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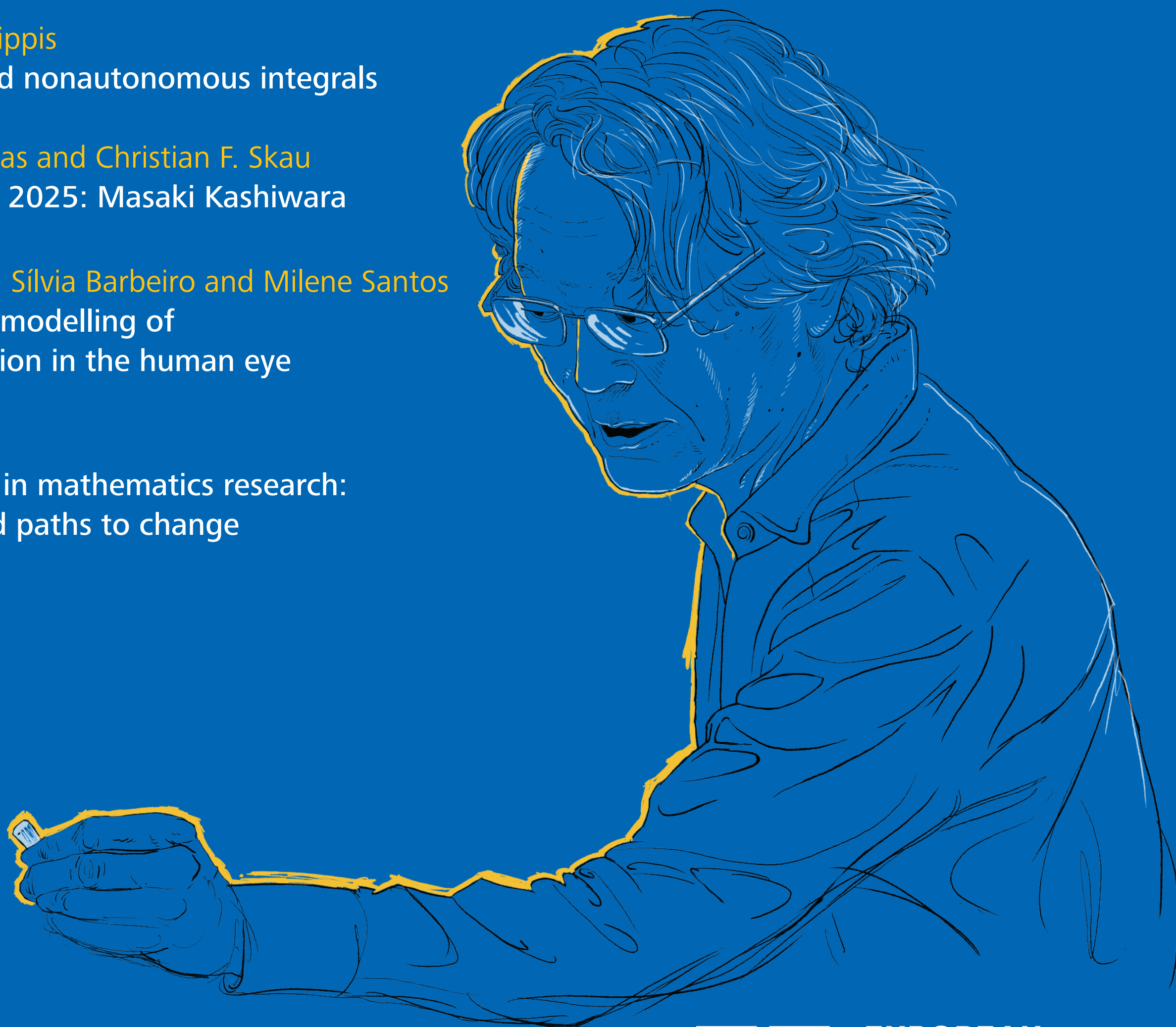
Cristiana De Filippis
μ-ellipticity and nonautonomous integrals

Bjørn Ian Dundas and Christian F. Skau
Abel interview 2025: Masaki Kashiwara

Adérito Araújo, Sílvia Barbeiro and Milene Santos
Mathematical modelling of
light propagation in the human eye

Anita Waltho
Mental health in mathematics research:
challenges and paths to change



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The cover illustration is a portrait of Masaki Kashiwara by António B. Araújo.

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We are all preparing for a new academic year and for many of us our focus right now is on our teaching. Personally, in my work I have always considered teaching as important as my research. The impact we make through our teaching cannot be underestimated, whether teaching first year calculus or advising PhD students or mentoring postdocs. Through our Education Committee the EMS stays deeply engaged in issues of teaching. Of course, much of the focus is on university teaching, but education at all levels concerns us. The EMS sees it as one of its main roles to highlight the importance of mathematics in all aspects of society. No one questions the general importance of mathematics. After all, it is probably the only subject taught in first grade (or even earlier) worldwide. But that this goes well beyond simple arithmetic may not be universally appreciated. The true importance of mathematics as a force behind many aspects of society at large including the economy, sustainability, and modern technologies, is something we must continuously emphasize. It is also something the EMS will be focusing on in the near future. In this message I would like to speak about education. If, indeed, mathematics is the foundation for much of our modern society, the value of a good mathematics education cannot be overemphasized. Unfortunately, the level of mathematical skills among secondary school students across Europe is declining, as evidenced by the latest PISA and TIMMS reports. Part of the problem seems to be a lack of qualified mathematics teachers in our primary and secondary schools. The problem is not that we do not educate excellent mathematicians at university level, but rather that being a teacher may no longer be as attractive a proposition—or at least not as appealing in terms of the salary offered and the work conditions—as many of the other jobs available to mathematicians. Indeed, mathematicians are in high demand, and on the whole this is of course a good thing for our field, but sadly the tendency has not had a positive influence on the quality of mathematics teachers employed in our schools. The EMS Education Committee has decided to take steps to address this problem. The first initiative is a lecture series by leading experts in mathematics education, to be streamed and hosted by the EMS. I am taking this opportunity to ask for your help with supporting the EMS Education Committee in these efforts and in identifying both good speakers for the lectures and the right audience and how to reach them. We are looking for policymakers, educators, professionals, and academics. If you have suggestions of any kind, please contact the Chair of the EMS Education Committee Ann Dooms (Ann.Dooms@vub.be). It is a wonderful initiative and I want to thank Ann and the whole Education Committee for taking the lead on this.

Let me now finish by wishing you all a great start to the academic year and great success with your teaching and research.

Jan Philip Solovej
President of the EMS

μ-ellipticity and nonautonomous integrals

Cristiana De Filippis

μ-ellipticity is a form of nonuniform ellipticity arising in various contexts from the calculus of variations. Understanding regularity properties of minimizers in the nonautonomous setting is a challenging task fostering the development of delicate techniques and the discovery of new irregularity phenomena.

The classical area functional, given by

$$w \mapsto \int_\Omega \sqrt{1+|Dw|^2}\,\mathrm{d}x \tag{1}$$

and its Euler–Lagrange equation, the celebrated minimal surface equation

$$-\mathrm{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = 0 \qquad \text{in } \Omega \tag{2}$$

are classical objects of study in the modern calculus of variations and in theory of elliptic partial differential equations.[1] Their peculiarities allowed to build a rich and large existence and regularity theory and have fostered generations of mathematicians to tackle difficult analytical questions. Equation (2) is intimately linked to the classical Plateau problem, which has historically driven the development of geometric measure theory. Foundational contributions by De Giorgi, Reifenberg, Federer, Fleming, Almgren, Simons, Bombieri, Miranda, and Giusti have shaped the field. In this note we are particularly interested in gradient estimates for solutions and minimizers of integral functionals featuring ellipticity properties connected to the ones of (1). As for (1), we would like to single out here a particularly elegant result of Bombieri, De Giorgi and Miranda [10], see also Trudinger [68], asserting the validity of the pointwise gradient estimate

$$|Du(x)| \lesssim_n \exp\Big(c(n) \sup_{y\in B_\varrho(x)} \frac{|u(y)-u(x)|}{\varrho}\Big) \tag{3}$$

for C^2-solutions u to (2). This a priori estimate is an essential tool in the proof of existence theorems of classical solutions, see [40, Theorem 13.6]. The functional in (1) is an example of a general integral of the calculus of variations of the type

$$w \mapsto \mathcal{F}(w,\Omega) := \int_\Omega \mathrm{F}(x,Dw)\,\mathrm{d}x, \tag{4}$$

where $\mathrm{F}\colon \Omega\times\mathbb{R}^n \to \mathbb{R}$ is a Carathéodory integrand[2] having linear growth, in the sense that $\mathrm{F}(x,Dw) \approx |Dw|$ for $|Dw|$ large, see [5]. Another such example is provided by the integral

$$w \mapsto \int_\Omega (1+|Dw|^m)^{1/m}\,\mathrm{d}x, \qquad m>1. \tag{5}$$

Next, consider the superlinear, p-growth classical model

$$w \mapsto \int_\Omega (1+|Dw|^2)^{p/2}\,\mathrm{d}x, \qquad p>1. \tag{6}$$

Also in this case we have a neat a priori gradient estimate for minimizers u, namely

$$|Du(x)| \lesssim_{n,p} \Big(\frac{1}{|B_\varrho(x)|}\int_{B_\varrho(x)} |Du|^p\,\mathrm{d}y\Big)^{1/p} + 1,$$

which can be derived as in the fundamental work of Ural'tseva [70] and Uhlenbeck [69]. Functionals with superlinear growth as in (6) are at the core of a vast part of by now classical literature. Here we shall mainly concentrate on a class of borderline integrals lying in between those with linear growth as in (1) and (5) and those with standard polynomial growth as in (6). These are functionals of the form (4) with so-called nearly linear growth, i.e., such that

$$\lim_{|z|\to\infty} \frac{\mathrm{F}(x,z)}{|z|} = \infty, \qquad \lim_{|z|\to\infty} \frac{\mathrm{F}(x,z)}{|z|^p} = 0 \quad \text{for all } p>1. \tag{7}$$

[1] Unless otherwise specified, in this note we shall always assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and $n \geq 2$. Moreover, we shall denote $B_\varrho(x) = \{y \in \mathbb{R}^n : |y-x| < \varrho\}$.

[2] That is, $x \mapsto \mathrm{F}(x,z)$ is measurable for every fixed z and $z \mapsto \mathrm{F}(x,z)$ is continuous for a.e. fixed x. This ensures that the composition $x \mapsto \mathrm{F}(x,\mathtt{D}(x))$ is measurable whenever $\mathtt{D}\colon \Omega \to \mathbb{R}^n$ is a measurable vector field.

A typical example belonging to such a class is the $L\log L$ functional

$$w \mapsto \int_\Omega |Dw|\log(1+|Dw|)\,\mathrm{d}x, \tag{8}$$

and its iterated versions

$$w \mapsto \int_\Omega \mathsf{L}_{i+1}(Dw)\,\mathrm{d}x, \tag{9}$$

where, for integer $i \geq 0$, the integrands L_{i+1} are inductively defined via

$$\begin{cases} \ell_0(|z|) = |z|, & \\ \ell_{i+1}(|z|) := \log(1+\ell_i(|z|)) & \text{for } i \geq 0, \\ \mathsf{L}_{i+1}(z) := |z|\ell_{i+1}(|z|) & \text{for } i \geq 0, \end{cases} \tag{10}$$

for all $z \in \mathbb{R}^n$. As a consequence of the superlinear growth in (7), the functionals we shall consider in the following pages will always be defined on the Sobolev space $W^{1,1}$, where in this situation direct methods of the calculus of variations apply. Indeed, we shall consider situations where

$$\mathsf{G}(|z|) \lesssim \mathsf{F}(x,z), \qquad \text{where } \lim_{|z|\to\infty} \frac{\mathsf{G}(|z|)}{|z|} = \infty, \tag{11}$$

which implies the first condition in (7); this allows to recover weak compactness in $W^{1,1}$ of minimizing sequences via the classical Dunford–Pettis theorem. This is for instance the case of (6), (8) and (9). Accordingly, a function $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ will be called a local minimizer[3] of the functional $\mathcal{F}$ if for every ball $\mathrm{B} \Subset \Omega$ we have $\mathsf{F}(\cdot, Du) \in L^1(\mathrm{B})$ and $\mathcal{F}(u,\mathrm{B}) \leq \mathcal{F}(w,\mathrm{B})$ holds for every $w \in u + W^{1,1}_0(\mathrm{B})$.

1 Anisotropic μ-ellipticity

In view of (7), a natural way to quantify the ellipticity properties of the functional in (4), and in such a way to cover all the models considered above, is to use the concept of (anisotropic) μ-ellipticity. We assume that $z \mapsto \mathsf{F}(\cdot, z)$ is C^2-regular and satisfies

$$\frac{|\xi|^2}{(|z|^2+1)^{\mu/2}} \lesssim \langle \partial_{zz}\mathsf{F}(x,z)\xi,\xi\rangle \lesssim \frac{(1+\mathsf{g}(|z|))|\xi|^2}{(|z|^2+1)^{(2-q)/2}} \tag{12}$$

for all $z,\xi \in \mathbb{R}^n$, $x \in \Omega$, where $\mu \in [2-q,\infty)$, $q \geq 1$ are fixed numbers and $\mathsf{g}\colon [0,\infty) \to (0,\infty)$ is a continuous, nondecreasing, possibly unbounded function, with at most power growth at infinity. Related equations of the type $-\mathrm{div}\, A(x,Du) = 0$ arising in connection with the Euler–Lagrange equation of the functional in (4), i.e.,

$$-\mathrm{div}\,\partial_z \mathsf{F}(x,Du) = 0, \tag{13}$$

can be also considered. In this case we can use assumptions (12) with $\partial_{zz}\mathsf{F}$ replaced by $\partial_z A$, where $A\colon \mathbb{R}^n \mapsto \mathbb{R}^n$ is a C^1-vector field. Note that the integrand appearing in (1) fits (12) with $\mu = 3$, $q \geq 1$, $\mathsf{g}(|z|) \equiv 1$, while the one in (8) verifies (12) with $\mu = 1$, $q \geq 1$ and $\mathsf{g}(|z|) \equiv \log(1+|z|)$.[4] The integrands L_{i+1} in (9)–(10) instead satisfy (12) for any $\mu > 1$, $q = 1$ and with $\mathsf{g}(|z|) \equiv \ell_{i+1}(|z|)$, cf. [18, 23, 33, 64]. Finally, (6) satisfies (12) with $\mu = 2-p$, $q = p > 1$ and $g(|z|) \equiv 1$. Functionals of the form (8)–(9) appear, for instance, in the theories of Prandtl–Eyring fluids and plastic materials with logarithmic hardening, [64], see also [8] for more examples and a detailed discussion. The Orlicz space $L\log L(\Omega)$, defined via $f \in L\log L(\Omega)$ if and only if $|f|\log(1+|f|) \in L^1(\Omega)$, directly connects to the functional in (8) and plays a crucial role in modern analysis, especially for its relations to Hardy spaces and maximal operators [67].

2 Nonuniform ellipticity and degeneracy

μ-ellipticity is a degenerate type of nonuniform ellipticity in the sense that the lowest eigenvalue of $\partial_{zz}\mathsf{F}$ might, in principle, admit no positive lower bound. This follows by considering the so-called ellipticity ratio, defined as

$$\mathcal{R}_{\mathsf{F}}(x,z) := \frac{\text{highest eigenvalue of } \partial_{zz}\mathsf{F}(x,z)}{\text{lowest eigenvalue of } \partial_{zz}\mathsf{F}(x,z)}. \tag{14}$$

The boundedness of such a quantity is the condition defining classical uniform ellipticity for equations and functionals [66], and, in that setting, it is crucial to derive a priori estimates for solutions. Here the situation is different. Condition (12) implies that the only a priori available bound on the ellipticity ratio is

$$\mathcal{R}_{\mathsf{F}}(x,z) \lesssim \mathsf{g}(|z|)|z|^{\mu+q-2}, \qquad \text{for } |z| \geq 1, \tag{15}$$

which yields no uniform control for $|z| \to \infty$ (when $\mu + q > 2$ and when $\mu + q = 2$ and $g(|z|) \to \infty$ for $|z| \to \infty$). This occurrence pushes μ-elliptic problems out of reach for regularity techniques of standard use in the uniformly elliptic setting [36–38, 69, 70]. Degeneracy represents another pathological feature of μ-ellipticity. As indicated by (12), the smallest eigenvalue of $\partial_{zz}\mathsf{F}$ – which characterizes the ellipticity of the operator – has a power-type decay at infinity with respect to the gradient variable z. This, due to severe loss of ellipticity, makes the regularity theory of μ-elliptic problems very challenging, rich and technically delicate.

[3] From now on, simply a minimizer.

[4] The integrand in (5) verifies (for $|z| \neq 0$ when $m < 2$)

$$\frac{\min\{m-1,1\}|z|^{m-2}|\xi|^2}{(1+|z|^m)^{2-1/m}} \leq \langle \partial_{zz}\mathsf{F}(z)\xi,\xi\rangle \leq \frac{\max\{m-1,1\}|z|^{m-2}|\xi|^2}{(1+|z|^m)^{1-1/m}}$$

so that (12) are satisfied with $\mu = m+1$, $q = 1$ and $\mathsf{g}(|z|) \equiv 1$ provided $|z| \geq 1$. Functionals as in (5) are studied for instance in [6, 62].

3 The autonomous case

The first general gradient regularity result for general μ-elliptic integrals available in the literature is the following theorem.

Theorem 3.1 (Fuchs and Mingione [33]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of the functional (4) with $\mathsf{F}(x,z) \equiv \mathsf{F}(z)$ satisfying $\mathsf{F}(z) \lesssim |z|^q + 1$, (11) and (12) with $g(\cdot) \equiv 1$, $q \in (1,2)$, $1 \le \mu < 2$ such that*

$$\mu + q < 2 + 2/n. \tag{16}$$

Then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

This covers the models L_i in (10). In most of the results on nonuniformly elliptic problems we shall consider, the core point is actually to prove that Du is locally bounded. Once this is secured, we are in a sense back to the uniformly elliptic setting,[5] and more standard, yet delicate methods can be adapted to obtain higher regularity of minima, and, in particular, local Hölder continuity of first derivatives of minima (see [22, Section 10], [23, Sections 5.9–5.11], [24, Section 5]). In the case of Theorem 3.1, the local Hölder continuity of Du follows, i.e.,

$$C^{0,1}\text{-estimates} \Longrightarrow C^{1,\beta}\text{-estimates} \tag{17}$$

and, in the case of Theorem 3.1, for every $\beta \in (0,1)$. In view of (15), conditions of the type in (16) obviously limit the growth of the ellipticity ratio $\mathcal{R}_{\mathsf{F}}(\cdot, Du)$ with respect to Du, while in fact proving that the gradient is locally bounded. In order to enlarge the rate of nonuniform ellipticity of the problem considered, that is, to allow a larger value of $\mu + q$, it is possible to incorporate interpolative information, such as, for instance, some a priori boundedness on solutions.[6] This has the ultimate effect of dropping the dimensional dependence on the growth of the ellipticity ratio, i.e., (16) can be replaced by

$$\mu + q < 4, \qquad u \in L^{\infty}_{\mathrm{loc}}(\Omega). \tag{18}$$

For results of this type we refer to [8, Section 5.2]. Theorem 3.1 rests on an anisotropic version of Moser-type iteration, whose convergence is ensured by (16). In the (18)-variant case, this method also involves a careful use of certain interpolation-type inequalities aimed at maximizing the integrability gain, eventually leading to the relaxed bound in (18). In both cases, the first step of the proof consists in differentiating the Euler–Lagrange equation (13), which unavoidably breaks down when considering nonautonomous integrands with nondifferentiable coefficients, like for instance, when $x \mapsto \partial_z \mathsf{F}(x,\cdot)$ is only Hölder continuous. In this case, scheme (17) is not viable using standard methods and novel ideas must be developed, as we shall see in the next sections. For further literature on the autonomous case we recommend the interesting work of Marcellini and Papi [59].

Remark (Vectorial problems). In this note we deal with the scalar case, i.e., when minima and competitors are scalar functions. Nevertheless, a large literature is available on the vectorial case, depending on the kind of regularity one is interested in. In general, and already in the uniformly elliptic case, solutions to elliptic systems and minima of vectorial functionals might exhibit singularities even in the most favourable situation of smooth, autonomous integrands. What is usually done in those cases is proving partial regularity, i.e., regularity of minima outside a negligible closed subset whose Hausdorff dimension can be eventually proven to be smaller than the ambient dimension; we refer to [34] for an account of this theory. Additional structural assumptions on the integrand allow to prove everywhere regularity in the interior. For instance, Theorem 3 extends to vector valued solutions provided L is assumed to have a so-called Uhlenbeck structure [69], i.e., $\mathsf{L}(Dw) = \ell(|Dw|)$. Partial regularity results in the vectorial case under μ-ellipticity conditions were established by Bildhauer and Fuchs [9]. Key advancements are due to Gmeineder and Kristensen [43], who developed a unified, sharp approach to the almost everywhere regularity of minima of anisotropic multiple integrals covering also nonconvex, possibly signed functionals; see also [41, 42, 63] for earlier results, and [17] where optimal partial regularity criteria are inferred via nonlinear potential theory. Finally, dimensionless bounds as $q < p + 2$ have been employed in the vectorial case in [13] by means of certain tricky penalization methods.

3.1 *Superlinear nonuniform ellipticity*

The bounds relating the size of μ and q in (16) and (18) are the natural counterpart of those appearing in the theory of nonuniformly elliptic problems with superlinear growth. Specifically, they parallel those available for so-called functionals with (p,q)-nonuniform ellipticity [57, 58], formulated as

$$\frac{|\xi|^2}{(|z|^2+1)^{(2-p)/2}} \lesssim \langle \partial_{zz}\mathsf{F}(x,z)\xi, \xi\rangle \lesssim \frac{|\xi|^2}{(|z|^2+1)^{(2-q)/2}} \tag{19}$$

for all $x \in \Omega$, $z, \xi \in \mathbb{R}^n$ and exponents $1 < p \le q$. Accordingly, (p,q)-growth conditions refer to similar conditions, but this time prescribed directly on the integrand, i.e.,

$$|z|^p - c \lesssim \mathsf{F}(x,z) \lesssim |z|^q + 1, \qquad c \ge 0. \tag{20}$$

Conditions (19) and (20) are often verified together when considering autonomous, convex integrands. In such situations uniform ellipticity in ensured only when $p = q$. Formally, conditions

[5] Indeed, by (15) the ellipticity ratio $\mathcal{R}_{\mathsf{F}}(x, Du)$ cannot blow up when $Du \in L^{\infty}$. Therefore, the problem behaves as it was uniformly elliptic when considered on Lipschitz solutions.

[6] This is for instance implied by the maximum principle, when minimizers are found solving Dirichlet problems with bounded boundary data.

(12) and (19) coincide letting $\mu = 2 - p$. Restrictions on the size of the so-called gap q/p are necessary for minima to be regular.

Theorem 3.2 (Giaquinta [35]; Marcellini [56,57]). *Let $\Omega \subset \{x \in \mathbb{R}^n : x_n > 0\}$ be an open, bounded set. With $n > 3$ and $q > 2$, the function*[7]

$$u(x) := \left(\frac{c_{n,q} x_n^q}{\sum_{i=1}^{n-1} |x_i|^2} \right)^{\frac{1}{q-2}}, \tag{21}$$

where

$$c_{n,q} := \left(\frac{n-1}{q-1} - \frac{2}{q-2} \right) \left(\frac{q-2}{q} \right)^{q-1}$$

is a minimizer of the functional

$$w \mapsto \int_\Omega \left(\frac{1}{2} \sum_{i=1}^{n-1} |D_i w|^2 + \frac{1}{q} |D_n w|^q \right) dx, \tag{22}$$

provided

$$q > \frac{2(n-1)}{n-3}. \tag{23}$$

The integrand in (22) satisfies condition (20) with $p = 2$, and the function u is obviously unbounded on the line $(0, \dots, 0, x_n)$. Similar examples can be produced with functionals satisfying (19) with $p = 2$, see [47] and [57], thus offering instances of convex, scalar, regular integrals, with nonsmooth minimizers. This stands in sharp contrast with the classical literature, where in the case $p = q$ solutions and minimizers typically have Hölder-continuous gradient [70]. On the positive side, in violation of (23) with $p = 2$, we have the following theorem.

Theorem 3.3 (Hirsch and Schäffner [46]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of the functional* (4)*, where the autonomous integrand $\mathrm{F} : \mathbb{R}^n \to \mathbb{R}$ is strictly convex and satisfies* (20) *with*

$$1 < p \le q, \qquad \frac{1}{p} - \frac{1}{q} \le \frac{1}{n-1}. \tag{24}$$

Then $u \in L^\infty_{\mathrm{loc}}(\Omega)$.[8]

[7] We denote points $x \in \mathbb{R}^n$ as $x = (x_1, \dots, x_n)$.

[8] In [46], Hirsch and Schäffner consider nonautonomous Carathéodory integrands $\mathrm{F} : \Omega \times \mathbb{R}^n \to \mathbb{R}$, and assume convexity of $z \mapsto \mathrm{F}(\cdot, z)$ and $\mathrm{F}(\cdot, 2z) \lesssim \mathrm{F}(\cdot, z)$. This is in line with the traditional De Giorgi–Nash–Moser theory, where, for the level of regularity of solutions considered here, coefficients are allowed to be just measurable. The version reported here can be obtained by combining the a priori estimates of Hirsch and Schäffner with an approximation argument as for instance the one in [24, Section 8].

Theorem 3.3 builds on earlier results of Bella and Schäffner [7]. Note that in this case no assumptions on second derivatives of the integrand F of the type in (12) are imposed, and nonuniform ellipticity is implicitly described by the growth conditions in (20). Accordingly, no gradient regularity of minima is involved. The interest in the previous result, making it standing out in comparison to the previously published literature, rests on the optimal condition on the exponents (p, q) in (24). Similar general sharp results remain unknown when switching to gradient regularity and considering assumptions (19), while positive results on gradient boundedness are in [7, 58]. However, for certain large classes of functionals it is possible to derive sharp bounds on q/p, as shown in the work of Koch, Kristensen and the author [19], who cover autonomous integrands $\mathrm{F}(x, Dw) \equiv \mathrm{F}(Dw)$ that are convex, even polynomials, with nonnegative homogeneous components and lowest homogeneity degree larger than $p \ge 2$. Indeed, the peculiar structure of convex polynomials allows for a finer nonuniform ellipticity measurement, referred to in [19] as Legendre (p, q)-nonuniform ellipticity, which quantifies the subtle interplay between the gradient of minima and the stress tensor. This is analysed via convex duality arguments and related regularity techniques. The following theorem is a model result.

Theorem 3.4 ([19]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of*

$$w \mapsto \int_\Omega \left(|Dw|^p + \sum_{i=1}^n |D_i w|^{q_i} \right) dx, \quad 2 \le p \le q_1 \le \dots \le q_n. \tag{25}$$

Assume that

$$q_n < \frac{p(n-1)}{n-3} \qquad \text{if } n \ge 4 \tag{26}$$

and no other condition if $n = 2, 3$. Then $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$.

Theorem 3.4 covers the models originally considered by Marcellini [57, 58]. If $p = 2$ in (25), the bound in (26) reduces to $q < 2(n-1)/(n-3)$, which is precisely the threshold violated by (23), so that Theorem 3.4 is sharp for polynomial-type integrals with quadratic growth from below. In two and three space dimensions no condition on p, q is needed, as implicitly suggested by (26). Concerning the superlinear counterpart of (18), Choe [15] and Esposito, Leonetti and Mingione [30] showed different gradient regularity results for a priori locally bounded minimizers provided that $q < p + 1$ and $q < p + 2$, respectively, consistently with (18) when formally letting $\mu = 2 - p$.

4 Uniformly elliptic Schauder estimates

The focus of Schauder theory for elliptic equations or variational integrals is to quantify the effect of external data, i.e., coefficients, on the regularity of solutions.

4.1 *Classical Schauder (a model case)*

By Weyl's lemma, L^1-regular distributional solutions to the Laplace equation $-\Delta u = 0$ are smooth. This easily extends to linear elliptic equations with constant coefficients. The subsequent question is how much of this regularity is preserved when plugging in non-constant coefficients, or, more precisely, how the regularity of coefficients affects that of solutions. Specifically, with $\mathrm{A}\colon \Omega \to \mathbb{R}^{n\times n}$ being a bounded and elliptic matrix, i.e., $\mathbb{I}_{n\times n} \approx \mathrm{A}$ in the sense of matrices, what can be said on the regularity of weak solutions[9] to

$$-\operatorname{div}(\mathrm{A}(x)Du) = 0 \qquad \text{in } \Omega? \tag{27}$$

Since A and Du stick together in (27), a natural guess is

$$\mathrm{A} \in C^{0,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^{n\times n}) \implies Du \in C^{0,\beta}_{\mathrm{loc}}(\Omega, \mathbb{R}^n), \tag{28}$$

which is in fact true for all $\beta \in (0,1)$. Results in the spirit of (28) were obtained by Hopf, Caccioppoli, Giraud and Schauder (1929–1934), including global versions. Later on, streamlined and different approaches were found by several other authors. All the methods available unavoidably exploit the quantitative information on the power-type decay of the modulus of continuity of coefficients (Hölder continuity), to show that energy solutions to (27) are close at all scales to harmonic-type maps, such as for instance their $\mathrm{A}(x_0)$-harmonic lifting v in B_r,

$$-\operatorname{div}(\mathrm{A}(x_0)Dv) = 0 \text{ in } B_r, \qquad v = u \text{ on } \partial B_r.$$

Indeed, ellipticity yields the homogeneous comparison estimate

$$\fint_{B_r} |Du - Dv|^2 \,\mathrm{d}x \lesssim r^{2\beta} \fint_{B_r} |Du|^2 \,\mathrm{d}x. \tag{29}$$

On the other hand, standard theory for linear elliptic equations with constant coefficients grants homogeneous decay estimates as

$$\fint_{B_\sigma} |Dv - (Dv)_{B_\sigma}|^2 \,\mathrm{d}x \lesssim \left(\frac{\sigma}{\varrho}\right)^2 \fint_{B_\varrho} |Dv - (Dv)_{B_\varrho}|^2 \,\mathrm{d}x, \tag{30}$$

for all concentric balls $B_\sigma \subset B_\varrho \subset B_r$. Estimates (29)–(30) can be matched and iterated to deliver

$$\fint_{B_r} |Du - (Du)_{B_r}|^2 \,\mathrm{d}x \lesssim r^{2\beta}$$

for all balls $B_r \Subset \Omega$, which implies the local β-Hölder continuity by certain integral characterization of Hölder continuity due to Campanato and Meyers.[10] This line of proof extends to $W^{1,p}$-regular distributional solutions to nonautonomous, quasilinear operators, such as

$$-\operatorname{div}(\gamma(x)|Du|^{p-2}Du) = 0 \qquad \text{in } \Omega, \tag{31}$$

with $1 \lesssim \gamma(\cdot) \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$, $\beta \in (0,1)$ and $1 < p < \infty$. This is due to the work of Manfredi [55], Giaquinta and Giusti [36–38], and DiBenedetto [26]. Also in this case Du is locally Hölder continuous.[11] Note that Schauder estimates obviously imply Lipschitz estimates:

$$C^{1,\beta}\text{-estimates} \Longrightarrow C^{0,1}\text{-estimates} \tag{32}$$

and this is in general the only way to get Lipschitz estimates in the presence of Hölder continuous coefficients, since the equations considered cannot be differentiated. The key point the above techniques rely on is that all the a priori estimates involved, such as (29)–(30), are homogeneous, and, as such, can be iterated. In turn, this is a feature of uniform ellipticity. When uniform ellipticity fails, a priori estimates are in general not homogeneous and these classical schemes fail as well.

4.2 *More uniformly elliptic Schauder*

The double phase functional[12]

$$\begin{gathered} w \mapsto \int_\Omega (|Dw|^p + a(x)|Dw|^q)\,\mathrm{d}x \\ 1 < p \le q, \qquad 0 \le a(\cdot) \in C^\alpha(\Omega), \qquad \alpha \in (0,\infty) \end{gathered} \tag{33}$$

was first considered by Zhikov in the context of homogenization of strongly anisotropic materials and of the study of the Lavrentiev phenomenon [49]. It only satisfies nonstandard growth conditions of (p,q) type as in (20) but it is still uniformly elliptic in the sense that, with $\mathrm{F}(x,z) := |z|^p + a(x)|z|^q$, the ellipticity ratio $\mathcal{R}_{\mathrm{F}}(x,z)$ remains uniformly bounded. Indeed, Schauder-type results hold as in the following theorem.

Theorem 4.1 (Baroni, Colombo and Mingione [4, 16]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of (33) with $\alpha \in (0,1]$. If*

- *either $q/p \le 1 + \alpha/n$,*
- *or $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and $q \le p + \alpha$,*

then Du is locally Hölder continuous.

[9] Although this is not strictly necessary in the linear case when coefficients are Hölder continuous, here we assume to deal with energy solutions, that is, distributional solutions that belong to the reference energy space $W^{1,2}(\Omega)$. These are usually called weak solutions.

[10] The one described here is in fact Campanato's classical approach to Schauder estimates [12].

[11] Gradient Hölder continuity of energy solutions holds but, in general, not with the sharp exponent $Du \in C^{0,\beta}$, due to the fact that the equation is degenerate.

[12] We denote $C^\alpha \equiv C^{[\alpha],\alpha-[\alpha]}$ when α is not an integer, and $[\alpha]$ denotes its integer part.

The key of the proof is that the uniform ellipticity of the double phase functional (33) allows to implement a few more refined, nonstandard perturbation arguments. Specifically, recalling the discussion in Section 4.1, minimizers to frozen functionals of the type

$$w \mapsto \int_{B_r(x_0)} (|Dw|^p + a(x_0)|Dw|^q)\,dx \tag{34}$$

have locally Hölder continuous gradient and enjoy good reference estimates. This is in fact a consequence of the fact that functionals as in (34) are uniformly elliptic for every choice of x_0. On the other hand, the aforementioned nonstandard growth conditions, impacting solely on the comparison estimates, can be compensated via certain delicate schemes of reverse Hölder inequalities and higher integrability lemmas. Eventually, the approach of [4, 16] was extended in [1, 44, 45] to treat large classes of uniformly elliptic integrals with nonstandard growth conditions, such as the double phase one; see also [20], where the bound $q < p + \alpha$ is proved to be effective also in the vectorial case.

4.3 Soft nonuniform ellipticity and hard irregularity

One might argue that Theorem 4.1 is incomplete since, the double phase functional being uniformly elliptic, Schauder-type results should hold with no restrictions on p, q, α. Surprisingly enough, as first discovered in [31], the conditions imposed on such quantities in Theorem 4.1 are necessary and the result is sharp. In fact, building on certain Zhikov's two-dimensional examples [49] in the setting of the Lavrentiev phenomenon, in [31, 32] a novel, sharp phenomenology was disclosed, demonstrating the failure of Schauder estimates in general, notwithstanding the uniform ellipticity of the problem considered.

Theorem 4.2 (Fonseca, Malý and Mingione [32]). *For every choice of the parameters*

$$\begin{cases} 1 < p < n < n + \alpha < q < \infty, & \alpha \in (0, \infty) \\ n \geq 2, \quad \varepsilon > 0 \end{cases} \tag{35}$$

there exist a double phase integral (33), a related minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, and a closed set $\Sigma \subset \Omega$ with $\dim_{\mathcal{H}}(\Sigma) > n - p - \varepsilon$, such that all the points of Σ are non-Lebesgue points of the precise representative of u.

In fact, in the same range (35), non-$W^{1,q}$-regular, yet bounded minima of (33) with one-point singularity were constructed by Esposito, Leonetti and Mingione in [31]. New constructions of singular minimizers eventually came, combining and improving the above features.

Theorem 4.3 (Balci, Diening and Surnachev [2,3]). *For every choice of the parameters*

$$q > p + \alpha \max\Big\{1, \frac{p-1}{n-1}\Big\}, \qquad \alpha \in (0, \infty), \qquad p > 1 \tag{36}$$

there exist a double phase integral (33) and a related minimizer $u \in W^{1,p}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$, such that $u \notin W^{1,d}$ for any $d > p$. If $p < n$, there exists a closed set $\Sigma \subset \Omega$ of non-Lebesgue points of u with $\dim_{\mathcal{H}}(\Sigma) = n - p$.

The occurrence of irregular minima is not only explained in terms of (p, q)-growth conditions. More is actually there, i.e., a softer form of nonuniform ellipticity hidden in (33), that cannot be detected by using the classical ellipticity ratio $\mathcal{R}_{\mathrm{F}}(x, z)$ in (14), but rather considering a larger, nonlocal quantity accounting for the contribution of coefficients to the ellipticity of the functional over sets of positive measure. Specifically, with $\mathrm{B} \subset \Omega$ being a ball, we consider the nonlocal ellipticity ratio [21] defined as

$$\bar{\mathcal{R}}_{\mathrm{F}}(z, \mathrm{B}) := \frac{\sup_{x \in \mathrm{B}} \text{highest eigenvalue of } \partial_{zz}\mathrm{F}(x, z)}{\inf_{x \in \mathrm{B}} \text{lowest eigenvalue of } \partial_{zz}\mathrm{F}(x, z)} \tag{37}$$

for $|z| \neq 0$. Observe that $\mathcal{R}_{\mathrm{F}}(x, z) \leq \bar{\mathcal{R}}_{\mathrm{F}}(z, B)$ for $x \in \mathrm{B}$ and that the best upper bound obtainable on $\bar{\mathcal{R}}_{\mathrm{F}}(z, B)$ is this time

$$\bar{\mathcal{R}}_{\mathrm{F}}(z, \mathrm{B}) \lesssim_{p,q} 1 + \|a\|_{L^\infty(B)} |z|^{q-p}.$$

Moreover, if $a(\cdot)$ vanishes at some point in $\bar{\mathrm{B}}$, then

$$\|a\|_{L^\infty(B)} |z|^{q-p} \lesssim \bar{\mathcal{R}}_{\mathrm{F}}(z, \mathrm{B}),$$

so that $\bar{\mathcal{R}}_{\mathrm{F}}(z) \to \infty$ as $|z| \to \infty$ if $a(\cdot)$ does not vanish identically in B. This could be considered as a weaker form of nonuniform ellipticity, eventually generating singular minimizers, although in the presence of regular coefficients and classical uniform ellipticity. Indeed, note that $\bar{\mathcal{R}}_{\mathrm{F}}(z, \mathrm{B})$ remains bounded when $a(\cdot)$ stays quantitatively away from zero on $\bar{\mathrm{B}}$, and in this case the same proof of Theorem 4.1 implies that Du is locally Hölder continuous in B, this time with no restriction on p, q, α.

4.4 Fractal cones and malicious competitors

The key to Theorems 4.2–4.3 (we concentrate here on the second one, case $p < n$) is in the blending of three main ingredients.

- A merely $W^{1,p}$-regular map u_* – the malicious competitor – attaining opposite values m and $-m$ on the top and the bottom of $\Omega = [-1, 1]^n$ and whose singularities can be distributed along a Cantor-type fractal $\mathcal{C}$ whose Hausdorff dimension $\dim_{\mathcal{H}}$ equals $n - p$. Here $m \geq 1$ is a large constant.
- A Lipschitz-regular boundary datum u_0, with $u_0 \equiv u_*$ on $\partial\Omega$.
- A nonnegative, α-Hölder continuous coefficient $a(\cdot)$ vanishing where $|Du_*|$ is positive, see Figure 1.

The last bullet point means that $a(x)|Du_*|^q = 0$ in Ω, and therefore u_* is a finite energy competitor in the Dirichlet problem driven by integral (33), with p, q, α as in (36), and boundary datum u_0. Basic direct methods of the calculus of variations yield the existence of a unique solution

$$u \mapsto \min_{w \in u_0 + W_0^{1,p}(\Omega)} \int_\Omega (|Dw|^p + a(x)|Dw|^q)\, \mathrm{d}x,$$

whose energy is set low, being controlled via minimality by the p-energy of u_*. Recalling that $u_0 - u_* \in W_0^{1,p}(\Omega)$ and that u_0 reaches opposite values on the top and the bottom of Ω, a sufficiently large choice of m ensures that the minimum u "does not have enough energy" to cover the gap between the lower trace $-m$ and the upper one m without developing discontinuities. In other, more accurate terms, a delicate combination of energy and trace estimates allows proving that Σ_u, the set of essential discontinuity points of u, contains a piece of fractal $\mathcal{C}$, thus forcing $\dim_{\mathcal{H}}(\Sigma_u) = n - p$. This implies[13] that $u \notin W^{1,d}_{\mathrm{loc}}(\Omega)$ for all $d > p$, showing that higher Sobolev regularity is in general unattainable under condition (36). This construction is paradigmatic of the idea that, once identified the right (bad) competitor, and a related geometry of the coefficient, minimality can be used to produce singularities rather than proving regularity properties. The strength of these examples lies in the following aspects:

- Minimizers, which are simply as bad as any other competitor.
- Scalar setting. This is a genuinely nonstandard growth conditions phenomenon, in contrast with standard cases, where to produce singularities one needs to look at vectorial problems [25] or to violate the initial energetic information [65].
- No degeneracy issues [70]. The integrand can be further made nondegenerate by replacing $|Dw|$ with $(|Dw|^2 + 1)^{1/2}$.
- Lipschitz domains and boundary data.

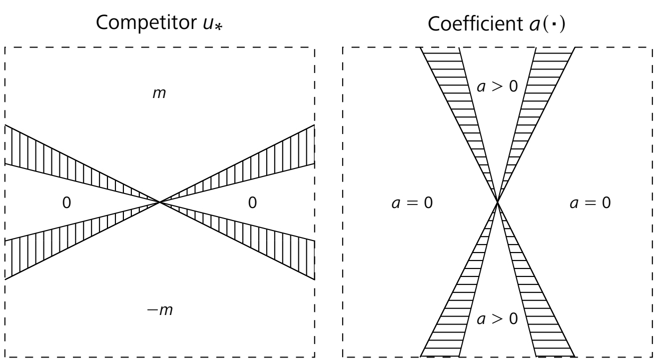

Figure 1. Competitor u_* vs coefficient $a(\cdot)$. Figure 1 is a modification of the one in [2].

[13] The Hausdorff dimension of the set of non-Lebesgue points of a $W^{1,d}$-regular function does not exceed $n - d$, $d \le n$.

5 Schauder estimates and μ-ellipticity

The perturbation-based circle of ideas and techniques discussed in Section 4.2 breaks down when genuine nonuniform ellipticity is involved: both reference and comparison estimates become nonhomogeneous, and the perturbative approaches, based on iterations, become unviable. The validity of Schauder estimates in the nonuniformly elliptic setting was a longstanding open problem raised at various stages in the literature: see, e.g., [48, Page 7] on classical results of Ladyzhenskaya and Ural'tseva [53], Giaquinta and Giusti's paper [38], and its MathSciNet review[14] by Lieberman. A complete solution eventually appeared in [22, 24] in the (p, q)-setting, and in [18, 23] in the nearly linear, μ-elliptic one. The novel techniques introduced in [18, 22–24] reverse the classical paradigm in (32) to obtain gradient estimates when dependency on coefficients is Hölder continuous. Indeed, for the first time gradient L^∞-bounds are not derived as a consequence of $C^{1,\beta}$-bounds (in turn obtained via perturbation), but are rather derived directly, and eventually used to prove $C^{1,\beta}$-estimates. In other words, we return to (17), although the functionals and the equations considered here are nonautonomous and nondifferentiable. We shall try to give an overview of some of the ideas leading to establishment of Schauder estimates for certain classes of functionals with nearly linear growth. As explained immediately after Theorem 3.1 and displayed in (17), we can concentrate on Lipschitz estimates.

5.1 *Nearly linear Schauder and intrinsic Bernstein functions*

The main models to initially keep in mind are the logarithmic energies (8)–(9), but now also featuring Hölder continuous coefficients.

Theorem 5.1 ([23]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of functional*

$$w \mapsto \int_\Omega (\gamma(x)|Dw|\log(1 + |Dw|))\mathrm{d}x, \tag{38}$$

where $1 \lesssim \gamma(\cdot) \in C^{0,\beta}(\Omega)$, $\beta \in (0,1)$. Then $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$[15].

Analogous results hold if in (38) the $L \log L$ integrand is replaced by the iterated logarithmic one in (10). Theorem 5.1 is actually a special case of a more general result covering nonautonomous

[14] Math. Rev. MR0749677.

[15] In fact, in [23] we proved that $u \in C^{1,\beta/2}_{\mathrm{loc}}(\Omega)$, but the improvement to the full exponent $u \in C^{1,\beta}_{\mathrm{loc}}(\Omega)$ can be easily reached, see [18], by arguing as in [24, Section 5]. The main point in Theorem 5.1 is as usual to get that $Du \in L^\infty_{\mathrm{loc}}$, although the adaption of the standard perturbation methods to get gradient Hölder continuity once Du is known to be locally bounded still requires care.

μ-elliptic functionals, like for instance those exhibited by nearly linear double phase integrals of the type

$$w \mapsto \int_\Omega (|Dw|\log(1+|Dw|) + a(x)(|Dw|^2+s^2)^{q/2})\,dx \qquad 1<q, \quad 0 \le a(\cdot) \in C^{0,\alpha}(\Omega), \quad 0 \le s \le 1. \tag{39}$$

This can be considered as the borderline configuration of (33) as $p \to 1$, while actually approaching nearly linear growth conditions. A key point here is that, in contrast to (33), the functional in (39) is not uniformly elliptic. This is easily seen at those points x where $a(x)=0$, where the integrand reduces to $|Dw|\log(1+|Dw|)$, which is nonuniformly elliptic. Nevertheless, also in this case it is possible to achieve maximal regularity for minima, and under optimal structural conditions regulating nonuniform ellipticity. Furthermore, this extends to larger classes of functionals, to which (39) belongs to, of the type

$$w \mapsto \int_\Omega (\gamma(x)\mathfrak{L}(Dw) + a(x)(|Dw|^2+s^2)^{q/2})\,dx,$$

where $\gamma(\cdot)$ is as in Theorem 5.1 and

$$\frac{|\xi|^2}{(|z|^2+1)^{\mu/2}} \lesssim \langle \partial_{zz}\mathfrak{L}(z)\xi,\xi\rangle \lesssim \frac{|\xi|^2}{(|z|^2+1)^{1/2}}$$

is assumed to hold for $\mu \approx 1$ (i.e., for $1 \le \mu < \mu_m \equiv \mu_m(n,q,\alpha) < 2$) and every choice of $z,\xi \in \mathbb{R}^n$, see [23] for details. For instance, all models featuring iterated logarithms as

$$w \mapsto \int_\Omega (\mathsf{L}_{i+1}(Dw) + a(x)(|Dw|^2+s^2)^{q/2})\,dx \tag{40}$$

are included, where $s \in [0,1]$, and $a(\cdot)$ and q as in (39). In this respect, the following holds.

Theorem 5.2 ([18, 23]). *Let $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ be a minimizer of functionals in (39) or in (40) with $\alpha \in (0,1)$. If*

- *either $q < 1+\alpha/n$,*
- *or $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and $q < 1+\alpha$,*

then Du is locally Hölder continuous in Ω. Moreover, in the nonsingular case $s>0$, we have $u \in C^{1,\alpha}_{\mathrm{loc}}(\Omega)$ for (39), and $u \in C^{1,\alpha/2}_{\mathrm{loc}}(\Omega)$ for (40).

The bound $q < 1+\alpha$ turns out to be sharp, as the following holds true.

Theorem 5.3 ([18]). *For every choice of the parameters $\alpha,\varepsilon > 0$, $q>1$, such that $q > 1+\alpha$, $0 < \varepsilon < \min\{q-1-\alpha, n-1\}$, there exist a double phase integral (39) and a related minimizer $u \in W^{1,1}_{\mathrm{loc}}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$ such that $u \notin W^{1,p}_{\mathrm{loc}}(\Omega)$ for all $p > 1+\varepsilon$. In particular, the Hausdorff dimension of the set of non-Lebesgue points of the precise representative of u is at least equal to $n-1-\varepsilon$.*

Although the outcome is formally the same, the approach to Theorem 5.2 is completely different from the one of Theorem 4.1. Indeed, while the functional in (33) is *uniformly elliptic*, the functionals in (39) and (40) are not. Therefore, perturbation approaches of the type considered in [1, 4, 16, 44, 45] fail to deliver results and more complicated, completely different routes are necessary. In this respect, note that the functionals in (39)–(40) globally satisfy assumptions (12) for any $\mu > 1$ (actually, we can take $\mu = 1$ in case (39)), but the simple appeal to such properties is not sufficient to prove Theorem 5.2 and more is needed. There are in fact three main points in the proof of Theorem 5.2 and in the following we shall restrict for simplicity to the case where the bound $q < 1+\alpha$ is considered. A first key idea is to fully exploit the specific structure of the functional to rebalance the significant loss of ellipticity due to degenerate nonuniform ellipticity. This is achieved via a novel, intrinsic version of the Bernstein technique combining fractional estimates and nonlinear potential theoretic methods. In the uniformly elliptic case of functionals of the type (6) one observes that a function of the type

$$v(x) = (|Du(x)|^2+1)^{p/2} \tag{41}$$

is a subsolution to a linear, uniformly elliptic equation, and, as such, it is bounded. This follows from the possibility of differentiating the related Euler–Lagrange equation and the fact that the functional is uniformly elliptic. Both things fail for (40). Indeed, recall that Euler–Lagrange equation to (40) is

$$-\mathrm{div}(\partial_z \mathsf{L}_{i+1}(Du)) - q\,\mathrm{div}(a(x)(s^2+|Du|^2)^{(q-2)/2}Du) = 0$$

and therefore is it not differentiable by the Hölder continuity of the coefficient a. The idea is then to replace the function v in (41) by another, more intrinsic Bernstein function, incorporating larger information on the structure of the integrand and its ellipticity, namely

$$\begin{aligned}\mathsf{E}(x) := {}& \frac{1}{2-\mu}\big[(|Du(x)|^2+1)^{1-\mu/2}-1\big]\\ &+ (1-1/q)\mathsf{a}(x)\big[(|Du(x)|^2+s^2)^{q/2}-s^q\big],\end{aligned}$$

where in fact $\mu \in [1,2)$ is the one for which (12) is satisfied. In turn, this function is shown to satisfy a renormalized, fractional Caccioppoli-type inequality,[16] i.e.,

$$\begin{aligned} r^{2\theta}[(\mathsf{E}-\kappa)_+]^2_{\theta,2;B_{r/2}} \lesssim {}& \mathsf{M}^{2\mathsf{b}_1}\int_{B_r}(\mathsf{E}-\kappa)^2_+\,dx\\ &+ \mathsf{M}^{2\mathsf{b}_2}r^{2\alpha}\int_{B_r}(|Du|^m+1)\,dx \end{aligned} \tag{42}$$

holds for any $\kappa \ge 0$, all balls $B_r \subset \Omega$ with radius r, suitable numbers $\theta \in (0,\alpha)$, $\mathsf{b}_1, \mathsf{b}_2, m \ge 1$ and M such that $\mathsf{M} \ge \|Du\|_{L^\infty(B_r)}$. On the

[16] Of course (42) makes sense as an a priori estimate and must be fixed via an approximation argument where original minimizers are the limit of minima of certain more regular, uniformly elliptic functionals.

left-hand side in (42) there appears the classical fractional Gagliardo norm, which is defined as

$$[v]^2_{\theta,2;A} := \int_A \int_A \frac{|v(x)-v(y)|^2}{|x-y|^{n+2\theta}}\,\mathrm{d}x\,\mathrm{d}y$$

whenever $A \subset \mathbb{R}^n$ is an open set and $v\colon A \to \mathbb{R}$ is a measurable function. The term renormalized accounts for the fact that inequality (42) is homogeneous with respect to E, despite the fact that the integrals (39)–(40) are not. This is exactly the feature allowing to apply the nonlinear potential machinery mentioned above. The price to pay is the appearance of multiplicative constants depending on $\|Du\|_{L^\infty(B_r)}$ (via $\mathtt{M}$). Such constants must be carefully kept under control all over the proof and reabsorbed at the very end. This will be a point where the bound $q < 1+\alpha$ assumed in Theorem 5.2 plays a crucial role. The validity of (42) is established via a nonlinear dyadic/atomic decomposition technique, finding its roots in [51], that resembles the one used for Besov spaces in the setting of Littlewood–Paley theory. Fractional Caccioppoli inequalities of the type in (42), first pioneered in [61] in the setting of nonlinear potential theory, eventually allow to prove boundedness of E via a nonlinear potential theoretic version of De Giorgi's iteration, which made its first appearance in [50]. In this respect, here a more delicate and quantitative form of such ideas is needed [22]. The boundedness of E obviously implies the one of Du. Back to the proof of the boundedness of E, we point out that the nonlinear potentials used in the estimates are of the type introduced by Khavin and Maz'ya [60], and deeply studied by Adams, Hedberg, Meyers and Wolff. Specifically, these are of the form

$$\mathbf{P}^{\vartheta}_{\sigma}(f; x, r) := \int_0^r \varrho^{\sigma}\Big(\frac{1}{|B_\varrho(x)|}\int_{B_\varrho(x)} |f|\,\mathrm{d}y\Big)^{\vartheta}\frac{\mathrm{d}\varrho}{\varrho}$$

for parameters $\sigma, \vartheta > 0$ and f being an $L^1(B_r(x))$-regular vector field. Suitable choices of σ and ϑ give back the standard Riesz potential $\mathbf{I}_1$ and the Wolff potential $\mathbf{W}_{1,p}$ [52]. The mapping properties of potentials among function spaces are known. Specifically,

$$\|\mathbf{P}^{\vartheta}_{\sigma}(f;\cdot,r)\|_{L^\infty(B_\varrho)} \lesssim \|f\|^{\vartheta}_{L^m(B_{r+\varrho})} \tag{43}$$

holds whenever $n\vartheta > \sigma$, $m > n\vartheta/\sigma$ and $B_{r+\varrho} \subset \Omega$, [22, 24]. In contrast with previous foundational contributions [50, 52], where potentials are employed as ghosts of the representation formula to derive optimal regularity of solutions from data, in [18, 22–24] potentials fit the fractional nature of (42) and sharply quantify how the rate of Hölder continuity of coefficients interacts with the growth of the terms they stick to towards L^∞-estimates. Another main idea in this setting is a fractional Moser's iteration in Besov spaces already employed for problems with polynomial growth in [24],[17] and which allows reading the Hölder continuity of the coefficient $a(\cdot)$ as fractional differentiability. This allows to gain arbitrarily high gradient integrability and therefore, in a sense, to quantitatively reduce the rate of nonuniform ellipticity of (39)–(40). However, further obstructions arise due to the severe loss of ellipticity in integrals at nearly linear growth. These require a limiting version of the aforementioned fractional Moser's iteration in [24] yielding hybrid reverse Hölder inequalities of the form

$$\|Du\|_{L^t(B_{r/2})} \lesssim \mathtt{M}^{\omega}\Big(1+\frac{\|u\|_{L^\infty(B_r)}}{r}\Big)^{\mathrm{b}}\big(1+\|Du\|^{1/t}_{L^1(B_r)}\big), \tag{44}$$

which is valid for all $1 \le t < \infty$, $\omega \in (0,1)$, some $\mathrm{b} > 0$, any ball $B_r \subset \Omega$, and, as in (42), again $\mathtt{M} \ge \|Du\|_{L^\infty(B_r)}$. Estimate (44) is a sort of borderline interpolation inequality, and, when combined with (42), allows working under the maximal ellipticity range $q < 1+\alpha$. Once again, the price to pay is the appearance in the bounding constants of $\mathtt{M}^{\omega}$, with ω that can be picked to be arbitrarily small, to compensate the loss of ellipticity and trade between an arbitrarily high power of the modulus of the gradient and its L^1-norm.

Remark (Obstacles). The techniques devised for Theorem 5.2 are flexible enough to deal with variational obstacle problems. Specifically, they allow to bypass the classical linearization procedure pioneered by Duzaar and Fuchs [28, 29] and to prove gradient regularity in nondifferentiable, nonuniformly elliptic variational inequalities. Duzaar and Fuchs's approach turns constrained minimizers of homogeneous integrals into unconstrained minima of forced functionals whose right-hand side is a function of the second derivatives of the obstacle and of the gradient of coefficients. This again requires that $x \mapsto \partial_z \mathrm{F}(x,\cdot)$ is differentiable, which is not the case in the present setting. Alternative techniques, as those used by, for instance, Choe [14], only work in the uniformly elliptic case. On the other hand, the scheme supporting Theorem 5.2, based on fractional differentiation and use of nonlinear potentials, can be tailored to account for obstacles in order to deliver sharp results also in the constrained case. For this we refer to [18].

5.2 *Back to the beginning*

We finally highlight a few formal connections between results of the type in Theorems 5.2–5.3 and some classical counterexamples to regularity for linear-growth functionals constructed by Giaquinta, Modica and Souček [39]. Consider the generalized[18] Dirichlet problem involving the area functional in one dimension,

$$\begin{cases} w \mapsto \min \displaystyle\int_{-1}^{1} \sqrt{1+\gamma(x)|w'|^2}\,\mathrm{d}x, \\ w(-1) = -w_0, \qquad w(1) = w_0, \end{cases} \tag{45}$$

[17] See [11, 27] for similar Besov spaces techniques in the context of degenerate integro-differential equations.

[18] By "generalized" we mean that problem (45) must be extended to BV and the functional appearing in (45) is actually replaced by a suitable relaxed form in which boundary data are penalized, [5, 39].

where $\gamma(x) := 1 + x^2(\log(2/|x|))^4$ and $w_0 > 0$ satisfies

$$\infty > w_0 > \int_{-1}^{1} \frac{1}{\sqrt{\gamma(x) - 1}}\, dx.$$

The minimizer has a jump at zero, making $W^{1,1}$-regularity fail. The function γ is not C^2-regular. In contrast, C^2-coefficients guarantee the possibility of a priori estimates [54] in the style of (3). This situation resembles the one of Theorems 4.2, 4.3 and 5.3. In this respect, the construction of Theorem 5.3 extends to linear-growth double phase integrals such as

$$w \mapsto \int_\Omega ((1 + |Dw|^m)^{1/m} + a(x)|Dw|^q)\, dx \tag{46}$$

with $1 < m, q$ and $0 \le a(\cdot) \in C^\alpha(\Omega)$.[19] This means that there is no hope of pointwise gradient regularity for minima of (46) whenever $q > 1 + \alpha$. An important point is that this information comes only from the growth conditions of the integrand, and not from its ellipticity, i.e., the growth of the eigenvalues of the second derivatives. On the other hand, the bounds relevant in order to prove a priori estimates come from conditions on second derivatives like (12) or (19). While these scale accordingly to the growth conditions of the integrand in superlinear growth regimes – like in the (p,q) case, cf. (19) and (20) – there might be a detachment when approaching the linear case: the integrand keeps growing linearly, but its derivatives can decay very fast (consider (5) with large m). In view of Theorem 5.2, and proceeding formally, the condition for a priori regularity gradient estimates looks as $q < 2 - \mu + \alpha$. In the case of anisotropic area-type functionals as in (46), it is $\mu = m + 1$. Coupling this with $q > 1$ leads to $\alpha > q + m - 1$, which, for m close to two, matches the need of C^2-regular coefficients in classical papers such as [54] and also aligns with counterexample (45). On the other hand, further imposing the restriction $\alpha \in (0,1)$ and then recalling that one must have $q < 1 + \alpha$, leads to $\mu \approx 1$, exactly as considered in [18, 23], cf. Theorem 5.2. This suggests that the functionals (39)–(40) might be the limiting configurations for the validity of Schauder theory in the presence of Hölder coefficients, convex anisotropy and μ-ellipticity.

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[19] Again, as above, one has to interpret this in a suitably relaxed way, considering competitors in BV and a relaxed form of the functional.

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Abel interview 2025: Masaki Kashiwara

Bjørn Ian Dundas and Christian F. Skau

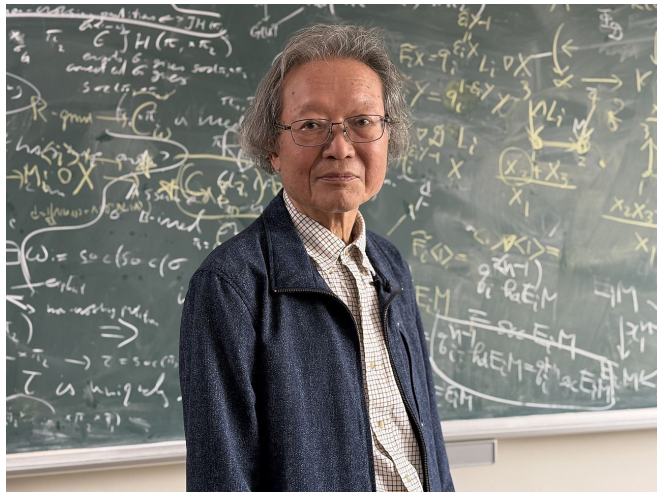
Bjørn Ian Dundas / Christian F. Skau: *Professor Kashiwara, we want to congratulate you for being awarded the Abel Prize for 2025.*

Masaki Kashiwara: Thank you very much.

[BID/CFS]: *The citation from the Abel committee reads as follows:*

> *"…for his fundamental contributions to algebraic analysis and representation theory; in particular the development of the theory of $\mathcal{D}$-modules and the discovery of crystal graphs."*

You will receive the prize tomorrow from His Majesty, the King of Norway.



Masaki Kashiwara, Abel Prize laureate 2025.
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Early life

You are the first Japanese national – and the first person based outside North America, Europe or Israel – to win the Abel Prize. Our first question to you is: what kindled or sparked your interest in mathematics in the first place, and when did you discover that you had a special talent for mathematics?

MK: That is not so easy to answer. I always liked mathematics, also when I was a kid, and perhaps I was much better at it than other kids. That's true, but I was not a child prodigy, as many talented mathematicians are. I'm not like that. I started with mathematics because I think that mathematics is somewhat very logical, but also somewhat very appealing to human beings. So that's how I started to like mathematics.

[BID/CFS]: *We talked earlier about the "tsurukamezan," or "cranes and turtles problem." Describe the problem, and tell us when and how that problem caught your interest.*

MK: As you know, the problem is about turtles and cranes. Cranes have two legs, and turtles have four legs, and we know the total number of legs and the total number of cranes and turtles. What is the number of cranes, and hence the number of turtles? I learned how to solve it, but it's very tricky, in fact. And when I learned about it first time I didn't like it at all, because it's too tricky.

Later I learned that you can solve it very easily using x and y, and then you don't need a trick. The solution comes about in a very natural way. I think that is in a way the aim of mathematics.

[BID/CFS]: *And you were fascinated by this?*

MK: Yes.

[BID/CFS]: *Were there persons that influenced you in your formative years, before you entered the university?*

MK: Not really. But some teacher made me aware of a book on projective geometry, which I could borrow from the library. I studied it and learned that in dimensions larger than three there is only one solution, while in dimension two there are several solutions to a certain problem. I remember this made a deep impression on me.

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The Abel Prize Award Ceremony 2025.
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Early cooperation and the 1970 master's thesis

[BID/CFS]: *Your prospective advisor at the university, professor Mikio Sato (1928–2023), said of you that you learned Bourbaki, Grothendieck and these things when you were 18 or 19 years old. He said that you studied it by yourself, with no teachers. Why were you so fascinated early on by this type of abstract mathematics?*

MK: So what happened is that I was taking a course in mathematics. I think it was an algebra course. And the professor told me of the existence of EGA, *Éléments de géométrie algébrique*, and of Bourbaki. I started to read the EGA and found it very easy to read, in fact. I think that is a very well written book. And I was very happy to have read that.

[BID/CFS]: *And you didn't have any problem with the text being in French?*

MK: I had a problem, yes. I learned some French when I entered the university. Besides English, you have to learn another foreign language, and I chose French. But I'm not very good at French.

[BID/CFS]: *Professor Mikio Sato has been described as a visionary mathematician. Your master's thesis from 1970 – you were 23 years of age at the time – where you developed Sato's idea of $\mathcal{D}$-modules was described as epoch-making. Please comment on this and also on how and why you got Sato as your advisor.*

MK: Professor Sato was in those days well known among mathematicians, and he was also a very charismatic mathematician. Hikosaburo Komatsu, a younger professor, came back from the United States, and I attended the seminar he and Sato organized. And Sato started to create so-called microlocal analysis.

Microlocal analysis is to study the functions not only on the space, but on the space with codirection; that is, on the cotangent bundle. So that was his idea. It was a very new idea. Sato thought that is possible to study functions microlocally. He started to create the theory, and, fortunately, I was there. I was participating in his seminar and started to collaborate with him. It was very lucky for me.

Together with Takahiro Kawai, who was also a student, we started to collaborate and to construct microlocal analysis. It was a very happy and propitious moment. Because of Sato I learned many valuable things, which in a way contributed to me being awarded the Abel Prize.

[BID/CFS]: *You once said that "Sato wanted to bring the equality world into analysis." This is the marriage between algebra and analysis?*

MK: Yes, you can say so. And also geometry. As you know, mathematics is often divided into geometry, algebra and analysis. Those are the three main divisions, and I think somehow the $\mathcal{D}$-modules combine all these. Sato himself started to study $\mathcal{D}$-modules because of that. So I was lucky to start my study with the theory of $\mathcal{D}$-modules.

[BID/CFS]: *They were absolutely happy circumstances! You told us that it was a lecturer at the university who suggested to you that you should read EGA. Was that in order to prepare you for studying with Sato?*

MK: No, not at all. I didn't think about that at all. Sato developed homological algebra in his own particular way. Of course, homological algebra existed already, and was developed by Grothendieck and others. But I think that Sato didn't know about that, but that he independently discovered things.

[BID/CFS]: *Then you write your master's thesis in 1970, which contains so much fundamental stuff. For instance, you prove the Cauchy–Kovalevskaya theorem, which has to do with existence of local solutions. But you had your own version, which, of course, fits in the framework we're talking about now. Could you describe that to us?*

MK: Some of that is already covered by the idea of Sato. Sato initiated the theory of $\mathcal{D}$-modules, but he didn't develop it, unfortunately. I think he was good to initiate a theory. But I think he never completed it himself. So I started from his idea, that's true, but once you know the idea, it's very easy to proceed. It's like climbing a mountain. You know the mountain, and you have to look for some route, and you can find it. I think that's what happened. Sato showed that there is a mountain over there. I think that up to that point, the influence of Sato is very big.

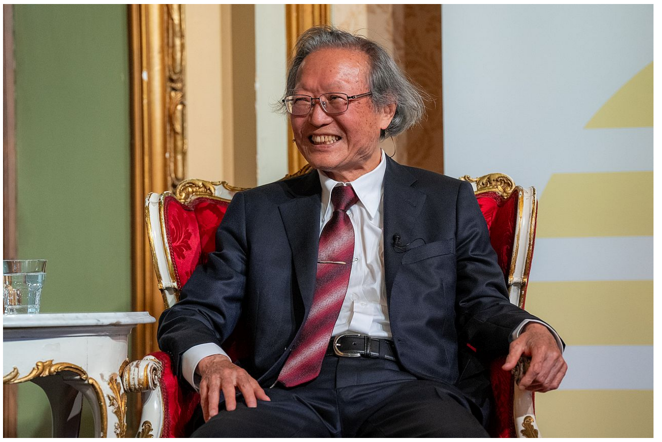
[BID/CFS]: *The statement, as you phrase it, is that "if the function is non-characteristic, then the associated inverse image functor commutes with the solution functor."*

MK: Yes, that's right.

[BID/CFS]: *We'll come back to $\mathcal{D}$-modules in more detail, but can you in more elementary terms connect your result back to the original Cauchy–Kovalevskaya theorem in any way, so that we see that there is a connection. Where are the differential equations and where are their solutions?*

MK: I think it's difficult to say. We have a real world, and so there's a real manifold. One of the basic ideas of Sato is that this is not a good point of view. A good point of view is that you have a real world, but that is surrounded by a complex world. And the real world is determined by this surrounding complex world. I think that is Sato's main idea. And the complex world is, in fact, more important than the real world. If you know the complex world, then you know the real world. That was the starting point of the hyperfunction theory of Sato. And as for the Cauchy–Kovalevskaya theorem, I think the main part is concerned with the complex world.

And the next idea is that you can connect it to the real world. Sato, Kawai and I succeeded to construct microlocal analysis, and that was vital to how we understand the real world through the complex world.



Reception at the National Theatre 2025.
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Early career; SKK-paper and the index theorem

[BID/CFS]: *Which culminated in the groundbreaking "SKK" paper of 1973, co-authored with Sato and Kawai. The cotangent bundle – or, as the physicist would say, the phase space, which is position and momentum – has a prominent role in this theory. You talk about a wave front, and you just mentioned the tension between the real and complex worlds.*

Can you explain to us why the cotangent bundle should play such an important role?

MK: That idea, of course, had existed for a long time, for example, in Fourier analysis. We can say that is some kind of incarnation of the symplectic geometry, and physics also, as you mention. The cotangent bundle is important not in a global sense, but in a local sense. Before we thought the idea of "symplectic geometry" works only in a global sense. Sato's big idea is that it works also in a local world, in the form of the cotangent bundle.

[BID/CFS]: *Could we interject with an interesting fact? We are talking about your master's thesis, written when you were 23 years of age, and you wrote it by hand in Japanese. It took 25 years before this important manuscript was translated into English, which is remarkable.*

MK: If I look at my master's thesis now, I can see that there is a germ of an idea there, which was important for the subsequent theory.

[BID/CFS]: *These ideas became influential, and, given that they were only in handwritten Japanese, how did the ideas come out? Was that through the SKK paper from 1973?*

MK: In my master's thesis, the microlocal point of view is very small. However, in the SKK paper the microlocal theory was developed. I worked with Sato and Kawai, and continued to do so for five years or so. During that period the microlocal point of view was fully developed.

[BID/CFS]: *In the paper from '73, Sato, Kawai and you established the involutivity of characteristics of microdifferential systems, which, of course, had been touched upon by Guillemin, Quillen and Sternberg in 1970, and proved totally algebraically by Gabber in '81. Explain to us: what is the involutivity of characteristics of microdifferential systems?*

MK: That part is very important. In the cotangent bundle you have some subspaces inside, but the important ones are not arbitrary subsets, just the involutive ones are important. These are the good ones with respect to the symplectic structure. Another word for this is integrability.

[BID/CFS]: *Also in 1973 – it must have been a very good year for you! – you proved an index theorem, which we understand you are especially proud of. You have revisited the index theorem* *several times, for instance in 1985 and in 2014 together with Pierre Schapira. Why do you like this result so much?*

MK: It is better to go back a little bit. So, in my master's thesis I proved the index theorem for the one-dimensional case. For higher-dimensional cases, I didn't know how to do that, and that concerned me. The index theorem is a global theorem, you seemingly have to know the local information at every point. But my index theorem is not like that. It is enough to know some generic part – the generic part is enough. I was very surprised when I realized this and thus proved the theorem.

Of course, some of it can be expected because of involutivity. Arbitrary subsets do not appear. Only the involutive ones. There is no small subset which is involutive, which means that the generic part determines the whole thing. When I look back afterwards, then perhaps one can expect this, but it is still surprising.

The Riemann–Hilbert correspondence

[BID/CFS]: *Let's move to another of your major achievements. In 1980, you prove the Riemann–Hilbert correspondence, which in your framework is an equivalence between the derived categories of certain $\mathcal{D}$-modules and constructible sheaves. The algebraic case had been done ten years before by Pierre Deligne; Zoghman Mebkhout did something very similar at the same time as you, and Alexander Beilinson and Joseph Bernstein should also be mentioned.*

Do we understand correctly that the hard part here is determining how to restrict so that your solution functor becomes fully faithful? Is that the key problem?

MK: There are many ways to look at it, but I think the most difficult part was regularity. So, the Riemann–Hilbert correspondence is a correspondence between two things, the topological part and the algebraic part: the algebraic part corresponds to the $\mathcal{D}$-modules, and the topological part corresponds to the monodromy. The algebraic part is not about all $\mathcal{D}$-modules, but more specifically about the so-called *regular* type $\mathcal{D}$-modules, a very difficult concept. In fact, regularity is not easy to handle. That part is hard. In fact, it took a long time for me to define properly the regular holonomic $\mathcal{D}$-modules.

[BID/CFS]: *Perhaps it's good to go back to Hilbert 21st problem and connect it to your result. Which part, in Hilbert 21st problem, corresponds to $\mathcal{D}$-modules, and which one corresponds to the constructible sheaves; and what is Hilbert 21st problem?*

MK: The original problem has to do with the so-called monodromy. The solution to a linear ordinary differential equation is not univalent, it is multivalent. Consequently, if you go around a singularity the so-called monodromy appears. The original problem is about the connection between the monodromy and the linear ordinary differential operators with regular singularities. The $\mathcal{D}$-modules correspond to the differential equations, and the constructive sheaves to monodromy.

[BID/CFS]: *So let's see, if X is your manifold, $\mathcal{D}_X$ is the algebra of differential operators, that explains the $\mathcal{D}$-modules, which then control the differential equations. But how do constructible sheaves at all look like monodromy?*

MK: They're exactly the same! Locally the structure is simple, and sheaves are exactly what connects the local simple structure to form the global picture. I think nowadays people understand monodromy in terms of sheaves. The sheaves patch things together so that you actually have the loop around.

[BID/CFS]: *The Riemann–Hilbert correspondence has been called a parallel, or perhaps, a generalization of the equivalence between the de Rham cohomology and Betti, or singular, cohomology. Is that a valid point of view?*

MK: No, no, no, I don't think so. A bit different point of view is better, I think. The de Rham sheaf figures prominently, but the setting is different. While you have the correspondence between de Rham cohomology and Betti cohomology, the $\mathcal{D}$-modules are so different, and the correspondence itself is important.

Grothendieck wanted to unify this in a theory of motives. So, there is something called motives, and one aspect is Betti cohomology, and another aspect is de Rham cohomology. You can view them this way or you can view them another way, and you can see two kinds of things. But I think it's a little bit different from what you suggested.

[BID/CFS]: *But, like the correspondence between de Rham and singular cohomology, the Riemann–Hilbert correspondence is between, on one hand, differentials and, on the other hand, topology in the form of constructible sheaves?*

MK: Somehow, what you're saying is right, but somehow it's not. Of course, nowadays they are mixed, for example, in the work of Morihiko Saito. They connect two things, and then you know more information.

[BID/CFS]: *Later on – and published in 2016 – you and Andrea D'Agnolo were actually able to remove the regularity condition at the price of replacing the sheaves with* `Ind`*-sheaves. So, how does* `Ind`*-sheaves help you "resolve" – if we may – the singularity?*

MK: So, now it concerns not only the regular, but the irregular case also. In the regular case the behaviour of the solution at a singular

Andrea D'Agnolo, professor at the University of Padova, Italy and Masaki Kashiwara enjoy the Abel Lectures at the University of Oslo.
(© Thomas Eckhoff / The Abel Prize)

point is rather easy to understand, but in the irregular case it is more complicated, and it takes a long time to understand how to control the singularities of functions in the irregular case. It's a rather tricky part – it took 20 years to reach a solution.

$\mathtt{Ind}$-sheaves have an origin already in my paper on the Riemann–Hilbert correspondence in the regular case. In that paper, I introduced some operation to construct regular holonomic $\mathcal{D}$-modules starting from constructible sheaves. Schapira noticed that we can develop this operation further. We have reached the notion of $\mathtt{Ind}$-sheaves to capture a phenomenon we cannot capture by means of "sheaves." To solve the irregular Riemann–Hilbert correspondence we needed an additional idea: to add one variable.

[BID/CFS]: *Replacing sheaves with* $\mathtt{Ind}$*-sheaves doesn't seem such a dramatic thing, but it seems to us very drastic to remove the regularity condition. Was it surprising to you that such a "slight" extension actually would resolve the problem?*

MK: Yes, but it's a long history, because in the beginning, in co-operation with Schapira, we did it to control the singularity of functions. So, that is a starting point of $\mathtt{Ind}$-sheaves, but gradually it developed, and the history is rather long.

[BID/CFS]: *On the other hand, the Riemann–Hilbert correspondence and* $\mathcal{D}$*-modules figure prominently in the geometric Langlands conjecture; most recently, in the proposed proof by Gaitsgory and his collaborators. What are your thoughts on this? It sits so centrally in there.*

MK: For the geometric Langlands, of course you need $\mathcal{D}$-modules. It's a very basic tool, but to solve the geometric Langlands it's not enough. You need much more.

The Kazhdan–Lusztig conjecture (and the role of co-operation)

[BID/CFS]: *You have authored more than 250 papers with about 50 co-authors. This is an unusually large number for a mathematician. Could you comment on this, and, in particular, on your cooperation with Jean-Luc Brylinski in 1979 which led to a proof of the Kazhdan–Lusztig conjecture?*

MK: Although the theory of $\mathcal{D}$-modules was constructed in the 1970s, I did not encounter a chance for its direct application. While in Paris in 1979, the mathematician Jean-Luc Brylinski called me and proposed for us to meet, since he had an idea pertaining to $\mathcal{D}$-modules. We met, and he talked about the Kazhdan–Lusztig conjecture which had been posed the year before. Brylinski suggested a possible application of $\mathcal{D}$-modules to prove the conjecture.

This initial suggestion was extremely fruitful, and $\mathcal{D}$-modules came to play an essential role in our proof of the Kazhdan–Lusztig conjecture. It is also noteworthy that a striking application of the Riemann–Hilbert correspondence is used in the proof. An independent proof was obtained by Beilinson and Bernstein using different methods.

The conjecture itself states that the so-called Kazhdan–Lusztig polynomials give complete information on how a canonical representation splits into irreducible ones.

As for your more general question about having many collaborators, I consider myself fortunate to have comparatively many joint papers. Through these collaborators I have learned a lot about other fields of mathematics.

Crystal bases and quantum groups

[BID/CFS]: *Around 1990, the name Lusztig appears again. He introduced his canonical basis for representations of quantum groups on a vector space over the field* $\mathbb{Q}(q)$ *of rational functions, and you introduced the notion of the crystal basis, which is a basis as the temperature* q *goes to zero. That results in a combinatorial description, by means of what you call a "crystal graph." This was an extraordinary combinatorial tour-de-force which has been dubbed the "the grand loop argument," involving about 20 interlocked steps. Can you tell us about that?*

MK: I worked with Sato on microlocal analysis, and he started to study so-called exactly solvable models in mathematical physics. He worked with Tetsuji Miwa and Michio Jimbo. I knew them both very well and collaborated with them sometimes. Jimbo introduced so-called quantum groups, independently of Drinfeld. The theory of quantum groups concerns itself with a parameter q, which corresponds to the temperature. I thought that at q equals zero things should simplify. That is what happened: As q goes to zero

Masaki Kashiwara lecture at University of Oslo, 21 May 2025.
(© Thomas Eckhoff / The Abel Prize)

something good happens. I spent at least several months to get it to work.

What comes out of this is the notion of a "crystal basis" – by analogy with physics, where matter crystallizes at low temperature. We can analyse it totally combinatorially, and by a combinatorial method, we can describe the representation itself. The representation is rather difficult to analyse, but combinatorially it is not very difficult.

[BID/CFS]: *How does this propagate from q, the temperature, being equal to zero? Does that lift to bases with positive temperature?*

MK: Afterwards, yes. So, in my case, I started at zero and at zero there is a good basis. And after that, I considered the extended case.

[BID/CFS]: *You've returned to this topic several times. For instance, in 2018 together with Seok-Jin Kang and Myungho Kim, you discovered a duality relating quantum affine algebras to certain quiver Hecke algebras. Do you foresee that you will continue working on this?*

MK: That's right. I'm still working on this.

[BID/CFS]: *Is this the main thing you're working on, or are there other ideas that you are pursuing?*

MK: Yes, I'm working on that, but also on so-called cluster algebras. Cluster algebras were introduced by Sergey Fomin and Andrei Zelevinsky. That is what I'm most interested in and working on right now.

Actually, that is what I will explain in my Abel lecture on Wednesday.

[BID/CFS]: *We're looking forward to that!*

Work style, outlook and legacy

[BID/CFS]: *We know it's not completely fair. But if we were to challenge you, which of your mathematical achievements are you most proud of?*

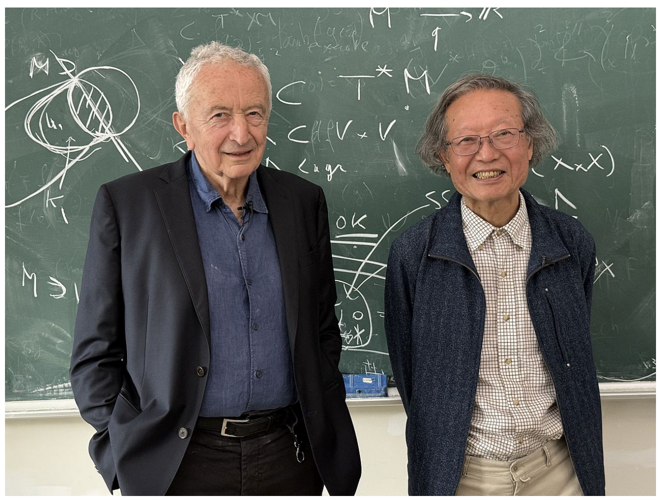
MK: I think it might be the Riemann–Hilbert correspondence and the crystal basis result.

[BID/CFS]: *Another question: working in isolation or working in collaboration, what do you prefer?*

MK: The short answer is: both. However, the meaning of collaboration is different now from when I was young. Today I am working with some younger mathematicians, and I get some energy from working with them.

[BID/CFS]: *Lennart Carleson, the Abel Prize Laureate in 2006, said the following during the Abel Prize interview with him: to prove something hard, it is extremely important to be convinced of what is right and what is wrong. You could never do it by alternating between the one and the other, because the conviction somehow has to be there. Do you agree with that, or…?*

MK: Partially, I agree. I can give you an example. I was working with Sato on microlocal analysis, and after that, with Pierre Schapira, I worked on the microlocal aspect of sheaf theory. When I was working with Sato, I thought that microlocal analysis can only be applied to the singularity of functions. That presumption was not good, and it stifled the imagination. After that I worked with Schapira, and we discovered that microlocal analysis can apply to other things as well.



With Pierre Schapira. (© Liwlig Norway AS / The Abel Prize)

Beyond mathematics

[BID/CFS]: *At the end of these interviews we generally ask: Do you have any special interests besides mathematics?*

MK: Not many, but I like music.

[BID/CFS]: *What sort of music do you enjoy?*

MK: Nowadays, I like Indian music very much. There are basically two types of Indian music, Carnatic and Hindustani. The Hindustani music is from the north, and Carnatic is from the south. The sitar is much used in Hindustani music. Carnatic music doesn't use sitar that much. I started to like Indian music when I was in Chennai.

In Chennai, there is a music season, a music month. All December was one festival, and every night I went to various theatres to listen to music (together with a lot of mosquitoes, I must add!).

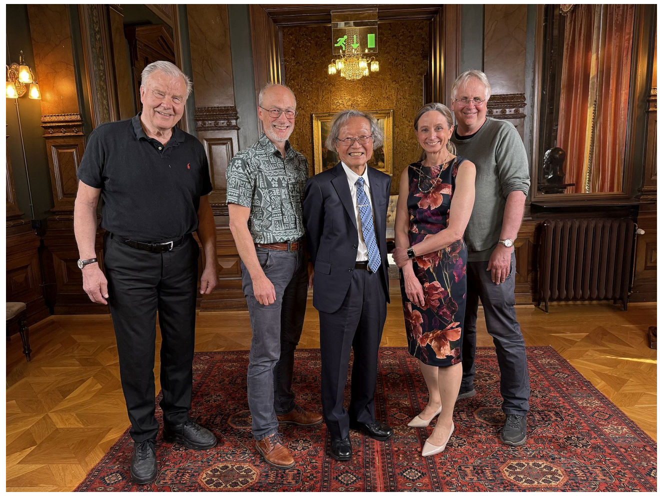
[BID/CFS]: *There is also a rumour that you played table tennis rather well. Do you play table tennis now?*

MK: No, not anymore. I have a bad knee.

[BID/CFS]: *But you were rather actively playing table tennis for a while, right?*

MK: Yes, for a while. Incidentally, I played with Jean-Pierre Serre. He was good; I was beaten by him.

[BID/CFS]: *On behalf of the Norwegian Mathematical Society, the European Mathematical Society, and both of us, we'd like to thank you for this very interesting interview.*



With current and future Abel Prize interviewers. From left to right: Christian Skau, Bjørn Ian Dundas, Masaki Kashiwara, Kathryn Hess and Tom Lindstrøm. (© Liwlig Norway AS / The Abel Prize)

MK: Thank you very much.

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The pre-history of the Abel Prize

Arild Stubhaug

Translated by Ulf Persson.[1]

In 1839, Scandinavian scientists started to meet occasionally in their capitals for both scientific and social reasons. In the summer of 1886 they met in Kristiania (Oslo), and the mathematical section was honoured by a visit of two professors of mathematics from Stockholm, namely Gösta Mittag-Leffler and his protégé Sonja Kovalevsky (Sofya Kovalevskaya), a Russian lady who was the world's first female professor in mathematics; widely celebrated not only for her mathematics, but also for her personal charm. The students applauded, and at the concluding dinner, the Norwegian professor of mathematics Carl Anton Bjerknes gave a speech in her honour.

Bjerknes (incidentally, the father of the more well-known meteorologist Vilhelm Bjerknes) had six years earlier published an ambitious biography of Niels Henrik Abel, and upon receiving it, Kovalevsky had written the author that in her eyes Abel had always appeared as the ideal of a true mathematician.

At the very same dinner, Mittag-Leffler took the stand, raised his glass in honour of Abel, and suggested that one should start collecting money for the erection of a statue of him in connection with his upcoming centennial sixteen years later in 1902. The suggestion was received enthusiastically, and even before the participants had dispersed from the meeting, a significant amount (three thousand crowns, roughly half a yearly salary of a professor) had already been collected, and national committees have been formed. The Swedish side of the project was led by Mittag-Leffler and Sonja.

In European mathematics, Mittag-Leffler played a central role, a veritable entrepreneur, the likes of which had never been seen before (nor after?) in the world of mathematicians. Supported by the Scandinavian ministries headed by King Oscar II, he had founded *Acta Mathematica* in 1882, which quickly became a leading mathematics journal (and still remains so). To set the high standards the journal was intended to hold, its first issue displayed a beautiful portrait of Abel on the front page. Thanks to the prestige of Mittag-Leffler as well as his engagement, mathematics became the subject of the most elevated status in the newly established Stockholm Högskola. Mittag-Leffler had also been instrumental in securing funds for Sonja's salary at the same time.[2]

Niels Henrik Abel. (Public domain, published in: C. A. Bjerknes, Niels-Henrik Abel: tableau de sa vie et de son action scientifique. Gauthier-Villars, Paris, 1885)

[1] All footnotes, save the present one, are comments and annotations by the translator.

[2] Högskola' is obviously cognate with 'high school,' but should be seen more as an institute of advanced study. It was meant to be an institute guided by idealistic principles based less on examinations than on the true joy of learning and research. Eventually it became more of a regular university, with its emphasis on conferring degrees, but achieving its formal status as a university would take until 1960, when it became the fourth university in Sweden after Uppsala (1477), Lund (1660) and Gothenburg (1954). For many decades, it was the centre of the leading mathematical tradition in early 20th-century Sweden, with an impressive list of professors, many of whom would have students of equal reputation.

1 Review

During his sojourns on the continent (during the period 1873–74) Mittag-Leffler (1846–1927) had met many of the foremost mathematicians of those times and thereby having realised the unique high regard Abel enjoyed among mathematicians. In Paris, he had listened to Charles Hermite (1822–1901) talk about Abel's groundbreaking work and tragic fate. Actually, it was the 'romance' of the tragically short life of Abel that served as an inspiration to Hermite to not only study mathematics, but also to devote his life to it.

Another Parisian mathematician, Joseph Liouville (1809–1882), considered it to be the greatest regret of his life to have met Abel in Paris without really getting to know him.

A third mathematician, Adrien-Marie Legendre (1752–1833), who was the world authority on elliptic integrals, which he had studied for almost sixty years, had admitted that thanks to the young man Abel he had been able to more successfully pursue his research by having been given a new fruitful point of view. In Paris Abel was talked about in the same breath as Isaac Newton, and it was not different in Göttingen nor in Berlin.

In Berlin, he met Karl Weierstrass (1815–1897) and Leopold Kronecker (1823–1891), who claimed that the entire edifice of modern mathematics rested on the shoulders of this Scandinavian giant. Abel had also met the engineer August Leopold Crelle (1780–1855) in Berlin, a meeting that resulted in the establishing of a new mathematical journal, known as *Crelle's Journal*[3]. This journal supplied Abel with an outlet, and not only established his reputation, but also that of the journal.

Abel was not a misunderstood genius, his greatness was never in doubt, and his exceptional talents and potential were immediately recognised already when he was a schoolboy. The problem was just that he was too big for little Norway. He was called as a professor to Berlin, but died in tuberculosis before the news reached him.

In 1839, ten years after his death, Abel's collected works were published. This was possible to a large extent thanks to the support of the king at the time – Karl XIV Johan (1763–1844), who was French and had great confidence in the judgment of French mathematicians who wanted access to all of Abel's work[4]. But this collection turned out to be incomplete, and work on an updated revision was initiated in 1874, financed by the Norwegian government and led by Norway's most well-known living mathematicians at the time, Sophus Lie (1842–1899) and Ludvig Sylow (1832–1918) (who should be household names for every mathematician). But the project was not just a Norwegian one: several Scandinavian mathematicians of renown and critically also with administrative influence, engaged themselves. Names to be noted were the Norwegian Ole Jacob Broch, the Finn Lorentz Lindelöf and the Swede Carl Johan Malmsten, and later on the Dane Hieronymus Georg Zeuthen. A truly Nordic project indeed.

Magnus Gösta Mittag-Leffler. (© DRM / Ferdinand Flodin, CC BY-NC-SA)

After seven years of hard work (1881), Lie and Sylow could finally present the complete collection of Abel's work. To publicise it widely, Sophus Lie travelled around, in particular, to Stockholm and Uppsala. In meeting Lie, Mittag-Leffler came up with the idea of founding a Nordic journal of mathematics. Mittag-Leffler being a man of action, already a year later the first issue of *Acta Mathematica*, referred to above, saw the light of day.

[3] At the time academic journals almost did not exist, the works of Euler and others appeared either as books, or as communications of academies, when not simply distributed by personal letters.

[4] As a result of the conclusion of the Napoleonic wars in 1814, Sweden was offered Norway, which had been a province of Denmark since medieval times, as a compensation for the Russian conquest of Finland, a part of Sweden. The Norwegians were not too happy about matters being decided over their heads and declared themselves an independent nation. The whole issue was resolved by forming a personal union with a common king, and common foreign policy, but where of course Sweden was the senior partner. The union was eventually dissolved in 1905 during the reign of King Oscar II (1829–1907), a grandson of Karl Johan. The background history of how a French commoner and Napoleonic marshal ended up on the Swedish throne starting a new dynasty – Bernadotte, is too long, and besides irrelevant to the present issue, to be explained.

The launching of *Acta* strengthened the reputation both of Mittag-Leffler and Stockholms Högskola, and more of the same were to come. In an audience with King Oscar II in the spring of 1884, Mittag-Leffler breached the idea of a mathematics prize, and shortly thereafter the mathematical prize of Oscar II was established. It was decided that the winner would be announced on the sixtieth birthday of the King, the 21st of January 1889, and that the winning work would be published in *Acta*, which went without saying.

Thanks to the extraordinary mathematical networking that characterised Mittag-Leffler, he was able to engage mathematicians of world rank not only to sit on a jury, but also to participate in the competition. Twelve contributions were submitted to be judged by people such as Weierstrass in Berlin and Hermite in Paris. The choice of the winning contribution was beyond doubt, namely that of Henri Poincaré (1854–1912), the young upcoming brilliant French mathematician.[5] His contribution was, according to Mittag-Leffler, of such a calibre, that this mathematical competition would forever be inscribed in the annals of the history of the exact sciences. As many readers may be aware of, it concerned the notorious three-body problem. However, the prophecy of Mittag-Leffler was close to come true in a sense he would not have welcomed; in fact, it turned out that a proofreader discovered a mathematical mistake in the first delivered version; but this was later fixed up by Poincaré, and what could have turned out into *un grande fiasco* was avoided.

In the end, the competition turned out to be an unqualified success, but it was a one time affair, and Mittag-Leffler was well aware that future competitions could not be expected to generate similar epoch-making results, as that of Poincaré's, yet he did not abandon the idea of competitions and their concomitant prize ceremonies, hopefully to be established on a permanent basis.[6]

In order to achieve this, he approached his contacts in Stockholm's financial world as well as wealthy expatriates abroad; notably Thorsten Nordenfelt in London (submarines and weapons), John Ericsson in New York (the inventor of the propeller among other things) and also, interestingly, Alfred Nobel (1833–1896) in Paris (the inventor of dynamite). Initially Mittag-Leffler suggested to set up a fund with a modest capital of about 10,000 crowns, the proceeds of which could be used to mint gold medals with portraits of the greatest mathematicians, and if so it was natural to start with the foremost Nordic one, Niels Henrik Abel.

[5] Incidentally, a cousin of his, Raymond Poincaré, was the French president during the First World War, and three times prime minister.

[6] One thinks of the latter Nobel Prize, where the prizewinning ceremony takes centre stage. Also, the original Nobel idea was that the prizes were intended for work done during the previous year, makes the analogy with competitions even stronger.

Ludvig Sylow. (Photographer: R. Ovesen; credit: Staatliche Museen zu Berlin, Kunstbibliothek / Public Domain Mark 1.0)

2 Abel

The King was also presented with plans about a mathematical prize to be awarded every fourth year to those who had made the greatest mathematical discoveries in the interim. The prize should be a gold medal with a portrait bust of the king, The king was excited by the idea and even wanted to participate in the forming of the jury. More specifically this medal should be awarded to a mathematician who had published an exceptionally good work in *Acta Mathematica*.

However, nothing of this came to fruition, save a small fund which made it possible to invite distinguished foreign mathematicians to Stockholm, and was subsequently used for that purpose. After the great accomplishment with the collected works of Abel (1881), Ludvig Sylow returned to his modest teaching position in Halden, while Sophus Lie went to a prestigious professorship in Leipzig (1886) where he, so to speak, created a new mathematical discipline; and where consequently students from all over the world flocked to sit by his feet. And for Mittag-Leffler at Stockholm the reputation and success of *Acta* as well as that of Stockholms Högskola were his main concerns. Thoughts about and plans for a mathematical prize were not revived and instead put on hold until the event of the grand testament of Nobel.

3 Nobel's glorious prizes for scientific research

One of Mittag-Leffler's tasks in Stockholm also included securing financial support to Stockholms Högskola, which was predominantly based on private funds. As already mentioned, this included, in particular, the funds for Sonja Kovalevsky's professorship, for which he worked hard to renew every New Year in order to prolong her presence in Sweden. In view of those efforts, Nobel's plans to stimulate scientific research became very interesting to him.

Rumours that Alfred Nobel wanted to donate considerable sums to scientific research became widely known at the very beginning of 1890. Mittag-Leffler then offered to create a professorship to hold the name of Nobel, and invited him to Stockholm to personally announce it. But the ensuing contact between Mittag-Leffler and Nobel was riddled with misunderstandings as well as actual disagreements, which soon caused rumours to the effect that the two men competed for the favours of the beautiful Sonja Kovalevsky.[7]

Nobel dismissed out of hand all arguments Mittag-Leffler presented to show the importance of Sonja for Stockholm and Sweden. Instead, Nobel claimed to know what was best for Sonja and wrote from Paris on March 1, 1889.

> *I believe that Frau Kovalevsky, whom I have the honour to be personally acquainted with, would be much better suited for St. Petersburg than for Stockholm. Fruntimmer*[8] *will find in Russia a wider horizon, and prejudices – that persistent European malaise*[9] *are there reduced to a minimum. Frau Kovalevsky is not only an excellent mathematician, but furthermore an exceedingly gifted and sympathetic personality, whom one wishes not to remain in a winged state, confined in a small cage.*

But the saga of Sonja came to a premature end: she died on one of the first days of 1891. For the scientific circles in Stockholm it was a great loss, for Mittag-Leffler also a great personal one, maybe also for Nobel; but the plan for a mathematical prize was given a new venue.

In Nobel's preliminary version of his testament (March 14, 1893) Stockholms Högskola was a major beneficiary. But at the subsequent death of Nobel (December 10, 1896) that institution turned out to be cut out of the will, nor was there any mention of a mathematical prize in it. On the other hand, there was money earmarked for three other sciences, physics, chemistry and medicine. That mathematics was overlooked in this way nourished speculations that there really would have been a case of sexual rivalry between Mittag-Leffler and Nobel.[10] However, everything indicates that the reasons were more prosaic. Nobel was attracted less by the theoretical aspects of science and the purpose of the prizes were to further practical applications.[11]

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[7] Note that Mittag-Leffler was only four years older than her, while Nobel was seventeen years older. She was in fact born in Moscow in 1850. Her father was a Russian general of the minor Belarus nobility, while her mother was ultimately of German descent. She grew up on her father's family estate in Polibino near the border to Belarus, and received an excellent education at home. At the age of 18 she married the young radical palaeontologist Vladimir Kovalevsky. It was a marriage of convenience (although a daughter issued from the union) for the purpose to be allowed to travel abroad to study, as the possibilities for women to pursue higher studies in Russia were, to put it mildly, highly constrained. She ended up in Berlin under the tutelage of Karl Weierstrass, who counted Mittag-Leffler as a student. The rest is history.

[8] A slightly archaic Swedish word for mature women, which nowadays, at least, has a derogatory quaint. It would be interesting to know whether that was also true at the end of the 19th century (possibly as at least Strindberg uses it when in a misogynic mood), which, if so, would be interesting as it contrast against the otherwise respectful references to her. Is it a case of a mixed message?

[9] Nobel actually uses the Swedish word 'surdeg' which literally means 'sour-dough,' but which often is used metaphorically, meaning a problem that never seems to go away but is perpetually masticated.

[10] As noted above, Mittag-Leffler was more or less a contemporary of Sonja (in fact, younger than her husband), while Nobel would have been more of a 'sugar daddy,' so it was not really the case of two older men seeking the favours of a young woman. Nobel was a bachelor, Mittag-Leffler had made an advantageous match with a Swede-Finnish lady in Helsingfors (Helsinki) while being professor there. From what one gathers from biographies, the marriage was not excessively passionate. Clearly Mittag-Leffler was strongly emotionally tied to her, but what that entailed, is really not our business to speculate upon.

[11] In his will Nobel specified that the prizes should go to those whose discoveries had benefited mankind the most during the past year, a stipulation that was to be ignored. True, in the early years prizes in physics were geared towards inventions and awarded to engineers such as the Swede Gustaf Dalén (improvements of lighthouses) and the Italian Guglielmo Marconi (wireless telegraphy), not to mention the spectacular discovery of Röntgen, who was the first physics laureate. This latter award gave, in fact, the whole thing a flying start: X-rays were something that the general public could immediately appreciate the usefulness of, requiring no contorted explanations, which have usually been the case afterwards. But also in the early years theoretical results of fundamental scientific importance were in fact awarded as well, and then came the quantum revolution in the 1920s setting the standards. It can be noted that the Swedish academy was not at first exactly thrilled at the prospects of administering the prize, they would have preferred to send the medals and checks by mail, but were persuaded to make a ceremony out of it, without which (as Mittag-Leffler well understood) it would never have become the unique institution it now is (and ironically it now provides the main rationale for the existence of the Swedish Academy of Sciences). Both Mittag-Leffler and Nobel were entrepreneurs and businessmen, Mittag-Leffler by necessity Nobel by inclination. Nobel was an inventor and engineer, while Mittag-Leffler was motivated by idealism, although the rhetorics with which he used to describe it, nowadays strike people as slightly ridiculous. Mittag-Leffler saw mathematics as the highest achievement of the human spirit. He must have been very disappointed by the exclusion of mathematics, but he put a brave face on it, regularly inviting the laureates to dinner at his sumptuous villa in Djursholm (a

Sophus Lie. (Photographer: L. Szacinski, owner: National Library of Norway)

4 Sophus Lie steps in

In the Russian city of Kazan a mathematical prize was instigated 1894 for geometrical contributions, in particular, for those in hyperbolic geometry. Not surprisingly it was called the Lobachevsky Prize.[12] This prize was awarded for the first time in 1897, and then to Sophus Lie as already noted residing in Leipzig.

This prize and the fact that in the testament of Nobel there was no place for a mathematical prize, seems to have provoked Lie's interest to instigate a prize to be named after Abel. For that purpose he presented plans for a foundation which would every fourth year give a prize to an outstanding work in pure mathematics. Lie had of course a large mathematical network and set out to mobilise it, and soon acquired moral support from many influential mathematical centres.

Luigi Cremona (1830–1903) and Luigi Bianchi (1856–1928) in Rome and Pisa assured him of support, Emile Picard (1856–1941) wrote from Paris not only that he and Hermite wanted to contribute to the fund, but also that France as a whole, through its universities and high schools, could come up with a significant sum of money. He could also envision more frequent prize ceremonies than every four of fifth year. Gaston Darboux (1842–1917) followed up with further positive reactions and was of the opinion that every mathematician of the Academy in Paris would surely support an Abel fund. Such assurances of support were also forthcoming from Andrew Russell Forsyth (1858–1942) in Cambridge, who thought that also Lord Kelvin would be prepared to join the ranks, Felix Klein (1849–1925) in Göttingen wrote that his support for the undertaking went without saying, and he thought that both David Hilbert (1862–1943) and Lazarus Fuchs (1833–1902) would be willing as well. The only reservations emanated from Berlin, where Georg Frobenius (1849–1917) and Hermann Amandus Schwarz (1843–1921) expressed the opinion that scientific prizes in general often distracted young people from the true path.

However, all those contacts and promises of support were attached to Lie personally, so when he returned to Kristiania to a position and died a few months later (in January 1899, at the age of 57) there was no one to develop and administer all those contacts and promises, still remaining on a preliminary stage, and it all petered out.

5 The centenary of Abel's birth, 1902

The next serious opportunity for a prize or a gold medal carrying the name of Abel, naturally arose in connection with the celebration of the centenary of Abel's birth.

As noted above, Mittag-Leffler had ever since the Scandinavian meeting in Kristiania the summer of 1886 kept the upcoming jubilee in his thoughts. However, the collection of the necessary funds to erect a monument in celebration of the young genius had been poorly managed, and most likely earmarked money had already been spent domestically in the participating countries in anticipation of the memorial festivities.

In Sweden, Mittag-Leffler wanted to devote a whole issue of *Acta* to the works of Abel. He sent out invitations to potential contributors, and the response was overwhelming. Instead of a single issue, three consecutive issues (26–28) of *Acta* were eventually published with the title 'Niels Henrik Abel in memoriam,' encompassing almost 60 articles and close to 1200 pages. However, only the first issue would be ready for the celebration in the Norwegian capital in September 1902.

In Norway, work had already proceeded far on a Festschrift when Mittag-Leffler's plans for Abel in *Acta* became known. It

wealthy suburb of Stockholm). He is also responsible for making Pierre Curie's wife Maria a co-recipient of the prize.

[12] Nikolai Lobachevsky (1792–1856) was the first one who made hyperbolic geometry publicly known, neither Gauss nor Bolyai had published anything on it. Lobachevsky did not view it as a theoretical construct, but as a possible alternative to Euclidean geometry to describe the large-scale universe. He called it astral geometry. According to stories, Gauss tried to measure the angular sums in large triangles during his survey work. However, assuming he expected a geometry of constant curvature, he would surely have realised that the task was not practical.

would contain letters by Abel (in all 44 letters to and from Abel) in addition to some documents pertaining to his life. The Festschrift would also include a biography of Abel written by the mathematician and collector of folk songs Elling Holst (1849–1915), together with an article about Abel's studies penned by Ludvig Sylow.

6 The memorial festivities, 1902

In the Norwegian capital much emphasis was on having the memorial put Norway on the map as a nation of culture, something that gained in importance as the issue of breaking up the union with Sweden was heating up.[13]

As the chairman of the festival committee the hero of Polar exploration, Fridtjof Nansen (1861–1930), had been chosen. The nestor of Norwegian literature Bjørnstjerne Bjørnson (1832–1910)[14] had been tasked with writing a cantata set to music by the composer Christian Sinding (1856–1941). The King, the Storting (the Parliament) and the ministry all participated in the general celebration of a Norwegian achievement, that the Abel festivities had turned into. Furthermore, the Norwegian university bestowed for the first time honour's degrees. In fact, of the 29 chosen, ten were present, namely Vito Volterra, David Hilbert, Heinrich Weber, Émile Picard, Andrew Forsyth, Georg Zeuthen, Hermann Schwarz, Oscar Backlund, Simon Newcomb, and of course Gösta Mittag-Leffler himself.

King Oscar hosted a late supper at the palace, the bourgeois partied in Logen[15]; and the students arranged the largest procession ever to walk the streets of the city, brandishing torches. The National theatre set up a performance of Peer Gynt by Ibsen, and the newspapers, not only those of the capital but all over the country, contained extensive reports on what was going on, as well as submitted poems and prologues lauding Abel.

About seventy foreign guests met up as delegates for their respective institutions. The climax was held in the great hall of the university, Oscar II obviously was present, but also his youngest son Prince Eugèn,[16] together with the teaching staff, the members of the Scientific Society, specially invited guests, and with the students and other curious of the general public, watching from the galleries.

King Oscar II of Sweden and Norway. (Photographer: L. Szacinski, owner: National Library of Norway)

The King and the prince were both presented with copies of the Norwegian Festschrift, which were given to the delegates as well, with a French version for non-Scandinavians. Thereafter, many of the foreign delegates delivered short accounts of Abel and his importance. Among the speakers one can note Schwarz from Berlin, Picard, from Paris, Forsyth from Cambridge, Gravé from Kiev and Zeuthen from Copenhagen. Mittag-Leffler had of course brought the first issue of *Acta*'s 'Abel in memoriam' along, and announced in his speech that two further issues would follow; and that fifty of the world's foremost mathematicians had written articles, all having in common of being based on Abel's work.

In spite of the exuberant speeches, Mittag-Leffler could not but sense that he was met with what he called a certain cold aloofness, even among former friends. And he also started to discern the reason. Rumours flourished to the effect that he wanted to make Abel a Swede and wanted to make *Acta*'s 'Niels Henrik Abel in memoriam' as the real Festschrift. He was also accused of taking the personal responsibility for the presence of so many distinguished foreign mathematicians in the Norwegian capital. The claim that Mittag-Leffler wanted to make Abel Swedish was founded on an incident at his fiftieth anniversary (1896), when the French mathematician Paul Émile Appell (1855–1930) had sent his best wishes attaching Abel to Mittag-Leffler's 'fatherland', and according to the rumour, Mittag-Leffler had not done anything to rectify the mistake, it had even been suggested that

[13] To be completed in 1905, as (foot)noted above.

[14] Henrik Ibsen, by far the internationally most well-known Norwegian writer was still alive, but Bjørnson (Nobel Prize in 1903) was more influential on the domestic scene, not only as a poet, but also as a polemicist and debater on cultural issues. Incidentally, he wrote the lyrics to the Norwegian national anthem.

[15] A concert hall and general music venue, which also hosts banquets and conferences, situated in Oslo and built and inaugurated in the 1830s.

[16] A highly respected landscape painter in Sweden, a testimony to the artistic temperament and talent exhibited by the Bernadotte dynasty.

Mittag-Leffler himself had supplied Appell with a sketch of the message.

That the cloud of suspicion still rested above his head in the Norwegian capital he once again experienced when he tried to have published in the papers a speech he had given to the Norwegian students. In order to have this accomplished, he had to appeal to Bjørnson to stand up for him and assure them of his pro-Norwegian stand, that, in fact, he belonged to the most pro-Norwegians in Stockholm.

To his old friend, the geologist Waldemar Christopher Brøgger, who was now the president of the university, he commented on the baseless accusation that he wanted to make the Abel festivities a Swedish one, and assured him that he would have done the same, had Abel been French, German or English.[17]

7 Nachspiel, 1902

Norwegian politicians and scientists formulated three principal tasks for the memorial festivities.

(1) To arrange a memorial event with a general cultural aim.
(2) To erect a dignified monument to the mathematical genius.
(3) To establish an international Abel Prize.

The first two of those objectives were resolved, even if the big Abel statue by Gustav Vigeland (1869–1943)[18] was not finished and displayed until six years later. But what about an Abel Prize?

After the official end of the celebrations, the king announced that he wanted to have gold medals minted to the memory of Niels Henrik Abel. It seems that the plan was the medal would be awarded by the university of Kristiania every third year to mathematical achievements of the first rank. Oscar II, who after his active support of *Acta Mathematica* and his eponymous mathematical prize (1889) was considered somewhat of a patron saint of mathematics, now seemed to go out on a limb in his predilection for mathematics.

On the Swedish side Mittag-Leffler feared that an Abel Prize would be overshadowed by the Nobel Prizes (which were awarded for the first time in 1901). He despaired of finding another patron who would be able to lift it up the level of the those prizes.[19] He also meant that it would be easier for a jury to come to an uncontroversial decision if they only had to judge solutions to a given problem, preferably connected to the works of Abel.

On the Norwegian side the mathematicians Ludvig Sylow and Carl Størmer (1874–1957) were appointed to head a committee to work out the statute of an Abel Prize. In the fall of 1904 they presented their suggestion, but the work had hardly begun when the political crises of the dissolution of the Union with Sweden erupted. Any plans for an Abel Prize were put on hold again, more or less permanently it appeared, as testified by the letter by Nansen to the mathematician Elling Holst.

> *That by the blessed King Oscar promised Abel Prize ascended to Heaven along with the Union.*[20]

8 Postscript

It would take another century until the idea of an Abel Prize was not only revived but even realised. This is another story which has already been told, see K. G. Helsvig, The Abel Prize: The missing Nobel in mathematics?, *Centaurus* **56**, 1–30 (2014).

Suggested further reading

[1] J. Gray, A history of prizes in mathematics. In *The millennium prize problems*, Amer. Math. Soc., Providence, RI and Clay Math. Inst., Cambridge, MA, 3–27 (2006)

[17] I guess meaning that in this case the accusations of usurping him as Swedish would have been too absurd to be taken literally. This bespeaks a certain sensitivity on the parts of the Norwegians, which is however quite understandable. The Scandinavian countries formed a Union (the Kalmar Union) in 1397, in which Denmark was the dominant member; Sweden broke out in the early 16th century, and centuries of military rivalry between the two countries ensued where Denmark found itself repeatedly on the receiving end. It was not until the early 19th century a feeling of Scandinavian brotherhood sentimentally formed. This was also the time when the notion of nationalism got established, namely the powerful but potentially pernicious idea of states being founded on natural ethnic principles, as opposed to formal corporate ones. This clearly inspired the modern sense of Norwegian nationhood that was further developed during the 19th century. Resentment of being the junior partner and the Swedish domination grew, and relations between the two brotherly people (in the rhetorics of Oscar II) soured to the point of almost coming to blows. However, what can most aptly be described as a potential civil war, was avoided by mutual diplomatic efforts. The pride of Abel was part of Norwegian independence, but even more so the arctic exploits of Nansen and Amundsen put Norway on the map.

[18] Norwegian sculptor who was widely employed in public assignments. There is a large part of the Frogner park in Oslo devoted to his sculptures.

[19] As noted above the choice of Röntgen caught the admiration of the public, and thus established its prestige right away, as hinted in a previous note.

[20] This letter by Nansen was obviously written after the split up between Sweden and Norway (in fact, on December 7, 1906) but *before* the King had died (1907). In Nansen's original Norwegian one reads *den af salig Kong Oscar lovede Abelpris…* using the word 'salig,' which literally means something like *blissful* and in Christian religion more like being *blessed*, more particularly it refers to someone who has been saved and hence been granted eternal life (which is assumed to be blissful). In particular, it means someone who has gone to Heaven (the ultimate goal). Thus it can be used as a euphemism for someone being dead. Whether this was a trivial slip by Nansen, or intentional, we can only speculate upon.

[2] K. G. Helsvig, *Elitisme på norsk: Det Norske Videnskaps-Akademi 1945–2007*. Novus, Oslo (2007)

[3] A. Stubhaug, *Niels Henrik Abel and his times: Called too soon by flames afar*. Springer, Berlin, Heidelberg (2000)

[4] A. Stubhaug, *The mathematician Sophus Lie: It was the audacity of my thinking*. Springer, Berlin, Heidelberg (2002)

[5] A. Stubhaug, *Gösta Mittag-Leffler: A man of conviction*. Springer, Berlin, Heidelberg (2010)

Arild Stubhaug (born in 1948) is a writer of fiction and biography with an honorary doctorate from Oslo University. His biography *Niels Henrik Abel and his times* was followed by biographies about the mathematicians Sophus Lie and Gösta Mittag-Leffler. He has won numerous awards, including the Norwegian Language Council Prize, the Norwegian Academy Prize and the Dobloug Prize awarded by the Swedish Academy.

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Mathematical modelling of light propagation in the human eye

Adérito Araújo, Sílvia Barbeiro and Milene Santos

This review paper surveys the application of Maxwell's equations to simulate light propagation in the human eye, using discontinuous Galerkin methods for spatial discretisation. Understanding this process is crucial for medical imaging and the early diagnosis of eye diseases. Case studies involving corneal opacity, diabetic macular edema, and retinal elasticity demonstrate the importance of simulating this phenomenon considering realistic geometries and material properties. Specifically, these simulations provide valuable insight into how structural changes in the cornea and retina affect light scattering and transparency, offering a useful tool for non-invasive diagnosis. Curved anatomical features, such as structures of the eye, require accurate boundary representation to avoid loss of order of convergence of the numerical schemes. High-order discontinuous Galerkin method combined with a polynomial reconstruction technique enable an appropriate enforcement of boundary conditions without relying on curved meshes.

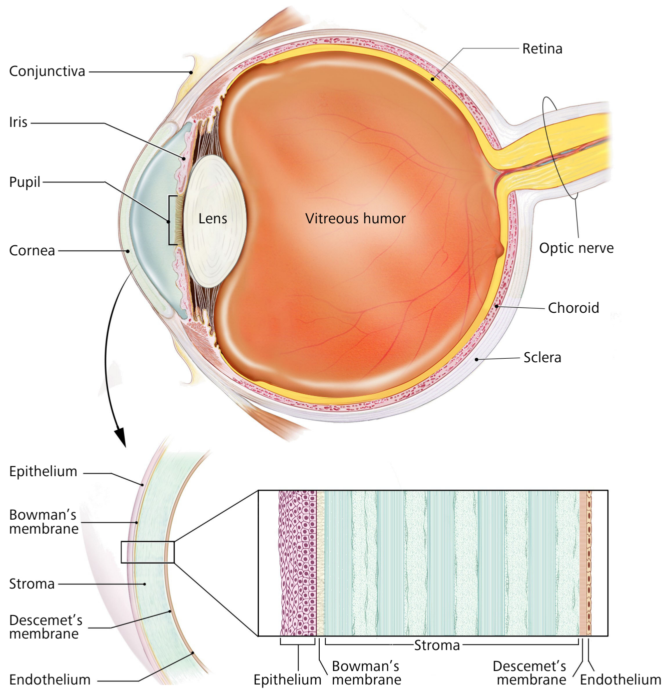


Figure 1. Structure of the human eye. (National Eye Institute, CC BY 2.0)

1 Introduction

Light entering the human eye undergoes refraction and transformation as it passes through layered media with varying refractive indices, including the cornea, aqueous humour, lens, vitreous humour, and retina [17] (see Figure 1). These structures collectively focus and guide light toward the retina, where photoreceptor cells convert electromagnetic waves into neural signals for vision. Accurately modelling light propagation through these structures is essential not only for understanding the fundamental optics of vision, but also for improving diagnostic imaging techniques and uncovering biomarkers of disease.

Among modern ophthalmic imaging modalities, optical coherence tomography (OCT) has become a widely adopted standard in both research and clinical use. Since its introduction in the 1990s [15], OCT has revolutionised the non-invasive evaluation of the retina by enabling cross-sectional imaging with high axial resolution. It operates on the principle of low-coherence interferometry: a beam of light is directed into the tissue and backscattered photons are captured to generate high-resolution cross-sectional images of the tissue. As light propagates through tissue, scattering occurs at refractive index discontinuities. The magnitude of the backscattered light is influenced by factors such as the size and shape of the scatterers, the incident wavelength, and local optical heterogeneity. The OCT signal at each point is captured in an A-scan, and multiple A-scans collected laterally form a B-scan; three-dimensional imaging is obtained by compiling a stack of B-scans across the azimuthal plane (see Figure 2 and Figure 3).

OCT plays a central role in the diagnosis, staging, and monitoring of major ocular diseases such as diabetic macular edema (DME), age-related macular degeneration (AMD), and glaucoma [12]. Beyond ophthalmology, OCT has also emerged as a valuable tool in neurology, with retinal changes being increasingly recognised as biomarkers of central nervous system disorders, including multiple

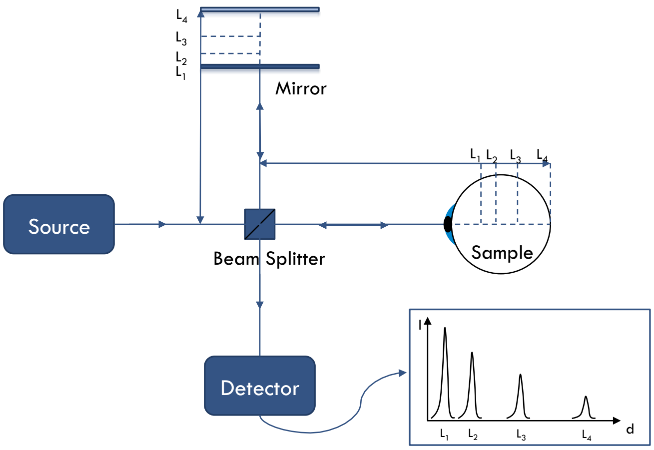


Figure 2. Scheme for the principle of OCT [29].

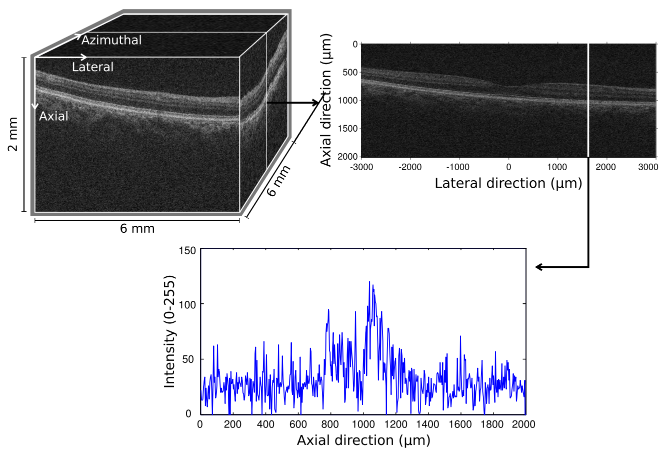


Figure 3. Example of an OCT volume (top left), B-scan (top right) and A-scan (bottom) of a human retina [9].

sclerosis, diabetic retinopathy, Parkinson's disease, and Alzheimer's disease [7, 24, 27]. However, despite its widespread use, OCT has intrinsic limitations. The technique is primarily sensitive to backscattered light intensity and does not provide direct access to sub-cellular features or to the underlying optical parameters (e.g., refractive index distribution, anisotropy) that drive contrast formation in scattering tissues.

Understanding how microscopic changes, such as extracellular fluid accumulation, cell swelling or microstructure organisation, affect the macroscopic OCT signal is therefore a crucial step towards improving the sensitivity and specificity of OCT-based diagnoses. Many early pathological changes in both ocular and neurological diseases are subtle and occur at scales below the resolution of OCT. Accurately linking these microscopic changes to OCT signal patterns requires detailed biophysical models of light propagation in the eye, capable of resolving scattering and interference phenomena in realistic anatomical geometries.

Maxwell's equations provide the most complete and physically consistent framework for modelling electromagnetic wave propagation in biological tissues. However, their numerical solution in full three-dimensional ocular domains presents significant challenges: the wavelength of light is several orders of magnitude smaller than the overall domain size; the anatomical structures involve curved, multilayered boundaries with sharp optical property contrasts; and the resulting wave fields are highly oscillatory. To address these complexities, we employ high-order discontinuous Galerkin (DG) methods for spatial discretisation. DG methods combine the geometric flexibility of finite elements with the high-order accuracy and local conservation properties of spectral methods, making them especially well suited for simulating time-domain wave propagation in heterogeneous biological media [14]. The use of low-storage Runge–Kutta time integrators further enables long-time simulations while preserving computational efficiency [31].

Anatomical accuracy is central to our modelling approach. Optical phenomena such as forward and multiple scattering, or subtle angular dependencies of the backscattered signal, can be strongly influenced by small geometric features. Errors in representing curved interfaces, such as the corneal surface or retinal layers, can lead to misleading conclusions. While high-order curved meshes offer one approach to improving geometric accuracy, they often introduce significant meshing and computational overhead. To address these challenges, we adopt a polynomial reconstruction approach that applies boundary conditions directly on smooth anatomical surfaces, maintaining high-order accuracy without requiring curved mesh elements. This enables precise modelling of the optical properties of the eye, including complex light paths and tissue-specific scattering behaviours.

In the following sections, we present an electromagnetic model of the human eye, considering both scattered-field and total-field configurations. We detail the DG spatial discretisation, the high-order time integration scheme, and the implementation of realistic anatomical geometries. Finally, we apply our methodology to three clinically motivated scenarios: changes in corneal transparency, light scattering in retinal layers affected by DME, and elastic-wave propagation relevant to optical coherence elastography. These case studies illustrate how high-order modelling can enhance the interpretation of OCT data, support early diagnosis, and contribute to a deeper understanding of disease mechanisms at the microstructural level.

2 Maxwell's equations

Maxwell's equations describe how electromagnetic waves propagate in a medium and are essential for modelling light transmission through the eye [18]. The electromagnetic field involves four vector fields: the electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, electric flux density $\mathbf{D}$, and magnetic flux density $\mathbf{B}$. Constitutive relations link these fields via $\mathbf{D} = \boldsymbol{\varepsilon}\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, where $\boldsymbol{\varepsilon}$ is the permittivity tensor and μ the permeability. In isotropic media, $\boldsymbol{\varepsilon} = \varepsilon I$ with scalar ε, and $\mu > 0$ is typically close to the vacuum permeability in biological tissues. These parameters determine the refractive index, wave speed $c = 1/\sqrt{\varepsilon\mu}$, and impedance $Z = \sqrt{\mu/\varepsilon}$, all critical for accurate modelling of light propagation through the eye's heterogeneous structures.

Assuming zero charge density and zero current density, and using the constitutive relations, Maxwell's equations in a source-free isotropic medium can be written as

$$\boldsymbol{\varepsilon}\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H}, \tag{1}$$

$$\mu\frac{\partial \mathbf{H}}{\partial t} = -\nabla \times \mathbf{E}, \tag{2}$$

where $\nabla\times$ denotes the curl operator, $\mathbf{E} = (E_x, E_y, E_z)^\top$ and $\mathbf{H} = (H_x, H_y, H_z)^\top$.

In the context of light propagation in the eye, the transverse electric (TE) mode of Maxwell's equations provides a simplified yet physically relevant representation of electromagnetic wave behaviour. In TE mode, the magnetic field is assumed to be polarised perpendicular to the plane of propagation (commonly taken as the xy-plane). So the electric field lies entirely in the transverse plane and has two components of interest, usually denoted by E_x and E_y, while the magnetic field has a non-zero component in the direction of propagation, usually denoted by H_z. This simplification suits many biological optics applications, including the eye, where light is often linearly or partially polarised with the electric field primarily oriented in the plane defined by ocular structures. The properties of tissues such as the corneal stroma and retina strongly influence these in-plane fields [21]. The TE mode also enables realistic simulation of polarisation-dependent phenomena, such as scattering and absorption, which are crucial for advanced imaging techniques like optical coherence tomography and polarimetry [12]. From a computational standpoint, the TE mode reduces Maxwell's equations to a coupled system involving only three components, namely E_x, E_y and H_z, facilitating efficient numerical implementations. In a 2D TE mode setting, time-domain Maxwell's equations (1)–(2) can be expressed as

$$\varepsilon\frac{\partial E_x}{\partial t} = \frac{\partial H_z}{\partial y}, \tag{3}$$

$$\varepsilon\frac{\partial E_y}{\partial t} = -\frac{\partial H_z}{\partial x}, \tag{4}$$

$$\mu\frac{\partial H_z}{\partial t} = -\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right). \tag{5}$$

Thus, the TE mode offers an effective balance between physical realism and numerical tractability for modelling light propagation in the eye. It accurately represents the in-plane electric field interactions with ocular tissues while simplifying the electromagnetic problem, making it a preferred choice in many optical analyses of the eye.

A comprehensive description of an electromagnetic problem should include both the governing differential equations and the corresponding boundary conditions. Since simulations are performed within a finite domain Ω with boundary $\partial\Omega$, it is essential to implement boundary conditions that accurately represent the behaviour of waves exiting the domain. Silver–Müller absorbing boundary conditions simulate open boundaries that allow outgoing waves to exit the domain without reflection. On a boundary with outward unit normal vector $\mathbf{n} = [n_x, n_y]^\top$, they are given by

$$\mathbf{n} \times \mathbf{E} = Z\mathbf{n} \times (\mathbf{H} \times \mathbf{n}) \quad \text{on } \partial\Omega.$$

These conditions are designed to replicate the effect of an open domain by minimising spurious reflections at the computational boundary. Derived from plane wave solutions of Maxwell's equations, they are particularly well suited for problems in which electromagnetic waves radiate outward from the domain [18].

3 Discontinuous Galerkin method

Discontinuous Galerkin (DG) methods are a class of finite element methods that combine the advantages of finite volume and finite element schemes [14]. They are particularly suitable for solving hyperbolic partial differential equations, such as Maxwell's equations, because of their ability to handle complex geometries, adaptivity, and high-order accuracy.

Let $\mathbf{U} = (E_x, E_y, H_z)^\top$ denote the vector of unknowns. The system (3)–(5) can be written in conservation form

$$\mathbf{Q}\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathbf{F}(\mathbf{U}) = \mathbf{0}, \tag{6}$$

where $\nabla\cdot$ denotes the divergence operator, and the flux tensor $\mathbf{F}(\mathbf{U})$ and the material matrix $\mathbf{Q}$ are defined by

$$\mathbf{F}(\mathbf{U}) = \begin{bmatrix} 0 & -H_z \\ H_z & 0 \\ E_y & -E_x \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} \boldsymbol{\varepsilon} & \mathbf{0} \\ \mathbf{0} & \mu \end{bmatrix}.$$

The domain Ω is meshed with K non-overlapping elements T^k, $k = 1, \dots, K$, leading to an approximate computational domain $\Omega_h = \bigcup_{k=1}^K T^k$. We seek a weak solution $\mathbf{U}_h$ by requiring the residuals to vanish in the following way

$$\int_{\Omega_h} \mathbf{Q}\frac{\partial \mathbf{U}_h}{\partial t} \cdot \phi \, d\mathbf{x} + \int_{\Omega_h} \nabla \cdot \mathbf{F}(\mathbf{U}_h) \cdot \phi \, d\mathbf{x} = 0,$$

where ϕ is a test function from a discontinuous (piecewise polynomial) finite element space. Next, we employ one integration by parts and replace the flux $\mathbf{F}$ by a numerical flux $\hat{\mathbf{F}}$. Applying integration by parts again yields

$$\sum_{k=1}^{K} \int_{T^k} \Big(\mathbf{Q} \frac{\partial \mathbf{U}_h}{\partial t} \, d\mathbf{x} + \nabla \cdot \mathbf{F}(\mathbf{U}_h) \Big) \cdot \phi \, d\mathbf{x}$$
$$= \sum_{k=1}^{K} \int_{\partial T^k} \mathbf{n} \cdot (\mathbf{F} - \hat{\mathbf{F}}) \cdot \phi \, ds,$$

where $\mathbf{n}$ is the outward unit normal vector of the contour.

Numerical fluxes are used in DG methods to manage discontinuities at element interfaces. A common choice is the upwind flux, which incorporates the wave propagation direction and ensures numerical stability. Alternatively, central fluxes with penalty terms may be employed. Introducing the notation for jumps of fields $[\![u]\!] = u^- - u^+$, where $-$ refers to the local cell and $+$ refers to the neighbouring one, we define the upwind flux as [14, 19]

$$\mathbf{n} \cdot (\mathbf{F} - \hat{\mathbf{F}}) = \begin{pmatrix} \frac{-n_y}{Z^+ + Z^-} (Z^+ [\![H_{hz}]\!] - (n_x [\![E_{hy}]\!] - n_y [\![E_{hx}]\!])) \\ \frac{n_x}{Z^+ + Z^-} (Z^+ [\![H_{hz}]\!] - (n_x [\![E_{hy}]\!] - n_y [\![E_{hx}]\!])) \\ \frac{1}{Y^+ + Y^-} (Y^+ (n_x [\![E_{hy}]\!] - n_y [\![E_{hx}]\!]) - [\![H_{hz}]\!]) \end{pmatrix},$$

with impedance $Z^{\pm} = \sqrt{\mu^{\pm}/\varepsilon^{\pm}}$ and conductance $Y^{\pm} = 1/Z^{\pm}$. These choices ensure that energy transmission and reflection at interfaces are physically accurate. These flux expressions help maintain consistency, stability, and physical fidelity of the numerical scheme. For Silver–Müller absorbing boundary conditions, we consider at the outer boundary

$$E_{hy}^{+} n_x - E_{hx}^{+} n_y = Z^- H_{hz}^{+},$$

which is equivalent to considering $[\![E_{hx}]\!] = E_{hx}^{-}$, $[\![E_{hy}]\!] = E_{hy}^{-}$ and $[\![H_{hz}]\!] = H_{hz}^{-}$. To complete the evaluation of fluxes at boundary edges, we mention that the material properties at the boundary are set as $Z^+ = Z^-$ and $Y^+ = Y^-$.

4 Time integration

The system of ordinary differential equations resulting from the spatial discretisation of Maxwell's equations using the DG method takes on the form

$$\frac{d\mathbf{U_h}}{dt} = \mathcal{L}_h(\mathbf{U_h}),$$

where $\mathcal{L}_h(\mathbf{U_h})$ includes the contributions from element integrals, numerical fluxes and the material matrix.

Explicit low-storage Runge–Kutta (LSRK) methods are particularly well suited for time integration in wave propagation problems due to their balance of efficiency, accuracy, and stability. Unlike standard Runge–Kutta schemes that require multiple storage vectors for intermediate stages, LSRK methods reuse just a few vectors – typically two – through recursive updates. This leads to significantly reduced memory usage, an important feature in large-scale simulations.

LSRK methods can achieve high-order temporal accuracy, typically fourth or fifth order, while maintaining favourable stability properties for hyperbolic systems. Their explicit formulation makes them easy to implement and naturally suited for parallel computation. Although there are several ways to implement LSRK methods, we adopt the formulation introduced by Williamson [31], which uses only two vectors – one for the solution and one for the residual – and is well suited to large-scale DG simulations. The method proceeds as follows:

$$\begin{aligned} \mathbf{U}^{(0)} &= \mathbf{U}_h^n, \quad \mathbf{R}^{(0)} = \mathbf{0}, \\ \mathbf{R}^{(i)} &= \alpha_i \mathbf{R}^{(i-1)} + \Delta t \, \mathcal{L}_h(\mathbf{U}^{(i-1)}), \\ \mathbf{U}^{(i)} &= \mathbf{U}^{(i-1)} + \beta_i \mathbf{R}^{(i)}, \quad i = 1, \dots, s, \\ \mathbf{U}_h^{n+1} &= \mathbf{U}^{(s)}, \end{aligned}$$

where α_i and β_i are method-specific coefficients chosen to achieve high-order accuracy and favourable stability properties, Δt is the time step, and s is the number of stages. The method reuses the vectors $\mathbf{U}$ and $\mathbf{R}$ at each stage, thus achieving low memory consumption. In this work, we consider the L(14,4) scheme by Niegemann, Diehl, and Busch [22], a method of 14 stages and order of accuracy 4. These methods were designed to minimise memory usage while maximising stability, making them particularly well suited for high-order spatial discretisations such as discontinuous Galerkin methods applied to hyperbolic problems.

An alternative to LSRK methods for time integration of Maxwell's equations is the Leapfrog scheme, which is commonly used due to its explicitness, simplicity, and time centred structure that matches the staggered nature of Maxwell's equations [3, 4]. In the Leapfrog scheme, the electric field $\mathbf{E} = (E_x, E_y)$ and magnetic field H_z are updated at staggered time levels. The electric fields E_x, E_y are computed at integer time steps $t^n = n\Delta t$ and the magnetic field H_z is computed at half-time steps $t^{n+1/2} = (n + 1/2)\Delta t$. This time-centred structure is second-order accurate and conditionally stable under a CFL condition, which imposes a bound on the time step Δt depending on the mesh size h and the wave speed. Theoretical results on stability and convergence may be found in [3, 4]. The Leapfrog method is particularly attractive for long-time simulations due to its low computational cost and symplectic nature, preserving energy in non-dissipative systems. However, it requires care when dealing with complex geometries, boundary conditions, and variable material properties, especially when combined with DG spatial discretisations.

5 Case studies

Early detection of eye diseases is essential for effective treatment and preserving vision. Conditions such as corneal disorders and diabetic macular edema often involve subtle alterations of the microstructure of the tissue. Optical coherence tomography (OCT) and optical coherence elastography (OCE) are two non-invasive imaging techniques used in ophthalmology to detect abnormalities in the eye. This section presents three case studies that perform simulations based on Maxwell's equations to model the effect of these changes on OCT/OCE signals.

5.1 Corneal transparency problem

The cornea is the transparent outer layer of the eye, and it plays a key role in directing and focusing light toward the retina. It is composed of five layers: the epithelium, Bowman's layer, the stroma, Descemet's membrane, and the endothelium (see Figure 1). Under healthy conditions, these layers remain thin and well structured, minimising light scattering. However, pathological modifications to any of these layers can disrupt this structure, leading to increased light scattering and a loss of corneal transparency [20]. Among the five layers, the stroma corresponds to nearly 90% of the total corneal thickness and is mainly composed of collagen fibrils with uniform diameters that are further gathered into collagen lamellae. Among other functions, these collagen fibrils are responsible for maintaining the structural regularity that supports transparency [21]. Specifically, what is maintained is the uniformity of the diameters of the collagen fibrils and the distances between adjacent collagen fibrils. Alterations in either of these properties may result in increased light scattering, compromising corneal transparency.

Corneal opacity problem is a common issue in vision diseases, particularly for ageing population, while decreased visual acuity is associated with a reduced quality of life and life expectancy. Diagnostic is based on cornea analysis using OCT.

Ageing of the ocular surface and corneal tissues causes major eye diseases and results in substantial costs in both medical and social terms. Furthermore, often corneal edema occurs after surgery for treating cataracts is performed. Understanding the mechanisms of transparency loss requires understanding the structural bases of corneal transparency itself, which ensure minimal scattering of visible light. Scattering takes place when an incident light wave encounters fluctuations in the refractive index of a material, characterised by the matrix of collagen fibrils. Modelling and simulating the cornea transparency loss is a critical digital tool to measure and prevent possible illnesses.

Results in [2] illustrate that an increase in the diameter of some fibrils causes an increase in backscattering. Therein, the problem was studied considering a two-dimensional model of backscattered light in two different scenarios (healthy and pathological).

To simulate light scattering in the cornea, the scattered field formulation for the transverse electric (TE) mode was adopted. This involves decomposing the total electromagnetic field into two components: the incident field $\boldsymbol{U}^{\mathrm{i}} = (E_x^{\mathrm{i}}, E_y^{\mathrm{i}}, H_z^{\mathrm{i}})^{\top}$, which represents the wave propagating in the absence of scatterers, and the scattered field $\boldsymbol{U}^{\mathrm{s}} = (E_x^{\mathrm{s}}, E_y^{\mathrm{s}}, H_z^{\mathrm{s}})^{\top}$, which accounts for the perturbation caused by the inhomogeneities. Thus, the total field is expressed as the sum of the incident and scattered components.

Assuming the incident field satisfies the Maxwell equations (3)–(5) with relative permittivity ε_0 and permeability $\mu_0 = \mu$ corresponding to the background (scatterer-free) medium, we substitute this decomposition into equation (3)–(5). This leads to the scattered field formulation, which isolates the effect of the scattering medium on the wave propagation:

$$\varepsilon \frac{\partial E_x^{\mathrm{s}}}{\partial t} = \frac{\partial H_z^{\mathrm{s}}}{\partial y} + (\varepsilon_0 - \varepsilon)\frac{\partial E_x^{\mathrm{i}}}{\partial t}, \tag{7}$$

$$\varepsilon \frac{\partial E_y^{\mathrm{s}}}{\partial t} = -\frac{\partial H_z^{\mathrm{s}}}{\partial x} + (\varepsilon_0 - \varepsilon)\frac{\partial E_y^{\mathrm{i}}}{\partial t}, \tag{8}$$

$$\mu \frac{\partial H_z^{\mathrm{s}}}{\partial t} = \frac{\partial E_x^{\mathrm{s}}}{\partial y} - \frac{\partial E_y^{\mathrm{s}}}{\partial x}. \tag{9}$$

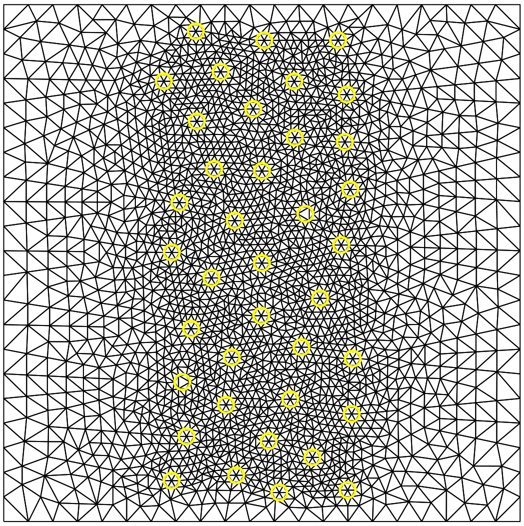
For a computational (dimensionless) domain we took $\Omega_h = [-1, 1]^2$ with circles that stand for the collagen fibrils of the cornea. In the healthy scenario, we considered that the diameter of each fibril is 31 nm. In the pathological situation, the positions of the fibrils were kept and eight fibrils were randomly chosen to have doubled diameter. For the numerical simulations, the magnetic permeability was set to $\mu = 1$, the electric permittivity of the free space to $\varepsilon_0 = 1.365^2$ and ε is such that

$$\varepsilon(x,y) = \begin{cases} 1.411^2, & (x,y) \in \mathcal{F},\ (x,y) \in \mathcal{F}', \\ 1.365^2, & (x,y) \in \Omega \setminus \mathcal{F},\ (x,y) \in \Omega \setminus \mathcal{F}', \end{cases}$$

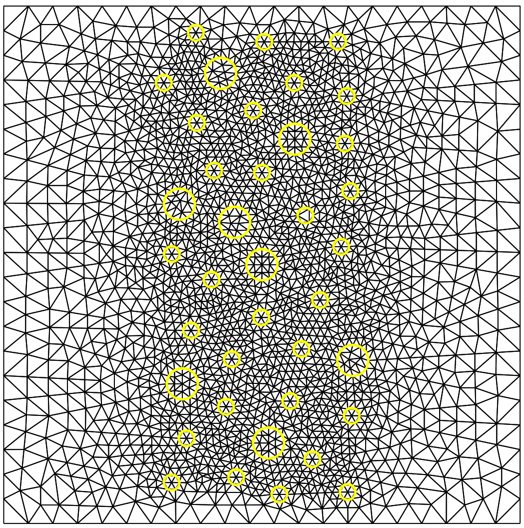
where $\mathcal{F}$ denotes the union of circles that model healthy collagen fibrils and $\mathcal{F}'$ represents the pathological situation. The scattered field formulation is completed with Silver–Müller absorbing boundary conditions and initial conditions defined by $\boldsymbol{U}^{\mathrm{s}}(x,y,0) = 0$ and $\boldsymbol{U}^{\mathrm{i}}(x,y,0) = (0, \cos(10(x-t)), 0)^{\top}$.

The spatial discretisation is done on meshes defined in Figure 4 using the DG method. Note that, when considering the scattered field formulation (7)–(9), the conservation form (6) would have an additional term $\mathbf{G}(\mathbf{U}^{\mathrm{i}})$ that results from the incident field. The low-storage explicit Runge–Kutta method, L(14,4), is applied for time integration with step $\Delta t = 10^{-3}$, respecting the stability constrains [5]. The intensity of the scattered electric field, $I^{\mathrm{s}} = \sqrt{(E_x^{\mathrm{s}})^2 + (E_y^{\mathrm{s}})^2}$, is represented in Figure 5 where the solution was approximated by polynomials of order $N = 4$ and plotted for different values of simulation time T.

As one can see in Figure 5, the planar characteristic of the wavefront is significantly lost in the situation where we double the diameter of 20% of the stromal fibres. One can also observe an increase in backscattering in the case where the organisation of the fibrils is not uniform, which subsequently leads to corneal swelling and a loss of transparency, as predicted in [21].

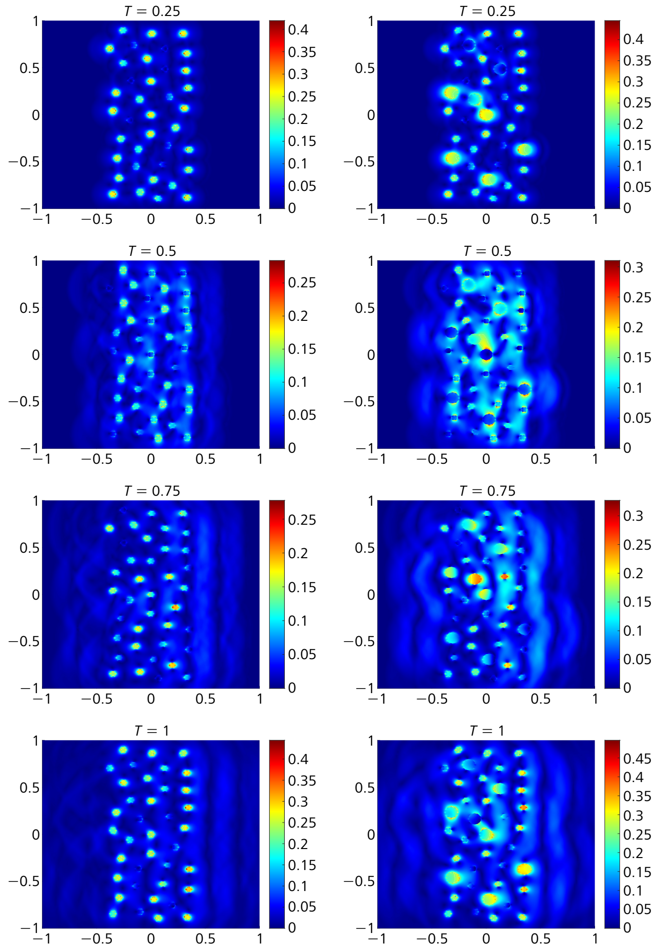


(a) Healthy scenario.



(b) Pathological scenario.

Figure 4. Spatial meshes for the healthy and pathological scenarios. The mesh in (a) is composed of $K = 5072$ elements, and the mesh in (b) counts $K = 4972$ elements [2].



(a) Healthy cornea. (b) Pathological cornea.

Figure 5. Scattered electric field intensity [2].

5.2 *Simulation of diabetic macular edema changes on optical coherence tomography data*

Diabetes mellitus is one of the most prevalent diseases in developed countries. According to the World Health Organization (WHO), approximately 74 million adults in the WHO European Region are living with diabetes, with prevalence rates of 11.9% among men and 10.9% among women [32]. This marks a significant increase from earlier estimates and underscores the growing public health challenges posed by diabetes in Europe. Projections indicate that by 2045 one in ten adults in the region will live with diabetes [16].

A major complication in diabetes is diabetic macular edema, one of the leading causes of visual impairment among diabetic patients [8]. DME is defined as an increase in retinal thickness due to fluid accumulation, which may occur intra- or extracellularly. In intracellular edema, cells retain excess fluid, becoming enlarged, whereas extracellular edema results from the accumulation of fluid outside cells, often due to a breakdown in the blood-retinal barrier [10]. Differentiating the type and severity of DME in its early stages is often challenging.

OCT is a widely used imaging technique that provides high-resolution views of retinal structure in vivo (Figure 6), making it essential for diagnosing and monitoring DME. However, standard OCT lacks the ability to directly capture microscopic changes at the cellular level, particularly the morphological alterations linked to intracellular and extracellular edema. Understanding how these microscopic features influence the macroscopic OCT signal is therefore crucial. To address this, Correia et al. [9] developed a hybrid

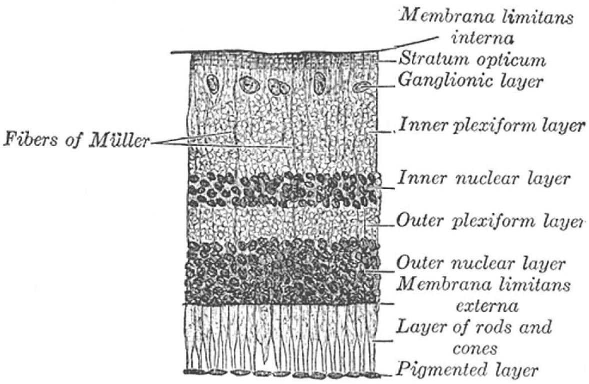


Figure 6. Section of the retina. From Henry Gray: Anatomy of the Human Body (Philadelphia: Lea & Febiger, 1918). (In public domain at bartleby.com.[1])

simulation framework that integrates physical models of light propagation with biologically-informed retinal representations, focusing on a 3D optical model of the outer nuclear layer (ONL), a region consistently affected in DME and amenable to modelling via spherical scatterers.

Various methods have been proposed to describe the interaction of light with retinal tissues. Most are based on single-scattering theory [28], which is insufficient to fully capture the structural complexity of the retina. The Mie solution to the equations in question is among the most widely used techniques for modelling tissue scattering at the cellular level [17]. However, Mie theory is limited to scattering by a single homogeneous sphere, restricting its applicability to scatterers of different shapes or aggregates. This limitation is partially overcome by the generalized multiparticle Mie (GMM) theory introduced in [33], which extends Mie's solution to account for multiple scattering in aggregates of spheres and enables more accurate modelling of biological tissue. Nevertheless, GMM is still confined to spherical structures.

More complex models need to be used when considering scatterers of arbitrary shapes. The finite-difference time-domain method is a numerical technique used to solve the Maxwell equations in the time domain that has been applied to a wide range of electromagnetic problems, including light scattering from biological cells [11,30]. In [25], Maxwell's equations were solved on a small 3D ONL domain using the DG method, coupled with a fourth-order, 14-stage low-storage Runge–Kutta integrator. Silver–Müller boundary conditions were applied to suppress spurious reflections. The model was validated against Mie's theory using identical parameters, showing good agreement with errors of 0.37% for the scattering anisotropy and 0.06% for the scattering cross-section [25].

An algorithm to simulate A-scans was also developed. A sinusoidal plane wave propagating in the z-direction excites the domain, and the DG model computes fields around a single spherical scatterer representing an ONL nucleus. Anisotropy (g) and scattering cross-section (σ_s) are obtained from the far-field scattering pattern in spherical coordinates. A Monte Carlo simulation then launches 10^9 photons, simulating distances between interactions (using σ_s) and angular deflections (using g). Photon paths reaching a 15 µm radius detector within a 5° acceptance cone are recorded. Comparing simulated and experimental A-scans allows identification of parameter sets that best reproduce the observed data.

The tool was applied to OCT scans (Cirrus HD-OCT, Carl Zeiss Meditec) collected from healthy subjects and two DME patient groups: DME I (increased ONL thickness) and DME II (no apparent ONL change). Literature-based parameters (Table 1) were used for healthy tissue, while parameters d (nucleus diameter in DME I) and ρ (nucleus density in DME II) were adjusted to fit the measured A-scans [9].

Group	Nuclei diameter (µm)	Nuclei density (nuclei/µm^3)	Nuclei RI (at 870 nm)	Medium RI (at 870 nm)
Healthy	7.0	0.002	1.39	1.35
DME I	d	0.002	1.39	1.35
DME II	7.0	ρ	1.39	1.35

Table 1. Properties of the medium used in Monte Carlo simulations [9].

For each group, the ONL was segmented and B-scans were aligned to the upper ONL boundary. Mean B-scans and A-scans were computed and normalised to the maximum intensity at the retinal pigmented epithelium. Final A-scans were cropped to the minimum ONL thickness observed to ensure consistency (Figure 7).

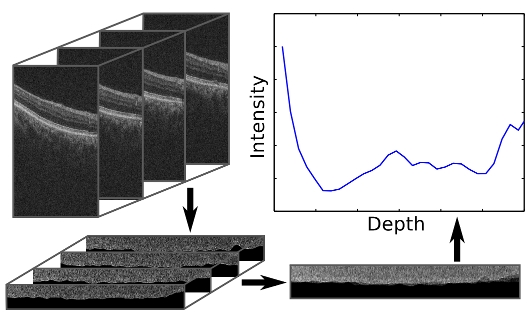


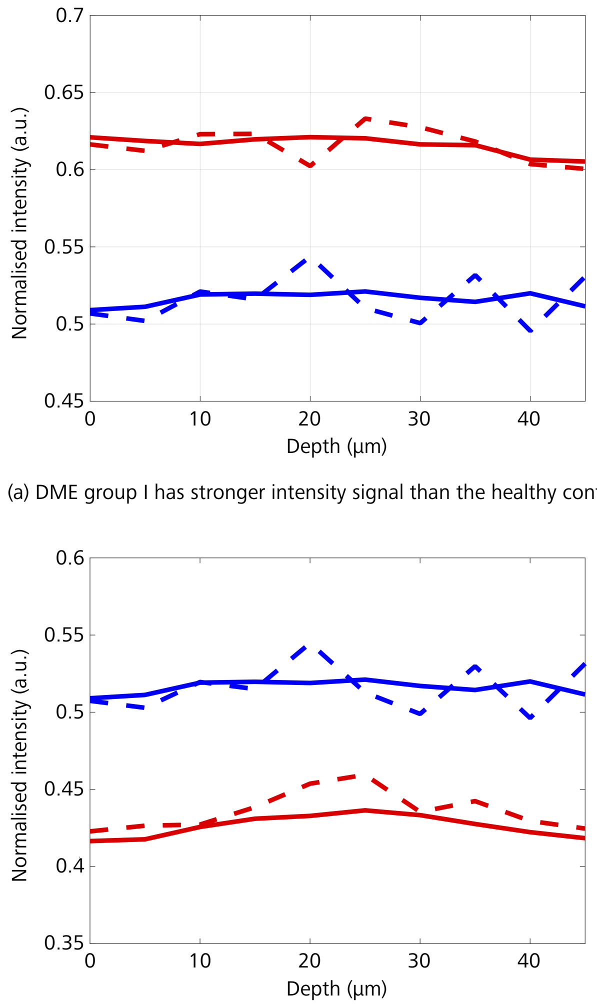
Figure 7. OCT processing algorithm (counterclockwise from upper left corner). The ONL of a group's B-scans are segmented and aligned to their upper boundary. The segmented B-scans are averaged to produce a demonstrative B-scan. The A-scans are averaged to obtain an A-scan that fully describes the group [9].

[1] https://www.bartleby.com/lit-hub/anatomy-of-the-human-body

It was observed that DME group I (Figure 8(a)) consistently exhibits a stronger intensity signal compared to healthy controls. To replicate this condition in simulation, it was necessary to increase the nuclei diameter by 14% relative to the healthy state. In [9], the data for DME group I were successfully reproduced by increasing the nuclear radius d from 7.0 μm to 8.0 μm, while keeping all other parameters unchanged.

In contrast, DME group II (Figure 8(b)) shows a reduced backscattered signal compared to healthy tissue, which is consistent with theoretical expectations. A thicker layer containing the same number of nuclei should result in a lower backscattering signal due to the reduced density of scatterers. In [9], this behaviour was replicated by reducing the simulated nuclei density by 40%. Specifically, the density ρ was decreased from 0.002 μm^{-3} to 0.0012 μm^{-3}, based on the assumption that the number of nuclei remains constant while the volume of the ONL increases. This adjustment reflects the observed ONL thickness increase from 70 μm to 117.2 μm.

The results reported in [9] support the hypothesis that the two types of edema, cytotoxic (intracellular) and vasogenic (extracellular), can be distinguished through OCT signal characteristics. In the case of DME group I, the increased OCT signal is best explained by an enlargement of the nuclei, which is compatible with intracellular swelling. While alternative mechanisms, such as changes in nuclear radius or the refractive index of the surrounding medium, could also affect the signal, this preliminary study restricted the analysis to physiological alterations consistent with known biological behaviour.



(a) DME group I has stronger intensity signal than the healthy controls.

(b) DME group II shows reduced signal compared to the healthy status.

Figure 8. Comparison between simulated A-scans (solid lines) and real A-scans, both normalised, for healthy controls (blue) and DME patients (red) [9].

5.3 Elastography problem in the human retina

Understanding the mechanical properties of ocular tissues is essential in the diagnosis of retinal and corneal pathologies. Optical coherence elastography (OCE) is a promising imaging technique that combines high-resolution structural imaging with mechanical characterisation, providing valuable insights into tissue elasticity [23]. There are different implementations of OCE, varying in the type of mechanical loading of the tissue. Using a piezoelectric actuator to induce tissue displacements is one of many possible techniques.

In OCE, waves from a source propagate through the eye, including the cornea, the lens, and the vitreous humour, to reach the retina (see Figure 1). When the acoustic pressure interacts with the retina, it generates an elastic wave within the tissue. This elastic wave then propagates through the retina, potentially causing measurable displacements and providing information about the mechanical properties of the tissue. By analysing how tissues respond to acoustic excitations, it is possible to non-invasively infer variations of mechanical properties that often indicate disease. In particular, the response of the retina to these excitations is of primary clinical interest, and the accurate modelling of wave transmission through the anterior layers is essential for a correct interpretation. An accurate interpretation of the measured displacements requires a reliable numerical simulation of the wave propagation through layered, curved, and heterogeneous ocular structures.

In [6], the eye is described as a narrow cylindrical layered domain, where each layer Ω_j, $j = 1, \ldots, n$, represents a different ocular medium with distinct acoustic or elastic properties. The upper layer Ω_n corresponds to the retina and is modelled using an elastic wave equation, and the other layers are governed by acoustic wave propagation.

For the case of a piezoelectric actuator, electromagnetic fields induce mechanical displacements. When considering a time-harmonic emission in the source, the governing PDEs simplify significantly.

Let us now consider the mechanical deformation induced by the piezoelectric actuator, via the piezoelectric coupling

$$-\omega^2 \rho \mathbf{u} = \nabla \cdot (\mathbf{c} \cdot \mathbf{S} - \mathbf{e}^\top \cdot \mathbf{E}),$$

where ω denotes the frequency of the wave, ρ is the density of the medium, $\mathbf{u}$ denotes the mechanical displacement, $\mathbf{c}$ is the stiffness tensor, $\mathbf{S} = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^\top)$ is the strain tensor, and $\mathbf{e}$ is the piezoelectric coupling tensor.

The time-harmonic acoustic pressure p_j in the layer Ω_j satisfies the Helmholtz equation

$$\Delta p_j + k_j^2 p_j = 0, \quad \text{in } \Omega_j, \quad j = 1, \dots, n-1,$$

where $k_j = \omega / c_j$ is the wavenumber, dependent on the frequency ω and wave speed c_j in the layer Ω_j. For the first interface, $\partial\Omega_0$, one prescribes the boundary condition

$$\frac{1}{\rho_1} \frac{\partial p_1}{\partial \nu} = \omega^2 \mathbf{u} \cdot \nu_1, \quad \text{on } \partial\Omega_0.$$

Between layers, one prescribes the relations

$$p_j = p_{j+1}, \quad \text{on } \partial\Omega_j, \qquad \rho_j \frac{\partial p_j}{\partial \nu_j} = \rho_{j+1} \frac{\partial p_{j+1}}{\partial \nu_{j+1}}, \quad \text{on } \partial\Omega_j,$$

where ρ_j is the density of the layer Ω_j and ν_j is the outward unit normal to $\partial\Omega_j$.

The Lamé equation describes the elastic displacement field $\mathbf{u}$ in the retina:

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}) + \omega^2 \rho \mathbf{u} = 0, \quad \text{in } \Omega_n,$$

where ρ denote the density in the retina, and the Lamé constants are given by

$$\mu = \frac{E}{2(1+\upsilon)}, \qquad \lambda = \frac{\upsilon E}{(1+\upsilon)(1-2\upsilon)},$$

with E the Young modulus and υ the Poisson's ratio. The acoustic-elastic transmission condition is given by

$$\frac{1}{\rho_n} \frac{\partial p_{n-1}}{\partial \nu_n} = \omega^2 \mathbf{u} \cdot \nu_n, \quad \text{on } \partial\Omega_{n-1},$$
$$p_{n-1} \nu_{n-1} = \sigma(\mathbf{u}) \nu_n, \quad \text{on } \partial\Omega_{n-1},$$

where the stress tensor is given by

$$\sigma(\mathbf{u}) = \mu(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top) + \lambda \nabla \cdot \mathbf{u} I.$$

In [6], the method of fundamental solutions (MFS) was employed to approximate the solution in each layer, assuming homogeneity in each layer. The method was tested using physical parameters representative for the human eye and the application of the MFS to simulate the process of elastography seems feasible, even in the presence of high frequencies.

For the case of heterogeneous layers, the MFS is no longer suitable (because the fundamental solutions are not available) and the use of other types of methods, such as the DG method, is required.

Although layered models with planar interfaces simplify the geometry, real biological structures, such as the cornea and retina, have curved geometries. To obtain a model that closely resembles a real scenario, it is essential to capture these anatomical features. However, dealing with domains with curved boundary presents additional numerical challenges. It is well known that meshing curved domains using polygonal elements introduces geometric mismatches along the boundary. This discrepancy between $\partial\Omega$ and $\partial\Omega_h$ can significantly reduce the accuracy of numerical schemes. In particular, standard finite element or discontinuous Galerkin methods will be at most second-order accurate, regardless of the polynomial degree used for the numerical solution, unless specialised techniques are employed to recover the optimal rate.

To address this issue, we propose a method called DG-ROD (reconstruction for off-site data), presented in [26]. This approach is based on specific polynomial reconstructions constrained for the prescribed boundary conditions on the physical boundary $\partial\Omega$. The method recovers the optimal order of convergence of the DG method without relying on curved meshes to approximate the physical domain Ω. The overall DG-ROD method is based on an iterative procedure that alternates between a classical DG solver and a polynomial reconstruction step. In each iteration, a polynomial reconstruction is performed to improve the accuracy of the solution near the curved boundary. More specifically, for each boundary element, a new polynomial is computed such that it is the closest polynomial to the numerical DG solution on the computational domain, and the new polynomial exactly satisfies the prescribed boundary condition at a set of R points on the physical boundary. This new polynomial is then used to correct the boundary condition imposed on the computational polygonal boundary. The process is repeated until convergence is achieved.

To illustrate the ideas and validate the approach, consider the 2D Helmholtz equation $\Delta u + k^2 u = 0$ on a curved strip domain given by

$$\Omega = \{(x,y) : -1 < x < 1,\ h_2(x) < y < h_1(x)\},$$

such that the solution satisfies a Neumann condition on the upper and lower parts of the boundary and a Dirichlet condition on the left and right parts of the boundary. In this benchmark, we take $h_1(x) = \sqrt{1 + 2\log(\cosh(x))}$ and $h_2(x) = h_1(x) - 0.2$. Consider the fundamental solution for the Helmholtz operator, with $k = 1$, which is given by the Hankel function in 2D, $u(r) = \frac{\mathrm{i}}{4} H_0^{(1)}(r)$, with $r = |\mathbf{x}|$ and i the imaginary unit [1]. Simulations are carried out with successively finer meshes generated by Gmsh (version 4.6.0) [13] (see Figure 9). The fixed point iterative procedure for the DG-ROD method stops when either the tolerance for the residual or the maximum number of iterations is reached. In order to determine the set of points used in the polynomial reconstructions, we consider the vertical projection of the nodal points located on the upper and lower computational boundaries onto the physical boundary.

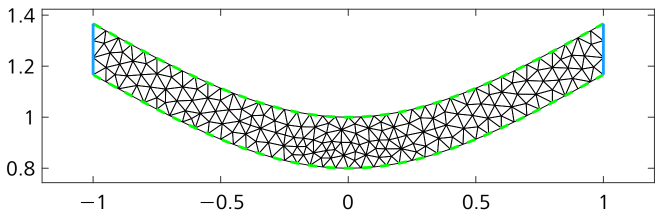


Figure 9. Unstructured mesh generated for the curved strip domain with Dirichlet (blue solid line) and Neumann boundary conditions (green dashed line).

Let u be the exact solution and u_h be the DG solution for a given mesh $\mathcal{T}_h$, and let $E_2(\mathcal{T}_h) = \|u - u_h\|_{L^2(\mathcal{T}_h)}$ be the L^2-norm of the error. The method has order of convergence p if asymptotically

$$E_2(\mathcal{T}_h) \leq Ch^p,$$

with C a real constant independent of h. The L^2-errors are assessed at the node points of the elements of the mesh. Let $\mathcal{T}_{h_1}$ and $\mathcal{T}_{h_2}$ be two different meshes, with different mesh sizes h_1 and h_2, respectively. Then, the order of convergence between two successively finer meshes is determined as

$$O_2(\mathcal{T}_{h_1}, \mathcal{T}_{h_2}) = \frac{\log(E_2(\mathcal{T}_{h_1})/E_2(\mathcal{T}_{h_2}))}{\log(h_1/h_2)}.$$

Note that each node on the computational boundary has a corresponding node on the real boundary where the boundary condition is prescribed. For the classical DG method, the value evaluated at the physical boundary point is used at the corresponding node on the computational boundary. The results for the classical DG method, reported in Table 2, demonstrate the deterioration of accuracy from the geometrical mismatch without any specific treatment for curved boundaries, and the error convergence is limited to the second-order. On the other hand, Table 3 reports the errors and convergence orders for the DG-ROD method. As observed, the convergence orders improve according to the polynomial degree N and the number of points used in the polynomial reconstruction in each element with a boundary edge. More precisely, this improvement occurs when the relation $R = N + 1$ is satisfied.

6 Conclusion

This review surveys a computational framework for modelling light propagation in the human eye using Maxwell's equations. The study addressed how structural changes in the cornea and retina influence light scattering and transparency, which are crucial for vision quality. Case studies explored clinically relevant scenarios, including corneal opacity, diabetic macular edema, and retinal elasticity. In each case, the numerical simulations provided insight into the structural changes, thereby facilitating early diagnosis.

K	h	$N = 2$		$N = 3$		$N = 4$	
		E_2	O_2	E_2	O_2	E_2	O_2
20	2.69E−01	9.79E−05	–	1.20E−04	–	1.26E−04	–
80	1.46E−01	2.55E−05	2.2	2.78E−05	2.4	2.85E−05	2.4
342	7.51E−02	6.42E−04	2.1	6.72E−06	2.1	6.79E−06	2.2
1454	3.94E−02	1.61E−06	2.1	1.65E−06	2.2	1.65E−06	2.2
5966	2.06E−02	4.02E−07	2.1	4.07E−07	2.2	4.08E−07	2.2

Table 2. Errors and convergence orders for the classical DG method in the curved strip domain.

K	h	$N = 2,\ R = 3$		$N = 3,\ R = 4$		$N = 4,\ R = 5$	
		E_2	O_2	E_2	O_2	E_2	O_2
20	2.69E−01	1.65E−04	–	2.85E−07	–	5.41E−08	–
80	1.46E−01	3.82E−05	2.4	3.81E−08	3.3	2.60E−09	5.0
342	7.51E−02	4.23E−06	3.3	2.31E−09	4.2	5.70E−11	5.8
1454	3.94E−02	8.31E−07	2.5	1.24E−10	4.5	–	–
5966	2.06E−02	1.79E−07	2.4	–	–	–	–

Table 3. Errors and convergence orders for the DG-ROD method in the curved strip domain.

Domains with curved boundary arise naturally in this problem setting due to the geometry of optical structures, such as the cornea and retina, which exhibit a curved layered composition. It is well known that curved domains require an accurate boundary representation to avoid a reduction of the order of convergence of numerical schemes. We considered a high-order discontinuous Galerkin method combined with a polynomial reconstruction technique. This approach enables an appropriate enforcement of boundary conditions without relying on curved meshes, preserving both computational efficiency and high-order accuracy.

Integrating accurate physical models with high-order numerical methods is a promising approach to simulate light propagation in the human eye. Future work will involve applying the method to curved interfaces within a layered medium in order to mimic the anatomy of the eye.

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[2] https://doi.org/10.54499/UI/BD/153816/2022

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Échos de la pensée: reconciling art and maths

Louis-Hadrien Robert and Paul Turner

Reflecting on our personal experience making the multimedia installation Échos de la pensée*, we discuss how the demands of artistic creation are not necessarily aligned with those of mathematical communication and attempt to dig a little into the reasons why.*

> *I once saw a wonderful painting of an owl by Picasso. Today, I suppose, an artist might just stuff the bird and put it in a case… But Picasso's owl is an account of a human being looking at an owl, which is a lot more interesting than a preserved specimen.*
> – David Hockney [4]

> *Créer le navire ce n'est point tisser les toiles, forger les clous, lire les astres, mais bien donner le goût de la mer…*
> – Antoine de Saint-Exupéry [5]

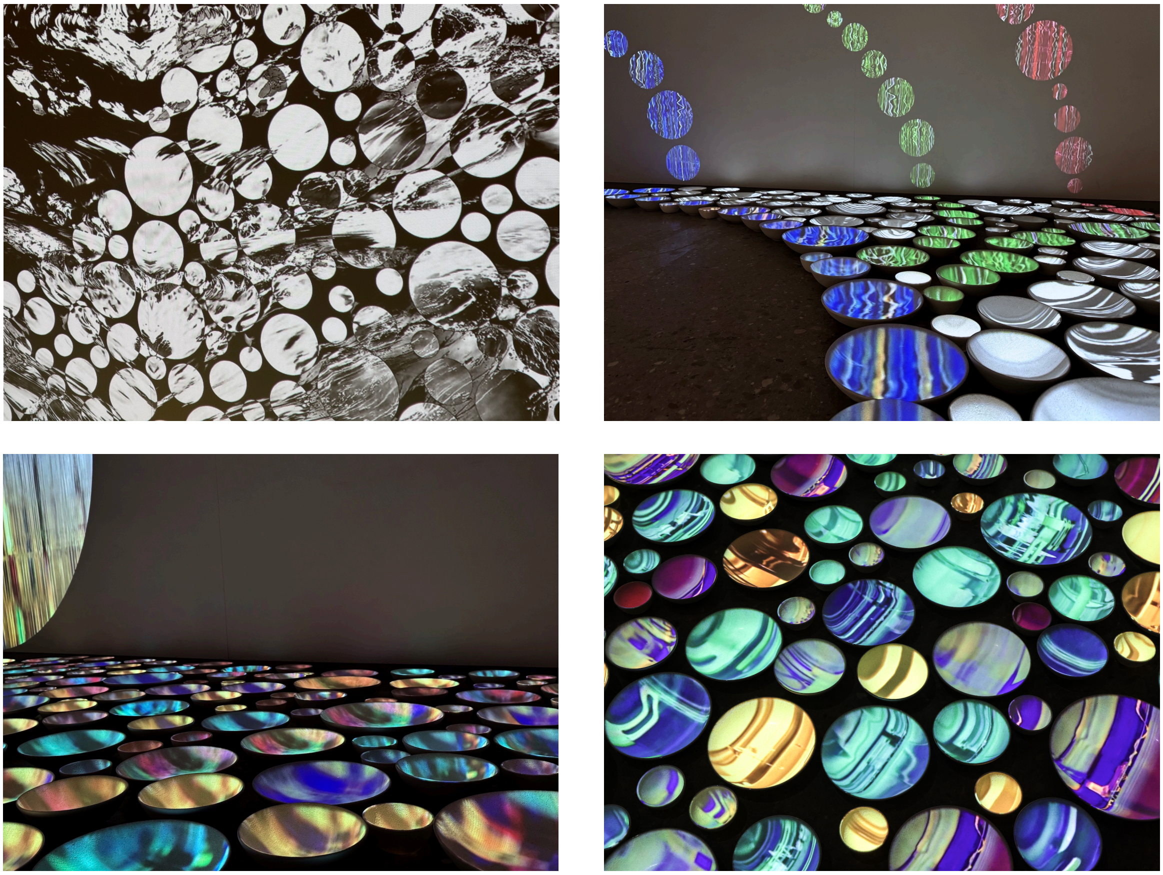
1 Being pulled two ways

In our guise as the Robert Turner Collective, we were recently commissioned to make an artwork related to mathematics for which the initial tacit expectations were that the piece would communicate something (hopefully accurate) about the maths and simultaneously "count" as art. In isolation, neither of these criteria presents a problem: on the one hand, the fruits of many art residencies in scientific institutions testify to the fact that artists can be inspired by science with results that definitely "count" as art; on the other, the many wonderful examples of visualising mathematics, both in a research context and in outreach, show us that the scientific community can produce extraordinary visual material. However, do the former tell us something accurate about the science? Would the latter hold the attention of the art-going public? The waters quickly get murky. In this article we hope to shed some light on these questions by looking at our own experience of creating a multimedia installation, *Échos de la pensée*, born out of discussions with Fields Medallist Maryna Viazovska, Professor at EPFL (Switzerland), celebrated for her work on sphere packings.

Échos de la pensée at EPFL Pavilions, 2025. (Photo © Julien Gremaud)

The expectations referred to above are intentionally a little vague, and it is not useful here to get distracted by attempting to define "art" or "mathematics." We ask the reader to accept that, in general, art is what professional artists do, mathematics is what professional mathematicians do, and members of each community are more or less able to identify what "counts."

As our project advanced, we found increasing unease marrying our artistic needs with mathematical accuracy and, in fact, these two imperatives were pulling us in different, mutually exclusive directions. There was indeed something to reconcile. If one cannot see a reconciliation – which is essentially the point we reached – it is reasonable to ask why not? Any answer too general will lead to probable failure, so we firmly base our discussion on a very personal perspective.



Photos © Robert Turner Collective.

2 Description of *Échos de la pensée*

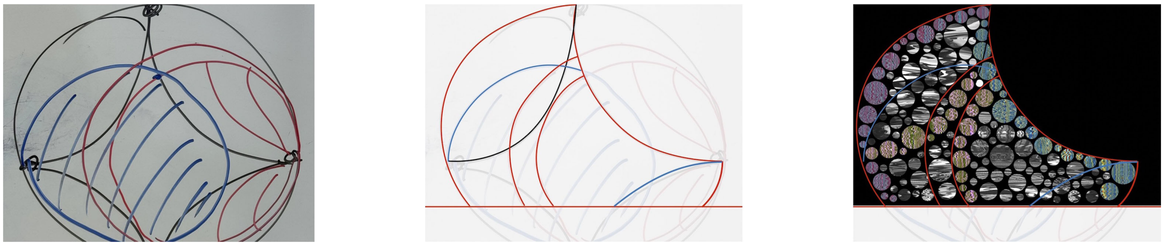
The installation, which was on display at EPFL Pavilions as part of the *Shapes* exhibition[1] consisted of a two-zoned projection canvas: one on the wall and the other extending to the floor, occupying the same space as the viewer. The floor zone was populated with 145 bowls of different sizes creating a non-standard projection surface. Only the interior of the bowls was used for projection. The scale was modest – each zone fitting in a 4 m × 2 m rectangle – as we wanted to retain an intimacy between viewer and physical elements. A 15-minute loop of moving images comprising five different three-minute segments was projected onto these surfaces.

Based only on the pictures shown here, it is perhaps already clear that to a large degree we sacrificed any mathematical accuracy for the art. There are some residue hints of the mathematics – the bowl placement is an obvious nod in the direction of circle packings and the global motif is based on a sketch of the hyperbolic plane taken from Viazovska's whiteboard – however we did not engage with mathematical communication as such.

The intention was to represent the human side of mathematical research, capturing the different emotions or states of mind involved in the process. Outside the mathematical community, there is a commonly held view that maths is cold and emotionless, and we wanted to confront this. Our aim was to portray a quest for truth guided by beauty and simplicity, whose practitioners are constantly confronting the unknown, driven by curiosity and influenced by a multitude of very human qualities. In an era in which the emergence of AI models appears to be removing our monopoly on complex thinking and deep reasoning, we believe it is all the more important to encourage this kind of reflection.

Focusing on the human side to mathematical research also naturally raises the eternal question of the nature of mathematical objects themselves. We use the two projection zones of *Échos de la pensée* to play with the dual nature of mathematical concepts. On the one hand, one can consider that they have a proper intrinsic existence in an abstract Platonic world – represented by the curved

[1] The exhibition *Shapes: Patterns in Art and Science* was open at EPFL Pavilions, Lausanne in 2025, from 17 January to 9 March. Under the scientific direction of Marc Troyanov, Hugo Parlier and Michael Herbst, it aimed to open a window to the richness of the natural and artificial patterns that surround us. https://epfl-pavilions.ch/en/exhibitions/shapes



From white-board sketch (Maryna Viazovska) to underlying motif. (© Robert Turner Collective)

surfaces and colourful, organic patterns in the bowls. On the other hand, since these objects are defined by humans, they are bound to our own understanding and this "shadow" is illustrated by the projection on the flat wall, where the images echo, but remain distinct from, those of the floor.

3 Discussion

Why did we make the choices we made and sacrifice mathematical accuracy for artistic expression? The quoted passage from David Hockney at the top of this article makes the point that Picasso has attentively observed the owl and is responding to it, projecting his own reaction and feelings onto his painting. Indeed, transmission of emotional content is a key goal. Perhaps an artwork carries some further intent or background message, a political statement for example, but whatever lies behind, the artist aims to elicit some response beyond the surface-level experience of simply taking in visual information. Pressing this point and sticking with Hockney, in [2] (also [1]) there is a photo of the artist painting "The Road to Thwing, Late Spring," in which one sees a landscape that is pretty, but that might be walked through without any particular second thought. The emerging painting, however, calls out to the viewer in a very powerful way. Hockney is sharing *his* experience. This added intensity in which an artist projects their thoughts, prejudices, fears and so on, onto the thing being portrayed is the secret sauce. The situation is a little less evident in more abstract art, but the principle remains the same.

How does one go about adding this intangible ingredient? There is no straightforward answer. In general, there is no escaping hard work which relies on intuition based on previous experience and learnt models, uses a good deal of trial and error, and requires a keen sense of self-criticism (not to mention a large rubbish bin at the ready). Nothing here is awfully surprising, but it leads us to a hint of the difficulty we are facing: throwing every method, technique or trick at the problem requires a high degree of flexibility. If one starts with some science or mathematics to communicate, the constraints are simply too restrictive. Conversely, if one starts with an artistic idea, it becomes nearly impossible to hammer it back into some realistic science context without sacrificing to an unreasonable degree the original idea. In a nutshell: the required "artistic licence" takes us too far from the accuracy one needs to communicate the science. Scientific understanding and artistic understanding both enrich us, and ideally we could appreciate both in equal measure, but they do nonetheless seem rather different to us. Indeed, the didactic impulse of mathematical explanation and the creative impulse of artistic practice pull in different directions.

Photo © Julien Gremaud.

For *Échos de la pensée* we did not really reconcile the two, but a way forward quickly emerged. During our interviews we did not observe any modular forms nor any Eisenstein series, nor for that matter any sphere packings. Rather, we were observing Maryna Viazovska – her enthusiasm, her passion for a part of a proof coming together beautifully, her curiosity and so on. Most readers

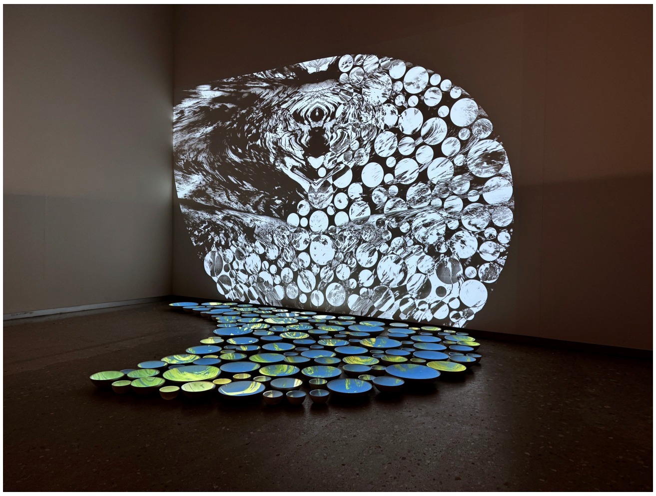


Photo © Robert Turner Collective.

of this article will be mathematicians, who will know from their own experience that notions such as curiosity, simplicity and beauty, frustration, inspiration, confronting the unknown… are all aspects of mathematical research. Yet a great many people will be surprised to learn this is how mathematics is seen by its practitioners. To us, this message seemed extremely important, in fact, more so than conveying any particular snippet of mathematics, and we felt equipped to address it artistically.

We should not over-generalise, and let us stress once again that we are writing from personal experience, but the points made are groping towards a more general understanding and so it is interesting to examine where this leaves us.

What, for example, can we make of the many artistic residencies at scientific institutions? Parachuting an artist into a lab, with intended output some art "related to" the science, is quite a fashionable thing. There is no doubt that science inspires some wonderful art: talk to an artist about infinity or 24-dimensional space, and you will see a fascination and openness that is very refreshing and discussions may well lead to great artistic output. On the other hand, rather little will be added to science, mathematics or its communication. It is not the point of art to do so, but it can be a frustrating source of misunderstanding between the two communities. The language used to describe what is being done can be misleading and the artist who says "I am investigating/exploring/interrogating the infinite" can come across as painfully naive to scientific ears. In fact, the artist will have little to nothing to say about infinity itself, but this misses the main point: the artist can hold up a mirror and say something interesting about our psychological or emotional response to being confronted with the infinite. We may learn something about ourselves from this kind of approach and, more importantly, others may simply be inspired by the art and later drawn to the actual science.

What about math/art? Recently, George Hart [3] wrote a very thought-provoking piece discussing a particular class of artworks, math/art, produced almost entirely by mathematicians. His interesting and frank analysis leads him to celebrate math/art wholeheartedly, yet – noticing a disinterest from the art establishment – he concludes that much of it "is not truly *fine art*." It is of course difficult to pinpoint why not. Suggesting there may be missing secret sauce is not exactly providing the kind of analysis needed. Nonetheless, if the art community are looking out for the kind of "added value" we are referring to above and not finding it, this might explain some of their reticence.

4 Reconciliation

While our approach to *Échos de la pensée* avoided conflict between our two worlds rather than reconciling them, the experience convinced us of the mutual benefit of engagement between maths and art in all its forms. Mathematics has a well-known image problem and this stems in large part from its intangible nature: one cannot feel or smell mathematics, nor can it be directly seen or heard. Traditional maths outreach which attempts to overcome these difficulties is evidently not touching everyone and the question of how to reach new audiences remains ever present. Any approach that targets new groups – however unusual – should be taken seriously. Art lovers, for example, may include people not usually found among the audiences of maths outreach. Art can hold up a mirror to the mathematical community and offer an inspiring window to others. If, as a mathematician, you find yourself talking to an artist, you may be best advised to detach your mathematical content from your enthusiasm and try to communicate the latter: allowing an artist to represent your passion may turn out to be a very valuable and effective contribution.

There was one final irony to *Échos de la pensée*. The last day of the exhibition came and, having agonised about how to include sphere packings in a more explicit way in our work (and then not doing so), we had to pack the bowls into storage crates … efficiently.

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The Robert Turner Collective is an artistic collaboration between Louis-Hadrien Robert and Paul Turner, founded in 2019. Louis-Hadrien lives and works in Clermont-Ferrand (France); Paul lives and works in Geneva (Switzerland). Both are also mathematicians – at the Université Clermont Auvergne and the Université de Genève, respectively.

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Interview with Giovanni S. Alberti

Marie-Therese Wolfram and Marc E. Pfetsch

Giovanni S. Alberti (born 1987) studied mathematics in Genoa and obtained his PhD from Oxford University in 2014. He was a postdoc at École Normale Supérieure Paris (2014–2015) and at ETH Zürich (2015–2016). Since 2016 he has been a professor at the Department of Mathematics of the University of Genoa, first as an assistant, then an associate (2022), and since 2024 as a full professor. He is a member of the Machine Learning Genoa Center (MaLGa). He received the Eurasian Association on Inverse Problems Young Scientist Award in 2018 and the Calderón Prize in 2025. His research interests lie in the analysis of PDEs, inverse problems, functional analysis, applied harmonic analysis, wavelets, compressed sensing and machine learning. He is currently heading an ERC Starting Grant (2022–2027).

Giovanni S. Alberti in Oberwolfach, 2023. (Photo credit: Archives of the Mathematisches Forschungsinstitut Oberwolfach)

This interview is part of a series in which ERC grantees are asked about their experiences and research.

Marie-Therese Wolfram: *Let us start with a question that I 'stole' from Martin Hairer: If you had a mathematical wand for fulfilling your wishes, what kind of result would you like to prove?*

Giovanni S. Alberti: I'm an applied mathematician and somehow applied mathematicians try to put together real life and abstract mathematical results. Those two things don't typically go too close together, in the following sense: Theories can be beautiful and perhaps very tough, but they often work in simplified settings. Reality is not always like this. My background is mostly in analysis, I work a lot in function spaces and I try to use those techniques to understand signals in the real world. Function spaces typically work very well for smooth functions and smooth signals. But as soon as you have signals with singularities, the problems become much tougher to analyse. Nowadays, people just use machine learning and neural networks, which can represent real world signals very well. But here the theory is very limited. On the other hand, if you use old style techniques and function spaces, then you know everything about them. But they don't match reality too well.

So if I had this mathematical wand, I think I would try to put these two things together to be able to fully understand models that are very close to reality from the theoretical point of view.

[MTW]: *You say your research examines simplified problems and reality is much more complicated. So in a way you would like to make your models more realistic and the analytical toolbox more advanced.*

GA: Yes. Consider, for example, Shannon's theorem: if you have a bandlimited function, you can sample it at a certain distance, and then you're going to be fine. However, signals in real life are not bandlimited. So what do you do? One popular technique was compressed sensing, where people considered sparse signals. This means that you have only a few nonzero coefficients. But these techniques have become completely obsolete with the use of machine learning. However, we still understand very little about that, and so the question is whether it is possible to have a nice model for functions that is on the one hand fairly close to reality,

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like perhaps neural networks aim to be, and on the other hand is also fully understood from the mathematical point of view.

[MTW]: *Coming back to the use of neural networks. You said that neural networks now outperform traditional approaches. I find this surprising. For example, I come from a PDE background and I have colleagues who work on sophisticated finite element solvers for PDEs, which is the 'classical approach.' Nowadays, people use neural networks to solve PDEs. But to be fair, they don't do very well in many applications. I'm not saying that using neural networks is per se bad. But are they really so much better?*

GA: If you think about PDEs, for example the Poisson equation in 3D with a certain source, then most likely a finite element solver would perform very well. It's linear, simple and fairly quick as well. But let's think about a much more complicated PDE, maybe time dependent, very high dimensional, and perhaps involving particles. In that case, classical solvers may take hours or days to solve the problem, while networks trained with training sets perform much better. The idea is that classical solvers and also classical function spaces are completely agnostic to the actual setting: If you solve a Poisson equation with finite elements, any source in the right-hand side will work. What neural networks do, and that's their power, is that they capture additional structures in the data. For example, with weather forecasts, your initial data will have some structure. This cannot be modelled by using classical function spaces, say Sobolev or Besov, which are well-designed to quantify smoothness. Neural networks, especially convolutional neural networks, have the power to better represent those structures.

[MTW]: *At the same time, if you then throw something at the neural network that it doesn't expect, it will probably perform very poorly.*

GA: Indeed. That's another discussion, the topic of generalisation: How neural networks are able to go beyond what they have learned.

[MTW]: *What do you think about AI in general? How will it change mathematics?*

GA: First, most of my research is still outside of AI and machine learning. It seems I am one of the few people doing inverse problems and applied harmonic analysis who haven't made AI their main business. Because I still like old-fashioned topics. I believe that AI will probably be useful for mathematics in the sense that it may give us tools for proving things and for developing ideas that are difficult. But this is something that I really don't know anything about.

On the other hand, one can use mathematical techniques to analyse AI. Mathematical analysis can be used to study the properties of the functions that appear in neural networks. This is, of course, oversimplifying things. Another key issue in mathematics is stability. If you change the input to your problem a little bit, what happens to the output? With neural networks stability is a big topic. So I suppose that in that case, mathematics can be useful.

At the Applied Inverse Problems 2023 conference in Göttingen. (Photo by Dr. Markus Osterhoff)

Marc Pfetsch: *Can you tell us a bit about your ERC project?*

GA: The title of my project is "Sample complexity for inverse problems in PDEs." So, instead of solving a PDE, you either know the solutions or something about the solutions, for example, boundary data, and you want to infer the coefficients of those PDEs. Classically these problems have mostly been studied using PDE theory or computational techniques.

Since most of the problems do involve signals, my project aims at putting signal analysis and applied harmonic analysis together with PDE theory. We use techniques of sampling and compressed sensing and also machine learning combined with PDE theory.

Consider the following simplified problem for an elliptic PDE: you want to find its unknown coefficients from its solution. You can either study this problem from a PDE perspective, or you can think about the map from the solution to the coefficient as a map between function spaces. One question could be: What if I know something about my unknown? What if it has some additional structure? Can I recover it with a lower number of measurements? This type of questions can be addressed by PDE techniques, but the assumptions you make on the unknowns are more common for applied harmonic analysis and sampling theory or compressed sensing, where you put assumptions in terms of sparsity.

[MP]: *You mentioned compressed sensing, which used to be very popular in the past, right? I think it somehow slowed down – where do you think this field will go?*

GA: I think that 99% of the people who used to work on compressed sensing type problems, are now working on machine learning problems. Compressed sensing research seems to have reached such a level of maturity that there is not much to say any more. But my feeling is that a lot of those ideas haven't been used in interesting settings. This observation is one of the key aspects of my ERC project.

The main idea of compressed sensing is to reduce the number of measurements by using a priori assumptions on the unknowns. This key idea could very well be applied to inverse problems. How do I leverage some a priori knowledge I have on my unknowns to reduce the number of measurements? Unfortunately, you cannot do this with the current compressed sensing methods. So what my project aims to do is to generalise, extend and possibly develop new compressed sensing results that are general enough to be applicable to inverse problems in PDEs.

Let me just give you one example. One of the main applications of compressed sensing is magnetic resonance imaging, which is based on the Fourier transform. The other very important inverse problem in computerised tomography is based on the Radon transform. To our surprise, there was not even one theoretical paper explaining why you could use compressed sensing for the Radon transform. That's something we finished about a year ago.

[MTW]: *There is a lot of research going on in medical imaging. You said you had some theoretical results? Do they help to improve the performance and results of algorithms in medical imaging?*

GA: Compressed sensing techniques have been applied in practice, for example in tomography. But there was not even a single theoretical result saying: If you assume that your signal is sparse, then you need a number of measurements that is related to this sparsity. There simply was no result of this kind for the Radon transform. Regarding magnetic resonance imaging, the answer is certainly yes: the theory of compressed sensing has indeed improved the performance.

[MTW]: *So how did you get the idea for the ERC project?*

GA: It was a slow process. It wasn't like I woke up one morning and said 'OK, I will write an ERC proposal.' As I said, I had studied inverse problems in PDEs in my PhD. Before, I had focused on wavelets in my master's. During my postdocs, I started exploring the idea of putting these two fields together, developing my own research path that wasn't explored by other people. And we realised that there were a lot of open questions. So I said, well, maybe I should apply for an ERC.

[MP]: *If you look back at the application process. Are there any special things that you noticed? Do you have any advice for colleagues who want to apply?*

GA: I suppose that if you are a top-class mathematician, then maybe you don't have to worry about the things I had to worry about. For me, it took a long time to design the project. It took maybe a couple of months to actually write it down and polish it. But the actual design of the project, I think that was the most important part. I prepared the story of the project with some slides which I would share with friends, colleagues and other people. This really helped me structure the whole thing as well as possible. But I had to apply twice. The first time was during COVID and I passed the first round. There were no interviews and I didn't get it. But then, since I passed the first stage, I could resubmit a revised version the following year. It was based on the feedback I received in the first round, and this gave me quite an advantage, I guess. When I got to the interview for the second time, I already knew a lot of the possible criticism.

I think the key aspect is time. You cannot just wake up one day and apply for an ERC. It's a long process. I would say that it takes at least a year from the moment you start to think about it to the moment when you press the submit button. I think a year is a reasonable time. You shouldn't rush.

[MP]: *I can imagine that some of the feedback is also contradictory. The more people you ask, the more different opinions you get, right? And they are all valuable, but you cannot follow all. I mean, it's a limited space, you have limited time. How did you deal with this?*

GA: Well, absolutely. This is what you also see from the referee reports. I don't know what the panel does, but I suppose that they have to find a midpoint somewhere.

I can give you one example. I didn't have much experience in machine learning. I had maybe two or three papers in machine learning in good journals and conferences. But still it was not my main topic. So one issue was: Do I decide to make machine learning a very minor thing in such a way that people cannot say "you're not an expert"? But then the problem by doing this is that people can say "what are you doing? It's not 2010, we are in 2022." I mean, you cannot say you do only compressed sensing, right? And so I had to find a way in between. What I did was to acknowledge that my background was not 100% from machine learning, and I included a very strong machine learning colleague in Genoa as an additional collaborator. This is what I did, but I don't think that there is an easy solution to this problem.

[MTW]: *Let us change subjects a bit. When and how did you decide to study mathematics?*

GA: During high school, I liked mathematics. But I also liked physics and more practical things such as engineering. In Italy, the system allows you to sign up for a university degree until the very last moment. I decided five days before the start of the classes. During the summer after my final high-school examination, I bought some books on physics, mathematics and other topics. I read those books and realised that I found mathematics easier than the others. This opinion has become stronger and stronger over the years – a lot of people believe that mathematics is hard, but I really think the opposite: Mathematics is simple. It is based on very few rules and as soon as I have to think about physics, I find it much more complicated. And if I think about non-STEM disciplines like economics, politics or medicine, I find those even more difficult. That's why I studied maths – because it's easy.

[MTW]: *And why not pure maths?*

GA: I started as a pure mathematician. I did my whole bachelor and master mostly as a pure mathematician. But I took some courses in applied maths, and I was always fascinated with things that are somehow in between. That's where I would put myself now. If you are a pure mathematician you would think I am a very applied mathematician. If you are an applied mathematician, you would think I am pure. So I would think I'm somewhere in between.

[MTW]: *Where do you think applied maths should go?*

GA: I don't know. I mean, as I said a couple of times, I am not too keen in following what everybody else does. I cannot say anything about what is going to happen in ten years. It is possible that nobody is going to be using neural networks any more. I don't know. So I don't know where applied mathematics is going.

[MP]: *What do you think about the surrounding fields of mathematics? I think applied mathematics is quite successful, and there is also a downside because the surrounding fields pick it up in their own fields. They also do mathematics, so it is hard to distinguish a paper written by people from signal processing for people in maths. What is your view on this? Is it important to keep mathematics together? Is it good that it somehow distributes?*

GA: I like that mathematics distributes and it's used by others. But I also believe that a deep knowledge of the mathematical abstract theory is important. What I mean by this is that of course you can do numerics of PDEs, for example, without knowing basically anything about functional analysis or weak formulations. You just discretise the PDE and solve the linear system. But still, I do believe that you need a deep knowledge to understand the behaviour of those methods.

I think this applies everywhere. For example, if you want to invert the Radon transform, you can just apply the classical filtered back projection. But it's important to understand the involved function spaces, what it means that the Radon transform is ill posed and that its inverse is discontinuous. Only then you can fully understand these problems.

Giovanni S. Alberti during his acceptance speech for the Calderón Prize at the Applied Inverse Problems 2025 conference in Rio de Janeiro. (Reproduction: FGV EMAp, Photo by Ana Santiago)

I am happy that people from engineering use mathematical tools. But I think that for us, it is very important to defend the importance of the theory in understanding those phenomena.

[MTW]: *This is a very beautiful answer.*

[MP]: *How would you value the cultural background? I have the feeling that mathematicians can be identified easily after saying a few sentences. Maybe an electrical engineer would use very different language. Do you think that this difference is important?*

GA: I wouldn't say that we are better than others, of course. But what I like about maths is that the abstraction allows mathematics to be much more general than other fields. So I think the key cultural advantage of mathematics is that with basic knowledge and tools we are able to understand and analyse phenomena of very different kinds. Maybe an engineer struggles to achieve this.

As a side note: I'm not an expert in mathematics education. So I don't know what's most useful for students from primary and secondary schools. But nowadays, at least in Italy, there's a trend to say we should avoid mathematical abstraction: Let's just make everything concrete and give many examples. Of course, I understand and I value examples and practical things in mathematics, but I do also value its abstraction.

[MTW]: *In a way I do agree. At the same time, I am often quite struck how engineers come up with ideas that work. They often have great intuition. They might not be able to tell you why on earth this works, but how do they come up with that? I am not defending or judging. I just think both approaches have their own merits.*

[MP]: *I would like to test your opinion on something related. I fully agree with the abstraction point, but that also has a disadvantage. Namely, that then mathematics is often hidden behind the application. Often only the applications are visible, at least to the outside. The groundwork that we do is not visible any more. This fact is also reflected, I think, in funding. If you look outside the ERC on the European level, it's all tied to applications and not to fundamental research. So what's your view on this?*

GA: That's a tough question. You should have sent this to me before. [Laughs.]

It makes me think that most parts of my ERC grant (I applied to the mathematics panel) are about theorems and understanding phenomena that I will certainly not use in practice. But whenever I explain my ERC project to people who are not experts, I would just start with medical imaging and other things, because that's the way people can understand it.

It is important for people to understand that the elementary mathematics we study at school, for example arithmetic and geometry, play an important role everywhere in our lives: those basic building blocks are fundamental and important. Now, the world, and the science with which we describe it, have evolved a lot. Then, in the same way, I argue that the new building blocks for this understanding are given by pure mathematics, as both a tool and a language. But having said that, I have no definite solution to that.

[MTW]: *What career advice would you give if you look back?*

GA: One simple and possibly obvious advice is to enjoy yourself as much as you can. Try to find the fun in what you do and don't look at it only as a job.

Maybe one thing you don't hear so often is the importance of independence. Even as a PhD student and especially as a postdoc, you should gain independence from your supervisor soon. I think it is important to distinguish yourself from your supervisors and to find your own directions.

I've always found this important for two reasons. First, because in this way what you do is different and so people will not identify you as the student of X or someone who has done similar things that Y has done. Moreover, this is the only way you can actually do really new things. You can try to explore new directions that others haven't tried.

Another advice is about communication of science: I think that young, but also senior researchers, tend to focus more on what we do, and not much on how we write things and how we explain things to others. I believe that it is very important for young PhD students and postdocs to understand the value of writing good papers. That the readers say: "Ah, I understand this." It is also very important to give good talks, in which the audience understands what's going on. People don't pay too much attention to this, especially in pure maths. I've listened to many talks and typically the focus is only on the results. So, if the theorem is beautiful 'that's it.' Perhaps 'that's it,' but it should be well communicated.

[MTW]: *What has helped you in the past to develop this independence?*

GA: In my case, what happened was that one day at the end of my second year of PhD, my supervisor comes to me and tells me: "Next year I will be in Paris [I was in the UK]. So we will meet three or four times and that's going to be it." So that's what helped me in my independence. Of course, it was an extreme approach. [Laughs.]

[MTW]: *Did you then collaborate more with other people?*

GA: Yes, for instance at conferences, I would meet people and start side projects, but maybe more as a postdoc.

[MTW]: *For the academic writing, I think, it's nice advice. But learning this is not very easy.*

GA: Well, you know, the modern machine learning approach would be to use a classification approach. So you see many examples, many bad papers and many good papers. You see many talks, many bad talks and many good talks. And from those examples you learn. But I agree it's not easy.

[MP]: *But if you don't have the motivation to do it well, then it will not happen.*

GA: Right, absolutely. Perhaps the risk is that the motivation stops whenever you finish the proof of the theorem. Then you just write it down and see if it's correct.

After I obtained my ERC project, I've read a few proposals of colleagues. In some cases, these were written by mathematicians who are much stronger than me and the projects themselves were, mathematically speaking, excellent. But if you write them in such a way that a reader falls asleep after the first page, then you probably don't get an ERC.

[MTW, MP]: *Thank you for the insightful interview.*

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Shmuel Agmon (1922–2025)

In memory of a beloved mathematician

Ehud de Shalit and Yehuda Pinchover

S. Agmon in AgmonFest 2007. (© D. Guthrie)

Professor Shmuel Agmon passed away on March 21, 2025. He was the eldest member of the Israel Academy of Sciences and Humanities, and one of the founding faculty members of the Einstein Institute of Mathematics at the Hebrew University. Together with A. P. Caldéron, E. De Giorgi, L. Hörmander, F. John, O. A. Ladyzhenskaya, P. Lax, and L. Nirenberg, Agmon has been one of the world's leading researchers in the field of partial differential equations (PDEs) over the past half-century – a field that has proven highly valuable not only in mathematics, but also in scientific disciplines such as physics, biology, chemistry, and engineering.

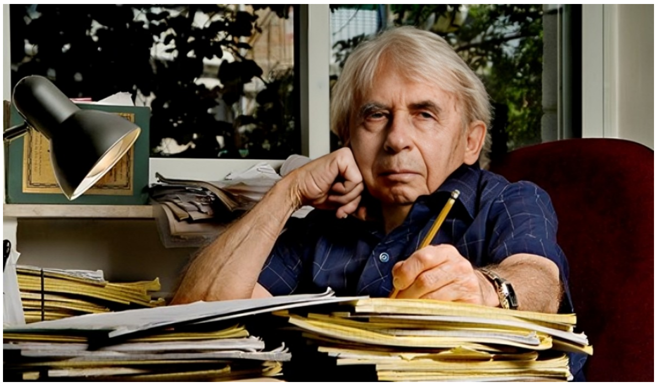
Shmuel Agmon was born on February 2, 1922, in Tel Aviv, to the writer and Zionist activist Natan Bistritzky and his wife Chaya (née Guttman). The family moved from Nazareth, where Chaya had worked as a dentist, to Jerusalem, where Shmuel completed his studies at the Hebrew Gymnasium. Although he was drawn to mathematics from a young age, he also found time for chess (he was the Jerusalem youth champion) and was a member of the "Machanot HaOlim" youth movement. In 1939, he went for training at Kibbutz Na'an, but within a year returned to academic studies at the Hebrew University. Among his teachers were Michael Fekete, Abraham Halevi Fraenkel, Benjamin Amira, and Yaakov Levitzki.

Agmon's studies were interrupted by World War II, during which he volunteered for four years in the Jewish Brigade. After being discharged, he completed his studies at the Hebrew University and was invited by Professor Szolem Mandelbrojt to pursue his doctorate at the Sorbonne in Paris. He completed his PhD in 1949 in the field of complex functions and then took a position at Rice University in Texas. At Rice, he transitioned into the field of PDEs, which would define his career. His encounters with the leading analysts Peter Lax and Louis Nirenberg – especially his collaboration and friendship with the latter – deeply influenced him and shaped his professional path.

In 1952, he joined the Hebrew University, pioneering the field of PDEs in Israel and mentoring many students. In 1964, he was elected to the Israel Academy of Sciences and Humanities.

Shmuel Agmon is one of the founders of the modern theory of linear elliptic PDEs. His research focused on the qualitative theory of linear elliptic differential equations. His groundbreaking and highly influential papers [5, 6] with Douglis and Nirenberg, published in 1959 and 1964, generalized the classical Schauder theory on regularity of second-order elliptic boundary value problems to general higher-order elliptic equations and even to systems of such equations. These papers covered the theory of L^p spaces for $1 < p < \infty$ and Hölder spaces while, for example, the Hörmander's classical 1963 book is restricted to the L^2 theory. These works, among the most cited in mathematical analysis, continue to influence the modern theory of linear and nonlinear elliptic equations. Agmon's 1965 book [1] on linear boundary value problems for elliptic PDEs of general order became a foundational text in the field (a revised edition was published in 2010 [4]).

From the mid-1960s, Agmon worked on the spectral theory and scattering theory of Schrödinger-type equations, publishing results of great significance in mathematical physics. His early breakthrough results on the spectral properties of Schrödinger operators and scattering theory were published as a book in the highly selective Lezioni Fermiane [Fermi Lectures] series [2]. Agmon's contributions in this field include, in particular, precise estimates on the decay of eigenfunctions of Schrödinger operators on unbounded Riemannian manifolds – expressed in terms of an ingenious metric he introduced, now known as the *Agmon metric* (see his widely



S. Agmon, a recipient of the EMET 2007 Prize. (© Yael Ben Haim)

influential lecture notes [3]) – as well as his results on the structure and behavior of positive solutions of second-order elliptic equations, and on the limiting absorption principle, had a profound and lasting impact on this extensively studied field. Furthermore, his work advanced the understanding of wavefunction scattering for both short-range and long-range potentials.

For his work, Shmuel Agmon was awarded the Weizmann Prize, the Rothschild Prize, the Israel Prize, and the EMET Prize. He was a speaker at the International Congress of Mathematicians (ICM) in 1962 and 1970.

Agmon was widely respected worldwide and known among his colleagues for his broad intellectual horizons, phenomenal memory, modesty, and sense of humor. In 1947, he married Galia Yardeni, a researcher of literature and the history of journalism in the Land of Israel, who sadly passed away in 1968. They had three sons, Noam Agmon, a theoretical chemist, Ariel Agmon, a neurobiologist, and Eytan Agmon, a musicologist. In 1972, he married Nechama de-Shalit, a psychiatrist who passed away in 1998.

Shmuel Agmon witnessed momentous events throughout his life and was blessed with longevity and good health. May his memory be a blessing.

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S. Agmon with his former students and H. Brezis, AgmonFest 2007. (© D. Guthrie)

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Mental health in mathematics research: challenges and paths to change

Exploring how systemic pressures affect researchers, and the growing efforts to support well-being across academia

EMYA column regularly presented by Vesna Iršič

Anita Waltho

Mental health has become an area of growing attention in academia. Researchers face a range of stressors, including excessive workload, intense competition, and imposter syndrome. Over the past decade, an increasing number of studies have shed light on this issue. A 2021 meta-analysis of 32 studies estimated that 24% of PhD students experience symptoms of depression and 17% anxiety [2].

These findings have served as a wake-up call, prompting many institutions and individuals to take action. Initiatives have emerged that aim to raise awareness and provide support for mental health within academia. In mathematics, where research is often solitary and intensely cerebral, and where the field has long cultivated the myth of the lone genius, more voices from the community are speaking out, and concrete steps are being taken to support researchers' mental health.

A research culture at odds with mental well-being

Across disciplines, academic culture is shaped by structural and social conditions that often undermine well-being. One serious concern is power abuse. "The structure of the academic system lends itself to power abuse, as it follows a hierarchical structure with multi-facetted dependencies of staff and students on their superiors," says Hendrik Huthoff, former researcher, and now a workshop facilitator as well as advocate for mental health and systemic changes in academia. He sees power-relationships as a key factor in many of the mental health challenges faced by early-career researchers.

Power abuse may arise in situations where supervisors exert undue influence over the academic lives of early-career researchers. Several recent investigations and exposés,[1] particularly in high-profile research institutions, have brought systemic abuse to public attention.

As researchers compete globally to publish in high-impact journals and secure prestigious limited positions, hard work and personal sacrifice often become the expectation. Mikael Vejdemo-Johansson, associate professor of mathematics, computer science and data science at College of Staten Island, part of The City University of New York, recounts a telling conversation he had at a conference early in his career,

"I was talking to one of the leaders in my field and at some point, I mentioned that I was going to spend the weekend with my wife and try to avoid doing too much work, so I could focus on her, and get some rest and relaxation. His immediate reaction was: 'But I thought you wanted a career in academia'."

Mikael's story also highlights how the pressure to prioritise work above all else can strain relationships outside academia. The demands of a research career can create additional challenges to personal relationships, particularly due to instability. As Barbara Betti, PhD student and former PhD representative at the Max Planck Institute for Mathematics in the Sciences, explained: "Finding permanent positions is very hard…, and you might have to move to another city or another country many times before stabilizing. If you want to have a family or be stable with your partner, that's challenging."

Mental health perspectives within mathematics

Mathematics research comes with its own set of challenges that can be detrimental to mental well-being. The nature of the work – often solitary, abstract and mentally taxing – can intensify feelings of isolation, imposter syndrome and mental fatigue.

"Mathematics is much more of an individual sport," says Mikael. "A lot of research boils down to sitting and staring at a blank sheet of paper. Everything happens inside your head. And unless you actively seek it out, you're unlikely to find yourself in much collaborative work. This isolation ends up being a huge problem in mathematics."

Additionally, mathematics has a culture that often glorifies brilliance and downplays the value of asking questions. The cult of the mathematical genius [1] puts a pressure on students and researchers that encourages competition and where struggle is seen as weakness.

[1] https://www.dw.com/en/abuse-elite-scientists-germany-max-planck-society-v2/a-71897800

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"[In a seminar or lecture] it's very common that people would say 'we don't need to present this proof. It's trivial'. And then no one dares to say, 'but I don't understand it'," recounts a master's student in numerical mathematics, who wished to remain anonymous, "It can very easily make you feel like you're actually too dumb for maths."

This imposter syndrome compounded by working in isolation can lead to individuals suffering in silence. "I heard stories of friends of friends who were puttering away as graduate students, utterly convinced they were the only one to struggle, and curled up under their desk crying on a weekly basis," said Mikael.

Whether stuck on a maths problem or struggling with your mental health, the first step towards a solution is the same: asking for help. But if people do not feel safe or empowered to speak up about their academic struggles, encouraging them to seek support for personal mental health issues becomes even more difficult.

The wave of mental health awareness

In the mathematics research world, as across Western society in general, the tides are changing and mental health is becoming a bigger topic of conversation. "What has changed in the last decade is that it's much more accepted to talk about [mental health struggles]," says Volker Mehrmann, professor at the Institute for Mathematics in the Technical University Berlin, and former president of the European Mathematical Society.

Mikael has worked in mathematics research in Sweden, Germany and the USA, and has been an advocate for mental health in research for nearly a decade. "My motivation for starting out my advocacy work was that I was watching one of my major life communities utterly fail everyone in the community. There was a complete silence and a complete lack of support structures," he recalls.

Mikael has contributed to raising awareness by sharing his mental health story in two articles for the American Mathematical Society's magazine [3, 4], and facilitating open conversation on the topic at panel discussions at the Joint Mathematics Meeting and his blog 'Depressed academics' cofounded with his former colleague Ian Gent.

Workshops are another way the conversation is expanding. Hendrik facilitates workshops on academic mental health and power abuse, and is the former mental health project lead at UniWIND/GUAT (German University Association of Advanced Graduate Training). He explains how workshops can act as an information provider and a discussion forum, "I have done workshops that really focus on what mental health is – that we all have mental health, and it fluctuates across our lives, even within a single week… [for the UniWIND mental health project] workshops were incredibly helpful for us to hear the voices of those on the ground – be it the doctoral researchers, coordinators of graduate programs or supervisors – to guide the direction where we should go."

Above all, maintaining momentum on mental health awareness is crucial. As researchers frequently change institutions during their career and new cohorts of early career researchers and students come in, *mental health awareness and advocacy is a continuous mission, not a one-time solution.*

"You can't just say 'OK, we had a mental health event in 2020. Subject closed'," says Hendrik. "It's something that needs to be continually talked about because you always have new generations of people coming in. It needs to remain at the forefront of the people's minds to create a culture where people feel safe to speak openly about it."

Supervisors must accommodate their students' mental health needs

Whether leading a research team or mentoring as a junior researcher, supervisors play a crucial role in supporting students' and researchers' mental health. This includes building trust, making accommodations, and directing students to appropriate support services.

Mikael has found that being open about his own experiences with mental health encourages his students to approach him for support. "I found it really useful in my teaching, so I usually end up using some sort of excuse within the first couple of weeks of a class to mention my struggles. This has the concrete effect that my students have a much lower threshold to seeking me out when they need accommodations or when they are struggling."

Volker also believes supervisors should offer the same support for mental health challenges as they would for other access needs. "In my opinion, it is like any other serious challenge a person might face, and one has to be accommodating. If I have a blind person in my lecture, I must ensure they have every chance to succeed. If I have a master's or PhD student dealing with depression, I should do everything I can to support them in continuing their research while managing their mental health."

Mental health accommodations will vary case to case and should be developed in consultation with the individual. They might include flexibility for therapy appointments, sick days, longer absences, or deadline extensions. Crucially, supervisors should also be ready to help students access institutional or local mental health services, by having the information on hand.

The power of grassroots action

Grassroots action, community-driven, bottom-up initiatives led by student and researcher representatives, plays a vital role in spreading mental health support, offering peer assistance, and advocating for positive change within or across academic institutions.

Former PhD representative Barbara describes some of these responsibilities, “Our role as representatives was, first and foremost, to listen and offer support when a PhD student came to us with a complaint or something they were struggling with. After that, we would direct them to the appropriate people with more expertise or authority to help. Whenever a new PhD student joined, we sent them an email with all the relevant contact information for reporting supervisor misconduct or seeking mental health support.”

Hendrik, who worked as a graduate coordinator for many years, comments: “What helped me enormously in driving forward my initiatives was the grassroots involvement from PhD and Postdoc representative organisations.”

Critical grassroots-led mental health advocacy also comes from initiatives that are independent of specific institutions and made up of advocates from several countries and career stages. One example is the former EU-funded ReMO (Researcher Mental Health Observatory) COST Action,[2] which brought together over 100 researchers and stakeholders from across Europe to map and improve mental health support for researchers. ReMO developed best practices, organised training events, and created a policy brief on mental health in academic settings.

Another is Dragonfly Mental Health,[3] a non-profit organisation run by academics that provides tailored workshops, consulting, and advocacy aimed at improving mental health in research environments. Dragonfly works globally, with a team representing multiple continents and disciplines, and partners with universities, funding bodies, and postdoc representative organisations.

Institutional support in accessing services

Long waiting times and overloaded mental health facilities are challenges that affect many people, not just researchers. For academics, these difficulties are often compounded by frequent relocations across cities and countries, requiring them to repeatedly adapt to new healthcare systems. This constant upheaval can disrupt ongoing treatment and add significant stress.

Mikael shares his personal experience of such disruption: “By the time I talked to a psychiatrist, it was three weeks before my moving date. When I arrived in Sweden, I had to start from scratch again – trying to build up and find someone to talk to. After about half a year, I was back in treatment.”

His experience at The City University of New York, however, was markedly more supportive thanks to their work-life balance service: “They had therapists, and various systems and mechanisms in place just to make things a little bit easier. I reached out with my needs and information, and I received back names and phone numbers of places that they had verified were actually accepting new patients and took my insurance,” Mikael recalls.

Institutional support in accessing local mental health services, or even on-site counselling, can be crucial in helping researchers maintain continuity of care and reduce the stress associated with relocation.

The benefits of mental health first-aiders

Hendrik recounts his first experience with mental health first aid whilst working in King’s College London: “One day we found a master student hyperventilating on the floor of the lab. Fortunately, he was seen within hours, if not minutes, by the mental health first-aiders that we had at the university.”

Mental health first-aiders can be likened to conventional first-aiders – they are voluntary first responders to mental health-related incidents or reports, who offer support and referral to appropriate services. Mental Health First Aid International[4] is the not-for-profit organisation that oversees and supports the official Mental Health First Aid Program worldwide. Currently, this program is available in 11 European countries: Austria, Denmark, Finland, France, Germany, Ireland, the Netherlands, Slovakia, Sweden, Switzerland and the UK.

Whilst working as a graduate coordinator, Hendrik set up the first mental health first aid programme at a German university, at the University of Jena. “It was a success in the sense that, on the one hand, staff were very keen, even on a voluntary basis, to participate and get their Mental Health First Aid qualifications. And on the other hand, we immediately had a lot of requests for consultations,” says Hendrik.

Mental health support facilities also benefit from mental health first-aiders. “We reduced waiting times for counselling, because we could filter out those in urgent need for a therapy referral from those that would benefit from other support, like conflict resolution with an ombudsperson, grants for financial hardship or simply assistance with overcoming cultural barriers,” explains Hendrik.

Looking ahead: support across career stages and addressing root causes

While mental health support for students is growing, postdoctoral researchers and faculty often remain overlooked. As Mikael recalls, “I can’t think of any university that doesn’t have mental health support for undergraduates – it feels old and well established. It’s

[2] https://www.cost.eu/remo-building-a-healthier-research-culture
[3] https://dragonflymentalhealth.org
[4] https://mhfainternational.org

newer, but widely accepted that graduate students may also need help. Postdocs, at least when I started out, had almost no support, and faculty even less. Once you got your PhD, you were basically on your own."

This highlights a pressing need to extend mental health resources and support systems to all career stages within academia, ensuring that no one navigating its unique challenges feels unsupported.

While mental health support is essential, it often addresses only the symptoms of deeper systemic challenges within academia. The current structure, with its hierarchical power dynamics, heavy emphasis on publication metrics, and limited tenure positions, can foster stress, unhealthy competition, and, at times, abuse. Of course, mental health struggles are complex and not always caused by the research environment, so there will always be a need for accessible, well-resourced support services. However, many of the widespread mental health challenges in academia are rooted in systemic issues that can and should be addressed.

Moving forward, efforts must focus on challenging and reforming these underlying conditions. Reforms, guided by robust data and shaped by voices from across all academic levels, can help build a healthier, more supportive environment for everyone in research.

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Thematic Working Group on Teaching and Learning of Discrete and Computational Mathematics, TWG11

ERME column regularly presented by Frode Rønning and Andreas Stylianides

In this issue presented by the group leaders Simon Modeste, Sylvia van Borkulo, Ulrich Kortenkamp, Janka Medová and David Zenkl

CERME Thematic Working Groups

We continue the initiative of introducing the CERME Thematic Working Groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be interesting or relevant for people working in pure and applied mathematics. Our aim is to disseminate developments in mathematics education research discussed at CERMEs and enrich the ERME community with new participants, who may benefit from hearing about research methods and findings and contribute to future CERMEs.

Introducing CERME's Thematic Working Group 11 – Teaching and Learning of Discrete and Computational Mathematics

This TWG was newly created for CERME14, emerging from the encounter of the interests of a previously existing group entitled *Algorithms* [3] and a prospective group project on discrete mathematics, based on common observations of recent trends in mathematics linked with computer science.

Indeed, the development of computer science in the last decades has generated changes in the contents and practices in mathematics, both in the advanced and theoretical dimensions and in the very applied and practical aspects. In particular, it has contributed to the recent emphasis and success of discrete mathematics, a classical topic in mathematics. Also, the development of computer science has led to the growth of computational approaches in mathematics, fostering applications and new ways of thinking and doing mathematics. These evolutions in the scientific field of mathematics, and the related needs for future generations, require and produce new contents in school that our TWG intends to address. The topics covered by the TWG take a growing place in many current curricula around the world and entered the common core of mathematics as described by international organizations like OECD. For this reason, research developments and insights are expected by teachers, teacher educators, and policymakers. Expert mathematicians and computer scientists have an important role to play, in collaboration with educators, to better understand the developments related to these trends and think about the changes in current and future curricula.

To address these challenges at CERME14, the TWG invited research papers discussing empirical, theoretical, methodological, or philosophical issues pertaining to the teaching and learning of discrete and computational mathematics, including the following themes: all aspects of discrete mathematics, i.e., both classical subjects, and subjects invoking new needs for computer science, namely, combinatorics, graph theory, discrete geometry, automata, game theory, cryptography, etc.; manipulation and representation of mathematical objects and data in computer science, simulations and discrete modelling, new computational views on classical mathematical objects, computational approaches in mathematics; links between mathematics and computer science, their foundations, concepts, ideas, and methods; algorithms in mathematics, the study and analysis of algorithms; computational thinking, and programming in mathematics; proof, proving, and problem solving in discrete and computational mathematics. Given the current curricular issues, our TWG has interest in these topics in pre-school, primary, and secondary levels, as well as studies at university level, and the implications for teacher education.

We were pleased that the 25 accepted papers and posters presented at CERME14 covered a wide diversity of the targeted topics. We organized them in four themes. The first theme was "Implementing computational thinking in mathematics." Computational thinking, although having various and sometimes blurry definitions [1], appears more and more in curricula [4], and focuses on the development of computational procedures and automation of mathematical concepts. This was explored at different levels and regarding various contents of mathematics, like geometry, functions, algebra, and also in programming for mathematics, and as a skill in itself that students must develop. The second theme was "Links between mathematics and informatics." In this theme, theoretical and experimental research contributed to discussing the relation between the two disciplines, including topics like the use of automatic proof checkers, algorithmics and programming in mathematical problem solving, the differences and similarities between mathematics and computer science activities and ways of thinking, and what concepts computer science can bring to think about mathematics education issues. The third and fourth themes were connected to discrete mathematics [2]. The third theme explored the "nature and diversity of discrete mathematics," questioning specific topics like graph theory, or tiling problems, but also the way discrete mathematics can connect or is connected with other mathematical topics. The fourth theme was covered by a group of communications specifically dedicated to "combinatorics," which seems to be the most developed part of discrete mathematics in school curricula in many countries. Combinatorics, more traditional in curricula, has been much investigated in the past, and contributions covered various questions in this area, such as students' recurrent difficulties and errors, specific strategies and approaches in combinatorics (like identifying "isomorphisms" between combinatorial structures in counting problems), and teaching and learning issues, including the use of programming/algorithmic skills in combinatorics.

Only one theme was underrepresented in the TWG: teacher education issues. This can be partly explained by the novelty of the matter addressed, which only starts to appear in the curricula, and although integrated in teacher education programmes, may not be ready for research and for the early stage of the TWG, which met for the first time at CERME14. We have also identified some relevant contributions that were submitted to other TWGs, which means that we should try to attract such contributions in the future. In any case, developing this dimension on teacher education is an identified goal for the next CERME. It became clear from the discussion that immediate actions that could affect teacher education must be developed, and that research is needed to support and develop it.

Apart from that, we were very happy to see that in the discussions, participants, whatever their principal interest, were equally involved in our two main topics, computational mathematics and discrete mathematics. The collective discussions and work showed rich connections between discrete and computational mathematics and confirmed that it is interesting to join and make them interact. Many perspectives have been opened, and we hope that the TWG will be able to develop long-term collaborations on these "hot topics" for mathematics education.

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The Klein Project – elementary mathematics from a higher standpoint

ICMI column in this issue presented by Hans-Georg Weigand

The Klein Project[1] *is an IMU and ICMI project with the aim of producing mathematics resources for secondary teachers on contemporary mathematics. It was inspired by Felix Klein's book "Elementary Mathematics from a Higher Standpoint," first published over 100 years ago. It is intended as a stimulus for mathematics teachers, to help them make connections between the mathematics they teach, or they can be asked to teach, and the field of academic mathematics, while taking into account the evolution of this field over the last century. In the following, the origin of this project, the aims and intentions, the present state and the upcoming aims are described.*

Felix Klein and the *Elementary Mathematics from a Higher Standpoint*

2008 marked the 100th anniversary of the founding of the ICMI (International Commission on Mathematical Instruction). The first president of this commission was Felix Klein (1849–1925). On the occasion of this event, the IMU (International Mathematics Union) and ICMI initiated a project to revive Felix Klein's vision, for which he wrote "Elementary Mathematics from a Higher Standpoint" in 1908 [1]. The aim of this three-volume work was to show mathematics teachers the breadth of mathematical research at the time, to provide them with background knowledge beyond the usual curriculum, and to strengthen the relationship with mathematics teaching at the secondary-school level. This work has become a classic reference both nationally in Germany and internationally; its title alone has become its programme.

Felix Klein always emphasized the great importance of teaching at the university, and he vehemently demanded a modernization of teaching at high school through a stronger emphasis on mathematics and the natural sciences. This was ultimately reflected in the "Merano Resolutions" (1905). In 1908, the German Committee for Mathematics and Science Education (DAMNU) was founded. Felix Klein was the chair of this committee for teacher education. In the same year, ICMI was founded, and Klein became the first chair of this commission until 1916.

The first volume of "Elementary Mathematics from a Higher Standpoint" was published in 1908 [1]. In the foreword, Felix Klein addressed the "mathematical public and especially the teachers of mathematics at our secondary schools" [2, p. IV]. After concentrating on the aims of learning and teaching – especially in the "Merano Resolutions" –, he now focused on the high school and the mathematics content. His aim was to "present the content and basis of the subjects to be covered in mathematics lessons, with reference to the actual teaching process, in the simplest and most stimulating way possible from the point of view of modern science." [2, Preface, p. IV].

For Klein, teacher education was an ongoing and constantly redefinable task, in which problems – similar to those in schools – arise again and again in the same or similar ways. For over 100 years, Klein highlighted a central problem in teacher education, expressed using the quite famous notion of "double discontinuity."

> *A mathematics freshman at the university is confronted with problems he had not been concerned with at school. After finishing university and becoming a teacher, he/she is expected to teach traditional elementary mathematics which he was not confronted with at university. Therefore, he teaches mathematics the way he was taught some years ago and his university studies remain only a more or less pleasant memory which do not influence his teaching.*[2] [1, p. 1]

The first discontinuity concerns the well-known problem of transition from school to university. The second discontinuity concerns the transition from university back to school. Felix Klein aims to address this with his "Elementary Mathematics." The first volume covers the areas of arithmetic, algebra and analysis, the second is about geometry, and the third is about precision and approximation in mathematics. Klein relates the concepts of these areas, often used intuitively at school, to the new findings from mathematical

[1] https://www.mathunion.org/icmi/projects/klein-project

[2] Cited from [3].

research on the fundamentals of mathematics at the beginning of the 20th century.

The original intention of the Klein Project

The "Klein Project," initiated by the IMU and the ICMI, aimed to publish a work that takes up the ideas of Felix Klein in a modern and up-to-date form and presents them in a way that teachers can understand. To this end, an international project team was formed with the task of presenting mathematical research and its applications in such a way that prospective and practising teachers are encouraged to convey an up-to-date and contemporary picture of mathematics to students in their lessons. The 2010 "Klein Design Team" consisted of the following members: the ICMI President Bill Barton (New Zealand), Michèle Artigue (France), Ferdinando Arzarello (Italy), Yuriko Baldin (Brazil), Graeme Cohen (Australia), Bill McCallum (USA), Tomás Recio (Spain), Christiane Rousseau (Canada), and Hans-Georg Weigand (Germany). Another idea of the project was to develop the contents and ideas of this new book concerning the mathematics of the 21st century in close cooperation with school teachers and in the context of the so-called "Klein Conferences." Following the first "Klein Conference" in Madeira (Portugal), 2009, there have been some more in different parts of the world: 2010 in Castro Urdiales (Spain), Oxford (UK), Belo Horizonte (Brazil) and Pittsburg (USA), 2011 in Palo Alto (USA), 2012 in Paris (France) and 2013 in Berlin (Germany). Some quite well-known mathematicians, experts in the areas of, for example, geometry, logic, discrete mathematics and statistics, and also in philosophy of mathematics, were asked to contribute to this book. However, immediately after the submission of the first drafts of articles, it became obvious that the big challenge around the transfer of the present state of these areas of mathematics to high school mathematics is to make it understandable for teachers. Moreover, what also became obvious, is the impossibility of writing the book at a consistent level if too many different authors contributed individual chapters.

The idea of Klein Vignettes

Right from the start of the project, the idea was – of course – to develop an attractive website in addition to the new book and to the Klein Conferences. After the not very encouraging experiences with the book articles, the idea was to collect first some central contents, aspects, theorems, connections and explanations of current mathematics in the form of shorter articles on the website. These articles were called "Klein Vignettes." Vignettes are intended to give teachers a sense of connectedness between the mathematics of the teachers' world and contemporary research and applications in the mathematical sciences. All Klein Vignettes should have the same or at least a similar structure. Thus, a Klein Vignette will start with something with which the teacher is familiar and then move towards a greater understanding of the subject through a piece of interesting mathematics. It will ultimately illustrate a key principle of mathematics. A Klein Vignette is a short piece of writing about a single mathematical topic. It assumes the reader has some undergraduate mathematical knowledge, and at least a willingness to engage with mathematical writing and reading. Klein Vignettes are not just "about" mathematics, they contain significant mathematics which is likely to be new to most secondary teachers and give them a perspective of current mathematics. In the following years many Klein Vignettes were produced and listed in a blog.[3] Some titles are as follows:

- How Google works: Markov chains and eigenvalues
- Goodstein sequences – the power of a detour via infinity
- Matrices and digital images
- A regular heptadecagon is constructible
- The enormous theorem
- Art and mathematics: knots and links
- How do I solve this equation? Look at the symmetries! – The idea behind Galois theory
- Dimension
- Fair voting: the quest for gold
- Banach's microscope to find a fixed point
- Benford's law: learning to fraud or to detect frauds?
- Map colouring and Gröbner bases
- Recurrence and induction
- Trying to predict a floating leaf: chaos and predictions
- The revenge of the infinitesimals
- The hairy ball theorem
- Calculators, power series and Chebyshev polynomials
- The shocking behaviour of moving fluids
- Public-key cryptography
- A tale of two triangles: Heron triangles and elliptic curves

The Klein Vignettes are available in English, and many are translated into Arabic, French, German, Italian, Spanish, Portuguese, Khmer and Mandarin.

When discussing these vignettes with teachers, they appreciated the information about some pieces of present mathematics. However, they also asked for support and help to transfer some of these ideas into the classroom. This was the reason to launch "Klein Bridging Vignettes," which bridge the gap between the mathematics explained in classical Klein Vignettes and their use in the classroom. We wanted to motivate especially mathematics educators to write Klein Bridging Vignettes for the Klein Project. Some examples are as follows:

- From synthetic geometry to dynamic geometry and back: the case of circular inversion

[3] https://blog.kleinproject.org

- What is the way of packing oranges? – Kepler's conjecture on the packing of spheres
- Higher dimensions
- Symmetry step by step

Besides the Klein Vignettes and the Bridging Vignettes, we also established a column "Site of the month" on the website. These sites present some short information about new published books, interesting projects and websites in relation to the aims and topics of the Klein Project. In this respect, some of the books are as follows:

- Étienne Ghys, La petite histoire des flocons de neige
- Richard Guy, Unsolved Problems in Number Theory
- Eli Maor, The Pythagorean Theorem
- Hans-Georg Weigand, William McCallum, Marta Menghini, Michael Neubrand and Gert Schubring (eds.), The Legacy of Felix Klein
- David Richeson, Euler's Gem: The Polyhedron Formula and the Birth of Topology
- Douglas Hofstadter, Metamagical Themas
- Renate Tobies, Felix Klein: Visions for Mathematics, Applications, and Education
- David E. Rowe, A Richer Picture of Mathematics: The Göttingen Tradition and Beyond
- Otto Neugebauer, The Exact Sciences in Antiquity
- David Hilbert and Stefan Cohn-Vossen, Geometry and the Imagination
- Maria G. Bartolini Bussi and Michela Maschietto, Macchine matematiche: Dalla storia alla scuola

And these are some of the projects:

- The German Imaginary project[4]
- The Global Math Project[5]
- The Mathematics News Snapshots for High School (MNS) project[6]
- 3Blue1Brown[7]
- Wolfram MathWorld[8]
- Mathematical Etudes[9]
- Accromath[10]

The dissemination of the Klein Project

So far, the ideas of the Klein Project have been disseminated at many conferences, such as IMPA in Rio de Janeiro (Brazil) 2014, EARCOME 7 in Cebu City (Philippines) 2015, ICME13 in Hamburg (Germany) 2016, EARCOME 8 in Taipei (Taiwan) 2018, PME 42 in Umeå (Sweden) 2018, PME 43 in Pretoria (South Africa) 2019, and ICME14 in Shanghai (China) 2020. At ICME13 in Hamburg, there was a "Thematic Afternoon" with the topic "What is and what might be the Legacy of Felix Klein?", organized by Hans-Georg Weigand, William McCallum, Marta Menghini, Michael Neubrand, Gert Schubring and Renate Tobies. The starting point of this afternoon was the point of view that, at present, the problems Felix Klein saw at university and high school are similar or even the same 100 years later. Speaking about Felix Klein's legacy meant discussing the solutions to problems that he suggested and hoping to get answers to some of the problems we are struggling with today. The results of the meeting are available in "The legacy of Felix Klein," a 2019 book by H.-G. Weigand and colleagues [4]. At a Felix Klein workshop at ICME14 in Shanghai, 2020, "Vignettes in Practice" were presented. Moreover, we are in contact with some other projects, such as:

- French project "Images des mathématiques"[11]
- "Liceo Matematico" project in Italy[12]
- Mathematics News Snapshots for High School (MNS) project in Israel[13]
- Cornerstone Maths project in England[14]

The transfer of the website to the IMU server

Since some years ago, the website of the Klein Project exists on the IMU server.[15] However, for many years the collected Klein Vignettes, the Klein Bridging Vignettes and the Site of the month were hosted by Bill McCallum on the server of the University of Arizona. During the years of the pandemics, the Klein Project did not progress. There have been only a few Klein Conferences for professional development organized, for example, by Yuriko Baldin in Brazil, Michèle Artigue in France or Ferdinando Arzarello in Italy. Moreover, it was obvious that the Klein Project needed some restructuring and revival. In December 2024, all the files of the Klein blog were transferred to the IMU server in Berlin. At the moment, the website is not yet well-structured and still remains under construction. The reconstruction of the Klein Project homepage will be used to start the project in a refreshed way.

4 https://www.imaginary.org
5 https://globalmathproject.org
6 https://mns.org.il
7 https://www.3blue1brown.com
8 https://mathworld.wolfram.com
9 https://en.etudes.ru
10 https://accromath.uqam.ca
11 http://images.math.cnrs.fr
12 https://www.liceomatematico.it
13 https://mns.org.il
14 https://www.ucl.ac.uk/cornerstone-maths
15 https://www.mathunion.org/icmi/projects/klein-project

About the ongoing project

The Klein Design Team nowadays comprises Ferdinando Arzarello (Italy), Michèle Artigue (France), Yuriko Baldin (Brazil), Samuel Bengmark (Sweden), William McCallum (USA), and Hans-Georg Weigand (Germany). At the CERME14 conference in Bolzano in February 2025, a Klein information meeting was organized to recruit new people who will contribute to the project development. In particular, we were looking for mathematicians, mathematics educators and teachers who are interested in the following activities:

- writing new Klein vignettes and bridging vignettes
- translating existing vignettes into any language
- writing a page of the month
- providing technical support for the homepage and blog
- making connections with other projects
- encouraging collaboration with teachers in designing concrete classroom activities based on Klein Vignettes
- organizing a Klein conference
- establishing contacts with organizations that support the Klein Project financially

We were surprised by the great interest in the Klein Project. We subsequently received positive feedback from interested parties who wanted to get involved in the project. We are currently in the process of integrating these people into the project, but we are also grateful for anyone who would like to work on it – in whichever form.

References

[1] F. Klein, *Klein Elementarmathematik vom höheren Standpunkte aus. Teil I. Arithmetik, Algebra, Analysis*. B. G. Teubner, Leipzig (1908)

[2] F. Klein, *Elementary mathematics from an advanced standpoint. Arithmetic–algebra–analysis*. Macmillan, London (1932)

[3] F. Klein, *Elementary mathematics from a higher standpoint. Vol. I–III*. Springer, Berlin, Heidelberg (2016)

[4] H.-G. Weigand, W. McCallum, M. Menghini, M. Neubrand and G. Schubring (eds.), *The legacy of Felix Klein*. ICME-13 Monographs, Springer, Cham (2019)

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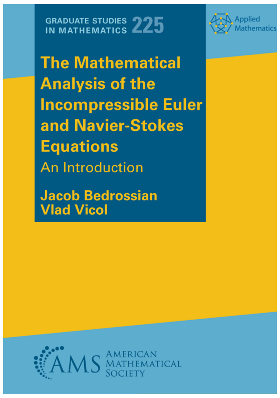
Book review

The Mathematical Analysis of the Incompressible Euler and Navier–Stokes Equations: An Introduction by Jacob Bedrossian and Vlad Vicol

Reviewed by Hao Jia



The book under review is an introduction to the mathematical theory of incompressible Navier–Stokes and Euler equations, with an emphasis on the regularity theory of solutions, and includes one chapter on recent research trends and important achievements. The authors, who are leading figures in the analysis of partial differential equation of fluids, have made important contributions in subjects like hydrodynamic stability at high Reynolds numbers, uniqueness of weak solutions to Navier–Stokes equations, statistical laws in fluid equations, shock formation in compressible fluids, stability of coherent solutions for Euler equations, among many other accomplishments.

The main topics covered include physical derivation and interpretation of key quantities that appear in the equations, local well-posedness and regularity criteria of classical solutions, existence theory of mild solutions, existence and partial regularity of Leray–Hopf weak solutions, and a survey of selected research topics. The appendix provides essential technical tools used throughout the book. Each chapter also contains a list of exercises which complement and extend the contents in that chapter. The book is essentially self-contained, with complete proofs of central theorems. The survey discusses important research directions not covered in the previous chapters, such as stationary solutions, self-similar solutions, potential singularity formation scenarios, hydrodynamic stability problems and Onsager's conjectures, and more. Many of these directions are still of great interest in the research community, to which the authors have made seminal contributions.

It is clear that this is not a comprehensive monograph on Euler and Navier–Stokes equations. Instead, the authors are sharply focused on the main ideas and key techniques in the simplest and most important cases. For example, the existence theory is presented in either the whole space or periodic domains, thus avoiding technical issues related to nontrivial boundaries. In the reviewer's opinion, for an introductory text aimed primarily at graduate students and mathematicians who have a general interest in equations of fluid dynamics, this is an excellent choice. It allowed the authors to present the core ideas in a direct and concise way. It also made it possible to cover both the incompressible Euler and Navier–Stokes equations, which is an important feature of the book. As a result, the physically significant issue of inviscid limit from the Navier–Stokes to Euler equations can be brought up naturally.

An extremely valuable contribution of the book, one that will be much appreciated by readers, is the survey of research topics in Chapter 6, where the authors provide a general review of twelve current research directions concerned with the equations of incompressible fluids. The surveys are not just compilations of references. Rather, the motivation, history, known results and open problems are all expertly presented with comments on promising future directions. This chapter will be useful not only for graduate students, but also for researchers, providing a broad view of the research landscape in incompressible fluid equations and a source of inspiration for new projects.

In summary, this is an excellent introductory textbook suitable for graduate students, instructors and general mathematicians with an interest in equations of fluids. Researchers on Navier–Stokes and Euler equations will also find it handy to keep in their library for quick references. By design, the book is *lightweight* and *focused*, which makes it a fantastic choice for a one-semester course on the subject. The readers will benefit from the clear presentation of the general material and deep insights in the survey, while having an enjoyable learning experience.

Jacob Bedrossian and Vlad Vicol, *The Mathematical Analysis of the Incompressible Euler and Navier–Stokes Equations: An Introduction.* Graduate Studies in Mathematics 225, American Mathematical Society,

2022, xiii+218 pages, Softcover ISBN 978-1-4704-7178-1,
eBook ISBN 978-1-4704-7177-4.

Hao Jia is an associate professor of mathematics at the University of Minnesota. His main current research interest is partial differential equations arising from the mathematical theory on the dynamics of incompressible fluid flows. He is on the editorial board of the journals Communications on Pure and Applied Analysis and Journal of Differential Equations.

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Report from the EMS Executive Committee meeting in Protaras (Cyprus), 28–29 March 2025

Enrico Schlitzer

The EMS Executive Committee held its meeting from 28 to 29 March 2025 in Protaras, Cyprus. This report focuses on matters of direct interest to EMS members, particularly the Committee's strategic objectives and positions on relevant issues, leaving aside technical and procedural details. The EMS warmly thanks the Cyprus Mathematical Society and its president, Gregory Makrides, for their generous hospitality in a beautiful and welcoming setting.

The meeting saw the participation of all Executive Committee members alongside invited guests, including Donatella Donatelli (EMS Magazine editor-in-chief), Ann Dooms (chair of the Education Committee), André Gaul (EMS Press managing director), Elvira Hyvönen (EMS Secretariat) and Enrico Schlitzer (Community Engagement Manager).

The Committee welcomed the International Centre for Mathematics in Ukraine (ICMU) as a new member of the EMS, reflecting the Society's continued commitment to supporting mathematics across all regions. The EMS Young Academy continues to flourish with innovative initiatives. Two EMYA members, Jelena Jankov Pavlović from the University of Osijek and Cristina Molero-Río from École Polytechnique, have successfully launched the Mathematics Online Seminar Series (MOSS). The series features distinguished speakers including EMS Prize winners such as Cristiana De Filippis, Thomas Hutchcroft, Adam Kanigowski, Jessica Fintzen and Richard Montgomery, with video recordings made available on the EMS YouTube channel. EMYA is also organizing an in-person meeting at the Banach Center.

The Indo-European Conference on Mathematics is taking shape nicely, thanks to the dedicated work of co-chairs Ramachandran Balasubramanian, Sudhir R. Ghorpade and Susanna Terracini. The event promises to be a truly international affair, bringing together mathematical excellence from across continents with ten carefully selected plenary speakers: four representing European mathematics, four from India's vibrant mathematical community, and two from other regions around the globe. Beyond the main lectures, attendees can look forward to a rich programme featuring ten to fifteen specialized mini-symposia and around 30 contributory talks, selected by the Scientific Committee to showcase cutting-edge research and foster cross-cultural dialogue.

Gregory Makrides and Jan Philip Solovej.

EMS Press continues to demonstrate the success of its Subscribe to Open (S2O) model, with all 26 journals publishing as open access for 2025. This marks the second consecutive year of full open-access publishing across the entire portfolio. Managing Director André Gaul reported steady journal publishing with reduced backlogs, particularly in the Journal of the EMS, achieved through increased online-first publishing. The company published ten new books in 2024, adding to a catalogue of over 250 titles. EMS Press welcomed two new team members to enhance technical capabilities: Arne Schlüter joined as Head of Technology in March 2025, and Goulven Schaal was hired as software developer starting in April.

Thanks to their valuable contributions, the EMS is set to significantly strengthen its technological infrastructure, ensuring the Society can keep pace with its expanding digital ambitions while delivering seamless operations across all platforms. For example, the EC decided to invest more substantially in the EMS job portal,

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Group photo of the participants at the Presidents Meeting, which took place in the days following the EC Meeting.

recognizing it as important infrastructure for the mathematical community. This aims to enhance the platform's functionality and better serve both jobseekers and employers in mathematical sciences.

The EMS Magazine under editor-in-chief Donatella Donatelli continues its comprehensive coverage of mathematical activities. The magazine successfully maintains its steady output with issues averaging 70 pages and continues publishing articles by the fourteen prize winners from the 9th ECM in Seville. The Problems Corner will be restarted from the September issue, and the Magazine plans to publish two to three articles related to the International Day of Mathematics in each March issue.

Following a comprehensive evaluation, the EC decided to reopen the Strategic Activities programme with enhanced structure and clearer guidelines. A maximum of one strategic activity will be funded each year, and only existing EMS Thematic Activity Groups will be eligible to propose activities. The process will involve two stages: expression of interest, followed by detailed proposals, with specific deadlines established throughout 2025.

The evaluation of the six existing EMS-TAGs revealed that all groups maintain high levels of activity and strong online presence. The Evaluation Committee, composed of Victoria Gould, Barbara Kaltenbacher, Susanna Terracini, and now joined by Alain Valette, will draft guidelines for the renewed Strategic Activities programme.

The Education Committee, under new chair Ann Dooms, has outlined ambitious plans including a series of webinars featuring world-class experts who have contributed through survey papers or extensive educational experience. These webinars will be open to members of local mathematical societies and affiliated institutions, serving as foundations for policy documents, while being recorded for wider accessibility. The Committee also plans surveys on how mathematicians are involved in secondary school education and will develop materials addressing mathematics and artificial intelligence.

The EMS Code of Ethics framework continues to develop through collaborative efforts involving the Ethics Committee, the Committee for Publications and Electronic Dissemination, and the Women in Mathematics Committee. Updated documents, including the Code of Practice for Mathematical Publication and procedures for raising complaints, have been approved and published on the EMS website. The Ethics Committee is expanding its scope to address broader aspects of professional conduct in mathematics, encompassing research integrity, publication practices, teaching, and practical applications of mathematics.

Community Engagement Manager Enrico Schlitzer reported significant developments across multiple areas of EMS outreach. The inaugural issue of EMS Updates, the Society's new email newsletter, was successfully launched with a structured approach that integrates with the EMS website's news section. A communication campaign is underway to increase EMS membership, focusing particularly on early-career researchers and developing new member benefits. The management of PopMath and the EMS job hub has been enhanced, with approximately 100 mathematics events advertised since the last report. Social media presence has expanded with new Bluesky profiles for both EMS and EMS Press, complementing existing platforms and utilizing improved scheduling tools. The successful implementation of the Good Grants platform has streamlined application processes for various committees, though initial challenges due to higher application volumes have been addressed.

Throughout the meeting, the Executive Committee emphasized its unwavering commitment to diversity in all forms – geographic, gender-based, and across mathematical fields – ensuring that the EMS continues to represent and serve the entire spectrum of European mathematical activity.

Enrico Schlitzer is a journalist and a science communicator with a background in mathematics (PhD from SISSA, Italy). He served as community engagement manager for the European Mathematical Society and its publishing house EMS Press.

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Resignation letter

To the president of the European Mathematical Society, the EMS Executive Committee and the editors of the EMS Magazine

Subject: *Resignation from the EMS in protest over the humanitarian catastrophe in Gaza*

Dear colleagues,

With a sense of deep sadness and moral conviction, I hereby tender my resignation from the European Mathematical Society, of which I have been a lifelong member. I do so in protest against the ongoing humanitarian catastrophe in Gaza, and in response to the EMS's silence in the face of such an unprecedented tragedy.

What we are witnessing is not a conventional military conflict, but a massive and disproportionate use of force against a civilian population. The death toll has reached devastating levels, with tens of thousands of civilians killed, including thousands of children. Entire families have disappeared from civil registries. Hospitals, schools, universities, refugee camps, and humanitarian convoys have been repeatedly struck. Access to food, clean water, medical care, and shelter has been deliberately obstructed. These facts have been documented in real time and are now widely recognized as evidence of possible crimes under international law, including by United Nations bodies and the International Court of Justice in its interim rulings.

As scientists, and as human beings, we cannot claim neutrality when faced with such large-scale human suffering. In such cases, silence is not a position of balance—it becomes a form of complicity. The EMS, which rightfully took a public stance and suspended cooperation with a Russian state institution after the invasion of Ukraine [following the general policy of the European Union], has not found the strength to express even minimal concern or condemnation in response to a much longer and even more destructive campaign against civilians.

This discrepancy raises serious ethical questions. Scientific communities have a responsibility to uphold not only intellectual values, but also universal human values. We cannot selectively decide when to express solidarity, nor remain silent when the scale of suffering is so overwhelming and so clearly documented.

For these reasons, I am resigning from the EMS as an act of protest, in the hope that it may contribute to opening a serious and responsible debate within the mathematical community. This community has, in the past, shown sensitivity and courage in speaking out against injustice and violence. I believe it is time to do so again.

I respectfully request that this letter of resignation be published in the EMS Magazine, as a call to conscience and an invitation to reflect on the role of scientific societies in the face of grave human suffering.

Yours sincerely,
Enrico Jabara
Lifelong member of the EMS
Venice, 12 August 2025

Response from the Executive Committee of the EMS

The European Mathematical Society Executive Committee and the editor-in-chief of the EMS Magazine have, on the request of Professor Enrico Jabara, agreed to publish his letter of resignation above. It is the policy of the EMS Magazine to publish letters to the editors that are considered to be of interest to its readership, without indicating that the views expressed are the views of the EMS or those of the editors of the EMS Magazine. The publication of Professor Jabara's letter should not be seen as an invitation to start an open discussion in the Magazine and the Magazine retains the rights not to publish such follow-up letters. The response below from the EMS Executive Committee was sent to Professor Jabara.

Dear Professor Jabara,

We have received your letter of resignation, and we are very sorry that you have decided to leave the EMS.

We all understand that the conflict in the Middle East has led to great suffering on both sides. As individuals of conscience, we agree with you that the humanitarian crisis in Gaza has reached unbearable and undefendable proportions.

The European Mathematical Society, as a legal organization under Finnish law, is not able to judge nor comment on the extent to which crimes may have been committed. We are not a political organization that speaks out on humanitarian crises, even those that are consequences of wars or conflicts. As you say, we are all free to do so as individual human beings and scientists.

You are making a reference to our actions in response to the attack of Russia on Ukraine. At that time, the European Mathematical Society decided to follow the European Union's recommended actions against Russia and freeze cooperations with Russian state institutions. Consequently, we suspended collaboration with the Euler International Mathematical Institute. At the same time, we used the opportunity to express solidarity with our colleagues on both sides of the conflict. Our actions against the Euler Institute were taken with some regret, as we fundamentally believe in the value and lasting benefit of apolitical scientific cooperation. In fact, while we were under some pressure to do so, we did not suspend collaboration with our two Russian society members, which are not state institutions: The Moscow Mathematical Society and the Saint Petersburg Mathematical Society. In line with this policy, we have not taken any action against institutions in the Middle East.

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European Mathematical Society

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- 80 € for persons not belonging to any EMS corporate member
- A particular reduced fee of 8 € can be applied for by mathematicians who reside in a developing country (the list is specified by the EMS CDC). This form of membership does not give access to the printed version of the EMS Magazine.
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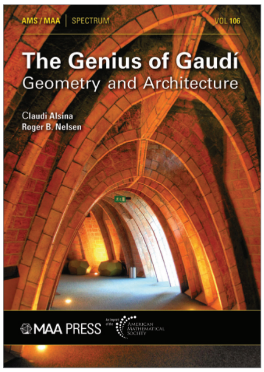

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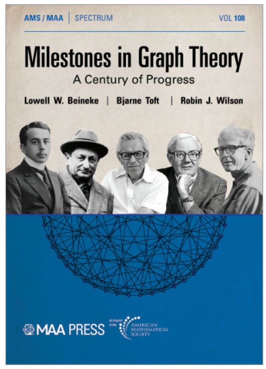

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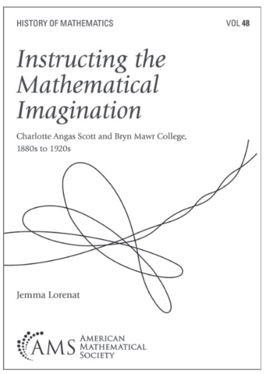

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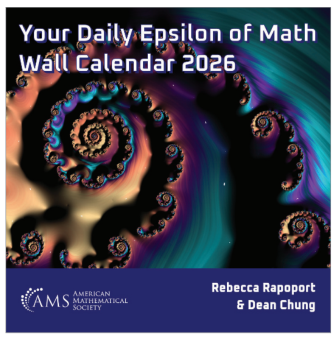

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