invariant under f. This in particular implies that all *Lyapunov exponents* are non-zero.

2.1 Uniformly hyperbolic systems

The main class of uniformly hyperbolic examples is given by *hyperbolic matrices* with integer entries. Rather than giving a general definition of a uniformly hyperbolic system, we will focus on these examples which should provide some intuition for the general situation.

Example 2.1. Let $C = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Note that $C\colon \mathbb{R}^2 \to \mathbb{R}^2$ and since C has integer entries, it descends to a map $\widetilde{C}\colon \mathbb{T}^2 \to \mathbb{T}^2$. Moreover, $\widetilde{C}$ preserves the Lebesgue measure since $\det C = 1$. The matrix C has eigenvalues $\lambda_1 = \frac{3+\sqrt{5}}{2} > 1$ and $\lambda_2 = \lambda_1^{-1} < 1$, with corresponding eigendirections E_1 and E_2 given by the vectors $v_1 = (\frac{1+\sqrt{5}}{2}, 1)$ and $v_2 = (\frac{1-\sqrt{5}}{2}, 1)$, respectively. This implies that for the derivative of $\widetilde{C}$ (which is C itself) the tangent space splits into the contracting E_2 and expanding E_1 directions. This hyperbolic matrix is called the *Arnold's cat map*.[2]

More generally, one can take a hyperbolic matrix $B \in SL(n, \mathbb{Z})$ (i.e., with no eigenvalues on the unit circle) and consider the action of the associated map $\widetilde{B}$ on $\mathbb{T}^n$.

Readers not familiar with dynamical systems may restrict their attention to hyperbolic matrices in what follows. The main mechanism that allows to study chaotic properties of uniformly hyperbolic systems is the existence of so-called *Markov partitions*, constructed in [5, 57]. They allow showing that a uniformly hyperbolic dynamical system is 'almost' Hölder conjugate to a *subshift of finite type* (SFT). SFTs are a class of symbolic systems which essentially exhibit the same statistical and chaotic properties as systems generated by sequences of independent coin tosses (Bernoulli shifts). Rather than defining Markov partitions in full generality, we will just present one simple example of a Markov partition for a non-invertible map. This should give the readers a general idea on usefulness of such partitions.

Example 2.2. Let $f(x) = 2x \mod 1$ on $\mathbb{T} = [0, 1)$ with addition mod 1. Consider a partition of $\mathbb{T}$ given by $P_0 = [0, 1/2)$ and $P_1 = [1/2, 1)$. Note that we can code the orbit of any x by a sequence $\omega \in \{0, 1\}^{\mathbb{N}}$ in the following way: x corresponds to ω if for any $i \in \mathbb{N}$, $f^i(x) \in P_{\omega_i}$. This coding is uniquely defined (and invertible) for points x whose orbit $f^i(x)$ avoids the discontinuities at 0 and 1/2. So for all but countably many points the system f is Hölder conjugate to the full shift σ on $\{0, 1\}^{\mathbb{N}}$. The countably many exceptions account for the word 'almost' in the paragraph above. Then $\{P_0, P_1\}$ is a Markov partition for f. One can then study statistical properties of f by studying σ, which is much easier.

Unfortunately, the powerful tool of Markov partitions is usually not available for systems that are not hyperbolic, i.e., for partially hyperbolic systems, or, in particular, systems with some Lyapunov exponents equal to 0. This is a much wider class of systems for which there might be some directions in the tangent space which do not contract or expand (as fast) when iterated by the derivative. The simplest examples would be an integer matrix like in Example 2.1, but for which some of the eigenvalues have absolute value one. Another class of straightforward examples would be time-1 maps of *Anosov flows* (think about the geodesic flow on the unit tangent bundle of a hyperbolic surface). The flow direction is then an isometry. Statistical and chaotic properties of systems for which not all Lyapunov exponents are non-zero are much less understood. There are two main classes for which one usually applies completely different methods: (a) systems with positive entropy, or equivalently, with some of the Lyapunov exponents different from zero (i.e., there are some directions in which we see contraction or expansion); and (b) systems of zero entropy (with all Lyapunov exponents equal to zero). We will discuss these two classes in separate subsections.

2.2 Systems with positive entropy

As mentioned above, the general theory of systems with some Lyapunov exponents equal to zero is much less understood and there is no general framework to study chaotic properties in this class. In the last century, significant partial progress was made on understanding qualitative (ergodic) properties of such systems that we will now describe.

Qualitative properties

Qualitative properties can be defined for abstract measure-preserving systems. The strongest such property is being *Bernoulli*, i.e.,

[2] See https://galileo-unbound.blog/2019/06/16/vladimir-arnolds-cat-map/.

being measure-theoretically isomorphic to a Bernoulli shift (a system generated by a sequence of independent coin tosses). A weaker property is called the *K-property* (K for Kolmogorov) which is equivalent to saying that every factor of the system has positive entropy [54]. Some weaker properties are mixing, weak mixing and ergodicity, but for simplicity we will focus here on the K and Bernoulli properties only. Anatole Katok [30], and then Brin, Feldman and Katok [6], have shown that every smooth manifold of dimension at least 2 supports a Bernoulli diffeomorphisms, and so there are no topological restrictions for Bernoullicity in the smooth setting. The main development for establishing K and Bernoulli properties was to use some geometric structures derived from the existence of non-zero exponents (stable and unstable manifolds). Sinai [56] has shown that the K-property follows from ergodicity of the so-called *unstable foliation*. This is still the main mechanism for showing K-property for smooth dynamical systems, used widely in literature, see for example [8]. The theory of Bernoulli shifts was developed in the second half of the last century, culminating in *Ornstein's theory* which allowed for their full classification in terms of entropy [42]. In particular, the *very weak Bernoulli* property introduced in [42] turned out to be a very useful tool in establishing Bernoullicity of smooth systems with (some) non-zero exponents. This came with the work of Ornstein and Weiss [44], where they used a geometric method to establish the very weak Bernoulli property for hyperbolic toral automorphisms, as well as for geodesic flows. Even though the systems they considered where algebraic, the introduced method proved to be applicable in a much more general setting. Specifically, the method is based on finding, for a given N, a matching between two pieces of typical unstable manifolds, i.e., a measure-preserving map Θ_N between these unstable manifolds such that the points x and $\Theta_N(x)$ are close for most of the iterates under the dynamics when the iterates are taken from the set $[0, N]$. This mechanism is to this day the main technique to study the Bernoulli property for smooth systems and has been used in various settings, e.g., [7, 10, 11, 36, 46, 48, 49]. Let us mention that despite considerable progress in the study of qualitative properties of smooth systems there are a number of open questions regarding whether a given dynamical system is Bernoulli or not. In particular, in [16] the authors showed that a smooth diffeomorphism which is exponentially mixing, is in fact Bernoulli. The methods used in [16] seem to suggest that there is a 'competition' between mixing rates of the system and growth on the center space which would determine if a system is Bernoulli or not. It is also not known if exponential mixing (for Hölder functions) implies Bernoulli in the non-smooth (e.g., symbolic) setting. Finally, let us mention that Kolmogorov initially conjectured that the K and Bernoulli properties are equivalent. This was shown not to be the case by Ornstein [43] in the abstract setting and independently by Katok [31] and Rudolph [55] in the smooth setting. In [29] the authors showed that K does not imply Bernoulli in dimension 4. It is known that it does in dimension 2 [45]. The problem in dimension 3 is one of the big open questions in the field [32].

Quantitative properties

Quantitative properties are those which use the fact that the system is a smooth diffeomorphism acting on a smooth manifold. Main quantitative properties are quantitative mixing (or decay of correlations), central limit theorem (CLT), large deviations, quantitative ergodicity. Quantitative properties are much less understood than qualitative ones. There are still open questions regarding interplay of quantitative and qualitative properties. In what follows we will mostly focus on quantitative mixing and central limit theorem. In fact, if the system has fast decay of correlations of all orders, then a CLT follows, see, e.g., [4]. The problem of establishing quantitative mixing rates for a given smooth system is difficult and widely open in general, even though a lot of progress was made for algebraic systems where one uses representation theory, see, e.g., [25, 34, 40]. The most successful methods of establishing (exponential) rates of mixing is by using *transfer operators* and their *spectral gaps*, initiated in the fundamental work of Dmitry Dolgopyat [13]. In fact, it seems that this method (or variants thereof) is the only approach for establishing rates of mixing in the smooth setting. It has been used by many authors in various situations, see, e.g., [37, 60] and references therein. In fact, this method also allows one to get other statistical properties, such as large deviations and CLT. It is based on translating the problem on (exponential) mixing into a problem concerning the spectral nature of a naturally associated operator (the transfer operator). One can then use tools from complex and harmonic analysis to study the spectrum of the system on so-called *anisotropic Banach spaces*, see, e.g., the survey [12] and references therein. As mentioned, existence of a spectral gap usually implies all statistical properties, and so if one wants systems with exotic behavior (enjoying some, but not all of the properties) one needs to introduce different methods. In particular, it is not known if there exists a smooth dynamical system which is exponentially mixing, but not exponentially mixing of order 3. In [14] the authors have produced exotic examples of smooth dynamical systems (with finite smoothness) which have zero entropy and still satisfy the CLT. A C^∞ flow of zero entropy and with CLT was constructed in [15]. Existence of a C^∞ map with zero entropy and with a CLT is still an open question. As mentioned in the previous section, it is known that every manifold of dimension at least 2 supports a Bernoulli diffeomorphism. It is not known if every manifold of dimension at least 2 supports a smooth diffeomorphism which satisfies a CLT as well as a smooth diffeomorphism which is exponentially mixing.

2.3 Systems with zero entropy

The world of zero-entropy systems in general is too wide to have any meaningful general structures. According to, e.g. [33], there are two rather different main classes of zero-entropy systems, *elliptic* systems and *parabolic* systems. The main model for elliptic systems is provided by irrational rotations on compact Abelian groups (irrational rotations of the circle being the main example), but there are also less rigid examples of elliptic dynamics [20]. Elliptic dynamics is usually characterized by a rather non-chaotic, tame behavior where *mixing* is often precluded. We will focus on the class of parabolic systems and briefly highlight typical chaotic features in this class. Unlike the hyperbolic world, there is no strict definition of a parabolic system. It is rather a set of common characteristics that is an indicator of parabolic behavior. The prime class of parabolic systems are *horocycle flows*. In algebraic language, the horocycle flow (h_t) is given by the action of the matrix $\left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$ acting on finite-volume quotients of the group $SL(2, \mathbb{R})$. Ergodic and statistical features of horocycle flows are the model behavior in the class of parabolic systems. More generally, unipotent flows on quotients of semisimple Lie group constitute a wider class of (algebraic) parabolic systems. Yet another class of (non-algebraic) systems that can be classified as parabolic (even though they also exhibit some elliptic-like behavior) is the class of smooth flows on surfaces, known as locally Hamiltonian flows (see Figure 2).

This is the class in which recently many tools from homogeneous dynamics were adapted to the non-algebraic setting. There are also systems which share both elliptic and parabolic characteristics: interval exchange transformations and translation flows, nilrotations on nilmanifolds, reparametrizations (or perturbations) of all the above-mentioned examples. One of the typical features of parabolic systems is that the orbit growth (or orbit divergence) happens typically at polynomial (or logarithmic) rate. This is the difference with hyperbolic systems, where the growth is exponential, and also with elliptic systems, where the growth is often sub-logarithmic. Moreover, typically the divergence of all nearby conditions occurs along well-understood, structured directions. Some characteristic features that one usually observes for parabolic systems are: small set of invariant measures (often a unique such measure), polynomial (or logarithmic) rates of mixing, which are usually obtained by the *mixing via shearing mechanism*, polynomial deviation of ergodic averages, absolutely continuous or Lebesgue spectrum, variants of Ratner's property (shearing in the direction of the dynamics), structured sets of *joinings*, 'nice' orbit closures. These properties for horocycle flows (or more generally unipotent flows) were established by a number of authors: Marina Ratner in her seminal work [50, 52] studied orbit closures and measure rigidity for unipotent flows, in [51] Ratner also studied rates of mixing for horocycle flows. Higher-order mixing was established by Marcus [39]. Deviation of ergodic averages was studied in [21] and [59].

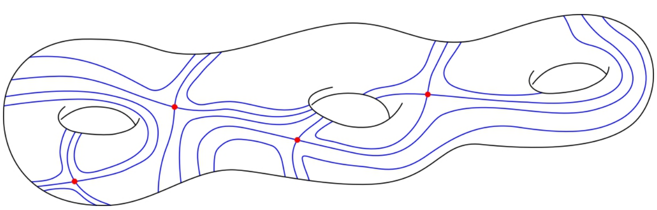

Figure 2. A locally Hamiltonian flow. The red points are fixed points of the flow.

Much less was known until recently for parabolic systems that are not algebraic. This has changed significantly in the last 20 years, in which major progress in the study of spectral, mixing and rigidity properties of these systems was made. Due to limited space we will again only provide a list of references on the topic: deviation of ergodic averages was studied in [22–24]. Mixing properties were established in [17, 53, 58, 61, 62]. Higher-order mixing was studied in [19, 26, 28]. Joinings and rigidity were studied in [2, 27]. Spectral properties were studied in [9, 18].

Even though there has been significant progress on general parabolic systems, there are still many directions which are quite open: spectral features (the spectral type and the multiplicity function) of non-algebraic parabolic systems are still not known in many cases. Optimal rates of mixing and higher-order mixing is another open problem in the area. And despite some partial results, the theory of joinings is another problem that is rather widely open.

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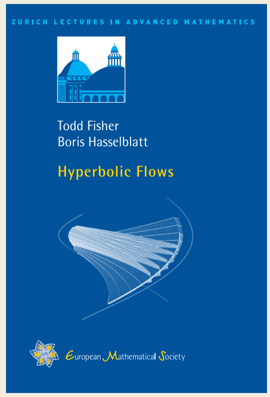
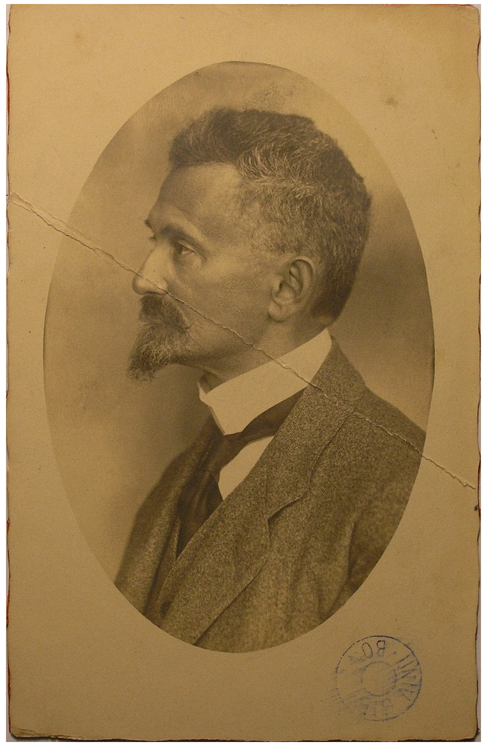


No survival for an elderly mathematician: Felix Hausdorff's failed emigration and death

Reinhard Siegmund-Schultze

This paper describes the failed attempts of the founder of modern set-theoretical topology, Felix Hausdorff, to save his life from the anti-Semitic Nazi threat by emigration to the United States in 1939. Although he was still mathematically productive, Hausdorff's advanced age was his undoing, making him less attractive to the Americans. The paper is a translation of the 2021 German original [Mitt. Dtsch. Math.-Ver. ***29*** *(2021), 132–136*].



Felix Hausdorff (1868–1942). (© ULB Bonn: Portr.)

Introduction

It has often been discussed why the German Nazis, who came to power in 1933, "harmed themselves" and left so much scientific and cultural capital to America, in particular, through the mass expulsions. This negative side ("push") of emigration has been partly explained by the fact that at that time, in the 1930s, the importance of basic sciences was still less visible worldwide. A second reason for the neglect of science by the Nazis was the priority of the ideology of anti-Semitism in the chauvinistic Nazi policy, which was aimed at preparing for war in the long term. On the other, positive side ("pull"), reference was made to the role of far-sighted foreign scientists (there were also short-sighted and xenophobic ones, even in the USA), who particularly supported the young and promising among the mostly Jewish immigrants, including mathematicians. However, it became clear that foreign countries set priorities early on, so that older scientists had poor chances of emigration from Germany and thus poor chances of survival. This was particularly true in the period after the outbreak of the war in Europe in September 1939, which led to increased political caution and isolation in all countries. Help that was given despite these difficult conditions often depended on individual humanistic acts in the host countries resisting bureaucratic limitations. In any case, we today have no reason to feel morally superior, especially not in view of the worldwide rise of nationalism since 1990 following the fall of the Iron Curtain and given the fact that thousands of refugees are drowning in the Mediterranean these days.

The fate of Felix Hausdorff (1868–1942), one of the great pioneers of modern set-theoretical topology with his 1914 book *Grundzüge der Mengenlehre*, may serve us mathematicians as an illustration of the above remarks and remind us of our responsibility in the present.

Hausdorff after the Nazis came to power in 1933 and until the November pogrom of 1938

In 2021, just over a century and a half after Felix Hausdorff's birth, the exemplary ten-volume German edition of Hausdorff's 'Collected Works' [7] has come to a successful completion. Within this edition, an extensive biography appeared in 2018 as Volume IB. It runs to over a thousand pages, was written by the late Egbert

Brieskorn and by Walter Purkert. A shortened version in English appeared in 2024 [2]. In it, "Hausdorff's life during the Nazi dictatorship" [2, pp. 415 ff.] is described as an example for the increasing disenfranchisement of Jewish academics after 1933 in material, as well as in general social and cultural terms. In the first years of the dictatorship, of course, Hausdorff had to bear his share of the public humiliation of German Jews, but his personal material situation was still satisfactory. At the beginning of 1935, at the age of 66, he was given emeritus status as part of a general Nazi law to streamline the administration [2, p. 422]. However, this put Hausdorff in a much better position than his colleague Otto Toeplitz in Bonn, for example, who was suspended in the autumn of the same year, at the age of 54, on the basis of the anti-Semitic "Nuremberg Laws." When the Nazi administration announced in December 1935 that Hausdorff's emeritus salary would be cut by around 50 per cent, they realized a few days later that this violated their own earlier laws, and they revoked this decision. The biographers report that Hausdorff was still active mathematically at the time and published seven more papers between 1933 and 1938, "all in Polish journals: six in *Fundamenta Mathematicae* and one in *Studia Mathematica*." [2, p. 429]. One might add that these papers were essentially devoted to set-theoretical topics, in which these two journals specialised. Moreover, like Hausdorff's works in general, these papers were all written in German. Hausdorff's undiminished creative vigour in his later years is also evident from unpublished manuscripts from those years. His passable material situation, which was also secured by private assets, his attachment to the German language, and his close mathematical relationships with the Poles may explain why he did not try to leave Germany at the time. His biographers add:

> Also his personal conviction – expressed in the presence of his daughter Nora – "that they probably won't do anything to us old people," helps to explain why Hausdorff made no attempt to emigrate during the years when that might still have been possible. He did not imagine, as few did at that time, that he and his family would witness a series of new decrees and orders, especially from 1939 onward, some of which directly affected Jewish emeriti and retired civil servants. The Nazi state passed these as part of its effort to oust Jews from economic life and gradually expropriate all Jewish property and wealth through "Aryanisation," a process that accelerated after the November 1938 pogrom. [2, p. 424]

Hausdorff's failed escape attempts

Indeed, the pogrom of November 9, 1938, often trivialised as "Reichskristallnacht," changed everything. A series of new repressive measures against German Jews, often demagogically referred to as "protective custody" or "atonement," was then initiated; Hausdorff was affected by the expropriation of a quarter of his saved assets, as were all his fellow Jewish citizens [2, p. 442]. In the months following the November pogrom, Hausdorff must have come to the conclusion that his life was in immediate danger and that only emigration could save him. The English language would not have been an insurmountable hurdle for the well-educated Hausdorff. He had already taken English lessons as a 25-year-old in 1893. Walter Purkert tells me in an email dated April 1, 2019:

> English sometimes went through his mind quite easily: in 1932 or a little later he studied Menger's book "Curve theory" very carefully and wrote down 132 sheets of notes on it (Nachlass, capsule 47, fascicle 985). He was particularly dissatisfied with Menger's proof of the so-called Menger's theorem in graph theory [*n*-Bogen-Satz]. He gave a much better proof in his manuscript; underneath he wrote [in English]: "That was a hard piece of work!" (sheet 116).

Hausdorff's attempts to emigrate at the beginning of 1939 are documented indirectly from contemporary correspondence of mathematicians expelled from Germany to America.[1]

On February 10, 1939, Richard Courant, the mathematician in New York who had been expelled from Göttingen, wrote a letter to Hausdorff in English. He had dictated the letter to his American secretary, and only a carbon copy has come down to us.[2] In it, Courant responds to an enquiry from Hausdorff dated January 31, 1939, which is not known to us in detail. Courant begins by saying: "Of course, every mathematician in the world is under a great obligation to you and I certainly always have felt this way." However, he immediately dampens Hausdorff's expectations:

> However, the circle of my personal influence is extremely narrow and offhand I do not see within it any concrete possibility, but I have immediately communicated with Weyl hoping that through a certain connection he has, something can be done. [2, pp. 442–444] and [1, p. 238]

Indeed, Hausdorff would hardly have been in a good place at Courant's new institute at New York University, with its strongly applied and industry-orientated profile. One might initially assume that Courant's allusion to Hermann Weyl refers to his position at the Institute for Advanced Study (IAS) in Princeton. Einstein and John von Neumann also worked there, pure mathematical and

[1] The following quotations are taken from documents that I found during archival visits to the USA in the 1990s and which were published partly in the exhibition catalogue [1] and partly in the Hausdorff biography [2] mentioned above.

[2] In the Courant Papers, New York University Archives.

physical research was in the foreground, and there were no first-year students. In the 1930s, many emigrants received research fellowships at this institute, which enabled them to make a start in the American host country (R. Brauer, Siegel, Artin).[3]

However, it is clear from a letter that Courant wrote to Weyl on the same day, February 10, 1939, that Courant regarded a scholarship for Hausdorff at the IAS as hopeless from the outset. The letter states, among other things:

> I just received the enclosed short and very touching letter from Professor F. Hausdorff (which please return), who is seventy years old and whose wife is sixty-five years old. He certainly is a mathematician of very great merit and still quite active. He asks me whether it would be possible to find a research fellowship for him. I refer the matter to you because it may be that Shapley, with whom you are in touch, might conceivably be interested in the case.[4]

Courant is referring here to the "Asylum Fellowship Plan" of the prominent astronomer at Harvard University, Harlow Shapley [3]. As the word "asylum" suggests, the aim here was to help older emigrants, in particular those who were no longer able to obtain regular academic positions. In contrast, fellowships at the IAS in Princeton were reserved for younger people. The mathematicians mentioned above were all considerably younger than Hausdorff. Even Einstein (born in 1879), one of the oldest at the institute, was eleven years younger.

Weyl seems to have responded immediately to Courant's letter, together with his colleague at the IAS, John von Neumann. In the "Refugee Files" in the Shapley estate at Harvard University, there is a document dated February 17, 1939, with the heading "Felix Hausdorff," where the following two expert opinions are quoted [2, p. 444]:

> Hausdorff is known the world over as the author of the classical work on theory of sets in general, and point sets in particular. On this foundation set-theoretic topology has built ever since. Much of his research work is along the same lines. His other important papers are on such diverse subjects as Waring's problem, bilinear forms of infinitely many variables, problem of momentum, astronomy, etc. In spite of his seventy years, he is still a creative mathematician.
> A man with a universal intellectual outlook, and a person of great culture and charm.
> H. Weyl

> Hausdorff is a many-sided mathematician who has made contributions in widely varying fields, so that his activities even outside of his main field – set theory – would put him in a very respectable place among mathematicians. His contributions to set theory are of the very first order; especially concerning the foundations of topology, point-set topology, theory of analytic sets, theory of measure, etc. His book on set theory is probably the best ever written on the subject. In spite of his age, he still keeps up production of absolutely first quality. I feel that the mathematical community is under great obligation to him.
> John von Neumann

All three, Courant, Weyl and von Neumann, thus emphasized in their statements that Hausdorff was still creative as a mathematician despite his age of 70.

Around the same time, there were other initiatives by mathematicians in Europe with the aim of helping Hausdorff. Weyl's former colleague in Zürich, George Pólya, who himself distrusted the relative safety of Switzerland and a year later fled to America, wrote to Weyl from Zürich on May 29, 1939:

> A case which is very near to me is Hausdorff. He had written a few lines first to Schwerdtfeger, then to me. From that anybody who knows him realizes that he is in a very bad situation. One hope that I had for him based on a communication by Toeplitz, and which I was incautious enough to relate to Hausdorff as well, has proved to be totally illusory. He is over 70 – and he is one of the nicest and most pleasant human beings I know – his direct and indirect students (through his book) are everywhere densely distributed [überall dicht verteilt]. Isn't there a chance of doing something for him? [4, p. 97]

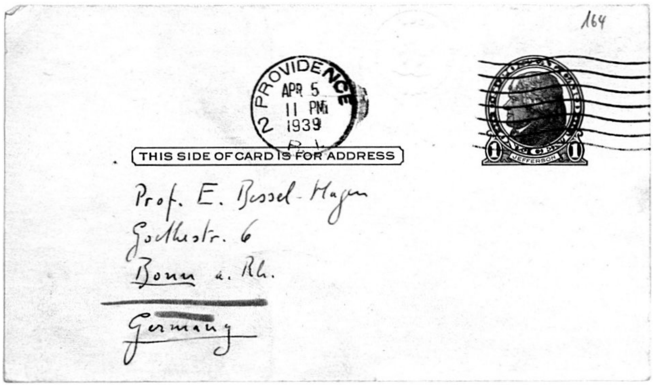
Another letter of support must have been written by Erich Bessel-Hagen in Bonn. This emerges indirectly from a postcard written to Bessel-Hagen on April 5, 1939 by Otto Neugebauer, the famous decipherer and interpreter of Babylonian mathematical and astronomical texts, who cooperated with Bessel-Hagen on historical issues [5]. Neugebauer, who had edited the *Zentralblatt für Mathematik* in Copenhagen as an émigré from 1934, had just emigrated to Providence in Rhode Island after the Zentralblatt crisis of November 1938. There he founded *Mathematical Reviews* in 1940. Bessel-Hagen, who (like Neugebauer) was not affected by the Nazi racial laws, was one of the few colleagues in Germany who remained in personal contact with Hausdorff, along with Toeplitz, who emigrated to Palestine in 1939.

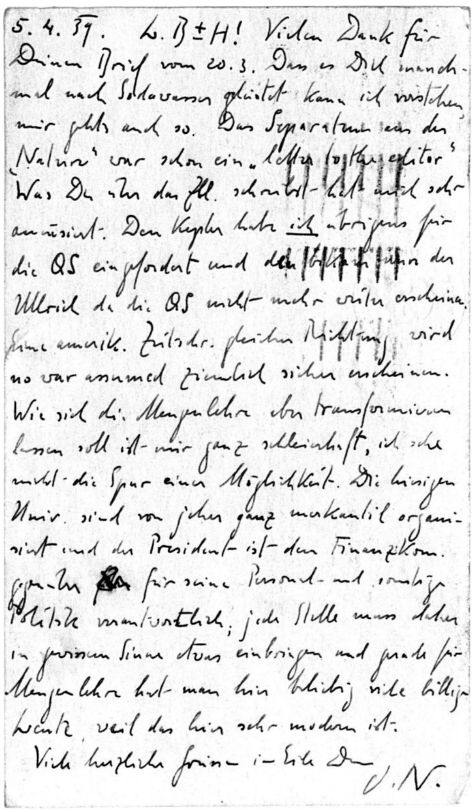
In the postcard, which we reprint here in facsimile as in [5, p. 94], Neugebauer reacted pessimistically to a presumed proposal by Bessel-Hagen:

[3] Emmy Noether gave special seminars at the Institute in 1934 and 1935, coming from her temporary position at Bryn Mawr College in Pennsylvania.

[4] R. Courant to H. Weyl, February 10, 1939. Veblen Papers, Library of Congress, cont. 31, f. Hausdorff.

How the theory of sets could be transformed is totally beyond me, I do not see the slightest chance. The universities here are traditionally commercial, and the president is responsible to the financial committee for his politics, particularly with respect to personnel. Therefore, each position has, in a way, to earn its own income, and as far as set theory is concerned, they have here plenty of affordable people, because the topic is very fashionable [modern] here.




5. 4. 39. L. B±H! Vielen Dank für
Deinen Brief vom 20.3. Dass es Dich manch-
mal nach Sodawasser gelüstet kann ich verstehen,
mir gehts auch so. Das Separatum aus der
'Nature' war schon ein „letter to the editor'
Was Du über das Ztbl. schreibst hat mich sehr
amüsiert. Den Kepler habe ich übrigens für
die QS eingefordert und [illegible] an
Ullrich da die QS nicht mehr weiter erscheinen.
Eine amerik. Zeitschr. gleicher Richtung wird
[illegible] ziemlich sicher erscheinen.
Wie sich die Mengenlehre aber transformieren
lassen soll ist mir ganz schleierhaft, ich sehe
nicht die Spur einer Möglichkeit. Die hiesigen
Univ. sind von jeher ganz merkantil organi-
siert und der President ist dem Finanzkom.
gegenüber [illegible] für seine Personal- und sonstige
Politik verantwortlich; jede Stelle muss daher
in gewissem Sinne etwas einbringen und gerade für
Mengenlehre hat man hier beliebig viele billige
Leute, weil das hier sehr modern ist.
Viele herzliche Grüsse in Eile Dein
O.N.

Postcard from Otto Neugebauer (Providence, RI) to Erich Bessel-Hagen (Bonn) April 5, 1939. (© ULB Bonn: NL Bessel-Hagen 053)

Given that Neugebauer had chosen to send an open postcard written in German to National Socialist Germany, the somewhat cryptic text is not surprising. In view of the efforts made at the same time in favour of Hausdorff as documented above, there can, in my opinion, be no doubt that "transformation of set theory" refers to a possible emigration of Hausdorff. On an earlier occasion Neugebauer and Courant had used the word "transformation" in a non-mathematical sense as part of a kind of secret political language. Courant had written the following to Neugebauer on September 12, 1933, apparently recommending that he emigrate to Copenhagen:

> Generally Harald [Bohr] tends to emphasize the transformation theory instead of fixed point theorems. That method is also more rewarding for your work. [4, p. 162]

The head of Neugebauer's mathematics department at Brown University in Providence was the influential secretary of the AMS, R. G. D. Richardson, who was often sceptical about the immigration of mathematicians. Thus, it is not surprising that Neugebauer, who had just arrived in Providence, saw no chance for Hausdorff.

A year after Neugebauer's postcard to Bessel-Hagen, the same Richardson responded, on April 16, 1940, i.e., between the German occupation of Denmark and Norway and that of France, to a letter from the German immigrant and mathematical statistician Emil Julius Gumbel. Apparently, Richardson saw the invasion of France coming and supported a lecture tour in America by the elderly French-Jewish mathematician Jacques Hadamard (1865–1963), who had helped Gumbel with his first emigration to France. Richardson wrote to Gumbel:

> While it is true that Brown University would be glad to welcome Professor Hadamard and pay him a small stipend if he were passing through, we recommend that he not be invited to come to this country, and I have said this to other persons. In his day, Hadamard was a great figure in mathematics, and he has visited this country and received a warm welcome, but he is now old and has not done anything significant in mathematics for a decade. There are persons in Europe whom the mathematicians of this country would much prefer to have come. [4, pp. 257–258]

Hadamard was three years older than Hausdorff. As a prominent Parisian mathematician, he was a "great figure," as Richardson writes, and certainly better known among older and influential American mathematicians than Hausdorff from the small town of Bonn. Hadamard finally reached the USA after the occupation of his country, but he never got beyond a temporary position at Columbia University in New York and returned to Europe at the end of 1944. Emigrants in America such as Weyl used his example, and that of other older immigrants such as Felix Bernstein and

Max Dehn, to discuss the particular problems of this age group, especially the lack of pensions for them in America [4, p. 258].

Hausdorff's death

Although Hausdorff was mathematically still more productive than Hadamard at the time, this was apparently not enough to save him. One last document that testifies to an attempt to save him is a circular letter from Courant dated May 21, 1941, to American scientific institutions "To Whom it May Concern," in which it is once again imploringly stated: "In spite of his age his presence would be an asset to any institution of higher learning." [1, p. 239]

Courant's letter also says prophetically: "His name will for a long time to come be unforgotten."

After the outbreak of war in Europe, however, both leaving Germany and travelling to countries of refuge became increasingly difficult, and the USA itself entered the war in December 1941.

Hausdorff's suicide in the spring of 1942 together with his wife and sister-in-law shortly before deportation has been vividly described by Erwin Neuenschwander, Brieskorn and Purkert. His poignant farewell letter bears witness to critical self-reflection, consideration for others, and sarcasm even in view of death. In the letter, he writes about the announced first station of his deportation, Endenich near Bonn:

Even Endenich is still perhaps not ["nich", for "nicht"] the end! [2, p. 408]

Translated by the author with help from DeepL and June Barrow-Green (London).

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A conversation with Volker Mehrmann

Mathematics brings happiness

Sílvia Barbeiro, Ana Isabel Mendes and Martin Raussen

Volker Mehrmann is a complete mathematician excelling in all areas: research, teaching, knowledge transfer, and management. He is a specialist in numerical linear algebra, algebraic-differential equations, and control theory, and he enjoys conducting research motivated by real-world problems and developing fundamental mathematics that has a significant impact on science and technology. With a striking personality, Mehrmann leaves a remarkable legacy as former president of GAMM (International Association for Applied Mathematics and Mechanics), MATHEON (Mathematical Research Center for Key Technologies), ECMath (Einstein Center for Mathematics in Berlin), and EMS (European Mathematical Society). He is attentive to how mathematics is communicated to fellow mathematicians and to the general public, and he takes care of the working environment of his students.

The three of us (Sílvia Barbeiro, Ana Isabel Mendes and Martin Raussen) held a videoconference meeting with Volker Mehrmann last October. We listened eagerly to his answers to all our questions. At the end, we discovered his secret: mathematics brings happiness.

Figure 1. Math makes you happy and a sign that says do not follow the typical procedures (20 October 2012).

1 High school and University of Bielefeld

Interviewers: *Dear Professor Mehrmann, dear Volker: What and/or who stimulated your interest in mathematics in the first place?*

Volker Mehrmann: Well, it happened in high school. I was at a special high school for math and physics and STEM students. I liked it a lot and I was good at it. This is why I decided I wanted to become a math teacher. And I am.

Int: *Can you tell us a bit more about your school experiences at that time?*

VM: That was in the sixties and seventies, when the German school system was still pretty rigid. Schools were separated into three layers. The Gymnasium or high school took care of the better pupils. Then there were the middle school and the basic school. A high school education took 13 years, the middle school ten, and the other one lasted eight or nine years. I was sent to high school on the advice of my teachers because I was good at math already in elementary school. My parents, who were working class people, agreed, although they were concerned that they would not be able to help me. It was not that easy: I had to go to a different city and had to go by bus every morning. But I liked it, and I was good at school and therefore it was quite clear that I wanted to continue.

Int: *Germany was separated into West and East in those times. What did that mean for you?*

VM: I lived in West Germany, where – at that time – school just began to become a little bit more individualized. Not yet when I arrived at the high school. But later on, there were discussions, and politics became really important. But since I was at a school with emphasis on math and physics, politics didn't play much of a role. We had more mathematics than other high schools, six hours per week, and I enjoyed it. As far as I know, the East German system was more like the Russian system, and it included a lot of persistence.

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Int: *Were you personally interested in politics already at that time, or did that only happen later?*

VM: That came a little bit later…

Int: *Can you tell us about your experiences when you started to study at the university, please?*

VM: Well, it started with a very bad experience. When I entered university, I was still 17, a very young student, which was unusual at that time. After the first semester, I was drafted into the army, so I had to leave my studies for 15 months of service. When I returned, I immediately went back into the third semester as if I hadn't been away. That was not good, since I missed a lot of important stuff that I had to catch up upon. I studied at the University of Bielefeld, which was established as a reform university, one of these newly founded universities in Germany under the social democratic governments. They were based on fantastic ideas. Bielefeld was meant to be highly interdisciplinary, it even has a Center for Interdisciplinary Research. But the department of mathematics at the time was, for 90%, very pure. The remaining 10% were asked to work together with economists, chemists, etc. I liked that approach, and therefore I ended up in applied mathematics. The study of mathematics was divided into two different programs, one for teachers' education, and the other, called "Diplom," for students who wanted to work with mathematics outside the schools. The teachers' program included didactics, philosophy, and so on. I was enrolled in that program since I wanted to become a teacher. But before that, I wanted to work as a student tutor at the university, and for that you had to pass an intermediate "Vordiplom" exam, a bit reminiscent of a bachelor's degree nowadays. This is why I took this exam after my third semester.

Int: *Did you have to write a thesis for this exam?*

VM: No, the thesis came only with the final diploma exam; this "Vordiplom" required four oral exams. After I had passed these exams, it happened that a new professor arrived; he was a specialist in combinatorics and graph theory. And as a new tutor, I was assigned to his course in these subjects, a course that I had never attended before. He asked me whether I would be interested in writing a thesis in graph theory with him. But I was already determined to do numerics because I liked that a lot. And I also liked programming.

2 Graduating in the US

VM: I started to work with Professor Ludwig Elsner, within numerics. But when I told him that I wanted to take a teacher's exam and to write a thesis in that program, he had difficulties imagining a teacher's thesis in numerics and suggested that I write a diploma thesis instead. He said that we can certainly have my thesis acknowledged as a teacher's thesis afterwards. By this accident, I ended up writing a diploma thesis. And then I told myself, I might as well take a full diploma exam as well. Shortly after, I also took the teachers' exam, and so I have both.

To work as a teacher at a high school required an additional step at a high school where you had to learn how to teach mathematics to pupils. When I applied, they sent me to a very remote place where I didn't want to go. Being adventurous, I asked Ludwig Elsner whether I could continue studying in a different country. He said: "That's very easy: You take the front page of the journal *Linear Algebra and its Applications*, and you look at all the people that are written on the front page. You write a letter to them asking whether you could go and study abroad." That's what I actually did. I wrote a handwritten letter.

Int: *To each of them?*

VM: Yes, to each of them!

Int: *Who was the first to accept you?*

VM: I got three acceptances: One from Stanford with Gene Golub, one from Calgary with Peter Lancaster, and one with Richard Varga from Kent State University. Richard Varga's answer was in German, and my English was not that good at that time, and his answer also came first. Hence, I agreed to go to Kent State University, a small Midwest college. Careerwise, probably Stanford would have been a bit better, but who knows? I enjoyed it at Kent State.

Int: *But you succeeded. Did you have to pay for tuition?*

VM: No, they offered me a teaching assistantship right away because I already had a diploma. And I also didn't have to do the qualifying exams, and I could right away start on a thesis. But then I had a very bad experience: I had a topic for a PhD thesis already, within perturbation theory for matrix pencils, and I started to work in this area.

Int: *Was this topic suggested by your supervisor at Kent State?*

VM: No, it came actually from my German master's thesis advisor, who had suggested it as an important open problem, and I was well prepared to work on it. When I came to Kent State, Richard Varga told me that he would not give a course that semester. I was his only student, and I took a reading course with him. I was on a one to one seminar for two hours every week with Richard Varga. Every week, he gave me a paper to read, and the next week I had to present it to him. For the third week, he gave me a paper by

Peter Stuart from Maryland, in which he had solved the problem that was supposed to be my topic!

Richard Varga gave me then a different topic concerning non-negative matrices, specifically, an important problem posed by Olga Taussky, the famous algebraist. This open problem consisted in unifying the theory of the three different types of non-negativity in matrices: matrices can be elementwise non-negative, non-negative with respect to quadratic forms, also known as *Loewner non-negativity.* The third option is total non-negativity, which means that every sub-determinant is non-negative. Olga Taussky had claimed that there must be a uniform way to treat all three types. She had a conjecture using determinental inequalities. I worked on this conjecture, and I proved it for $n = 5$. Then I started trying to find a counterexample with the computer, but I couldn't manage. As we always do in math, instead I proved some other results. I think it was only in 1999 that the conjecture was proved to be false, by a counterexample using 22×22 matrices. Between 5 and 22, the question is still open.

Int: *Who found this counterexample?*

VM: It was found by Olga Holtz at Berkeley. She had the ingenious idea to investigate a special class of matrices, banded Toeplitz matrices, so that she didn't have to worry about that many parameters. I had worked in the general class of $n \times n$-matrices with n^2 parameters and n! determinantal inequalities!

3 Directions in mathematical research

Int: *Can you tell us a bit more about your research, be it as a beginner or quite recently? Are there results that you are particularly fond of?*

VM: Well, there is the saying that the murderer always returns to the place where he committed his crime. In my master's thesis I developed an algorithm towards eigenvalue calculations over the symplectic group, the unitary group, and the orthogonal group with respect to indefinite inner products and the like. I had proved that the only place where the numerics works well was for the unitary group. The rest was unbounded and ill-conditioned. About four years later, after my PhD, I was a second time in the US, and I heard a talk by Ralph Byers who was using the symplectic group in optimal control. And I told myself: "Hey, here's an application for my useless algorithm!" And so, I started to work in control theory, and I have been working in control theory on top of the classical numerical stuff that I was familiar with. That is a lot of fun because it means collaboration with engineers. One learns a lot of different thinking when cooperating with control engineers, since they work with everyday life problems. In every car, in every airplane and in every bicycle, there are issues for which control theory is essential.

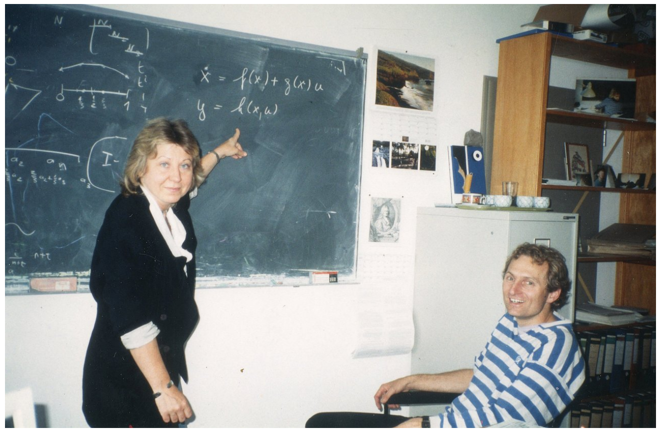

Figure 2. Together with Angelika Bunse-Gerstner in Bielefeld 1991.

Over the years, I have always been working in control theory and matrix theory, and also in high-performance computing. These were the key directions.

In 2011, I received an ERC advanced grant for modeling coupled systems from different physical domains. In the beginning, it was a complete failure. I couldn't manage. But then, I heard a talk by the control theorist Arjan van der Schaft from the Netherlands. He suggested that one should go energy based and hence closer to physics. Since 2015, I'm very much into this topic, port-Hamiltonian systems modeling, numerics, linear algebra, and it's really magic. You tell me a question and I put it into the port-Hamiltonan framework. There is nice mathematics in the background, geometry, and physics, of course. Whatever you try, it turns out to work much better than if you don't have this approach. A physicist would certainly say: "Oh well, we knew this all along." But the engineers didn't!

Int: *May we come back to the ERC grant? I imagine that you were already very confident about your research since you applied for such a grant. Did you already have a group of people working with you?*

VM: Yes, I had five people on the grant. Each of them had been working on different physical domains, modeling with differential-algebraic equations, that is, differential equations with constraints. In principle, the mathematics for each of them was the same theory. But when you couple mechanical and electric systems, you quickly learn that the variables that are used are so differently scaled that when you start the coupling, the electric system is typically gone. It's just so small or so negligible compared to the mechanical system. This has to do with the scales of variables, but also with the way the coupling is done. Coupling is usually done via position variables or velocity variables, which are not the variables in the other domain, and one had to apply various tricks. But in this port-Hamiltonian framework, you take a new approach: The only uniform quantity in physics is energy, or entropy. Your coupling works via the transfer of energy, which means that you must write your equations in a different way, so that the energy is present in your equations. Usually when you describe the mechanical system, there is kinetic plus potential energy, but you never write it down this way. You use velocity and positions, et cetera, but you forget about the energy. But as soon as you start coupling via the energy, everything fits together in a much nicer way.

Int: *Was that already a key point convincing the jury when you applied to the ERC, or what was the key point actually?*

VM: Well, I don't know, because I was not on the committee myself. Around the year 2000, there arose an initiative to couple different physical domains. That happened in physics, but also in engineering. At that time, we had computers that were good enough for this purpose. The main idea was to establish monolithic systems. Before that time, there were research groups in mechanics and separately in electrical systems. Each of them had their own software. And when you had systems encompassing fluid and structure interaction, you had two codes, and you had to couple the codes. That didn't go too well! A consequence was an initiative towards finding a monolithic modeling the fluid and structure interaction in one type of system. That was my project, and it was open at that time. By the way, I didn't get this grant in mathematics, but at the interdisciplinary panel.

4 Differential-algebraic equations

Int: *Could you please explain to us what differential-algebraic equations (DAE) are all about? They are the ingredients in your most cited work, co-authored with Peter Kunkel…*

VM: A differential-algebraic equation is in essence a differential equation on a manifold. You have a flow associated with a differential equation, but it is constrained. A typical example would be a satellite in orbit, or a robot which has constraints because of the motors in the joints. This came up in the seventies as a new way of investigation. But if you go back in history, it was initiated by Kirchhoff back in 1870 at the Berlin Academy of Science, when he presented the equations for the description of electricity, followed up by Kronecker and Weierstrass, who settled the theory for the linear case already in the late 19th century. But this approach came completely out of fashion. We mathematicians have the great tendency that when we have something that we know how to do, then we take up other things and twist it. ODE theory and dynamical systems were well developed.

Hence, this DAE approach was forgotten until it came back in the 70s when circuit simulation made it necessary to talk about millions of equations that you had to handle automatically. Nobody could resolve the algebraic equations and put them into the ODE framework. William Gear and Linda Petzold from the U.S., actually computer scientists, started research in that direction. It took off pretty fast, but then a lot of difficulties showed up when you need to take constraints into account. For example, take a mechanical system whose position is constrained to a surface. That could be a train on a track. Its position is constrained, but the velocity cannot leave the track either. That means the velocity is also constrained, but that is not written down. These so-called hidden constraints drive the numerical methods crazy. In most realistic systems, there are many hidden constraints.

The collaboration with Peter Kunkel started while I worked at the IBM Research Center in Heidelberg for a year. My boss came with an application from a customer. It was actually an optimal control problem. I implemented it, and I got a wonderful solution. I went for lunch with Peter Kunkel, who worked in the same office. He quickly said, I have a code too. He implemented it and ran it and got a wonderful solution as well. But they were very different!

Int: *What happened then?*

VM: One of them was exponentially growing and the other one was decaying; not a good result! But as a mathematician, you immediately know that the solution is probably not unique. And so, we started to analyze, and that started our cooperation. Actually, it is a wonderful cooperation: we have just finished the second edition of our book that just appeared at EMS Press. This is the most natural way to model. If you want to couple, you write down the coupling as an algebraic or differential or whatever constraint, and then you have more constraints. This was the topic: to get hold on this coupling with differential algebraic equations. But the difficulties are immense and there are not too many good codes available, still today.

Int: *It's the codes rather than the speed of computers that are limiting…*

VM: The problem is that you must differentiate. Numerically, differentiation is bad. Integration is nice, but differentiation is not. There are more changes to consider. If you change the direction a little bit, the derivative comes out completely different, given you implicitly have to differentiate – and the codes have problems. Here is a very nice story: In the seventies, Germany designed ICE trains or high-speed trains, and France did that as well, at the same time. The modeling was performed with ODEs in Germany and with DAEs in France. The German trains would be flying because the constraints weren't there, and at least numerically, the trains would lift off. This is why the German ICE trains were much heavier than the TGVs, at least in the early days.

5 TU Chemnitz

Int: *Interesting… You have been working in many different places. In the U.S., you have, apart from Kent State, been at Madison. You have worked in Aachen and in Chemnitz. And then finally in Berlin for many years. It would be quite interesting to learn a little bit about these different workplaces and how they are characterized. I am personally interested in hearing about Chemnitz, because Chemnitz is situated in the former GDR, which had a very different university system.*

VM: When I was on the job market after my habilitation, it was extremely difficult to get a position. In Germany, hardly any professorship opened up. That's why I took a two-year temporary professorship in Aachen, and I worked a year at IBM in Heidelberg. But then the wall came down. I was still in Aachen, and I got a call from the group in Chemnitz, which I had met before. In Chemnitz, they had had big conflicts in the department between numerics and analysis.

Int: *Going back to GDR times?*

VM: Many of the numerical people were party liners and had important positions, like rector, head of the Disciplinary Committee, and so on. These were the party guys, and the analysts were not of that kind. After unification, the state of Saxony, where Chemnitz is situated, had essentially fired everybody. But they opened new positions, and people could reapply. The people in charge wanted to make sure that the conflict between analysis and numerics wouldn't immediately pop up again. The most important question in my interview became: "Can you imagine working together with the operator theorists and the analysts and building a bridge between the two fields?" But that was what I had been doing before! I was lucky because that's exactly what they wanted.

I moved there in early 1993, just after the whole rehiring had been done. I would say it's one of the best places that I've ever been to for work. The students were fantastic because Chemnitz had been one of these hubs for gifted students. In the GDR, they had schools for gifted students in sports, in math, in music, and in other fields. If you managed to get into such a school, you could go to the university at grade 11. A wonderful example is the Heinrich Hertz Gymnasium in Berlin, with pupils like Peter Scholze and Yuri Tschinkel, now at the Simons Foundation, and similar people. Hence, the students from that particular high school in Chemnitz came to the university at a very young age. And they were fantastic. There were only a few students, around 30 during the first semester. But one half of them were professor candidates!

Int: *Did that system still exist in 1993?*

VM: Yes, it still existed in '93. But then Saxony got a conservative government, and they deleted everything that had a taste of socialism. The elite schools did not have any taste of socialism, but they were related to the old system. The schools are still there. But at least in Saxony, it is no longer possible for gifted students to enter university at a very young age. In Berlin, this is still possible. Gifted students can follow courses during their first year at university while still attending high school.

Int: *Did you move to Chemnitz with your family?*

VM: Yes, I moved there with my partner at that time, and we had a baby just after I arrived there. But she couldn't stand it. She was also in math, in graph theory. It was not a good place; Chemnitz was a dreadful city. When I left Chemnitz in 2000, I was asked: "Why? You have a great career here!" And I said: "Look, there are other parts of life beyond mathematics and if you cannot make friends…" Chemnitz was a very special place because it was the role model town in GDR. They had changed the name to Karl Marx Stadt. The party and the Secret Service were dominant everywhere. Due to this, professors didn't have much social contact with each other.

Int: *Fearing that there might be an informant.*

VM: Yes. And there was.

Int: *Even after '93?*

VM: No. Because they were fired and as soon as they found something in the records that a person had been an informant, they were not letting them in. Very different in other states.

6 TU Berlin and MATHEON

Int: *Berlin was a completely different story, I believe.*

VM: It's just the opposite. Open, international… Chemnitz wasn't international. Speaking English was the exception. Bringing in foreign students, or post-docs was not easy. Berlin was just the opposite. I must say, I was afraid. Because Berlin is a completely different scale of town, and I was already 45 then in 2000. But the people in Berlin are so open. I was immediately welcomed.

Int: *And then MATHEON came into place.*

VM: Yeah, that was funny. In 1999, I was elected as chair of the Math Committee of the German Science Foundation, the panel that decides about projects. I was elected as an Easterner because I was from Chemnitz. Just after I arrived in Berlin, the administration called me and said, there is this proposal upcoming for new research centers. And it would be great if there would be one in mathematics.

It was clear that they didn't want to have pure math. They wanted to introduce something more directly relevant to society.

After only four weeks in this place, this was a good challenge to start. I caught up with a few of the people: Peter Deuflhard, Jürgen Sprekels, and Martin Grötschel. We had six weeks of time to write a shorter exposé. We met in my office, and after about 20 minutes, I became a typist because of these guys, the ideas just jumped out of them. In the first round, we didn't make it. It went from medicine to physics and to ocean sciences. But they immediately opened a second round, and in the second round in 2002, we got it. I must have been a good typist, since Martin Grötschel asked me to be the vice-chair.

Int: *And MATHEON carried on for many years.*

VM: Yeah. We had 12 years of MATHEON. During six years, Martin was the chair, and then I became the chair. I liked being Martin's vice-chair because Martin is a fantastic politician. And he knew how to present mathematics to the public in a much better way than I could have done it. But when he said that we should switch now, I became the chair. Next step was the Einstein Center for five years. That continued to the excellent initiative. But then I stepped down.

Int: *Now it's called "Math+".*

VM: It's a new program, and they always want a new name. Math+ is a little bit more than MATHEON because it includes a mathematical graduate school, an American style graduate school. You can come from a bachelor's and go directly for a PhD. Both MATHEON and the Einstein Center for Mathematics, and also Math+, are a joint effort of the three universities in Berlin, and of two research centers: the Zuse Institut Berlin, which was originally a high-performance computing center and the Weierstrass Institut, which is a former Academy of Science of the GDR. Our biggest success is that we managed to get the entire Berlin math community together. Which looks great to the outside! But I can tell you, it caused a lot of friction by the different university administrators. They didn't like this always. They liked that we got it, but they wanted to make decisions, but they couldn't.

7 Teaching mathematics

Int: *Let us talk about your teaching career. You told us already that you were trained as a high school teacher, and you then became a university teacher. I know that you have taught Linear Algebra to first year students many times. You have even written a book about linear algebra according to your philosophy. But you have also trained a lot of PhD students.*

VM: Yes, indeed.

Figure 3. MATHEON received the prize "Germany land of ideas" (2007).

Int: *And there's a lot in between. May we have some comments, please?*

VM: Right away from the beginning, I always had the ambition to make mathematics digestible, not only for the three or four geniuses, but for the average, good mathematicians, as well. And that means that you must think in a slightly different way. How does mathematics enter our brains? As a high school teacher, it's clear you have to teach everybody. But when I came to the university, most of the teachers had only the three or four geniuses in mind. They taught linear algebra in a very abstract way, immediately in terms of modules and finite fields and things like that. That took a long time to enter my brain. And it's getting worse because high school education has been degenerating since then. And, in particular, abstract linear algebra has a problem. With analysis, it is a little bit easier because functions and derivatives and things like that, you can grasp that. But the abstract concept of a group or an algebra or things like that? I had difficulties producing a picture in my brain.

I find it very important in this learning process that you get a picture of what the abstract thing you're talking about, what these pictures mean, before you can learn about them. You can certainly learn to operate the whole apparatus. But does it sit there somewhere?

For the average high school student who arrives at the university now, linear algebra is a pain. That's why I went back and reflected about what would you do with such a high school student, and that's the approach that this linear algebra book takes.

We start with matrices rather than with vector spaces and maps, homomorphisms between vector spaces. That all comes, but it comes after the image of a linear mapping being a matrix times a vector is already encoded. And the feedback is really encouraging. Just recently, we made the fourth edition of the book, and the English version has 4.5 million downloads. And the next linear algebra book is in the hundred thousand. I think this is a very important step. I know that many of my colleagues don't like that.

It's far too concrete to have a matrix and then be forced to be coordinate based. But I think it enters the thinking like that a little bit better than the fully abstract theory right from the beginning. For the very good students, it doesn't really matter. I use the same philosophy in my courses on numerics and differential algebraic equations. I always tried to have the concrete application on the side so that one can imagine what it would mean to have a method working. On the other hand, I like the abstract stuff very much. So, I also teach it. But I think from the point of view of communication about mathematics, it's helpful to start far more concrete.

Int: *You mentioned to us that you were recently "recycled", you are teaching again.*

VM: Yes. I retired a year ago. But at the moment, we have a shortage of numerical analysis teachers. So, they asked me to come back, I volunteered, and I enjoy it.

Int: *We just want to add that even for a genius, it's an advantage to have good examples in mind. You have not only taught freshmen, but also many Ph.D. students. The Mathematics Genealogy Project counts 34. What is your philosophy for them?*

VM: It's actually fourty PhD students already. In the German system you can be an advisor only if you are a professor. I advised several students, but the professor had to sign. A PhD is the first step in becoming an independent researcher who is standing on his or her own feet. On the other hand, the field of mathematics, like any other scientific field, is growing exponentially in all kinds of directions. I always found it very important that PhD students have somebody to talk to all the time so that they don't end up in this black hole of not making any progress. Moreover, I also tried to encourage them to talk to other PhD students or postdocs. As a PhD student, already as a master or teacher student, you should be curious to learn new things, curious to make connections, curious to learn more. You shouldn't just be focused on a single direction. This is what I tried to teach to my students, that they should be curious and that they should also discuss with other PhD students and other groups. To foster that, I tried to provide a good working climate, organising parties and hiking events and the like.

Int: *Spend time together.*

VM: Yeah. And that helps a lot! In particular, when you're in this crisis, which we all from time to time drop into, when we don't make progress on the stuff that we're working on. And this has worked very well. They all talked to each other, and they wrote papers together, and I support that very much. So again, pretty much the same philosophy, working together. Try to show your curiosity about science and math and their developments and, if necessary, learn new directions that you haven't learnt. And my

Figure 4. MATHEON received the prize "Germany land of ideas" (2007).

commitment is that I'm always there for them. I still have four PhD students at the moment, very different ones. One of them is at the Potsdam Institute for Climate Impact Research, working on stabilization of power networks. One is together with the mechanics department, working on dynamical systems and instabilities in cars and brakes and things like that. A third, who is working on saving energy in automated train driving for the subway system of the city of Nuremberg. It is an advantage if you are in a more applied field, it's usually much easier to get funding for a PhD student.

Int: *But in these applied topics, do you also develop fundamental mathematics?*

VM: Yes, the thesis always includes fundamental mathematics. My experience is, if you get a really nice problem from applications, the mathematics that you need to tackle it, is not well developed. I can give you a very nice example. So, in 2004 a guy from a company working on the layout of the high-speed train tracks from Frankfurt to Cologne approached me. They wanted to travel at 350 kilometers per hour, and they were not at all sure that this was possible. The vibrations of the tracks might lead to derailing and things like that. They performed a lot of simulations and calculations, but they couldn't get any correct digits with the experiment; then you get really nervous as an engineer! They came to me with the problem. And I said, hey, the eigenvalues are lambda and lambda bar. There must be some symplecticity there. We investigated the situation, and it became very clearly essential. And then I said, there must be a generalization of symplecticity. These are polynomial eigenvalue problems. There must be a generalization of symplecticity to matrix polynomials. We wrote a few papers, and then it exploded immediately. Now, there are lots of papers, in particular in Spain and Italy. This came out of an entirely applied problem and ended up with entirely pure mathematics. It's now done over finite fields and the like…

8 EMS presidency and communication in mathematics

Int: *Can we spend a few minutes on your presidency of the European Mathematical Society? You told us already that it was difficult times for two reasons. There was the COVID period during which in-person meetings were impossible. And then, at the end of your presidency, we witnessed the aggression of Russia into Ukraine. That must have taken emphasis from the subject itself.*

VM: Yeah.

Int: *What could you achieve despite all of these complex difficulties?*

VM: I certainly had an agenda. The first point on the agenda involved unity of mathematics. I very much believe in the unity of mathematics. But I do believe that some of our colleagues don't, and they don't like it. Because I think it's much more comfortable to stay in your own community of 20 or 30 researchers worldwide who understand each other. Communicating about mathematics is not their strength typically. And they also feel threatened by it. All of a sudden, you have to talk to engineers or physicists. This was one of my key objectives. Let's try to keep the unity intact as much as possible! But let's also try to make an effort to be able to communicate better with each other. I think I partially succeeded, but only partially.

Int: *You gave us already many examples of the importance of the pipelines between application-driven mathematics and curiosity-driven mathematics. You have problems that come from applications and to tackle them you need to apply tools from the pool that already exists. On the other hand, you have also given examples showing that pure mathematics pops up as soon as you are curious enough to take problems for applied areas seriously. Back in history, a lot of mathematics has arisen from origins in physical considerations.*

VM: I think we need better communication, and I don't think that we have too many mathematicians in the world who can be translators. Take an abstract result in algebraic geometry or take Peter Scholze perfectoids, or the ABC conjecture claimed proven by Mochizuki. I do not think there are many people who can take what is in there. I am sure there are a lot of things in there translated or transferred to something that can be used by engineers or physicists. The strength of mathematics is abstraction. And it becomes more and more abstract. The problem arises if you stay up there and do not do the translation that brought you to the abstract level. This is what is strongly missing. And it is a pity. This is what I would like to have the community realize and to stop being arrogant if somebody else does not understand what you're talking about because you haven't studied algebraic geometry for 15 years. Make an effort and try to explain!

Int: *We talked already about the European Congress of Mathematics. You were the main organizer of the Congress that took place in Berlin eight years ago. And the experience that you had was probably more or less the same. At least some of the talks were not understandable to a large audience. Do you have ideas for what can be done to change this attitude?*

VM: I really do not know how to convince some of the people who are top stars to leave their ivory tower and try to explain at least the idea behind what they're doing. We tried it at that meeting, we sent everybody a memo with what we would like them to do. But…

Int: *You did not have too much success…*

VM: Let me tell you a story. My son is an algebraic topologist. Very abstract: categories and homology. He applied for stipend, a PhD stipend of the State of Baden-Württemberg. He asked me: "Can you have a look at my proposal?" I answered: "Sure. But you have to take my criticism." And then I had to tell him: "Look, I've read this three-page proposal. Several times I felt offended because you wrote: 'It's obvious that if you do this and that, then it's clear that blah, blah, blah, …' Think about it. You want to sell your proposal to a committee that wants to give you money. A committee for all fields: math, physics, sociology, philosophy, with at most one mathematician in that committee. If this guy is more applied or a statistician and he doesn't understand a single word, then you lost right from the beginning. Why don't you try on at least one of the three pages to explain what your overall way of thinking is? Nobody will blame you then for going into the details on the remaining pages."

There used to be an unwritten rule for talks: the first part for everybody, the next part for experienced people, and the last part is for yourself. It is not like that anymore. Many lecturers seem to expect that you know everything beforehand.

Int: *But we have to say that your legacy as EMS president was great! One of us participated in the EMS congress this year and met many people from applied mathematics. Engaging lots of people, also applied mathematicians, in the EMS, is a fine achievement.*

VM: True, but I went to all the invited and plenary talks at the Seville conference that I could go to. And I would still say that about 30 or 40% gave talks that made no effort whatsoever for the audience to understand, and that is talking to mathematicians. I don't understand it. If you do that, you drive away anybody outside of your inner circle…

Étienne Ghys gave a wonderful talk, a master example. He talked about something completely abstract, but he still could convey the message. This is what I would like people to do. But there is still a certain arrogance.

Int: *It should be perhaps part of training as a mathematician.*

VM: I would think so. You always have students who tend to speak over the heads of the audience. When that happens, I criticize it.

9 EMYA and TAGs

Int: *What else was important for you as EMS president?*

VM: The second point on the agenda was to make EMS a more balanced society, balanced in the sense of gender, but also balanced in the sense of participation of smaller and less wealthy countries and societies. Martin, you were a vice-president, you know, when the committees were formed, and names were thrown on the table these were almost always French, German and British, and sometimes somebody from Denmark or Poland or so. Some countries never showed up in committees. My goal was to get a little bit better together, over all Europe. The third important point on my agenda was to bring the young people into the Society and make sure that they felt at home in the mathematical community and that they contribute to bringing the mathematical community forward. These were the points at the head of my agenda.

Let us start with the balance. I think I partly succeeded. During the last EMS congress, EMS prizes were given to four women out of ten, which was pretty good. In the editorial boards, improvements are still needed when it comes to participation of people from Southeastern Europe or the former East. I think the war completely killed that initiative because their representatives wouldn't talk to each other anymore. And you can only be elected if you have supporters. That didn't work out well! The Russian community was very strong in the EMS, it is terrible that they are really set aside now.

The EMS Young Academy (EMYA) seems to work well. They have now a seat on the executive committee. They have to compete against others, but they can nominate one position. I have talked to a lot of these young people and they're eager to get active. I have actually stolen this idea from GAMM.[1]

Int: *You were the president of GAMM.*

VM: Yes, I was the president of GAMM from 2011 to 2014. We established the GAMM Juniors. Becoming a member is in fact a title, providing some prestige. But these young members get active. They organize many events, and they push us old guys in front of them. For example, the annual GAMM meeting was always scheduled in Easter week. But Easter week is the worst week for a meeting if you have small children. What can you do with your small children? They have organized career events; not only for academic careers, but also for careers outside of the university. They took action.

I hope that this will happen in a similar way in the EMS. The members already talk to each other. I think that is the most important thing, that the community already starts to talk to each other when they're young and not established superstars already.

Another goal I wanted to implement was to establish groups working in a certain subfield of mathematics within Europe so that they meet each other regularly. We formed these topical activity groups (TAG)s. I have to admit that I am pretty disappointed that there was no proposal for an activity group in anything from pure math. I think, we inside Europe should get together and know each other better and better. We have so much to offer.

10 EMS Press and S2O

VM: The biggest success was in fact handed over to me by my predecessor, Pavel Exner. In the last years when he was president, he was badly trying to find a succession for the EMS publishing house. There were at least five or six different options that were investigated, and in the end none of them worked out. I had this young rebel André Gaul, who was a student from our numerically linear algebra group. He said: "Let's forget about all this publishing. Let's do it differently, let's do it grassroots, let's do it open!" He started a company where you could download a paper in an Internet forum, you could make comments on papers and books, and so on. I had lots of discussions with him. My point was that this might work in fields that are read by a lot of people, but I do not think it would work in a highly specialized field. People would not write bad or critical comments. They might do it in a referee report, but not in the forum.

I asked this guy to apply to become the new director of the publishing house. And the committee chose him! We planned how to set up EMS Press in a completely different way. Now, it's a company that is owned by the EMS as the only shareholder. The company is oriented towards the community, but it should also make some money, that then helps the EMS.

And since André Gaul is so much in favour of open access and everything is open, we immediately started to look into the current ways of publishing by commercial publishers, which is really terrible. They essentially try to make as much money as possible and drop quality, if necessary, when it comes to getting another paper accepted with which they make money. André Gaul came up with the idea of going to Subscribe to Open (S2O). And I think it is a real success! He imagined it would work this way. On top of that, he's a fantastic organizer, he does a lot of great stuff.

[1] Gesellschaft für Angewandte Mathematik und Mechanik.

Int: *Could you please explain this S2O concept further?*

VM: Subscribe to Open is a new publishing model where the community, in this case, the math community, the mathematicians, the societies, and the libraries work together. The idea is: the libraries subscribe to the journal like they did in the past. When the journal has enough subscribers, it goes open, completely.

That means that the subscribers pay for the rest of the world, and then science is open! This is completely different from the model of article processing charges where you pay per article, where some government organization or you must pay. To give you a figure, if you want to subscribe to all the 22 journals of EMS Press, this costs you as a package something like €6,000 per year. If you want to publish an article in Springer or Elsevier, you must pay €2,500 per article.

The success of this approach is clear. More and more societies and publishing houses are moving to S2O. We are getting requests every week that a journal wants to come under the umbrella of EMS Press and to join Subscribe to Open.

Why is S2O so successful? It is in the key interests of the libraries that their old-style subscription model is kept. With processing charges, libraries will become obsolete. It will all be on the internet, but it's better if the library provides it.

Moreover, one should never sacrifice quality for money, particularly not in science. This is what this model can achieve. The publishing house is really doing well, making money for EMS and at the same time changing the world of publishing.

Int: *We wish you well, that is really very important for our community! Also, in relation to money: When you started as president of the EMS, one of your goals was to go to Brussels to get better funding for mathematics.*

VM: That is one of the goals where I did not succeed, for several reasons. When I started, or actually a few years before, several new hypes were created: artificial intelligence, quantum computing, et cetera. Math is just not as sexy as artificial intelligence or quantum computing.

Int: *Although it underpins both.*

VM: Yeah, exactly, it completely underpins both. But the money is going to these sexy topics. We had tried very hard to play digital twins as a topic in the European Commission. No success. If you know a little bit about how Brussels works, it is a permanent struggle to find somebody to talk to you. Unfortunately, if you want as much money as we mathematicians need, we are not considered seriously. We want a few millions. The quantum guys request billions, and this is a completely different level.

11 AI and computing

Int: *Talking about AI, do you think that it will influence the future of professional mathematicians? Will we still work in Math as before AI?*

VM: Certainly, some things will change. We had this discussion in the Congress in Seville. AI can certainly help with automatic proving. Image recognition in medical fields, will not be possible without AI in the near future, I think.

One big hype has already died. It's PINN: Physics informed neural networks. It was completely hyped by a guy from Princeton collecting all the prizes. PINN was everywhere.

Essentially, the idea was you take the PDE's from physics and then you use neural networks to solve the problem, but it didn't work out. For linear ODEs it worked out, but we don't need AI for that purpose. We can do that ourselves. [laughs]

What would change? I think we have to insert topics about the mathematics of artificial intelligence into our curricula, and we must train our students. I always tell my numerical analysis students, that, apart from doing nice algorithms and solving problems, they are the good conscience of the engineer. Because they can analyze. I tell them: You can analyze, you can trust the result, you can do an error analysis, you can do a convergence analysis. You can do estimates about how good the result would be. You can do uncertainty quantification.

In the near future, I think it's the mathematicians who have to be the good conscience of AI, because somebody has to tell when and where AI is really giving you something successful or whether AI is making a lot of money and giving you shitty results.

Int: *That would probably require deeper mathematical understanding around how and when AI actually works well, as it seemingly often does.*

VM: That's why we must train our students in this topic. Several universities have started already. But, you know, the students vote with their feet when they must choose between just applying AI to solve a problem or analyzing AI. Just applying it to solve a problem is so much easier; it is also sexier!

I'm just teaching this course numerical analysis. I am struggling, because a lot of people believe that what is taught in this course can be done by AI right away. Moreover, we are creating new computers for AI. These computers don't follow the standards that we're used to in numerical analysis, e.g., IEEE arithmetic, so that we can analyze the problem. So, what does that mean?

That means that a typical computer for AI, which the big companies produce, uses four digits. You can't do reasonable numerics with four digits. I would bet that in ten years from now a regular computer on which you can run MATLAB with 16 digits will be expensive.

Mainly, because all the money is going into the development of hardware which is written for AI and not for standard engineering, mathematics, or physics. This worries me.

Int: *Four digits can be a real danger…*

VM: Yes, it is.

Int: *So how can we convince people that for some problems, it is really bad…*

VM: As long as we have alternatives available, this is not a problem. I'll give you another example: Around 2000, people started to work on graphic cards (GPU computing). The graphics card doesn't follow the IEEE standard, only to a certain part. They do not have overflow and underflow, and they have bitflips. So, they have an array where small units do the calculation, and you will never find out whether one of the small units has flipped a bit into the wrong way. That means you cannot verify the results, and if you run the same program twice, you can get different results. Now, we as mathematicians know how to deal with this! If we have uncertainty in our data and in our algorithms, we do statistics, right?

You run thesame program 100 times and plot a Gauss curve, you may say, it's pretty nice, has a small tail, and then you know that this is probably the result that you should go for. But that would mean that you must do all the calculations 100 times, and they are already wasting energy like crazy for AI. Ecologically and sustainably, this is really a bad idea, and I am worried about this.

I tell my students: "Look, it is your job later on. You will work in a company and the boss will tell you to use this AI computer that was just bought. It is your job to tell them: 'Did you solve this linear system or this optimization problem?'" Only mathematicians can do that, because in the background there is deep approximation theory, deep stochastics and optimization. I have no clue about automatic proving or verification, what is behind. But for the numerics, I'm worried. I cannot teach numerics in the usual way, and we will have to integrate AI somehow.

At the University of Erlangen, they started a program in mathematics of data science. And now they have 10% of the new students in mathematics and 90% in mathematics of data science.

Int: *How will we see this in the future? How much mathematics and statistics will there be in data science courses?*

VM: It really depends. The simplistic idea of AI is as follows: you have data, and you want to find a function in several variables that interpolates the data so that you can evaluate it in between. In numerics, we call that interpolation. We have points and we want to find a curve which supports them. If your function that you started with is arctangent of an exponential function, for example, then we have polynomials or piecewise polynomials and Weierstrass tells us that if you take a degree that is high enough, you get convergence.

And what is AI? It uses a neural network. And what is a neural network? It is a function $Ax + b$, then $G(Ax + b)$, and then you iterate eight times or more. That is a neural network! There is a theorem using very nice mathematics from functional analysis that tells you that every continuous function can be arbitrarily well approximated by a neural network. You got it. It is no longer based on polynomials but another class of functions. It all relies on an approximation result!

Now, 99% of the computing time is spent fitting the parameters. To do this, you use the trial-and-error method, the stochastic gradient. You go in one direction, you try to find the minimum in that direction, then you choose a different one. You waste tons of computing time by trying to do this. It will finally work if you run 100 million times in all different directions in your city. At some stage, you might end up in the shop that you wanted to go to in the first place.

Int: *Don't you think that it is extremely important that people understand exactly what you said?*

VM: I am hundred percent sure that it is important, and that we have to teach this.

Int: *People have started talking about this now in the newspapers, in particular about the waste of energy…*

VM: But go to a standard math department and suggest that, instead of Algebraic Topology II or Algebraic Geometry II or III, students could follow a course on the mathematics of artificial intelligence. I'm not so sure that you'll be able to get it accepted by the faculty.

I think it is a big problem that the mathematical community drove away the computer scientists 40 years ago. In many universities, also the statisticians were driven out of the math departments. And I am afraid that we will do the same with artificial intelligence.

There is a new player in town, and we will have to give something from us to integrate this new player.

Int: *And the new player has a lot of support. From the outside, it's much easier to get support if you are in this part of science…*

VM: And you may get way more recognition by the president of the university.

12 Mathematics globally

Int: *Leaving European mathematics for a moment, countries that we have considered as developing countries have become important players in mathematics. With China, that's very obvious. But there are also other countries that have become driving forces in mathematics by now. How do you see this development and how can we support it if necessary? How can we tackle Chinese mathematics, for example?*

VM: This is a very difficult question. Twenty years ago, we could have integrated them much better, perhaps. But now they're so strong and they have so much money and they put far more money into science than we do in Europe or in the Western world. Similar things will happen in India, and perhaps also in Northern Africa.

We have to be open, but there is a real problem…. For example, in my journal, *Linear Algebra and Applications*, we get about 1400 submissions per year. About half of them come from China. But the refereeing is done to 80% by people from the West. Because we do not know these people. We have editors in China, but it is a totally different market, and it is completely money driven.

Some of these universities have incentives, if you write a paper in the top tier journal, if you publish in a journal such as *Annals* or *Inventiones,* you get a year salary on top.

Int: *In China?*

VM: In China, yes! What will scientists do as a consequence? They will try to get in and bombard the system with papers. And if you want to be non-biased, you should review them but, as a consequence, we will waste a lot of time of our colleagues. I don't know how to resolve this dilemma. The number of mathematicians in China is exploding and the number of incremental papers is also exploding.

I am hoping that we will at some stage know people in these communities better and we can then integrate them better into our evaluation system. After all, they all want to go to Europe or to the US, at least for the career jump. Then we know them! On the other hand, we cannot use most of them in teaching because their German, French, Portuguese is typically not adequate. It is a weird transition period; I don't know where it goes. EMS has for a while tried to organise conferences with China, with Japan and with India. Covid stopped all that, and I do not know how it will develop.

I will tell you another story. A very famous colleague, in Princeton, was once visited by the FBI because he had so many contacts with China, with Chinese people. Now he has left Princeton. He went to the Chinese Academy of Science and got money for hundreds of people to do PDEs and AI. If you have 100 people, even if there are only ten very good ones, that should be enough!

13 Private interests

Int: *Let us finally leave mathematics! I would like to ask you about your personal interests outside of mathematics. What are you delving into in your spare time?*

VM: In the recent five years, I have started gardening, growing my own vegetables. As you know, I am a vegetarian. It is easy in Berlin to be a vegetarian, but it's not easy if you're not in Berlin. Portugal was okay, but not spectacular. In Spain I spent a very difficult time. Now, I am growing my own vegetables. I have a nice house in the countryside, where I grow my vegetables and do mathematics.

Also, I am doing a lot of cycling. This is my hobby for my body. Gardening is actually not so good for the body. You always have a pain in your back. [laughs]

I have a lot of friends and like to go to the movies and music and to cultural events, which is so easy in Berlin.

A friend of mine, who started training on the trombone at age 52, asked me: "Why don't you, now that you retired, start music?" Well, I said: "I would be interested in playing rock guitar, but I have no time."

I was not too unhappy when I was asked to teach again. Moreover, I have one extra job that is on hold at the moment. In 2011, I applied for an interdisciplinary research center. Something like the MATHEON with a building where groups of people from different fields can come together. It was actually granted in 2013. And so, we are building a new house. And since we are in Berlin, we are a bit slow as usual. The building is finished now, and we will move in, at the beginning of next year. The university asked me to coordinate that enterprise, and to facilitate so that the people really do interdisciplinary research projects together. In a way, this is one of the babies of my career.

Int: *Integrating research, science and administration which you have been into for many, many years.*

VM: Yes. And I managed to set it up in such a way that the administration of the university has no say [laughs]. It is kind of a dream that we can make this work. And it goes back to the time when I studied, when I was at a university at which interdisciplinary research was very highly valued. That's why I'm saying the murderer always comes back to the place where he committed the crime.

Int: *How were you able to combine high level research, high level teaching, a high level of science and administration and a family life in one life? This is very impressive. What is your trick?*

VM: Well, the trick is very simple. All these are things that I like to do. It's fun… When you do things that you like and that you enjoy, and where you get recognition and people say: Great, you managed! Then I think we can harvest a lot of energy out of that.

Figure 5. Hiking in the Saxonian Switzerland with the Chemnitz and Berlin numerics groups (June 2005).

That is my kind of secret. Almost everybody who is successful in this respect has this experience. You cannot do it in a job that you don't like. It must be fun!

Int: *This is very inspiring indeed. I think this is a good moment to finish this interview. But if you have something else that you'd like to share with us…*

VM: Unfortunately, I don't have this T-shirt with me, but I have a wonderful T-shirt from the Math Museum in Giessen that states: "Maths makes you happy."

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Interpreting convex integration results in hydrodynamics

Gregory Eyink, Heiko Gimperlein, Nigel Goldenfeld, Michael Grinfeld, Ilya Karlin, Robin J. Knops, Florian Kogelbauer, Ondřej Kreml, Colin McLarty, Simon Markfelder, Mikhail Osipov and Marshall Slemrod

The aim of this article is to encourage debate of issues of the applications of modern methods of mathematical analysis in fluid dynamics. A recent surprising result derived by convex integration techniques shows non-uniqueness of weak solutions in initial value problems of the Navier–Stokes equations. The question of relevance of such a result to physical observed flows allows a variety of answers, some of which are discussed below.

1 Introduction

Convex integration techniques, introduced in the context of differential geometry by Nash [39] and Gromov [27], have been later applied to fluid dynamics by De Lellis and Székelyhidi in the path-breaking papers [14, 15]. Subsequently, these techniques have been used with great success by many researchers to prove *inter alia* non-uniqueness of weak solutions of Euler and Navier–Stokes equations. A recent review of the technique is provided in the monograph by Markfelder [37]. Below we refer to weak solutions obtained by convex integration techniques as *wild* solutions.

The question of present concern is what relevance these outstanding mathematical results have for the prediction and explanation of observed physical phenomena. For definiteness, consider the striking result proved by Buckmaster and Vicol [10]:

Theorem 1.1 ([10, Theorem 1.2]). *There exists $\beta > 0$ such that for any non-negative smooth function $e(t)\colon [0,T] \to \mathbb{R}_{\geq 0}$, there exists a weak solution of the Navier–Stokes equation in $C_t^0([0,T]; H_x^\beta(\mathbb{T}^3))$ such that its velocity $\boldsymbol{v}(\boldsymbol{x},t)$ satisfies*

$$\int_{\mathbb{T}^3} |\boldsymbol{v}(\boldsymbol{x},t)|^2 \, d\boldsymbol{x} = e(t)$$

for all $t \in [0,T]$.

A remarkable feature of this result is that the function $e(\cdot)$ does not have to be decreasing in time. The Buckmaster–Vicol theorem implies non-uniqueness of weak solutions of the Navier–Stokes equations on the 3-torus $\mathbb{T}^3$ as $e(\cdot)$ can have compact support. These wild velocity fields exhibit a strong degree of irregularity.

Although this theorem is clearly a great achievement of 21st century analysis, its relevance to turbulence and other observed fluid behaviour is not obvious and indeed is disputed by some. It is a potentially rich topic for discussion.

Thus, with the aim of making available to the wider scientific community differing views on this important subject, we[1] approached for their opinion a representative cross-section of leading practitioners not only in mathematics and physics, but also in philosophy of physics and mathematics.

Accordingly, we invited answers to the following specific question:

How, in your opinion, does the theorem of Buckmaster and Vicol relate to the prediction of observed fluid flows?

Section 2 contains the replies we have received. Section 3 summarises the discussion and contributes some concluding remarks.

To give the reader some guidance, we now describe how the contributors see their scientific outlook. Gregory Eyink and Nigel Goldenfeld are theoretical physicists with a wide range of interests, who, to quote Eyink, try "to talk with everyone and understand everyone, including philosophers, mathematicians, physicists, chemists, engineers, etc."; Ondřej Kreml is primarily a mathematician with a background in physics and mathematical modelling; Ilya Karlin and Florian Kogelbauer are applied mathematicians and mathematical physicists; Colin McLarty is a philosopher and a mathematician; Simon Markfelder defines himself as a mathematician, Mikhail Osipov is a theoretical physicist and an applied mathematician. We (understood in the sense of footnote 1) see ourselves as mathematicians who worry about the connections of mathematics with the natural world.

2 Expert opinions

This section reproduces in full the replies received to our invitation.

[1] Here and in Section 3 "we" means H. Gimperlein, M. Grinfeld, R. J. Knops and M. Slemrod.

2.1 Gregory Eyink and Nigel Goldenfeld

The question posed is whether there is any physical relevance of the convex integration techniques of Nash and Gromov, as successfully applied by De Lellis and Székelyhidi [14, 15] to the mathematics of the partial differential equations of continuum fluid mechanics. A particular focus is the remarkable result of Buckmaster and Vicol [10] proving non-uniqueness of incompressible Navier–Stokes solutions for fixed initial data. In fact, they proved the existence of weak Navier–Stokes solutions whose kinetic energy is any prescribed non-negative function. Here, as requested, we point out what aspects of this body of work are likely to be physically relevant to hydrodynamics, and which are of purely mathematical interest. We focus on turbulence in particular. Our conclusion is that the potentially important application of convex integration to hydrodynamic turbulence is to show the non-uniqueness of solutions of the Euler equations, not the Navier–Stokes equations! The reason is that it is the Euler solutions which describe the inertial-range cascade at asymptotically large Reynolds numbers [40], and the convex integration proofs of their non-uniqueness establish a prerequisite for the conjectured phenomenon of spontaneous stochasticity induced by thermal fluctuations [4, 21].

As physicists, our perspective is based on the physical relevance and structural stability of the starting assumptions, specifically the deterministic incompressible Navier–Stokes equations. These equations have been traditionally assumed in the fluid mechanics community to be valid at all scales down to the mean free path, where hydrodynamics must break down. However, molecular noise is required by the assumption of local thermal equilibrium and such noise produces many experimental effects in laminar flows that are missed by the deterministic equations [50]. It was realised already by Landau and Lifschitz [34] that hydrodynamic equations need to incorporate thermal fluctuations together with viscous dissipation, and they formulated a successful theory of "fluctuating hydrodynamics" consistent with the fluctuation-dissipation theorem of statistical physics. Using a shell model of these equations, we recently showed, in a high-Reynolds turbulent flow at parameters relevant to the atmospheric boundary layer, that thermal fluctuations become significant already at the Kolmogorov scale, orders of magnitude larger than the mean free path length [3]. Our result was subsequently confirmed by a numerical simulation at much lower Reynolds numbers, corresponding to a gently stirred vial of water, using the full Landau–Lifschitz equations [7]. Thus, we doubt the physical importance of the non-uniqueness results of Buckmaster and Vicol [10] because the deterministic Navier–Stokes equations which they study are based on an unphysical continuum approximation. The Landau–Lifschitz equations, on the contrary, are easily proved to have strong, pathwise-unique solutions (e.g., see [1]), because they are understood to be a mesoscopic effective field theory with an explicit high-wavenumber cutoff.

Our claims may appear in contradiction with rigorous mathematical derivations of incompressible Navier–Stokes as a deterministic PDE by means of a hydrodynamic scaling limit, e.g., in stochastic lattice gas models [41]. These rigorous results support the so-called "macroscopic fluctuation theory" [8], according to which velocity fields evolve by deterministic Navier–Stokes, with probability going to one. This theory attempts to incorporate thermal fluctuations by probabilistic large deviations theory, in which spontaneous fluctuations arise in thermal equilibrium by "instanton solutions" governed by time-reversed Navier–Stokes equations with negative viscosity [8]. In that setting, the Buckmaster and Vicol wild solutions might be accorded physical significance, because, under time reversal, they could describe a very rare fluctuation in which the fluid spontaneously cools down and the lost thermal energy is converted into macroscopic kinetic energy. The Buckmaster–Vicol theorem suggests that such macroscopic thermal fluctuations could arise in a strictly finite time from the rest state of the fluid and likewise could decay back to rest in a finite time, whereas smooth Navier–Stokes solutions require infinite time [43]. See Gess et al. [24] for a recent discussion of the Buckmaster–Vicol theorem in the context of macroscopic fluctuation theory. However, this interpretation of the Buckmaster–Vicol theorem is also unphysical in our opinion, because the "hydrodynamic scaling limit" invoked in macroscopic fluctuation theory is essentially impossible to achieve in any realistic molecular fluid. Indeed, the numerical counterexamples of Bandak et al. [3] and Bell et al. [7] showed that the thermal fluctuation effects can never be removed in practice even if the flow were scaled by ridiculously large factors of millions or billions!

In our opinion, convex integration techniques have physical relevance for turbulent fluids not when applied to Navier–Stokes but instead to Euler equations. Unlike the Navier–Stokes equations, the inviscid Euler equations have a precise physical regime of validity, since their weak solutions plausibly describe the inertial range of turbulent flows at very high Reynolds numbers ([40]; for a detailed review, see [20]). The inertial range is described by continuum PDEs not because the size of the smallest turbulent eddies shrinks below the molecular scale, but instead because the Kolmogorov length is much smaller than the outer scale of energy injection in the limit of large Reynolds numbers. Of particular physical significance are the convex integration proofs of non-uniqueness of weak solutions of the incompressible Euler equations with fixed initial data, even after imposing reasonable admissibility conditions such as dissipation of kinetic energy [12, 13]. The non-uniqueness of the Euler solutions suggests an intrinsic unpredictability or "spontaneous stochasticity" of turbulent flow at high enough Reynolds number. In fact, in recent work we have provided strong numerical evidence in the context of a shell model of the turbulent cascade that such stochastic behaviour with fixed deterministic initial data may be triggered by vanishingly small thermal noise [4]. Furthermore, we have proved using the physical Landau–Lifschitz equations that such persistent stochasticity requires non-uniqueness of the Euler solutions [21], exactly the condition established by the breakthrough methods of convex integration theory.

2.2 *Ilya Karlin and Florian Kogelbauer*

Convex integration is a powerful mathematical tool that allows for the construction of exotic solutions to partial differential equations – solutions that defy classical intuition and exist even when standard methods fail or when uniqueness is lost [11]. Originally introduced by Nash in the study of isometric embeddings [39], the technique was later expanded by Gromov and Eliashberg [27] in the realm of geometric PDEs.

Its impact on fluid dynamics became evident with the groundbreaking work of De Lellis and Székelyhidi [14, 15], who applied convex integration to construct weak solutions of the unforced Euler equations. Building on their insights, Buckmaster and Vicol revealed something striking: the existence of weak solutions to the unforced Navier–Stokes equations with arbitrary energy growth, as described in Theorem 1.1 above.

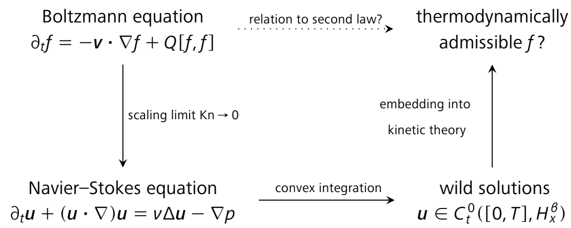
The existence of wild solutions ties deeply into the phenomenon of anomalous dissipation in turbulence. Onsager's famous conjecture states that weak solutions of the Euler equations with regularity above the critical $C^{1/3}$ threshold conserve energy, while rougher solutions may dissipate it anomalously [40]. Could a similar Onsager-type condition emerge for the Navier–Stokes equations, delineating when turbulent energy cascades lead to dissipation [18]? Despite their mathematical validity, wild solutions pose another important question: Are they physically observable? Can their bizarre behaviour – such as unbounded kinetic energy growth – be reproduced in a real-world experiment? Or is there some physical obstruction that prevents their manifestation in nature?

It seems that the puzzling properties of wild solutions, especially the loss of uniqueness, are contradictory to physically observable fluid flows and thus require further restrictions on the fluid initial conditions, i.e., a selection criterion similar to stability for dynamically observable states [46]. A recent variational admissibility principle proposed for the Euler equations [25] may serve as a step toward addressing this issue in the Navier–Stokes setting as well.

From a broader perspective, the Navier–Stokes equations arise as a scaling limit of the Boltzmann equation in the regime of vanishing Knudsen number – the ratio of the mean free path to a system's characteristic length scale [5, 26, 35, 36]. At the kinetic level, entropy translates into kinetic energy at the fluid scale, meaning that unrestricted energy growth in the Navier–Stokes framework would be in conflict with the second law of thermodynamics [47].

The previously raised question can thus be reformulated as: How do wild solutions of the Navier–Stokes equations relate to solutions of the Boltzmann equation? We know from the iterative structure of convex integration that these solutions involve extreme oscillations at very fine spatial scales [11]. However, at any finite Knudsen number, the hydrodynamic description is constrained by a critical wave number beyond which the Navier–Stokes model may break down [32].

Building on these insights, we propose the following conjecture: There exists a critical Onsager-type regularity threshold for Navier–Stokes solutions beyond which anomalous dissipation is no longer possible. Moreover, wild solutions of the Navier–Stokes equations cannot be derived from thermodynamically admissible kinetic states within the Boltzmann framework. In other words, the extreme irregularity and energy growth of these solutions may be fundamentally incompatible with the underlying kinetic theory governing physically observable fluids.



Schematic depiction of the relation of the Navier–Stokes equation to the Boltzmann equation: How do wild initial conditions translate to thermodynamically admissible kinetic states?

2.3 *Ondřej Kreml*

Recent convex integration results in the analysis of systems of partial differential equations describing fluid flows remind us that the equations we work with are merely simplified models of reality, and scientists should treat them as such. These models are highly useful in some scenarios, while in others, they may simply be unsuitable. This is evident in the case of the Euler equations, as physical fluids always have some nonzero viscosity, but to a certain extent, it also applies to the Navier–Stokes system.

Theorems stating the existence of infinitely many solutions to systems of equations used as models of reality suggest that the very definition of a solution may need careful reconsideration. It is well known that in the case of hyperbolic conservation laws, merely satisfying the weak formulation of the equations is insufficient to single out a unique solution. Additional admissibility criteria must be considered to determine what can be regarded as a physically meaningful solution. For the compressible Euler system in multiple space dimensions, identifying the correct notion of a solution that possesses the desired properties remains an open problem.

It may well be that, similar to hyperbolic conservation laws, the theorem by Buckmaster and Vicol for the Navier–Stokes equations establishes the existence of solutions that are not physically relevant. Only one – or perhaps none – of these solutions may accurately describe the behaviour of real fluids. While the theorem is a fascinating mathematical result with significant implications for the analysis of PDEs, its most important contribution to predicting real fluid behaviour may lie in reminding us that further work may be needed to identify the correct notion of a physical solution for some models describing physical processes.

2.4 *Colin McLarty*

It is an understatement to say convex integration gives beautifully motivated proofs of surprising geometric results. Beyond surprising, these results have been called "wild," and "paradoxical," and a Quanta magazine headline says they mark "where shapes give way" [10, 42, 48]. So the question arises: have these results any physical relevance? In particular, De Lellis and Székelyhidi [14], and then Buckmaster and Vicol [10], use convex integration for striking, mathematically revealing, physically shocking solutions to the Euler and Navier–Stokes equations. A fluid completely stationary for some time, and subject to no external force, suddenly bursts into rapid motion, and after some time moving with however high kinetic energy you choose, it returns to complete rest. No one expects to see this in real fluids. Does that make the results irrelevant to real fluid dynamics?

History and philosophy of science have developed terms for discussing this kind of novelty, as for example in Thomas Kuhn's *Structure of Scientific Revolutions* [33]. This book is not all about revolutions. To the contrary, Kuhn says a successful science *normally* has a widely accepted theory, together with recognized means of interpreting that theory in observational data, and specific research problems, accepted research methods, and landmark successes. He calls the bundle of a theory, problems, methods, and exemplary achievements a *paradigm*.[2] A productive science normally faces some empirical problems applying the theory to observed facts, and also conceptual difficulties extending the theory and saying exactly how the theoretical terms are meant to connect with observations. Kuhn [33, Chapter IV] compares these problems to *puzzles* in this sense: Facing a new crossword or jigsaw puzzle you cannot immediately solve it, but you assume it has a solution of known form, which can be found by skilful use of the known rules. Most often in a successful science that is what happens. Every once in a while a problem, which might not even seem like a big problem at first, grows in prominence to a point where practitioners feel it cannot be ignored, yet neither can it be settled by any of the standard means. It becomes a *crisis* and may eventuate in a *revolution* where the previously normal theory, methods, and questions give way to fundamentally new ones.

For our purposes the theory of fluid motion would be the Euler and Navier–Stokes equations [23, p. 3]:

> Till a few decades ago (early 1930s), there was unanimous opinion that the Navier–Stokes equations were useful (in agreement with the experiments, that is) only at "low" velocity regimes. It is also thanks to the efforts of outstanding mathematicians such as Jean Leray, Eberhard Hopf, Olga Ladyzhenskaya, and Robert Finn that they are nowadays regarded as the universal foundation of fluid mechanics.

In the vast literature on this theory the top questions are: Do the equations have smooth global solutions? Can this framework model turbulence? Convex integration points away from smoothness, but bears directly on turbulence [10, 14, 42, 44, 48].

The theory is not the whole paradigm. The Euler and Navier–Stokes equations per se do not specify what kind of functions count as physically meaningful (weak) solutions [38]. This is wide open today. Restricting to smooth functions is too narrow for applications. Taking all Schwartz distributions is too broad. Moderately rough functions produce the paradoxical flows.

Villani notes the most direct way to rule out these paradoxical solutions to the Euler equation would be to explicitly require kinetic energy be conserved. Or more likely just require it not to increase, since a long tradition suggests turbulence dissipates kinetic energy. This is related to requiring solutions to be at least moderately smooth. Villani discusses the problems at length [48, pp. 105, 127f., 130]. These have got sharper with the recent work on Navier–Stokes, where viscosity also dissipates energy [6, 10].

Shnirelman offers an opposite approach: Instead of requiring moderately smooth solutions, allow even rougher initial conditions than Schwartz distributions. He suggests finer analytic tools (e.g., Young measures) might make the known paradoxical solutions less paradoxical, by revealing the fluid in them is not really at rest initially, but is just so "infinitely-fast oscillating in space" that even Schwartz distributions cannot resolve the motion [44, p. 3].

From our Kuhnian point of view, then, yes, the paradoxical uses of convex integration are an integral part of current fluid dynamics. They are at least a *puzzle*. A given researcher in either pure theory or applications might not be interested in the wild solutions—if they do not need any theory of turbulence. But the decisive points are: First, we have so far no articulable paradigm for fluid mechanics doing justice to current applications while either ruling out the paradoxical ones or clearing them of the air of paradox. And second, the paradoxical solutions fit within the current paradigm in the sense that they are publicly discussed by the normal means in the subject. This note will not speculate on whether this puzzle could one day grow to a *crisis* requiring a revolution in fluid dynamics.

2.5 *Simon Markfelder*

The question of the physical relevance of the theorem by Buckmaster–Vicol is strongly related to the question what is the right solution concept for the PDEs of mathematical fluid mechanics. The same holds for all the other results achieved by convex integration in the context of fluid dynamics.

Since the equations are supposed to model the real world (i.e., they should describe and predict what happens in nature), one at least expects existence and uniqueness of solutions. For the 2-D incompressible Euler and Navier–Stokes equations, strong solutions

[2] Kuhn refined his terminology in later editions, but we can use the original.

(i.e., solutions which are sufficiently differentiable) exist globally in time. Consequently, one may regard strong solutions as a good solution concept in this case. However, for the compressible Euler and Navier–Stokes equations, as well as the 3-D incompressible Euler equations, strong solutions only exist for small times and may blow up afterwards, see e.g., [45], [49] and [19], respectively. Thus, in these cases the study of weak solutions (or even weaker solution concepts, see below) is unavoidable. Note, that for the 3-D incompressible Navier–Stokes equations whether smooth solutions exist globally in time or smooth solutions may blow up in finite time is an outstanding open question and is one of the famous millennium prize problems.

When considering weak solutions, an energy or entropy inequality has to be imposed in order to rule out unphysical solutions. This is a well-known fact in the context of systems of hyperbolic conservation laws like the compressible Euler equations. Moreover, energy/entropy inequalities can be justified from physics by thermodynamical explanations. Interestingly, both for the incompressible and the compressible Euler equations, the resulting concept of an admissible weak solution (i.e., a weak solution which complies with the corresponding energy/entropy inequality) does not lead to a unique solution, as shown by convex integration, see e.g., [14, 15]. The result by Buckmaster and Vicol on the incompressible Navier–Stokes equations is of similar type, with the difference that their solutions do not satisfy the usual energy inequality (i.e., they are not Leray–Hopf weak solutions). This is due to the fact, that in the context of the Navier–Stokes equations, one needs some minimal regularity in order to make sense of the energy inequality, as it involves terms on the gradient of the solution. And the solutions obtained by Buckmaster and Vicol do not have this minimal regularity. Still they comply with any prescribed energy profile.

So, strictly speaking, there is still hope that the non-uniqueness issue raised by the Buckmaster–Vicol theorem does not have great relevance. Namely, if it turns out that strong solutions exist globally in time, then one may question the importance of weak solutions. Moreover, Leray–Hopf solutions might still be unique. And in this case, one may argue that the solutions obtained by Buckmaster and Vicol are just too weak, and thus irrelevant. On the other hand, in view of the aforementioned results, one should better not bet on either of the two possibilities. In addition to that, for the Euler equations we know already that neither the consideration of strong solutions, nor the study of admissible weak solutions leads to existence and uniqueness. So in my opinion, one has to discard the classical notion of admissible weak solutions, and look for novel solution concepts instead. I will discuss some of them in the sequel.

In the context of the Euler equations, people have studied additional criteria with the goal to select one among the infinitely many admissible weak solutions, e.g., one has considered the solution whose energy dissipation is maximal. So far, such additional criteria have however not led to a satisfactory solution concept. What people find quite promising in this direction, is the vanishing viscosity method. This would however require well-posedness for the corresponding viscous model, e.g., the Navier–Stokes equations. Another problem is that sequences of approximate solutions (like vanishing-viscosity sequences) usually exhibit oscillations and concentrations. Thus, the sequences in general only converge with respect to weak topologies. Moreover, by understanding the (weak) limit as a conventional function, one forgets the oscillatory and concentrative behaviour of the corresponding approximate sequence. A nice way to capture oscillations and concentration is to understand the limit as a (generalised) Young measure, which yields a probability distribution at each point in space-time instead of a particular value. This leads to the notion of a measure-valued solution, see e.g., [2, 17]. It is in many situations not difficult to prove that approximate sequences converge to measure-valued solutions, which yields existence of the latter. This is in contrast to admissible weak solutions, which are—at least for the Euler equations—not known to exist. Moreover, the generalisation of weak solutions to measure-valued solutions also seems more plausible when turbulent flows are considered, since the statements of turbulence theory are of statistical nature.

However, the consideration of measure-valued instead of weak solutions does not solve the non-uniqueness problem. Quite the contrary is true, namely there are even more measure-valued solutions than weak solutions. For this reason, one has tried to impose selection criteria similar to the ones mentioned above in the context weak solutions, in order to identify the relevant solutions among the possibly many measure-valued solutions, see e.g., [9, 22]. Even though a satisfactory selection criterion has not been found yet, I feel that this approach is the most promising in order to find a good solution concept.

2.6 Mikhail Osipov

From the theoretical physics point of view, the theorem of Buckmaster and Vicol does not take into consideration that the Navier–Stokes equations are not a law of nature, but are approximate equations which are derived from the more general framework of molecular-statistical theory (e.g., from the BBGY chain of equations) using some important assumptions. In particular, these equations (as all classical hydrodynamics) are formally derived in the limit $k \to 0$, where k is the wave number. More subtle assumptions involve a properly imposed thermodynamic limit as discussed in classical works of Khinchin [30, 31]. In practice, such equations are used at finite k to describe real physical flows, but they are most likely incorrect even qualitatively in the opposite limit of large k. One notes also that the macroscopic fluid velocity is not an average over some small volume but is actually a statistical average of molecular velocity with a one-particle distribution function which is generally smooth. Thus, the average velocity is also smooth at small scales and can behave violently only on the macroscopic scale.

Therefore, any solutions of these equations that vary strongly on smaller and smaller scale are simply outside the limit of applicability of these macroscopic equations and are of purely mathematical significance. They do not have any direct relation to real physical flows.

This situation can be illustrated by a simple example from liquid crystals (my field of expertise; in this example this is elasticity but in the hydrodynamics the situation is similar). Indeed, continuum equations of elasticity of nematic liquid crystals yield point singularities in the orientational director field which correspond to point defects observed in real systems. However, continuum theory cannot describe the radius of the defect core as there is no length scale. In fact, the defect core has finite radius which is calculated approximately from a statistical theory.

It is possible to obtain formal mathematical solutions of the continuum equations for the director distribution sufficiently close to the point discontinuity. However, these solutions have no relation to the real physical defect because in this domain we are inside the finite defect core with a completely different structure. The macroscopic theory simply breaks down because of large gradients which increase on approach to the singularity and the mathematically correct solutions in this domain lose their physical validity because they are obtained outside the range of validity of the continuum equations. One has to describe the system in this domain within a different framework of a molecular statistical theory which yields the results which correspond to the experiment.

3 Discussion

We suggest that the Buckmaster–Vicol theorem can be viewed in three main ways.

The first is that the result is solely of mathematical interest and whether or not it has physical relevance is of no concern. It is unnecessary to consider the requirements of fluid dynamics. It is only essential that the result is rigorously proved.

The second way is to accept that the result is a contribution to fluid dynamics but the implications, especially in relation to turbulence, are not yet fully understood. What, for example, is the connection to the work of Kolmogorov and Onsager? A discussion of this aspect is presented by De Lellis and Székelyhidi in [16].

The third point of view accepts that the Buckmaster–Vicol results might have no physical relevance. The difficulty here is that there are many potential reasons for this lack of physical relevance, by which again we mean observability in nature or in experiments, and some of these are discussed below. Being able to rule out some of the reasons and deciding between the ones that still remain (if any), appears to be an important task for mathematics, physics, and the philosophy of science. In particular, we assert that this debate is a fruitful one for philosophy of science in its rôle as the logic inspector of the scientific enterprise, and that issues raised in this particular debate have a broader domain of application.[3]

Below we shall briefly review what seem to us as, for now, legitimate reasons to doubt physical relevance of the Buckmaster–Vicol result. The list is as exhaustive as we could make it; we can envisage it becoming both longer and shorter in the future: longer due to the failure of our imagination, and shorter as arguments become available to rule out some of the possible grounds for criticism.

3.1 Classification of reasons

The possible reasons for lack of physical relevance are of different levels of depth. At the "shallowest" (not used pejoratively) end of the spectrum are grounds that agree with the approach and its use of the object of enquiry (the Navier–Stokes equations), but find fault with the result itself in the sense of not accepting the wild solutions obtained as being of physical relevance. At the next level of depth lie objections that have to do with the applicability of convex integration to the specific object of interest. The deepest objections have to do with the structure of matter (fluids in particular) at very short scales that the convex integration technique as a technique of analysis, does not respect *a priori*.

3.2 Result-level reasons

Here we are dealing with arguments that accept that the convex integration techniques can be applied to the Navier–Stokes equations but still find the results (such as the Buckmaster–Vicol theorem) wanting in physical plausibility.

As an example of such an argument, let us consider admissibility criteria. While there certainly are wild solutions that satisfy the entropy admissibility criterion [37], there is not one known instance of a situation where any admissibility criterion yields a *unique* preferred wild solution. This is trivially true as wild solutions are created in physically indistinguishable (uncountable) equivalence classes. Hence, if one accepts that there should exist a mechanism that selects a unique solution of the Cauchy problem, such a mechanism will never select a wild solution; see for example the discussion in [25].

However, first of all, this suggestion invites the rejoinder that there is no *a priori* binding principle that requires such a property of solutions in order to force uniqueness; perhaps "God plays dice" at the macro-level. At the same time, the question arises *why* in all known cases even equivalence classes of wild solutions do not satisfy admissibility criteria. In other words, at this level objections are consequences of objections at a deeper level.

[3] An example would be a discussion of the gelation phenomenon in coagulation-fragmentation equations derived under the assumption of solute diluteness.

Another point of view, articulated in the contributions by Kreml and by Markfelder, is that the concept of weak solution is not appropriate here. Again, the questions arise why this framework, so often productively used in PDEs, is inappropriate in the particular case of Navier–Stokes equations and what other framework (measure value solutions, or considering ensembles of weak solutions as opposed to working with individual ones) should come to replace it.

3.3 Object-level reasons

In this class of reasons one claims that the possible physical implausibility of results such as the Buckmaster–Vicol theorem follows from flaws in the Navier–Stokes equations and/or the application of convex integration techniques to it. This class naturally splits in two.

For the first subclass, one could claim that convex integration is applicable but that the flaw lies in the Navier–Stokes equations themselves, and that there exists another version of the equations, e.g., one involving thermal noise modifications as advocated by Eyink and Goldenfeld, or incorporating higher-order spatial terms or nonlocal/peridynamics ones, in which results would be more intuitively realistic.

In the rest of this subsection and in the next one we discuss arguments based on the conviction that the convex integration techniques should not be in principle applied to *inter alia* the Navier–Stokes equations or any other continuum formulation of laws of fluid motion; see also the contributions by Karlin and Kogelbauer and by Osipov.

To do that we need a one-sentence summary of the convex integration technique; the reader should consult [37] for all the necessary definitions and details. It is this: starting with a suitable initial function (a sub-solution of an equation), we iteratively obtain a (family of) weak solutions to the equation in question in the limit by corrugating the sub-solution at ever decreasing scales.

It is therefore debatable if modifications at every scale can be made to a solution representing the velocity of a fluid, i.e., one may ask what it means for two such solutions to differ at the sub-Planck length scale. Observe that the incompressibility condition that assumes a well-defined macro-variable (density) and the definition of viscosity (that involves density as well) do not make sense at too short scales [29].

The main question that arises here is *how an equation derived by making the assumption of continuum can be manipulated at scales where the assumption no longer holds*.

Note that the formal object of enquiry bears no indication of the cut-off scales beyond which it cannot be manipulated.

We also observe that any analysis, such as the fundamental work of Onsager [40], that assumes that the velocity field contains information about the flow at *all* scales, appears open to the same criticism. See [20, 21] for a defence against this criticism.

This argument may harbour a hint to what concept of solution is appropriate for matter made of atoms and voids that is being analysed using a continuum description.

3.4 Structure of matter level reasons

The real numbers $\mathbb{R}$ provide a suitable setting for differential geometry, which is scale-free. However, it is arguable that a description of matter needs a fundamentally discrete setting at very short length scales (Planck length is of order 10^{-35} m and Planck time is of order 10^{-42} s). Without entering into details, we quote C. J. Isham [28, p. 189]:

> The general assumption is that something "dramatic" happens to the nature of space and time at these fundamental scales. Precisely what that dramatic change might be, has been the source of endless speculation and conjecture. However, there is a fairly widespread anticipation that insofar as spatiotemporal concepts have any meaning at all in the "deep" quantum-gravity regime, the appropriate mathematical model will not be based on standard, continuum differential geometry.

3.5 Final remarks

We are aware that not all the possible views on the subject have been presented. Hence, reader comments and opinions are welcome, for example, on computational aspects of wild solutions. These comments can be sent to m.grinfeld@strath.ac.uk. If a lively debate ensues, a continuation paper on the subject will be considered by the EMS Magazine.

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Reviving Ukrainian science: International Centre for Mathematics in Ukraine (ICMU)

Andrii Khrabustovskyi

For more than three years, Ukraine has been making headlines around the world. A bloody war, incessant bombing of cities, destroyed infrastructure, millions of refugees… And yet, even in the midst of this tragic news, there is a place for science. In 2022 a group of academics initiated an ambitious project – a new centre for mathematical sciences with the mission to support top-level research in mathematics. The position of its scientific director is held by Maryna Viazovska, a 2022 Fields Medal laureate.

Masha Vlasenko – the main driving force of the project. (Photo by Piotr Achinger, Courtesy of Masha Vlasenko)

How it all began

In the summer of 2022, a team of mathematicians of Ukrainian origin came together with the vision of establishing a "Ukrainian Oberwolfach" – a hub where mathematicians from around the world will gather to train, teach, collaborate, and share ideas.

"I've been thinking about the creation of a mathematical centre in Ukraine for a long time, apparently, since I first visited the Max Planck Institute for Mathematics in Bonn. That first visit made a huge impression on me, in particular, the difference between how everything is organised in the Ukrainian academy and there – said Masha Vlasenko, associate professor at the Kyiv School of Economics. – What we ended up with is not exactly what I thought about at the time, but rather the result of the work of a group of people who brought their ideas."

Discussions on the concept did not last long and the team quickly moved from plans to their implementation – already in autumn 2022 ICMU was officially registered in Ukraine. Soon after, the centre acquired its founding donor, the British algorithmic trading company XTX Markets.

In January 2023, Institut des Hautes Études Scientifiques (IHES) at Bures-sur-Yvette hosted the first support meeting of ICMU, which was attended by representatives of European mathematical institutes and several leading figures in mathematics philanthropy. Six Fields medallists (Vladimir Drinfeld, Hugo Duminil-Copin, Peter Scholze, Maryna Viazovska, Cédric Villani, Efim Zelmanov) took part in the event. State authorities were represented by Mykhailo Podolyak (advisor to the Head of the Office of the President of Ukraine) and Claire Giry (director general of research and innovation at the French Ministry of Higher Education and Research).

ICMU officially

The legal status of ICMU is a non-profit organisation. It is run by the Co-ordination Committee consisting of mathematicians who founded ICMU. This Committee is led by Maryna Viazovska (scientific director of ICMU) and Masha Vlasenko (managing director of ICMU). The scientific activities are overseen by the Scientific Board, which involves renowned experts from around the world. The Board makes decisions regarding the selection of the best proposals for organising thematic programmes, schools, and other events. Further implementation lies on the shoulders of the Coordination Committee operating on a voluntary basis. ICMU also has a Supervisory Board, whose mission is strategic planning of the centre's development and attracting funding. It is headed by the distinguished French mathematician Jean-Pierre Bourguignon, president of the European Mathematical Society (1995–1998) and president of the European Research Council (2014–2019).

ICMU

INTERNATIONAL CENTRE FOR MATHEMATICS IN UKRAINE

The logo of ICMU involving Voronoi cells. (Courtesy of ICMU)

ICMU is funded via charitable contributions and grants. In addition to XTX Markets, the centre got a great support from the French government, the London Mathematical Society, the Klaus Tschira Foundation from Germany, the Finnish Society of Sciences and Letters, the Finnish Academy of Science and Letters, and École Polytechnique Fédérale de Lausanne. Many companies and individuals chipped in with smaller donations. For example, Maryna Viazovska donated her "Talents for Ukraine" award from the KSE Foundation; MIT professor Daniel W. Stroock, along with many other mathematicians, also contributed to ICMU.

It is symbolic that a Voronoi diagram was chosen as the centre's logo. Invented by the Ukrainian mathematician Georgy Voronoi more than 100 years ago, these diagrams are used in various fields of natural sciences, engineering, architecture, humanities and many other fields.

In search of a permanent home

The centre is still looking for a permanent location. Masha Vlasenko: "…Several state authorities are willing to help us [with finding suitable premises]. We are collaborating with the Ministry of Education and Science and the Office of the President of Ukraine. We also received several offers of space from academic institutions. A location in the downtown of Kyiv is considered by our team as the most convenient. We hope to have a home soon."

In the meantime, the partner institute – Kyiv School of Economics (KSE), kindly provided the centre with a temporary workspace on its campus. KSE also became a temporary home for the centre's library. Its core is more than two thousands books donated by Springer Publishing House, including the 120 volumes series "Springer Collected Works in Mathematics." A bunch of books were donated to the new library by Société mathématique de France.

In the World Wide Web the centre set its home at the address https://mathcentre.in.ua.

The grand opening: Numbers in the Universe

In August 2023 ICMU organised its first scientific event – the school and conference "Numbers in the Universe." The event took place in the premises of KSE and the Banach Center in Warsaw, with a live connection between the two audiences. The programme included lecture series by two Fields Medal laureates, Maryna Viazovska and Terence Tao.

Maryna Viazovska lecturing in Kyiv during "Numbers in the Universe." (Photo by Gregory Bob, Courtesy of ICMU)

"It was a great event, thanks to which we managed to attract the attention of many people to our project, – said Vlasenko. – In fact, initially we doubted whether the Ukrainian audience would be interested in it." Indeed, the event was aimed to present the latest breakthroughs in number theory and its applications – a rather exotic topic for the Ukrainian mathematical community. Vlasenko continues: "We assumed that most would come to see the famous mathematicians, but would not go to the other lectures because they would find it difficult. Fortunately for us, the exact opposite happened – it turned out that the Ukrainian audience longed for such an event. A lot of young people stayed after the lectures asking questions and solving exercises until late in the evening." Jakub Byszewski, professor at the Jagiellonian University in Kraków, came to Kyiv specifically to run problem sessions. Among other brilliant speakers, the event featured lectures of two other Ukrainian mathematicians, Danylo Radchenko and Oleksandr Marynych.

Perhaps the main result of the event was the revealing of the Ukrainian mathematical community to the world. Inspired by the conference, Terence Tao wrote on his blog[1]: "The Ukrainian mathematical community was a lot more substantial than I realised, from its eager students all the way to its two Fields medallists (Drinfeld and Viazovska). In retrospect, I think a lot of this strength was historically obscured during the Cold War era due to the conflation 'Soviet Union == Russia' that was common in the West."

Done so far: schools and lecture series

During the first period of its activity, the centre mainly focused on organising various educational events. The first such direction of activity of ICMU concerns summer and winter schools intended for students and young scientists, aimed to involve them in various areas of contemporary mathematics. They are organised in a safe environment in the west of Ukraine (in particular, on resorts in the Carpathian Mountains). These schools are planned as international events – with lecturers from across the world being leading experts in their fields.

Anna Lytova, professor at Opole University in Poland, took part in one of the summer schools. Anna is from Kharkiv, the second-largest Ukrainian city and an established scientific centre, where she got her PhD before moving abroad. She shares her impressions: "The event was absolutely beautiful. Four excellent researchers from America, Canada, Poland, and Hungary introduced students to modern trends in mathematics and made them feel a part of the international mathematical community. We were happy to see so many strong students from different places all over Ukraine gathered together, working on maths, and socialising in a creative atmosphere. Ukraine lacks such events, and these ICMU schools are a big breakthrough."

Anna was also one of the organisers of the ICMU winter school "Random matrices, random analytic functions, and non-linear PDEs" in Zakarpattia region. She concludes: "I hope in the near future there will be a possibility to organise such events in my beloved Kharkiv."

Olena Atlasiuk, a postdoc from the University of Helsinki, who took part in that winter school, shares her experience: "Amid the profound darkness of war, the participants and organisers brought to life the joy of connection, the exchange of ideas, new friendships, and positive emotions. Throughout the week, the theory of random matrices was taught to everyone – even the cat, who made occasional appearances during the lectures. Only Ukrainian dumplings and lard could rival the excellence of the mathematical presentations! The experience felt seamless, enriching, and inspiring – a breath of fresh air for all involved."

The participants of the ICMU Summer School "ATA XVI: sub-Riemannian geometry and optimal transport" taking place in Kolochava.
(Courtesy of ICMU)

The other educational programme at ICMU comprises a series of online mini-courses designed primarily for Ukrainian undergraduate and graduate students, as well as for very well-prepared high school students. These courses included not only lectures but also practical classes with the need for homework to successfully pass the course.

Many lectures given at the centre's events were recorded and are now available to a wider audience at its quickly growing YouTube channel.[2]

Done so far: visitors

The focus of ICMU is the attraction of foreign researchers to visit ICMU for an extended period of time. The first distinguished visitor of ICMU was Stephan Klaus, a professor at the University of Mainz and scientific administrator at the Oberwolfach Research Institute for Mathematics (MFO). During his visit in May 2024, he conducted a week-long crash course on algebraic topology.

The restrictions caused by the war make such long term visits challenging, if only because not every university is ready to send its employees on a business trip to a war zone. But, despite all

[1] Source: https://mathstodon.xyz/@tao/110874054691602461

[2] https://www.youtube.com/@the_ICMU

these difficulties, in 2024 ICMU launched two programmes for the external visitors.

The first programme "LMS Distinguished Visiting Fellowships @ICMU" is run jointly by the London Mathematical Society (LMS) and ICMU, and is aimed for established researchers, who can spend a period from one week to one month and give a series of lectures or a colloquium for the benefit of the Ukrainian mathematical community. Alex Iosevich, a world-renowned expert in harmonic analysis, was the first mathematician to visit Ukraine under this scheme. He delivered a series of lectures in Lviv at the Ukrainian Catholic University and Ivan Franko National University, as well as in Kyiv at the working space of ICMU. The other visitors last year were Augusto Gerolin from the University of Ottawa and Francis Brown from the University of Oxford, both visiting ICMU in Kyiv. Every visitor has a Ukrainian hosting scientist.

The second programme "Collaboration @ICMU" resembles the well-known "Research-in-Pairs" programmes existing in many mathematical centres. It aims to bring together a group of researchers from different institutions who wish to organise a short, intensive period of research. The necessary condition – at least one of these researchers should have a Ukrainian affiliation. The ICMU provides a working space and arranges lodging for all involved.

Reception

From the very beginning ICMU got a lot of positive feedback from the worldwide mathematical community. "A lot of people, both Ukrainian and non-Ukrainian mathematicians, started contacting us with good ideas, – speaks enthusiastically Masha Vlasenko. – For example, Borys Kadets from the Hebrew University of Jerusalem, who proposed a series of lectures introducing a wide audience to algebraic geometry. These lectures went extremely successfully, attended by many students, high schoolers and teachers."

Professor Pavel Exner from the Doppler Institute for Mathematical Physics and Applied Mathematics in Prague, the president of the EMS during the period 2015–2018 and a current member of the centre's Scientific Board shared his opinion about the project: "When I was asked to join the board of ICMU, I accepted with pleasure, and not only because of my professional relations with Ukrainian mathematicians, some dating more than three decades back. I felt at the same time that it is vital to keep and develop high-level scientific activities in the times when the country faces a military aggression. I am glad that the ICMU project found a wide support and the first two years of its activity were successful; I am convinced the centre will keep this spirit in the years to come, for the good of Ukraine and mathematics."

It often happens that ideas generated by the diaspora do not find a response among locals. Couldn't this have happened to ICMU, which was founded (mostly) by those working abroad? Vlasenko responds: "When our team was organised, there was no need to explain to each other what it should be, there was an obvious consensus on what we are up to build. On the other hand, local mathematicians often have no experience of visiting such centres, so the benefit of our project was not obvious at the beginning – there was even some controversy in social networks. However, the people from Ukraine in our team said "You just move. They will see and join." And at the end of day, this is exactly what happened: for example, the summer school in Kolochava was initiated by mathematicians working in Ukraine."

Dmitry Shepelsky, the chair of the Department of Differential Equations and Geometry at the B. Verkin Institute for Low Temperature Physics and Engineering of the National Academy of Sciences of Ukraine, a member of ICMU Scientific board, explains the role of the centre: "For students from Kharkiv and other cities where education is conducted exclusively online, the schools established by ICMU provide a rare opportunity for live interaction – both with professors and within a community of fellow students. This is especially relevant for mathematical education, which is currently facing enormous challenges in Kharkiv – a traditional centre of mathematical education and research. Students are reluctant to enrol in such programmes, as they find them difficult, especially when taught online. As a result, the remaining professors are under constant pressure due to the decreasing number of positions. Thus, the centre's programmes offer at least some sense of 'normality' in studying mathematics."

What's next?

The main regular activity of the centre will be the holding of thematic programmes lasting from a month to a full semester. Such a thematic programme includes a number of activities such as lecture series by invited guests, weekly seminars, conferences, workshops and science schools to introduce young scientists to the chosen subject. The first call is planned for 2026.

The collaboration programme involves small groups of researchers spending one to two weeks at the centre to work intensively on a specific task. In the future, long-term visiting scholarships for postdoctoral researchers and scientists are also planned.

Another planned activity concerns the translation of modern mathematical textbooks into Ukrainian. ICMU is going to give grants to the translators and organise further editorial and publication processes. Iryna Yehorchenko from the Institute of Mathematics in Kyiv explains its importance for the Ukrainian mathematical community: "There is a generally accepted myth that Soviet mathematical education was brilliant – it is wrong as very few high school students (but only these students were visible) got that good education. This myth persisted into independent Ukraine, and education authorities, as well as publishers, dismissed the need to support popular mathematics and to publish both popular mathematics books and books for students. Less than ten books were published in each category – at a time when Ukrainian book publishing was developing fast. Now it is high time to promote popular mathematics and have good books in Ukrainian, at least for first-year students."

ICMU also aims to support students in their early steps in science. At the end of 2024, a competition was held to award outstanding student research papers in the mathematical sciences. Such awards are planned to be presented annually.

Speaking with journalists at the grand opening of ICMU, Maryna Viazovska said: "The new centre will serve as a place of intensive collaboration between Ukrainian scientists and their foreign colleagues, a place where aspiring researchers will be learning about the latest mathematical discoveries. While there are reputable mathematics schools and institutions with history in Ukraine, our infrastructure is insufficient. It is essential to provide favourable conditions for professional interaction and exchange of ideas, so that it would be possible to host gatherings with the participation of scientists of different generations, from different countries. Such gatherings are inspiring and there one can learn things one won't read about in any book or hear about at a university. I believe that knowledge resides in the human heart, and the exchange of knowledge is carried from heart to heart."

Masha Vlasenko summarises: "What is our activity in wartime? ICMU does everything possible and everything that is good… I would like Ukrainian mathematicians from all over the world to invest in this project – that is what it should be based on."

Acknowledgements. The author is deeply grateful to Masha Vlasenko for inspiring him to write this text. He also thanks Pavel Exner, Anna Lytova, Olena Atlasiuk, Dmitry Shepelsky and Iryna Yehorchenko for taking the time to share their thoughts on ICMU.

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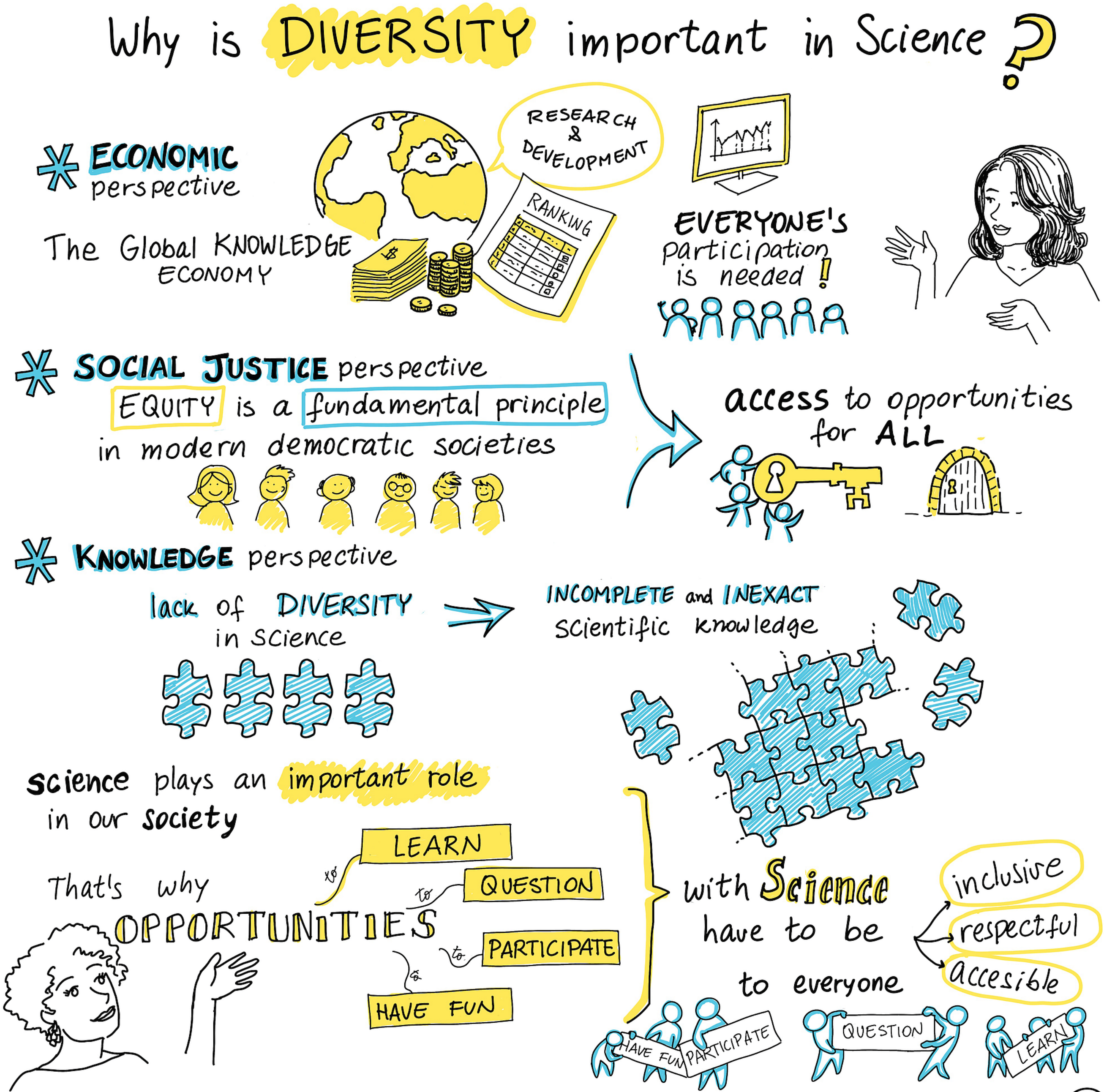
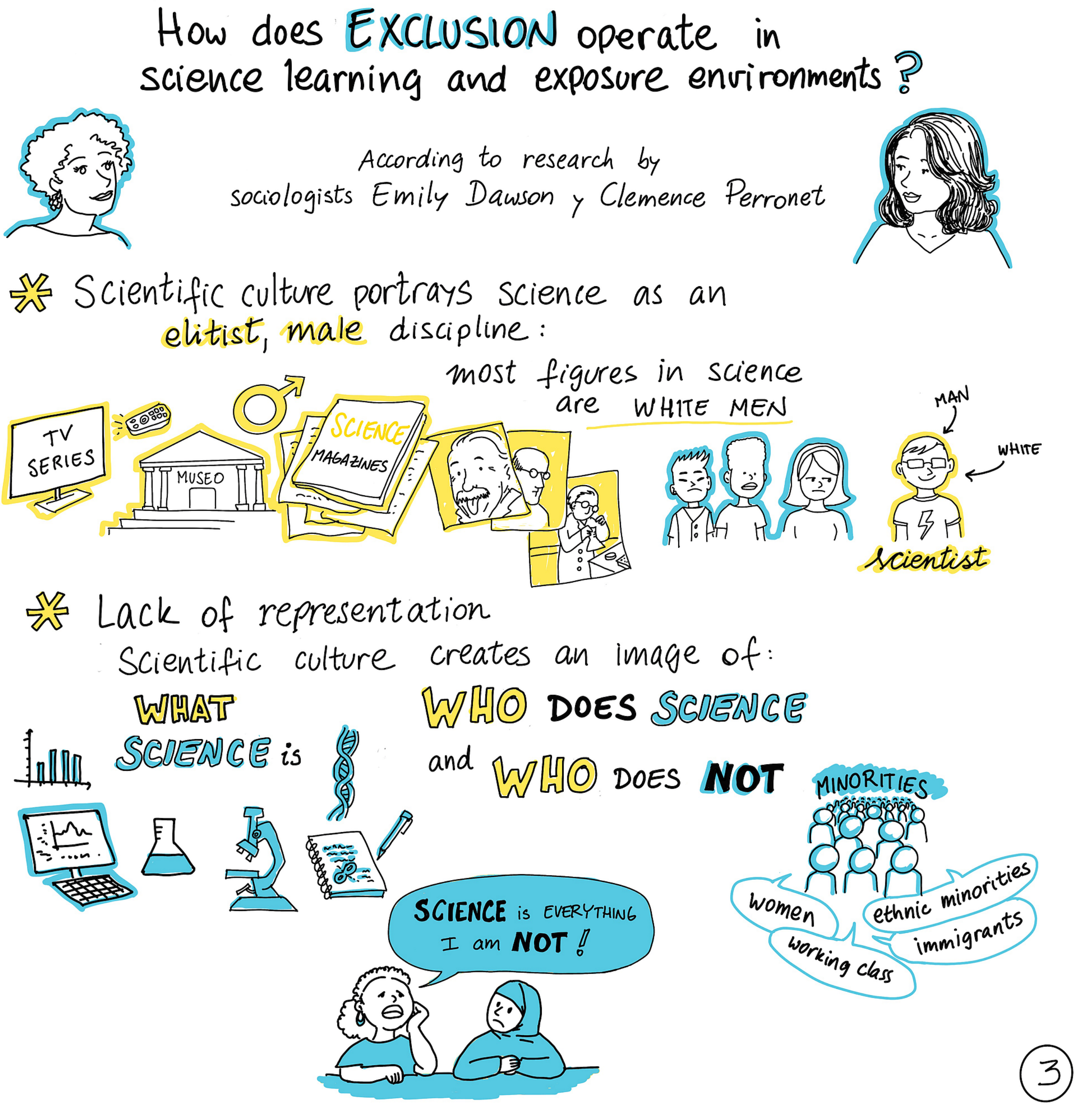
Why is diversity important for science?

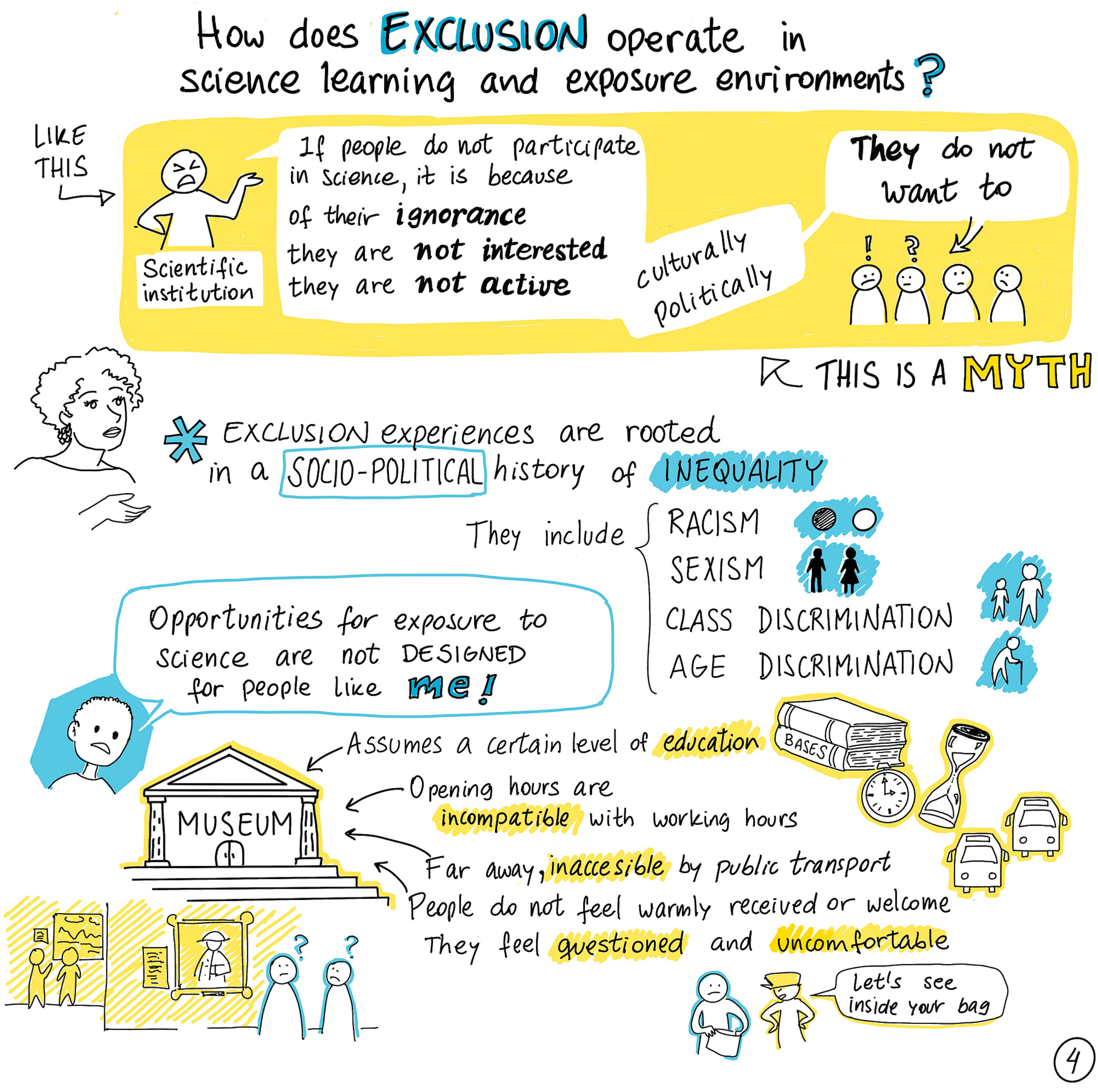
Constanza (Coni) Rojas-Molina

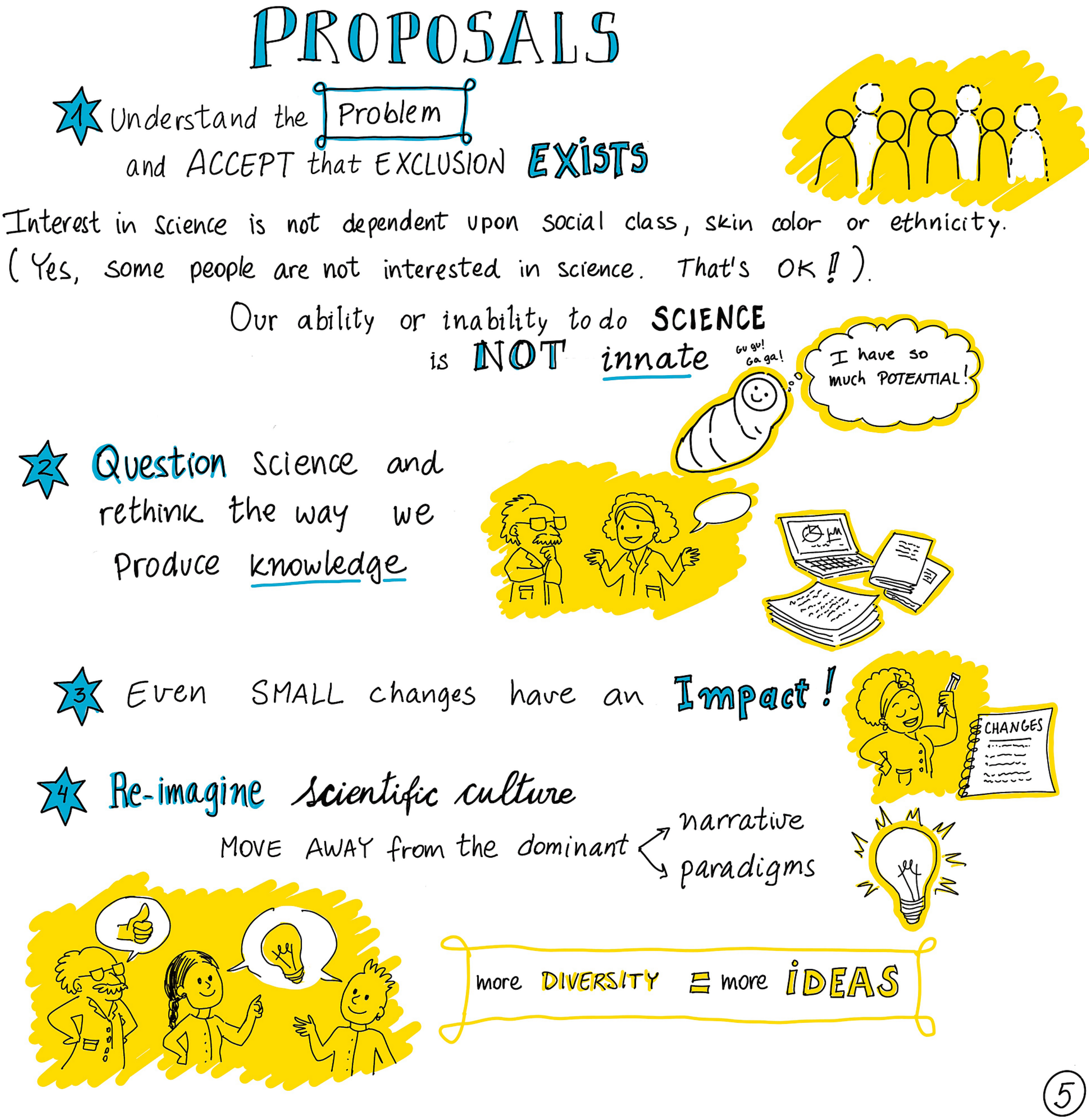
The panel discussion "Social justice, equity and science education" was organized by the project *Anillo Matematicas y Genero*. This interdisciplinary project, funded by the Chilean research agency ANID, brought together mathematicians and sociologists to study mathematics as a sociological field and in particular, the place of women in mathematics, in Chile. For the panel discussion, sociologists Jeanne Hersant, Emily Dawson and Clémence Perronnet met to answer the question of why diversity is important in science and how exclusion operates in this field.



Why is DIVERSITY important in Science ?
ECONOMIC perspective
The Global KNOWLEDGE ECONOMY
RESEARCH & DEVELOPMENT
RANKING
EVERYONE'S participation is needed !
SOCIAL JUSTICE perspective
EQUITY is a fundamental principle in modern democratic societies
access to opportunities for ALL
KNOWLEDGE perspective
lack of DIVERSITY in science
INCOMPLETE and INEXACT scientific knowledge
science plays an important role in our society
That's why OPPORTUNITIES
to LEARN
to QUESTION
to PARTICIPATE
to HAVE FUN
with Science have to be
inclusive
respectful
accesible
to everyone
HAVE FUN
PARTICIPATE
QUESTION
LEARN
2


How does EXCLUSION operate in science learning and exposure environments?
According to research by sociologists Emily Dawson y Clemence Perronet
Scientific culture portrays science as an elitist, male discipline:
TV SERIES
MUSEO
SCIENCE MAGAZINES
most figures in science are WHITE MEN
MAN
WHITE
scientist
Lack of representation
Scientific culture creates an image of:
WHAT SCIENCE is
WHO DOES SCIENCE
and WHO DOES NOT
MINORITIES
women
ethnic minorities
working class
immigrants
SCIENCE is EVERYTHING I am NOT!
3


How does EXCLUSION operate in science learning and exposure environments?
LIKE THIS
Scientific institution
If people do not participate in science, it is because of their ignorance
they are not interested
they are not active
culturally
politically
They do not want to
THIS IS A MYTH
EXCLUSION experiences are rooted in a SOCIO-POLITICAL history of INEQUALITY
They include
RACISM
SEXISM
CLASS DISCRIMINATION
AGE DISCRIMINATION
Opportunities for exposure to science are not DESIGNED for people like me!
MUSEUM
Assumes a certain level of education
BASES
Opening hours are incompatible with working hours
Far away, inaccessible by public transport
People do not feel warmly received or welcome
They feel questioned and uncomfortable
Let's see inside your bag
4



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Statistics and data science education as a vehicle for empowering citizens – short summary of a survey

Rolf Biehler, Takashi Kawakami, Erna Lampen, Travis Weiland and Lucía Zapata-Cardona

This article is a short summary of the report of survey team 3, presented to the 15th International Congress on Mathematical Education (ICME-15) in Sydney in July 2024.

1 Introduction: Data in society, data science education and citizen empowerment

We face a new challenge in determining which knowledge, skills, and dispositions about data and statistics should and could be taught in secondary education. Data science, as a new field embracing statistics, touches several disciplines, and its scholarly knowledge is interdisciplinary, dynamic, and unstable. This is already a challenge for the process of "didactic transposition" [10] of scholarly knowledge to knowledge to be taught. Moreover, the transposition has to take into account the role of data in society, which affects all communities and individuals. Therefore, the use of and discourse about data in society are more and more pertinent for those who reflect on the relevance and uses of knowledge to be taught in secondary and primary schools. School statistics has not kept pace with how citizens engage with increasingly pervasive data, such as navigating X feeds, using artificial intelligence (AI) to identify photos, and streaming GPS data as a live feed into Google Maps to estimate travel times. Data science and data-driven AI have led to breakthroughs in science and society. Data science can be used for social good, including work to protect the environment and address climate change. But it can also promote the economic and political interests of a few while failing to serve the interests of most citizens. Its use has raised massive concerns about privacy, misuse of data, ethics, and surveillance of citizens, to name a few. Awareness of the non-objective nature of data, such as the underlying gender/racial bias in how and whose data are used to train algorithms, is also at the forefront of these discourses – see, for example, the journals *Big Data & Society* and *AI & Society*.

These developments create the need to redefine what it can and should mean to empower citizens through education. The survey identified various conceptions of citizen education, such as aiming at personally responsible, participatory, justice-oriented citizens, or global citizenship and promoting the UN's sustainable development goals [7, 29, 33, 34]. In the case of expanding data science to empower citizens, these aims compete with forces on the educational system, such as economic, military and political forces as part of international competitions that require workforce education for different purposes.

The survey identified literacy conceptions for citizen education from different disciplines and perspectives, which enhance and transform more traditional conceptions of statistical literacy [15, 17]. These include (critical) statistical or data literacy [24], civic statistical literacy [29], and data-driven mathematical, computational, and algorithmic modeling [20]. Data literacy is also conceived as part of more general literacies, such as digital humanities literacy, media, news, information literacy, digital literacy, Artificial Intelligence (AI), and machine learning literacy [9]. The datafication of disciplines has led to subject-specific data practices in the social and natural sciences. New interdisciplinary approaches have emerged, particularly in addressing socio-scientific problems within the framework of citizen science. Moreover, conceptions such as critical datafication literacy [31], personal data literacy [27], data awareness [18], data acumen, conscience, ethics, activism, and feminism [11] are put forward in the discourse.

These complex and demanding developments impact all school subjects and collide with fully packed curricula in all subjects and with an abundance of problems in many educational systems, for instance, a very high number of low achievers.

The survey identified recent books and special issues of journals that help to orient the discourse in the field. Special issues on data science education have been published since 2020: in the *Journal of the Learning Sciences* (2020); *Teaching Statistics* (2021); *Statistics Education Research Journal* (2022); *Educational Technology and Society* (2022); *Information and Learning Sciences* (2024); *Computers and Education Open* (2024). This shows that the survey went far beyond the usual sources for mathematics and statistics education.

Given this complexity, the survey team focused on four topics.

2 Topic 1: Civic statistics and humanistic perspectives on data literacy education in the U.S. and Europe

Humanistic perspectives and citizenship development have been prevalent themes in statistics and data science educational research in the U.S. and Europe for several years (e.g., [23, 29]). Scholars in these areas also work transdisciplinary, infusing data literacy across the curriculum. Much of this work has been done in small-scale qualitative projects as these fields try to make sense of the rapidly evolving nature of data literacy in our information-centric societies. Several themes have emerged, including (1) reading the world with data, making sense of the data-based communication of others, including data viz and data journalism [19, 30]; (2) writing the world with data, using data practices to investigate the world around us with students authentically engaging in activities of the discipline through data investigations or creating data stories (e.g., [22, 35]; (3) data structures and handling, focusing on data moves and clean data to make raw data accessible to analysis particularly in the form of "tidy data" (e.g., [13]); and (4) technology, including the development, interaction with, and learning from technology.

3 Topic 2: Critical perspectives on data literacy emerging from Latin America

Data literacy in Latin American countries has taken a critical perspective in the form of critical data literacy, which is the skill set that enables people to use and produce data critically concerning the reality behind the data [4]. It requires a combination of technical skills and the ability to reason critically about data and context. Critical data literacy in Latin America is strongly influenced by the critical pedagogy and popular education of Paulo Freire [32], which seeks to help people to develop their ability to read and write their world. The work of Giroux and Skovsmose has also been influential in this perspective. Critical data literacy in a region with high levels of social, cultural, and economic inequality is needed to help people to (1) make sense of the important data that affects their lives, (2) make informed decisions, (3) participate in public life, (4) recognize the harm that powerful interests can inflict with data, (5) recognize that data is not neutral, (6) recognize that biased algorithms can exacerbate social and economic inequalities, (7) expose systematic social injustices, (8) understand the inequalities within the work system and how disparities between nations exacerbate the oppression of deprived populations [28].

4 Topic 3: Joint discourse between mathematical modeling and statistics/data science communities

More researchers have recently worked "at the boundary" of mathematical modeling (MM) and statistics/data science communities (e.g., [3, 21]). This section aimed to report new trends in the joint discourse between the two scientific communities, focusing on three relevant discourses on data-rich MM since 2020. The first discourse proposes a data-rich MM process with statistics and mathematics at its core to develop statistical and/or mathematical literacy and/or disciplinary learning (e.g., [12]). The second discourse discusses interdisciplinary data-rich MM, involving not only statistics and mathematics but also other disciplines/subjects, to promote STEM literacy as essential for citizens (e.g., [2, 25]). The third discourse focuses on societal data-rich MM, which uses global, social, political, ethical, and everyday contexts to promote critical thinking and citizenship (e.g., [16, 36]). All three discourses emphasize the process of modeling as a cycle and seek to understand the relationship between discipline-specific approaches to modeling and the role of data herein.

5 Topic 4: What can mathematics/statistics education contribute to Artificial Intelligence/Machine learning literacy

The discussion on AI literacy for secondary students has rapidly expanded in a few years, with several review articles emerging in this evolving field, primarily from computer science education (e.g., [1]). Machine learning (ML), which intersects with statistics and mathematics, is recognized as a key area of focus. It involves predicting outcomes through mathematical and statistical modeling and a broader form of inference than sample-to-population inference. A major concern is the opacity of ML models [8]. A consensus is forming that at least one type of machine learning should be taught in schools using a "white box" approach, i.e., a machine learning algorithm not as a black box, but making it partially transparent to learners ("gray box" or "white box") [26]. Researchers propose decision trees [14] and *k*-nearest neighbors [26] as promising candidates to reduce the opacity of machine learning algorithms. The reviewed discourse on fundamental concepts of ML includes distinguishing between regression and classification, understanding misclassification types, and differentiating training and test data to address overfitting, bias, and fairness (see [6] for an overview with examples for teachers). Research projects were identified that make these concepts accessible to secondary students through tools like data cards [14], CODAP, and Jupyter notebooks (e.g., [5]).

6 Conclusion

This survey clearly indicates that statistics has evolved into data science with new tools, discourses, and various source domains. The data demands that pervade most societies require renewed attention to statistics education at school, data science education across disciplines, and research-informed decisions about curriculum and pedagogy.

An extended version of this report is available. To order, please email biehler@math.upb.de.

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ICMI Study 27: Mathematics Education and the Socio-Ecological

ICMI column in this issue presented by Kate le Roux and Alf Coles

The roots of ICMI Study 27: Mathematics Education and the Socio-Ecological

As a named topic in international mathematics education forums, "Mathematics Education and the Socio-Ecological" is relatively new compared to many other topics in the field. Yet there is wide recognition of the urgent need for thought leadership to consider what is and might be the role of mathematics and mathematics education in multiple, intersecting, social, political, ecological, and economic issues such as climate change, poverty, inequality, health crises, discrimination, and marginalisation. Attention to this concern from within mathematics education is not new and has commonly drawn resources for research and practice from traditions such as critical mathematics education, decoloniality, ethnomathematics, feminist and new materialist thought, indigenous ways of knowing, and mathematical modelling (see [2] for a fuller description and acknowledgement of these traditions). These empirical and theoretical contributions, in their diversity, have only recently been brought into conversation together and with more established topics of mathematics education such as curriculum, teacher education, technology (for example, in journal special issues and conference discussion groups), towards overarching questions about both mathematics education and the entanglement suggested by the hyphenation in the term socio-ecological.

This growing move in mathematics education internationally was well illustrated when, prompted by the Executive Committee of the International Commission on Mathematics Instruction (ICMI), on March 20 (GMT) we collaborated with a group of scholars to host a one-day online symposium entitled "Mathematics Education and the Socio-Ecological" [7]. The organising team included: Kate le Roux, Alf Coles, Richard Barwell, Marcelo Borba, Anna Chronaki, Rochelle Gutiérrez, Lauren Hennessy, Mariam Makramalla, Aldo Parra, Milton Rosa, Armando Solares-Rojas and Jayasree Subramanian. This was the first such event for ICMI. Running over a full 10 hours to cater for different time zones, it brought together 170 people: mathematics education researchers (established and early career), mathematics teachers, mathematicians, and scientists from other disciplines. Some participants attended for the full time. There were 40 presentations from 18 countries, as well as one plenary presentation and two plenary panels.[1] Abstracts are represented in the Symposium Proceedings [6].

Conceptualisation of ICMI Study 27

ICMI Study 27, "Mathematics Education and the Socio-Ecological," announced in September 2023, builds on this interest, commitment and energy, from a range of contexts. As with all ICMI Studies, the task of participants is to analyse and describe the research and practice on the topic to date, and to harness such contributions towards anticipating new possibilities. The latter requires building community (including interdisciplinary relations) towards new directions in mathematics education, as it relates to the socio-ecological across local and international levels. An ICMI Study is marked by three key events: the publication of a Discussion Document and Call for Participation; a Study Conference (with published, peer-reviewed Proceedings); culminating in the publication of a Study Volume, published open access by Springer as part of the New ICMI Study Series.[2]

The ICMI Study 27 process is led by a 13-member International Programme Committee (IPC). This comprises 11 scholars, selected by the ICMI Executive Committee, who have been working in this topic area and who come from a variety of theoretical backgrounds and geographic areas. In addition, two ex-officio ICMI Executive Committee members have provided clear guidance to the team, based on extensive experience of the ICMI Study process, while also allowing us to give meaning to this specific Study, on a relatively new topic. These are the members:

- Alf Coles (United Kingdom), co-chair
- Kate le Roux (South Africa), co-chair
- Marcelo Borba (Brazil)
- Arindam Bose (India)
- Vince Geiger (Australia)

[1] See the programme: https://www.mathunion.org/fileadmin/ICMI/Conferences/Socio%20Ecological/ICMI_Symposium-announcement.pdf.

[2] https://www.springer.com/series/6351

- Rochelle Gutiérrez (United States of America)
- Mariam Makramalla (Egypt)
- Nathalie Sinclair (Canada)
- Armando Solares-Rojas (Mexico)
- Paola Valero (Sweden)
- Catherine Vistro-Yu (Philippines)
- Frederick Leung (Hong Kong SAR), president of ICMI until December 2024 (ex-officio)
- Merrilyn Goos (Australia), president of ICMI since January 2025 (ex-officio)
- Jean-Luc Dorier (Switzerland), secretary-general of ICMI (ex-officio)

The design and implementation of ICMI Study 27 launches from engagements with notions of the social and ecological (what we refer to as the "socio-ecological"). These engagements include, for example, recognition of the interdependent relations between all humans (individual and community, bodies, thoughts, emotions) with the living and inanimate natural world; material technologies; languages; and concepts (in this vein we, personally, have been particularly influenced by [3–5]). Our conceptualisation of the socio-ecological does not include: metaphorical definitions of socio-ecologies of mathematics education (within education) such as a socio-ecology of curricular reform, policies, etc.; socio-ecological issues as a neutral backdrop to mathematics education; and mathematical analysis of issues in nature as if they are distinct from the cultural.

The Study is organised into four Study themes, which identify and analyse tensions emerging from mathematics education's positionings in the complexities of the social, the ecological and their relations. We view these tensions as creative, as generating new questions about and for mathematics education, for example, questions related to: knowledge, curriculum, pedagogy, learning materials, professional development, language, philosophy, theory, methodology, and so on. And hence prompting new imaginaries of presents and futures. Briefly, the themes and subthemes, announced in January 2024 in the Discussion Document [2] (see also [8]), and the members in charge, are:

Theme A. *Aims of mathematics education*: Examining the aims of mathematics education; examining mathematics as a subject of education. (Led by Nathalie Sinclair and Paola Valero.)

Theme B. *Scales of mathematics education*: Relations between the local/global, historical, ecological, and political; curriculum innovations and different voices located in the socio-ecological; learning from site-specificities. (Led by Rochelle Gutiérrez, Mariam Makramalla, and Armando Solares-Rojas.)

Theme C. *Resources for and of mathematics education*: What and how resources are/may be used in relation to socio-ecological concerns; how resources are embedded within histories, values, and ideologies. (Led by Marcelo Borba and Vince Geiger.)

Theme D. *Mathematics education futures*: What contexts and communities of education can/have yet to be imagined; what knowledges, curriculum and pedagogies can/have yet to be imagined; practices and ethics of mathematics education research that can/have yet to be imagined. (Led by Arindam Bose and Catherine Vistro-Yu.)

The second key feature of ICMI Study 27 is the recognition that knowledge building in the field requires diverse voices, experiences, and knowledges, produced in collaborations involving researchers, teachers, teacher educators, and other stakeholders from diverse contexts and sites of education [9]. In April 2024 the IPC offered two interactive, online webinars during which participants could learn more about the Study aims, processes and themes. The IPC commitment to inclusivity has included:

- A review and revision process supportive in relation to writing in English.
- Giving visibility to all accepted contributions in the Study Conference Proceedings.
- Hosting the Study Conference in a developing, lower-middle income country, thus harnessing local and regional knowledge.
- Creating a solidarity fund to support participation in the conference.
- Considering relevant Study Proceedings contributions in the production of the Study Volume. This volume is taking shape after the conference, with authorship centred around conference participation.

Study conference submissions, reviews and proceedings

In planning the Study, we grappled with the affordances and constraints of in-person events to which those involved in the Study would have to travel. Many on the IPC were involved in the activities of the 15th International Congress on Mathematical Education (ICME-15) in Sydney, Australia, in July 2024. So we took that opportunity to be together – our first in-person IPC meeting of the full Study – at the Australian Catholic University, Sydney Campus [10]. Two IPC members contributed online. Most of the meeting time was dedicated to reading submissions, and writing and discussing reviews. Over 70 submissions, authored from 29 countries, were received. In an open review process, each submission was reviewed by two or more members of the IPC. Reviews contained detailed, constructive feedback on the papers, with a particular focus on the (potential) contribution of the paper to thinking about both mathematics education *and* the socio-ecological.

The IPC members attending ICME-15 also used the time together in Sydney to offer a discussion group "Mathematics Education and the Socio-Ecological" at the congress. This was structured

Nine members attending an IPC meeting, July 2024, Sydney, Australia. (Photo credit: ICMI Study 27 IPC)

according to the Study framing, and was an excellent opportunity to explore the potential of the individual themes and their coherence, to build our team to lead sessions at the actual Study Conference, and to prompt interest in the topic. Session 1 (75 participants) and session 2 (85 participants) confirmed the high level of interest in the topic.

Following the review process (which included, for some authors, the option of submitting a revision for further review by IPC members), 63 papers were published in the ICMI Study 27 Conference Proceedings [11]. This represents contributions by 135 authors from 24 countries, across six continents: Argentina, Australia, Austria, Brazil, Canada, Colombia, Egypt, Germany, India, Indonesia, Iran, Israel, Italy, Japan, Mexico, Nepal, Netherlands, New Zealand, Norway, Philippines, South Africa, Sweden, United Kingdom, United States of America. As noted, early in the process, we committed to representing all accepted contributions in the Study Proceedings, even if authors were not able to attend in person. The number of such cases was relatively small.

ICMI Study 27 Conference, at Ateneo de Manila University, Quezon City, Philippines, 22 to 25 January 2025

In January 2025, 65 Study Conference participants were warmly welcomed by the Local Organising Committee (LOC), led by Catherine Vistro-Yu, and comprising Angela Fatima H. Guzon, Lester C. Hao, Maria Alva Q. Aberin, Errol Matthew C. Garcia and an enthusiastic, energetic and efficient team of 14 student assistants [1]. The successful hosting of the conference in the Department of Mathematics at Ateneo de Manila University was in no small part due to the generosity and hospitality of the university as well as the Philippine Council of Mathematics Teacher Educators (MATHTED).

A proportion of the paid conference registration fee was allocated to a solidarity fund, which was run according to ICMI principles of supporting participation of individuals from developing countries (as classified by the Centre for Developing Countries, CDC). That allocation, as well as an ICMI contribution, enabled the

ICMI Study 27 Conference participants. (Photo credit: ICMI Study 27 Local Organising Committee)

IPC to disburse a total of €4,150 as partial conference attendance support amongst 13 participants.

As noted, in planning the Study we grappled with the environmental impact of an in-person Study Conference. Feeling strongly about the need for building community – to share knowledge and experiences, and to build collaboratively towards something bigger in the Study Volume – we chose to emphasise the importance of an in-person conference. Having co-chaired the event, we feel that the social and ecological practices and commitments of the location of the conference at Ateneo de Manila University played an important part in developing a collaborative, thinking community necessary for an ICMI Study Conference. We committed to calculating the carbon cost of air travel to the conference. Based on an estimate of carbon emissions done by the LOC, we arrived at a figure of 70,000 tones CO_2e. There were efforts to reduce the

Merrilyn Goos, ICMI president, welcomes ICMI Study 27 Conference participants. (Photo credit: ICMI Study 27 Local Organising Committee)

ICMI Study 27 Conference participants visiting Ateneo Art Gallery, Ateneo de Manila University. (Photo credit: ICMI Study 27 Local Organising Committee)

impact of all other activities: deliberate choices about paper usage; no conference bag and additions; and no single use materials for food and drink. The regular practices of members of the Ateneo de Manila University community – 'zero-food waste,' 'clean-as-you-go (CLAYGO),' and 're-use/re-cycling' – provided good modelling for our efforts. To avoid additional travel in Manila, the conference outings (named "Conference Engagements") took place on foot on the university campus, either a sustainability walk or tour of the Ateneo Art Gallery. The balance of the conference budget is going towards supporting a community-based environmental education initiative in the Philippines.

The Study Conference activities were arranged around theme working group sessions of 14 to 18 participants each, led by the IPC theme leads, all of whom had contributed (co-)authored papers. Here, participants engaged with the existing scholarship on mathematics education and the socio-ecological as represented in all the proceedings papers on a theme, with the session activities requiring engagement across perspectives towards knowledge building in the Study Volume to come.

Theme working group sessions were framed by four plenaries. The first plenary, facilitated by us as co-chairs, aimed to build the scholarly community comprising participants with diverse experiences of and perspectives on mathematics education and the socio-ecological. Actively – using drawing, writing and talking – the participants visualised and shared mappings of both their current socio-ecological contexts and their imagined future socio-ecological contexts, and importantly what their particular Study Proceedings paper contributed to realising this future. The mappings were displayed throughout the conference, and served as a productive reference point in the closing plenary, when participants reflected on the progress that had been made within themes, in just four days.

ICMI Study 27 Conference participants on a sustainability walk on the campus of Ateneo de Manila University. (Photo credit: ICMI Study 27 Local Organising Committee)

ICMI Study 27 Conference participants in discussion in a theme working group session. (Photo credit: ICMI Study 27 Local Organising Committee)

ICMI Study 27 Conference plenary 2 discussion: Jett Ramon Villarin, Nathalie Sinclair and Liz de Freitas. (Photo credit: ICMI Study 27 Local Organising Committee)

The second and third plenaries were carefully planned to prompt, challenge and enrich the discussions in the theme working group sessions. They took the form of a presentation, each by two invited guests, followed by a dialogue, chaired by an IPC member. We were delighted that three of the guests were able to be with us for the full duration of the Study Conference, participating fully in a theme working group of their choice.

Plenary 2 was addressed by Elizabeth (Liz) de Freitas (USA) and Jose (Jett) Ramon Villarin (Philippines). In the first plenary, Jett presented his work at the Manila Observatory involving climate change modelling. Amongst many insights, he raised the importance of an understanding of "rate" as a concept. The presence of chlorofluorocarbons (CFCs) in the atmosphere (which are linked to the destruction of the ozone layer), for example, is measured in parts per *trillion* (ppt), where differences of only a few hundred ppt, in CFC concentration, entail the difference between a liveable and unliveable earth. In reflecting on the philosophical dimension of work such as that done by Jett, Liz brought a vital "Science Technology and Society" (STS) perspective to the conference. STS points to the limits of thinking based on humanism and the need to think with the more-than-human. Liz's inclusive materialist ideas bring into question binaries such as nature-culture, and prompt us to reflect deeply on the historical and political entailments of the concepts we act with and the need for speculating with alternatives.

Plenary 3, by Adailton Alves da Silva (Brazil) and Pedro Walpole (Philippines), brought together two scholars' work with Indigenous communities. Pedro's work includes the development of regenerative forestry practices, which have now led to those communities being the only ones in the valley with reliable water supplies all year around. Indigenous maps, in this region, involve both space and time relations and there has been a need to integrate such maps with ones recognised by government, in making the case for land and water rights. Adailton described the work he does with the A'uwẽ/Xavante people in Brazil and the "ethnopedagogy" lens he brings to this work. This ethnopedagogy involves a web of three dimensions: cosmological (the construction of the world), socio-ecological (the construction of space and place) and socio-educational (the construction of being). A clear point of connection between the two plenary presentations was the importance of collectivity in the work of both communities – an idea that featured strongly in the work of the conference across all four of our themes. Adailton spoke in Portuguese and his slides and talk were ably translated by Marcelo Borba. The presence of multiple languages and translations added richness, as meaning was built in both English and Portuguese, and the additional time given to translation allowed individuals time for moments of contemplation.

Since the January conference, the IPC is working closely with Study Conference participants to conceptualise the chapters and author teams for the Study Volume, informed by our ongoing practising of inclusivity. As we finish writing this report, we are at the stage of having received chapter outlines for the whole book. We hope author teams will complete their first draft writing by August 2025 and, allowing time for a review process, that the finished book will be ready in 2026. The pressing nature of the issues being discussed compel us to move as quickly as we can to publication.

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ICMI Study 27 Conference plenary 3 discussion: Pedro Walpole, Alf Coles, Adailton Alves da Silva and Marcelo Borba. (Photo credit: ICMI Study 27 Local Organising Committee)

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Wikipedia and zbMATH Open: Connecting several layers of mathematical information

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Wikimedia and zbMATH Open share a vision of open knowledge accessible to all. While Wikimedia's mission is to provide a broad audience with an overview of well-established knowledge, zbMATH Open is oriented toward new knowledge for working mathematicians. In this article, we describe the intersection of these two platforms, compare their mathematical formula presentation methods, and outline future directions to deepen collaboration between them.

1 Introduction

Open platforms like Wikimedia (including Wikipedia, Wikidata, and many more) and zbMATH Open provide moderated spaces tailored to their respective communities. While differing in culture, policies, and content curation practices, each community creates their unique habitat and generates digital objects. These digital outputs might be of value to other communities, and thus linking similar content in different digital spaces has huge benefits.

Wikimedia has a broader range of audience than zbMATH Open (which is a platform exclusively dedicated to mathematical data). Yet, Wikipedia has a large and active community of contributors with a strong subcommunity of mathematics editors [3]. Aimed at a wider range of audiences, Wikimedia has its weaknesses and strengths, especially when it comes to presentation of mathematical formulae. For instance, Wikipedia misses some depth in certain mathematical fields. Wikipedia articles use references to scientific papers and, in rare cases, links to zbMATH Open or MathSciNet reviews of those papers and support this by reference macros. A further challenge arises from the inconsistency in notation and even content in different language versions of Wikipedia. The vision of a semi-formal, language-independent Wiki [2] has not yet materialised.

Despite these limitations, Wikimedia has its strengths as an open platform for mathematical data. For example, the Encyclopedia of Mathematics[1] uses the same Wiki software (albeit under a different license and editorial policies regarding who can contribute and how) and focuses on mathematics. In addition, Wikimedia's central database, Wikidata, holds a large amount of curated information on authors and publications, which are cross-referenced with zbMATH Open.

Although Wikimedia and zbMATH Open differ, both platforms support mathematical formulae as an integral part of the text content. Therefore, the presentation of mathematical formulae and the rendering methods play an important role for both Wikimedia and zbMATH Open. This is an important area where both platforms can be interlinked and use each other's presentation methods. We will delve further into this in the following sections.

In this article, we will address:

1. the connections between Wikimedia projects and zbMATH Open,
2. a comparison of mathematical formulae rendering methods,
3. plans to deepen the integration of the two platforms.

2 Linking Wikimedia and zbMATH Open

In this section, we discuss some shared entities across Wikimedia and zbMATH Open, emphasising author/public figure profiles. Other shared entities, such as conferences and journals, are essentially similar to public figure profiles in structure and function. Although Wikimedia and zbMATH Open may have different content and use different technologies for presenting those shared entities, the overlap provides opportunities for mutual benefits. We give examples and statistics showing how the two platforms benefit from each other. We briefly introduce Mathematical Research Data Initiative (MaRDI), a platform that uses Wikimedia's technology, but the content from zbMATH Open and similar platforms that are specialised in mathematical content. We conclude this section by presenting some facts and statistics on a framework that is used for connecting platforms via mathematical entity linking.

2.1 Connecting and improving common entities

A frequently shared content type in both zbMATH Open and Wikimedia projects is the public figures profiles. In zbMATH Open,

[1] https://encyclopediaofmath.org

mathematicians' profiles focus on zbMATH Open-indexed related documents – as authors, editors, subjects, or reviewers. These links allow statistics, timelines, and networks of co-authorship and co-co-authorship to be computed and visualised.

A mathematician covered in zbMATH Open might be missing in the Wikimedia ecosystem due to different criteria for public figures and the broader scope of Wikimedia, which often includes biographical information and extracurricular details. Additionally, a public figure may appear on several Wikimedia pages and its sister platforms, such as the multilingual platforms Wikidata and Wikimedia Commons, the various language versions of Wikipedia, Wikisource, Wikiquote and so on.

Despite structural differences, the Wikimedia ecosystem and zbMATH Open share similarities in handling public figure profiles. For example, both platforms link to external resources such as Open Researcher and Contributor ID (ORCID) or the Mathematics Genealogy Project (MGP).[2]

Beyond public figures, entities such as software, journals, conferences, and theorems appear on both zbMATH Open and Wikimedia. Similarly to public figure profiles, zbMATH Open has dedicated and more structured pages for publications, journals, and software, while Wikimedia provides a broader, albeit less comprehensive, coverage. Different approaches to shared entities on the two platforms can be considered complementary and provide room to link and improve each other.

Disambiguation is crucial for creating and maintaining content for public figure profiles, journals, etc. Both communities have developed workflows for resolving ambiguity, using different approaches, which are continuously being refined. Such disambiguation workflows benefit from expertise in relevant mathematical fields, linguistics (e.g., concerning naming conventions in certain cultures), bibliographic data and potentially other fields, for which diversity in the curator community is helpful. As the result of a disambiguation process often manifests itself in distinct links to external resources, the mutual links between zbMATH Open and Wikimedia platforms foster quality assessment and coordination.

While the advantages of disambiguation are clear, challenges remain, which often require manual corrections by experts. Several dedicated tools for profiling and disambiguating scholarly entities have been developed in Wikidata, such as Scholia [10], the Author Disambiguator[3] and the ORCID Scraper.[4]

The author identity workflow and the disambiguation process of the profiles in zbMATH Open have been investigated in [8, 11]. As an example of the content exchange between Wikimedia and zbMATH Open on authors, Wikidata Q IDs for zbMATH Open authors are listed automatically and manually. Currently, there are 75641 Wikidata IDs listed in zbMATH Open profiles – 15740 manually and 59901 automatically. Other than author data, various other information is harvested constantly from Wikidata, e.g., 2874 bio-events come from Wikidata. For transparency, the source of information is clarified in zbMATH Open.

Obviously, such an interconnected framework has significant advantages, since information, corrections, and data improvements can be shared among the platforms. However, such a system must be carefully curated to minimise error propagation. For instance, due to a raw ingest of data from MGP, there are many duplicated identifiers for mathematicians in Wikipedia (though some of them have been merged since then, often along with zbMATH Open curation). Many people have several registered ORCIDs, as well as some ORCIDs encompassing many people. Manual curation has also shown a surprising amount of improper ID matching between MGP and MathSciNet for ambiguous names. Even completely correct ID matches may be spoiled by future misassignments.

Therefore, beyond automated sanity checks, such a framework constantly needs feedback and corrections from the community, which is relatively easily possible through the corresponding interfaces in Wikipedia and zbMATH Open.

2.2 Mathematical Research Data Initiative (MaRDI)

An initiative to systematically profile mathematical entities of various types is currently underway at MaRDI.[5] the Mathematical Research Data Initiative [17] that forms the mathematical arm of Germany's National Research Data Infrastructure (NFDI),[6] MaRDI operates a portal[7] that provides information about various types of mathematical entities, from mathematical publications to mathematicians, formulae, algorithms, theorems and beyond.

The technical setup of the portal [14] is closely aligned with that of Wikidata. MaRDI also operates a fork of Scholia and is building Scholia-inspired profiles that reside on the portal wiki and are automatically generated based on the information the portal has about the entity to be profiled (e.g., a publication), as well as related ones (e.g., that publication's authors). Whenever possible, such profiles link to zbMATH Open, Wikidata, and other relevant resources, such as DBLP Computer Science Bibliography. This can serve as a fertile ground for coordinating curation workflows regarding the profiled entities. MaRDI can be considered as an example of interaction between zbMATH Open and Wikimedia, built on top of the Wikimedia set-up, although content-wise closer to zbMATH Open.

[2] https://mathgenealogy.org
[3] https://author-disambiguator.toolforge.org/
[4] https://orcid-scraper.toolforge.org/
[5] https://mardi4nfdi.de/
[6] https://nfdi.de/
[7] https://portal.mardi4nfdi.de/

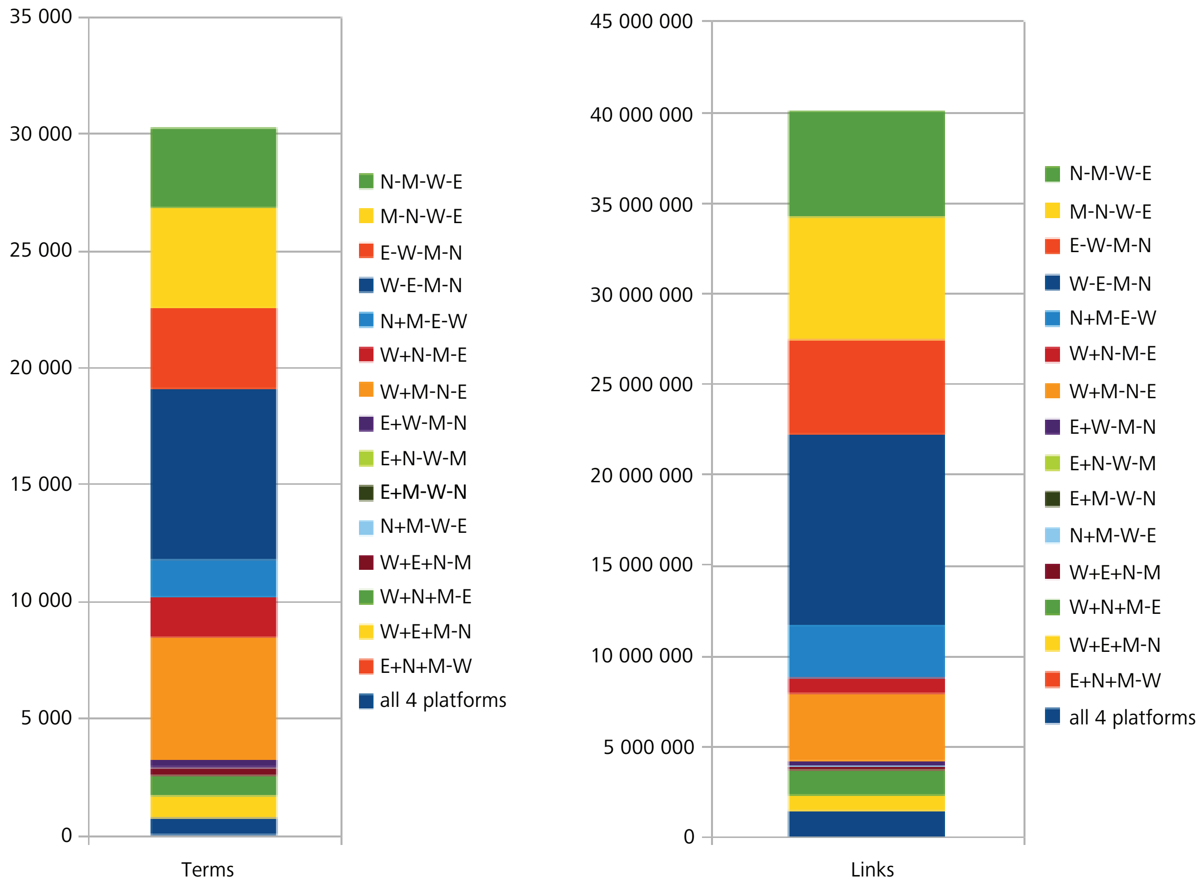

Figure 1. Distribution of terms and links from zbMATH Open to four different vocabularies: Wikipedia (W), Encyclopedia of Mathematics (E), NLab (N), and MathWorld (M) and their intersections.

2.3 Connecting platforms through mathematical entity linking

In [6] a framework is described for a phrase-based entity linking at zbMATH Open. Since then, it has been applied to generate links to four platforms: Wikipedia, Encyclopedia of Mathematics, NLab,[8] and Wolfram MathWorld.[9] While all links are made available via the zbMATH Open API [5], the zbMATH Open interface currently displays only a fraction of them for user convenience: Common elementary phrases, which are assumed to be familiar to zbMATH Open users, are not linked explicitly. However, their availability in the API enables a wide range of possible applications, such as reusability in knowledge graphs, backlinking from the platforms, recommender systems, or automated classification [15]. Certainly, more applications deriving from this will be implemented in the future and discussed in detail in forthcoming columns.

To give a brief impression about the scope and overlap of the platforms and the frequency of their phrases occurring in the mathematical literature, in Figure 1 we illustrate with a statistic the distribution of the currently identified 40 027 388 links to 30 216 unique phrases from the four services.

We can observe some effects here: Most links, as well as most phrases, come from Wikipedia alone (naturally, since it has the largest vocabulary), although with a reduced frequency. On the other hand, entries from NLab (individually or when included on other platforms) occur much more frequently in research-level mathematics.

3 Mathematical formulae

Mathematical typesetting is an essential part of the presentation of mathematical knowledge. TeX (and LaTeX) has been the main typesetting tool among mathematicians for decades due to its flexibility and precision. However, LaTeX has several limitations when used

[8] https://ncatlab.org/
[9] https://mathworld.wolfram.com/

in web-based environments such as zbMATH Open, particularly regarding consistency, long-term stability, and machine readability.

In this section, we compare the mathematical formula rendering methods in Wikimedia and zbMATH Open. We briefly introduce WikiTexVC [12, 16], the grammar-based method used in Wikimedia, and propose adapting a grammar-based presentation in zbMATH Open.

3.1 *Formula rendering: Wikimedia vs. zbMATH Open*

In Wikimedia, mathematical formulae are handled via the Math extension using WikiTexVC [12], which is a grammar-based approach to parsing LaTeX formulae into a structured internal representation and then translating them into MathML – a standard mathematical formula representation tool for the web. This approach creates a clear and consistent internal structure supporting long-term stability, semantic analysis, and interoperability (see [13] for a discussion of some examples).

Traditional mathematical presentation methods follow a standard structure in mathematical logic that consists in a set of rules applied to the given set of variables and functions to generate mathematical expressions. This approach makes such systems expressive, and hence powerful, but at the same time ambiguous on occasions. In contrast, recognition-based systems such as parsing expression grammars (PEG) [4] that are used in WikiTexVC include rules that decide whether a given formula is well-formed. Recognition-based systems avoid ambiguity.

In contrast to Wikipedia, zbMATH Open previously used MathML for formula search [1, 7, 9] and display, now supplemented with MathJax. While MathML is an established standard for formula presentation on the web, its usage typically relies on external LaTeX input that is parsed less systematically. This makes the input rely on many (custom) macros in LaTeX, which can cause inconsistency, especially over time, problems with searchability and semantic interoperability.

WikiTexVC mitigates these issues by offering a canonical structure, as it handles almost all (if not all) standard LaTeX macros. WikiTexVC also allows formulae to be rendered reliably and searched more effectively.

3.2 *Grammar-based LaTeX for zbMATH Open*

Given the challenges faced by zbMATH Open in handling LaTeX formulae, adopting a grammar-based presentation will provide several advantages.

- Searchability: enhances easier search, even across variants.
- Stability: ensures long-term interpretability for formulae, even years after publication without being affected by the macros version change.
- Interoperability: facilitates content sharing between Wikipedia and zbMATH Open.
- Longevity: allows durability, hence not being affected by obsolete or deprecated LaTeX macros.
- Balance between Machine Readability and Mathematical Expression: making formulae accessible to the machine and automated tools, easing parsing and mathematical content analysis.

WikiTexVC represents an alternative promising direction for web-based mathematical publishing. While implementation details used in parsing language grammar might not be of interest to all users, the long-term benefits of adapting WikiTeXVC to zbMATH Open for the mathematical community are clear.

Naturally, beyond phrases, a significant part of mathematical information is encoded in formulae. Automated extraction of relevant formula entities and interlinking them with the proper semantics is an even more sophisticated issue.

4 Conclusions

We discussed that Wikimedia and zbMATH Open have active communities, both of which engage in processing primary literature to enhance its reusability for their respective target audiences, the general public and professional mathematicians. While both platforms are open and their visions align, the links between them are limited. In this article, we presented ideas on sharing technology between the two platforms for rendering mathematical formulae and proposed future collaboration paths for improving connections. Strengthening these ties will benefit the mathematical community and digital communication at large.

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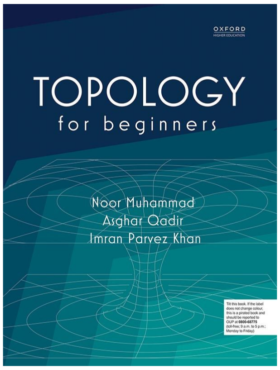
Book reviews

Topology for Beginners by Noor Muhammad, Asghar Qadir and Imran Parvez Khan

Reviewed by M. Ali Khan



There are wonderful books in topology and considerable material on hand on the internet, and so this particular reviewer may be forgiven for approaching this volume with some skepticism even if, or rather especially if, note is taken of the word "beginners" in its title. The question as to what the proverbial beginner would/could get from this book initially loomed large when texts such as those of Dugundji, Engelking, Kelley, Munkres, Schubert, Simmons, Ward, Willard are all easily available. There is also Wilder's classic piece on "Topology: its nature and significance," in *Mathematical Teaching*, **55(6)**, 462–475 (1962). The authors cite Steen–Seebach on examples – why does it not suffice as a text for a term? Why the need for another book? Can none of the texts listed above adequately serve the beginner? Do they require more commitment than a beginner can give? What is then unique and singular to what this book does offer?

The authors furnish a five-fold answer to the one question that underlies this medley already in their preface. They point to their use of examples (and counterexamples) to illustrate definitions, and to their explications of theorems by illustrating their use on well-chosen applications, excursions into applying the theory. Their third reason nods to history in the assertion that the "*raison d'être* for any definition really comes out by exploring changes in it." Yes, every theorem has its past. Surely, none of these reasons are controversial, let alone original, but the final two reasons concerning style and signature stand out for this reviewer: Wittgenstein's pictorial theory of meaning, and to his emphasis on the language of the everyday. However, there is a note of mild defensiveness when the authors note these.

> Early intuition is based in geometrical concepts which are best explained by "pictures," or diagrams, which may often be quite misleading. Students need to be gradually "weaned" away from using them rather than plunged into a bewildering world of great abstraction.

And as they move on to style and language, the authors rely on anthropomorphizing the book:

> [The] style of presentation is extremely *informal*, not to say *downright chatty*. We, as serious mathematicians, would not have it so, but that is the way the book insisted on being written, and we had to go along with it.

One imagines that they have the style of more demanding classical texts like that of Hardy–Littlewood–Pólya's on inequalities, or Fremlin's treatise on measure theory, in their minds as a counterpoint. However, speaking only for myself, I am glad that their book asserted itself. It knew its own strength.

The introductory chapter is most interesting. As the authors write, "it is as brief as the first chapter is long, and as simple and heuristic as that chapter." With the aid of 19 pictures/diagrams, the chapter is sectioned into twelve parts, and the individual titles succinctly convey what all is in the chapter. If permitted a little gloss on the titles, they may be listed as follows: (i) topology in the broad context of mathematics, (ii) the development of mathematics, (iii) the advent of topology, (iv) the mathematical background for it, (v) a psychology aside, (vi) topological constructions, (vii) the need for a language of sets, (viii) a diversion into logic, (ix) more on the language of sets, (x) cardinal numbers and counting, (xi) transfinite numbers and uncountability, (xii) more general topological transformations. This reviewer found some sections especially fascinating and provocative, and crying out for serious engagement for all interested in mathematics as language and its unreasonable effectiveness, not only in the natural sciences but also in the social sciences and the humanities. I shall resist this temptation in this short review, other than to send the readers of this newsletter to the standard references. I do so in response to the authors saying that "mathematical argument is there for all to see and judge without any ambiguity and without inducing any prejudices." If only the sociology of knowledge was as simple as that.

The material in Chapters 2 to 7 is standard. The concluding Chapter 8 is to whet the beginner's appetite by talking in the language the beginner has so far been talked to. In the concluding paragraph of the first section on further topological directions, while asserting that topology is "a *must* for theoretical physics and can be expected to rapidly extend its domain of influence," the authors write:

> Topology is required in Economics and is becoming important in Game Theory and Decision Making. Part of the reason lies in its use of the Bolzano–Weierstrass theorem, which is used for finding optimal solutions, or proving that there is no optimal solution available.

Relevant to optimization and economics is also the earlier statement:

> Since all Dynamics derives especially from the minimization of a function called the Lagrangian, the study of Dynamics and Dynamical Systems is based on the study of the connectedness of the space of permissible functions.

They might also want to confront the algebraic and the topological approaches to additive representations, as is done by Peter Wakker in the *Journal of Mathematical Psychology* **32**, 421–435 (1988).

I have only one minor quibble: it is that *compactness* and *upper semicontinuity* of the associated topology on the choice set, and the objective function respectively for the existence of an optimum, and the need for a *convex* structure for its uniqueness, may be as important as that for *connectedness*. Pushing a bit further, the applications of topology in the social sciences have by now gone considerably beyond the Bolzano-Weierstrass theorem which, strictly speaking, belongs more to analysis than to topology. This then leads to a plea to the authors, with their admirable expository style, to follow up this volume, if not with another written on topology for beginners in social sciences, at least to include another chapter in a second edition. The authors might enjoy Paul Samuelson's 1944 *Foundations of Economic Analysis* and his 1960 tribute to Harold Hotelling on the "structure of a minimum equilibrium system" in Ralph Pfouts (ed.) *Essays in Economics and Econometrics*, North Carolina Press. Samuelson was an avid reader of classical theoretical physics. They might also want to look at the reviewer's work with Metin Uyanik on a "deconstruction and integration of the continuity postulate" and on "On an extension of a theorem of Eilenberg and a characterization of topological connectedness": the first in the *Journal of Mathematical Economics* and the second in *Topology and its Applications*.

Let me conclude my strong recommendation of this book to interested beginners by drawing attention to its thirty-three references: fourteen are on physics, ten on topology, nine to articles of general interest, five of which concern mathematical issues. This reviewer, coming as he does from a background in the human sciences, found invaluable the references to Cabrera on superconductivity, to Glashow–Weinberg on unified theory, to Guth–Linde on an inflationary universe, to Misner–Thorne–Wheeler on gravitation, to Penrose on laws, and to Qadir himself on Einstein's relativity. What more could a beginner want in 160 pages of symbols, pictures and prose?

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Noor Muhammad, Asghar Qadir and Imran Parvez Khan, *Topology for Beginners.* Oxford University Press Pakistan, 2022, 210 pages;
Softcover ISBN 978-9697-34182-5.

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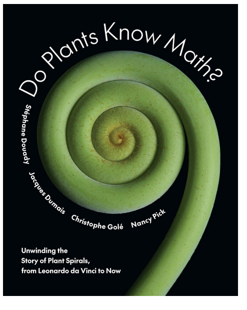
Do Plants Know Math? Unwinding the Story of Plant Spirals, from Leonardo Da Vinci to Now by Stéphane Douady, Jacques Dumais, Christophe Golé and Nancy Pick

Reviewed by Adhemar Bultheel



Since Nancy Pick, an American science writer, heard Stéphane Douady, a French physicist, discuss the Fibonacci numbers and the golden ratio during his lecture, they joined forces to start this book project. Later, they were able to convince Jacques Dumais, a Canadian biologist, and Christophe Golé, an Algerian-born mathematician, to join. All of them are fascinated by the fact that Fibonacci numbers continuously appear in phyllotaxis, and more generally, in nature. This book reports their search for an answer to the intriguing question: why is nature so 'obsessed' with these numbers?

The book starts by explaining the terminology. There are, of course, the Fibonacci numbers and the golden ratio, appearing in the spiral—or the helix—that can be connected to how plants grow leaves along their stem, but also to the spirals observed in the seeds of sunflowers, pine cones, or pineapples. These are called *parastichies* in phyllotaxis and are characterized by two integers, n and m: new leaves appear after a turn of n/m of a complete circle (the divergence angle) around the stem. These numbers turn out to be Fibonacci-like, approaching a golden angle of about 137.5° in the limit. They also describe the presence of n left-turning spirals and m right-turning ones. Later in the book, concepts such as dynamical systems, fractals, grids, and circle packing will be introduced. There are also biological terms, such as *meristem,* which is the tip where the plant grows. It contains the stem cells that differentiate and grow into primordia, which will develop into the plant's organs, like leaves, petals, etc. From the beginning, the authors take the reader along on their journey, encouraging them to observe patterns in nature, explore the many marvelous illustrations in the book, and even complete homework assignments to create crafts that verify the claims presented.

The authors tell the historical development of the subject from antiquity up to their own involvement. Each chapter starts with a 'Fibonacci poem' having seven lines, where the number of syllables in each line follows a Fibonacci sequence. Between the introduction and the conclusion, the main body of the book is arranged into five parts, each consisting of several chapters: (1) the early recognition of Fibonacci numbers and the golden ratio in mathematics and nature, (2) how science became interested in the topic and what additional information was revealed by the introduction of the (3) microscope, and later the (4) computer, and (5) how biologists investigated new pathways at a cellular level, eventually attempting to answer the why-question.

So, we start with the number sequence recognized long before Fibonacci in ancient Egypt and in Sanskrit poetry, move on to Leonardo da Vinci (who had a notebook classifying different kinds of leaves growing on plants), and of course Fibonacci (with his legendary example of population growth in rabbits), to Kepler (whose mother was a herbal healer, accused of being a witch), who tried to fit the planetary system into the mathematical rules of Platonic solids and who, in his treatise on the six-pointed snowflake, observed that the ratio of Fibonacci numbers approached the golden ratio, a concept traditionally attributed to Luca Pacioli.

Charles Bonnet (1720–1793) explained the placement of leaves as being optimal because he thought that plants grew by absorbing the dew coming from below. The word *phyllotaxis* seems to have been coined by Karl Friedrich Schimper in 1830, who observed the spirals in the placement of the leaves on a stem and in scales on a pine cone. But it was Alexander Braun who introduced this concept in his book, marvelously illustrated by drawings of his sister Cécile. In 1837, the Bravais brothers (one of whom was trained in crystallography) linked these spirals to plane grids (representing the surface of the cylindrical stem) defined by the n/m ratio. On this basis, Bonnet developed a continued fraction that converged to the irrational number $(3-\sqrt{5})/2$, and this is the portion of the circle giving the golden angle.

Wilhelm Hofmeister (1824–1877) rejected the prefixed spirals and looked at cell growth with the microscope. Swiss biologist Simon Schwendener (1829–1919) took a mechanical approach, considering cells on the stem surface to be circles. The stacking of these circles, that grow as they age, determines that a leaf starts growing where there is the most space available. The Dutch botanist Gerrit van Iterson Jr. (1878–1972) described possible solutions using a bifurcation diagram (1907), which has become a standard concept in today's phyllotaxis. One may notice these different branches as irregular transitions occurring, for example, at where the scales of the pine cone have different sizes. The Fibonacci branches are chosen by minimizing the energy.

With Alan Turing (1912–1954) we arrive in the computer age. He applied a theory of diffusion to explain biological patterns like the stripes of a zebra or the spots of a leopard. Much later, this evolved into chaos theory and (nonlinear) dynamical systems. Near the end of his life, Turing worked on morphogenesis and phyllotaxis, but, unfortunately, he did not live to see this work published. His notes, including the hypothesis of geometrical phyllotaxis, were only released 40 years after his death, when Douady published related results at approximately the same time.

Meanwhile, Aristid Lindenmayer (1925–1989) and Arthur Veen studied spirals in sunflower seeds. Lindenmayer modeled plant growth using what became known as *L-systems,* which established a formal computer language with an alphabet representing different elements and rules for their interaction. Their simulations of the diffusion of growth inhibitors produced very realistic images of sunflowers.

Douady shows with lab experiments on repulsing magnetic droplets and numerical simulations that, because the Schwendener circles grow as they move along, there are actual gaps at the bifurcation points of the Iterson diagram. This implies that always, the Fibonacci branch is taken, as earlier explained by Iterson on the basis of an energetic argument. Primordia in plants appear not only where, but also when the occasion is favorable.

That did not work for corn, which Douady considered a monster generated by extensive breeding, but here the work of Dumais and Golé comes in to explain the zigzagging growing front. Near the center of the meristem, the cells have different sizes, which explains an irregular (i.e., not hexagonal but rhombic) grid in the circle packing and hence the zigzag front. The teeth of the zigzag line are formed by the m left and n right spirals intersecting the growing front at different angles. Therefore, the rhombic pattern will, at some point in the growing process, produce a degeneration in the form of a triangle or pentagon, explaining the choice of Fibonacci branches.

What follows is a relatively short *intermezzo* on fractals and kirigami (the art of folding paper and cutting it to achieve particular effects when unfolded). Fractals describe accurately the shape of broccoli, ferns, leaves, etc., and kirigami is applied to show how leaves are packed in buds before they unfold.

But then they return to spirals from a biologist's point of view. As in physics, there are the empiricists who observe and do experiments, and there are the ones who try to explain everything through mathematics. Here, cell division is studied using a mathematical soap bubble model, something observed by the Belgian botanist Leo Errera (1858–1905), who was inspired by the work of Joseph Plateau (1801–1883) and popularized by D'Arcy Wentworth Thompson in his famous book *On Growth and Form* (1917). Cells divide in such a way that they form minimal surfaces, but there are small deviations. Depending on their shape, divisions of cells can happen at local minima rather than at the global one. Then Douady used dynamical systems again to show that cell division converges to an attractor and that generations of offspring cells will arrange again in spirals, bringing us back, in a fractal-like way, to the same story all over.

The authors add a chapter on animal analogs of the same story, like the spiral of the nautilus shell, the spiral patterns of the scales of fish or snakes, the tail of a peacock, or the multi-faceted eye of a fly. However, the Fibonacci sequence is not so frequent here, which might be explained by the mirror symmetry in animal bodies.

So, what is the conclusion? Do plants know math? The authors' answer is that these mathematical patterns are just the result of morphogenesis, and that there is no mathematical god that imposes them on nature. Each plant cell is just following the basic laws of science, and as the plant grows, the patterns spontaneously arise.

To finish with a celebration dinner, the last chapter presents several recipes to cook the plants with all their fantastic patterns, to make use of what they are really good at—i.e., to feed us.

This is a whirling journey through history and through different, seemingly unrelated, scientific topics. It is brought to the reader in a most entertaining and readable way.

Stéphane Douady, Jacques Dumais, Christophe Golé and Nancy Pick, *Do Plants Know Math? Unwinding the Story of Plant Spirals, from Leonardo Da Vinci to Now.* Princeton University Press, 2024, xiii+352 pages, Hardcover ISBN 978-0691-15865-5, eBook ISBN 978-0691-26108-9.

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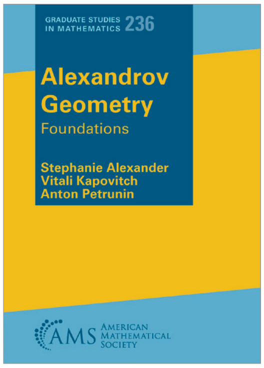

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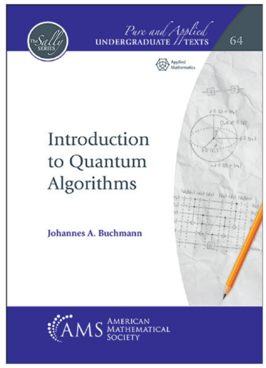

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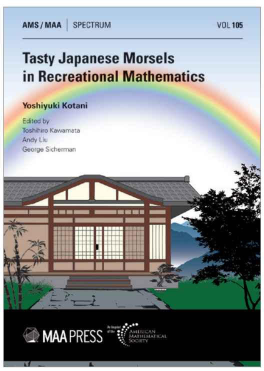

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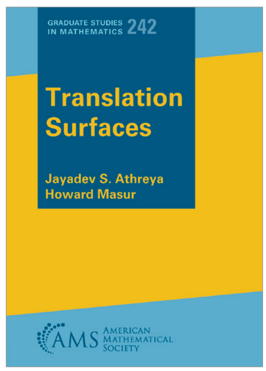

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