

EMS Magazine

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Interview with Sir John Kingman

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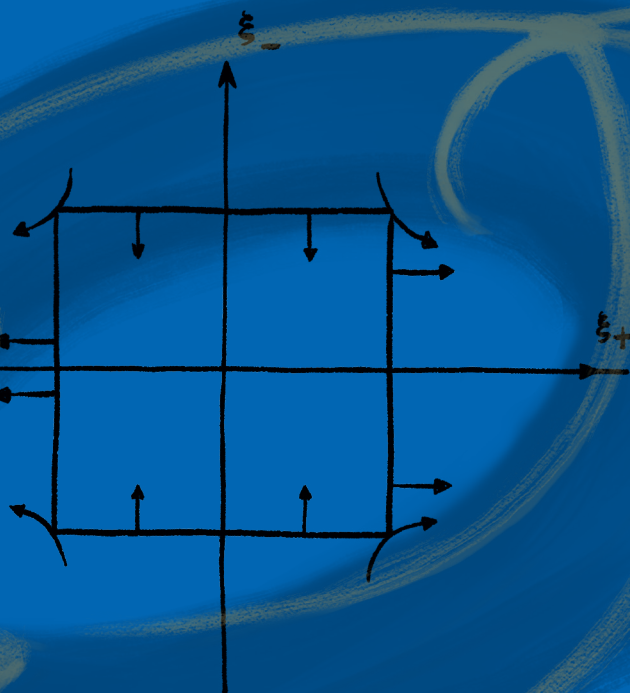
Side by side: interview with

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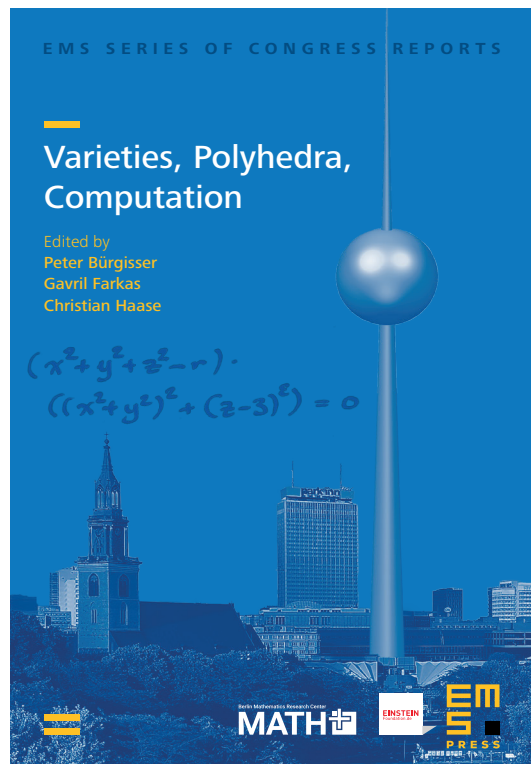
Fraudulent publishing in mathematics:
a European call to action and how
information infrastructure can help

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Varieties, Polyhedra, Computation

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Peter Bürgisser (Technische Universität Berlin, Germany)

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Christian Haase (Freie Universität Berlin, Germany)

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This volume collects contributions from the thematic Einstein semester on algebraic geometry that took place in the winter semester 2019/2020 in Berlin, Germany and was devoted to interactions of algebraic geometry with other fields as well as applications with a potential impact from/on algebraic geometric methods. Following the semester's topics, the present volume includes a selection of important survey and research articles by leading researchers in the field, which will be of interest to both young and experienced mathematicians working in algebraic geometry and its applications.

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Editor-in-Chief

Donatella Donatelli
Università degli Studi dell'Aquila
donatella.donatelli@univaq.it

Editors

António B. Araújo (Art & mathematics)
Universidade Aberta
antonio.araujo@uab.pt

Karin Baur (Raising public awareness)
Ruhr-Universität Bochum
karin.baur@ruhr-uni-bochum.de

Jean-Bernard Bru (Contacts with SMF)
Universidad del País Vasco
jb.bru@ikerbasque.org

Krzysztof Burnecki (Industrial mathematics)
Wrocław University of Science and Technology
krzysztof.burnecki@pwr.edu.pl

Jason Cooper (Maths education)
Weizmann Institute of Science
jason.cooper@weizmann.ac.il

Michele Coti Zelati (Book reviews)
Imperial College London
m.coti-zelati@imperial.ac.uk

Jezabel Curbelo (Research centers)
Universitat Politècnica de Catalunya
jezabel.curbelo@upc.edu

Ralf Krömer (History of mathematics)
Bergische Universität Wuppertal
rkroemer@uni-wuppertal.de

Youcef Mammeri (Features and discussions)
Université Jean Monnet Saint-Étienne
youcef.mammeri@math.cnrs.fr

Octavio Paniagua Taboada (Zentralblatt column)
FIZ Karlsruhe
octavio@zentralblatt-math.org

Ulf Persson (Interviews and discussions)
Chalmers Tekniska Högskola
ulfp@chalmers.se

The cover illustration by A. B. Araújo is inspired by a figure from the 1983 paper of Conley and Zehnder, in which the Arnold conjecture is proved in the case of symplectic tori. A central idea of their work is to reinterpret fixed points of a Hamiltonian diffeomorphism as 1-periodic orbits of the associated time-dependent Hamiltonian flow, realised as critical points of an action functional on the loop space of the torus. Conley and Zehnder then introduce a Lyapunov–Schmidt reduction of this infinite-dimensional variational problem to a finite-dimensional one, leading to the isolating block depicted here and yielding the lower bounds on the number of periodic orbits and hence fixed points predicted by the Arnold conjecture.

Jesse Railo (Young mathematicians column)
Lappeenranta-Lahti University of Technology
jesse.railo@lut.fi

Marie E. Rognes (Features and discussions)
Simula Research Laboratory, Oslo
meg@simula.no

Bruno Teheux (Features and discussions)
University of Luxembourg
bruno.teheux@uni.lu

Valdemar V. Tsanov (Problem corner)
Bulgarian Academy of Sciences
valdemar.tsanov@math.bas.bg

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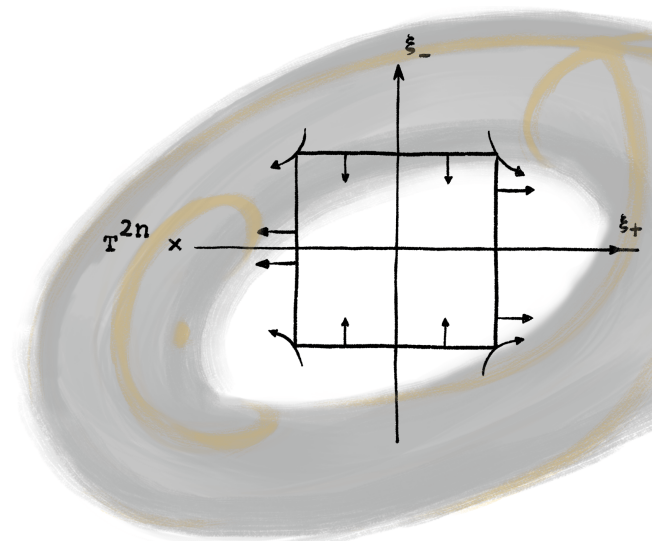
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A message from the president



Photo by Jim Høyer,
University of Copenhagen.

This is the final issue of 2025, but it also marks the start of the new year, so happy new year to all. A new year means that we say both welcome and goodbye to colleagues that work tirelessly to support the EMS. Several of our standing committees will get a new chair. I would like to thank Carola-Bibiane Schönlieb who chaired the Committee for Applications and Interdisciplinary Relations (CAIR), Balázs Szendrői who chaired the Committee for Developing Countries (CDC), and Mikaela Iacobelli who chaired the Committee for Women in Mathematics (WiM) for all the hard work they offered the EMS and I would like to welcome the new chairs: Robert Schichl (CAIR), Conall Kelly (CDC), and Marjeta Kramar Fijavž (WiM). I am looking forward to working closely with you in the new year.

One of the really big changes in the EMS is that we have had to say goodbye to our community engagement manager, Enrico Schlitzer. Enrico, who was the first to be appointed in this new position in 2024, decided for personal reasons to leave the EMS. We were very disappointed to lose Enrico, who had lots of ideas about how EMS could improve its communication and engagement with the European mathematical community and how it could support all aspects of mathematics in Europe. In an earlier message I had mentioned some of Enrico's ideas, particularly those that engage younger researchers. I want to thank Enrico for the efforts he made for the EMS and for shaping the role of the community engagement manager. The role is among many other things to create awareness of the EMS and of our publishing house EMS Press within the broader European mathematical community through social media and other communication initiatives. Luckily, we were only without community engagement manager for a short time as we were able to hire Markus Juvonen who is about to finish his PhD at the University of Helsinki. Markus has made a fantastic effort to secure a smooth takeover of the responsibilities both in what happens behind the scenes, e.g., in facilitating our many calls for support and what happens more openly concerning our engagement with the community. It is therefore a great pleasure to use this opportunity to thank Markus and introduce him to all of you. We have decided to expand the community engagement manager role to include administrative support for our officers and for EMS Press. All the EMS administration has been run through the secretariat in Helsinki, until recently almost single-handedly by Elvira Hyvönen. It has never been quite clear to me how Elvira manages all that work. As the

EMS is expanding its activities, we have been relying more and more on the volunteer work of devoted members of our society. This is not sustainable, and for this reason we have decided to increase the responsibilities of the community engagement manager.

The main goal is still to create greater awareness and visibility of the EMS. Our hope is that these efforts will also increase our individual membership. While we are very happy with the number of institutional members we have, we would value many more individual members. It is wonderful that the national mathematical societies that are institutional members of the EMS each have a substantial membership base, but many are not members of the EMS. I think this is unfortunate not only because the EMS need their support but also because I strongly believe it is very important that we unite not only on a national level but also internationally. Many of the challenges we face regarding both education and research are the same or at least similar independently of where we live. We should of course not focus only on challenges but also on opportunities. Mathematics is a striving field with applications that have important implications in all aspects of society. The role of mathematics in society may often be hidden, but it is the ambition of the EMS to showcase it. We can only do this if we stand together. If you are not a member of the EMS, please join today. It is easy (and fairly cheap). If you are a member, ask a friend or colleague to join.

The Society has or is in the process of launching new initiatives to strengthen the ties within the community and our visibility to stakeholders.

The EMS has launched—and continues to develop—new initiatives to strengthen ties within the mathematical community and improve our visibility among politicians and stakeholders. I hope to tell you more in the new year. Let me just mention here that one initiative has been to engage with the European Research Council. Another initiative that I also highlighted earlier is the EMS Lecture Series on Mathematics Education¹ organized by the EMS Education Committee. I hope you will join us for this impressive list of speakers for this Spring. This should be of interest broadly, so please help us spread the word. Everyone can watch the live stream or see the recordings later. Finally, let me point you to my joint contribution with Moritz Schubotz to this issue of the Magazine on *Fraudulent publishing in Mathematics: a European call to action and how information infrastructure can help*. I hope you will also enjoy reading everything else in the Magazine.

With this I wish you all a happy and productive 2026.

Jan Philip Solovej
President of the EMS

¹ <https://euromathsoc.org/news/ems-lecture-series-on-mathematics-education-190>

Brief words from the editor-in-chief



Dear readers of the EMS Magazine,

This is the last issue published in 2025. The overall balance of the year is highly positive: the articles we published span a wide range of topics, including interviews, scientific insights, education, gender issues, generational perspectives, and equity.

As usual, this issue will contain many interesting articles on a wide range of topics. Without diminishing other contributions, I would like to particularly highlight the new series of interviews whose title is “Side by side.” In this new interview series for the EMS Magazine, each article brings together two mathematicians who share something—such as a research area, a passion, or a professional context—but differ in other ways, as for instance age, gender identity, background, or geographical location. Sitting “side by side,” they reflect on their personal and professional journeys, their views, and the worlds they live in. The series is conducted by Raffaella Mulas and was born sharing some ideas together with Apostolos Damialis (editorial director of EMS Press).

With the current issue, two new editors are starting their duties in the editorial board: Jesse Railo, who will take care of the Young mathematicians column and Boris Adamczewski, who will be responsible for the contacts with the French Mathematical Society. A short biographical note of each of them can be found at the end of the issue. I thank them, and I am grateful to them for accepting to dedicate part of their time to our Magazine.

Our contributions continue to draw interest from other societies, confirming the relevance and impact of our work. This growing visibility encourages us to maintain the same level of commitment and quality in the coming year. I am grateful to all authors, reviewers, and readers who have supported the Magazine and helped it evolve. As we look ahead to 2026, I am confident that new ideas, collaborations, and voices will further enrich our publication and strengthen our community.

Finally, I warmly wish all of you a new year rich in inspiration, collaboration, and discoveries. Let us continue to share ideas openly, engage in meaningful dialogues, and build a truly inclusive space where every voice can contribute to advancing knowledge.

Donatella Donatelli
Editor-in-chief

Cycles and expansion in graphs

Richard Montgomery

Cycles are fundamental objects in graph theory, where their inherent simplicity belies the depth of even some simply stated questions. In this article, I will discuss three problems on cycles in graphs and recent progress on them. In each case, the progress has been made by new and different tools involving graph expansion, itself an important topic in extremal graph theory.

1 Eulerian graphs and the Erdős–Gallai problem

The advent of graph theory is often pinned to the Königsberg bridge problem from the 18th century. At the time, Königsberg had seven bridges connecting either side of the Pregel River and the two islands within it (see Figure 1). Was it possible to walk through the city while crossing each bridge exactly once? In 1735, this problem reached Euler, who comprehensively solved it in full generality. Representing each connected land mass by a vertex and each bridge by an edge between the two vertices it connects, we get a graph. Euler showed that there is a walk in a graph passing through every edge exactly once if and only if it is connected¹ and at most two vertices have odd degree² (if two such odd-degree vertices exist they must be the start and end vertices of the walk). Thus, there is no solution to the Königsberg bridge problem, as the corresponding graph has four vertices with odd degree.

A slightly neater equivalent formulation of this general problem is to ask when a graph has an *Eulerian tour*, which is a walk through the edges of the graph, covering each edge exactly once, and

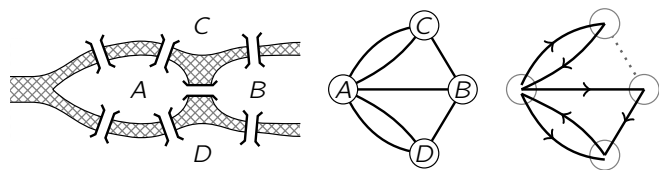


Figure 1. The Königsberg bridge problem, its representation as a (multi)graph and a walk crossing all but one edge/bridge.

arriving back at the start. As Euler showed, there is an Eulerian tour if and only if the graph is connected and every vertex has even degree. Note that the general solution to the original bridge-crossing problem can be deduced by applying this, after first adding a fictitious edge between two odd-degree vertices when they exist.

Behind Euler's work is a simple result: if all the vertex degrees of a graph G are even, then its edges can be decomposed into cycles, i.e., its edges can be exactly partitioned into cycles (see Figure 2 for a similar decomposition). As removing the edges of a cycle maintains the parity of each vertex degree, this can be proved easily by induction, as every graph with even vertex degrees and at least one edge has at least one cycle (and on the other hand any graph decomposable into cycles has even vertex degrees). To reach Euler's result, then, first decompose any connected graph with even vertex degrees into cycles, C_1, \dots, C_k . Then, start off walking around the edges of C_1 but follow the rule that whenever we first encounter a vertex on a new cycle C_i , we break off and start walking around that cycle, giving an iterative process that can be seen to produce an Eulerian tour. On the other hand, the existence of an Eulerian tour in a graph easily shows that it must be connected and have even vertex degrees.

Above, in the space of a few lines, we have characterised with proof exactly those graphs which can be decomposed into cycles. However, with a slight change, we can reach a much deeper and more challenging problem. First, let us note the following: so far we have quietly been dealing with *multigraphs*, where a pair of vertices can have more than one edge between. From now on, every graph we consider will be assumed to be a *simple* graph, with at most one edge between each pair of vertices. Now, what if we ask for a decomposition of the graph into *few* cycles? In 1966, Erdős and Gallai [17] conjectured that every n -vertex Eulerian graph should have a decomposition into $O(n)$ cycles, in the following equivalent form.

Conjecture 1.1 (Erdős and Gallai). *Every n -vertex graph has a decomposition into $O(n)$ cycles and edges.*

Of course, if true, then this conjecture is tight up to the implicit constant, as demonstrated, for example, by the n -vertex graph with

¹Any vertex can be reached from any other by a path in the edges.

²The number of edges containing that vertex.



Figure 2. A graph on the left decomposed into three cycles and four (dotted) edges on the right.

all possible edges: when n is even any such decomposition needs at least $n/2$ edges and at least $(n-2)/2$ cycles. Erdős observed that a construction of Gallai can be improved to show that at least $(3/2 - o(1))n$ cycles and edges may be needed. Interestingly, the number of cycles and *paths* required to decompose any graph is well understood, thanks to an old result of Lovász [32], who unimprovably showed that any n -vertex graph can be decomposed into at most $\lceil n/2 \rceil$ paths and cycles.

As observed by Erdős and Gallai, $O(n \log n)$ cycles and edges can easily be seen to suffice for decomposing any n -vertex graph. Indeed, given an n -vertex graph G , you can iteratively remove a longest cycle until no cycles (and thus at most $n-1$ edges) remain. Any graph with average degree $d \geq 2$ has a subgraph with minimum degree at least $d/2$ and thus a cycle with length at least $d/2$ (seen, for example, by considering a longest path and the neighbouring vertices of one of its endpoints, which must lie within that path). Therefore, if we iteratively remove longest cycles from a graph with average degree d , after removing $O(n)$ cycles the average degree will be below $d/2$. Tracing the decrease in the average degree of G as longest cycles are removed, we therefore will remove $O(n \log n)$ cycles before only edges remain.

It took around 50 years for this simple bound to be improved, despite Erdős highlighting the problem in many of his problem collections. Finally, in 2014, it was shown by Conlon, Fox and Sudakov [9] that $O(n \log \log n)$ cycles and edges suffice to decompose any n -vertex graph. As discussed below, a critical concept behind this breakthrough was *expansion* in graphs. More recently, Bucić and I [7] were able to use a much more delicate form of this, known as *sublinear expansion*, to push this bound lower. Specifically, this allowed us to improve the $\log \log n$ term to the iterated logarithm function $\log^* n$, defined as the least k such that the k -fold logarithm of n , $\log(\log(\dots \log(n)))$, is at most 1. That is, we showed the following.

Theorem 1.2. *Any n -vertex graph decomposes into $O(n \log^* n)$ cycles and edges.*

Thanks to the result of Lovász quoted above, we know that any n -vertex graph decomposes into $O(n)$ cycles and *paths*. A potential strategy – rooted in a variety of combinatorial techniques from the last 50 years – would be to start by setting aside a small, perhaps randomly chosen, selection of the edges. Then, decomposing the remaining edges into a collection of few paths and cycles using

Lovász's result, we could try to use the edges set aside to join the paths in the collection into cycles. Hopefully, then, we would have few cycles, and few edges remaining as well if we did not set many edges at the start.

This strategy is hard to achieve, but essentially was done by Conlon, Fox and Sudakov using a form of *graph expansion*. Expanders – graphs satisfying some type of graph expansion condition or other – are an important topic in their own right, both in combinatorics and in their application to computer science. Here we will concentrate on the use of graph expansion as a practical tool in extremal graph theory, and in particular for the study of cycles in graphs. For details on expanders more generally, we recommend the survey of Krivelevich [26] and its references.

In the simplest formulation, a graph expansion condition in a graph G might be that, for every set U of vertices which is not too large, there are plenty of vertices not in U which have at least one neighbouring edge to U (see Figure 3). More specifically, we might have, for some parameters m and a , that $|N_G(U)| \geq a|U|$ for every set U of at most m vertices, where $N_G(U)$ is the set of vertices of G not in U which have a neighbouring edge to U (the *neighbourhood* of U).

The possible utility of expansion here is that it can allow us to find paths connecting any pair of vertices. For example, if G has n vertices, $m = n/3$ and $a = 3/2$, we can consider iterative neighbourhoods $N_G(N_G(\dots N_G(x)))$ and $N_G(N_G(\dots N_G(y)))$ around any vertices x and y , respectively, and use the expansion condition to show that they grow in size exponentially until they each have more than $n/2$ vertices and thus overlap (see Figure 3). Thus, there must be a path from x to y , and moreover one which has length $O(\log n)$. This is only one path, however, and in the above very light sketch we would like to be able to connect many pairs of vertices (from the endpoints of the paths) by paths simultaneously while not using any edge more than once. Therefore, this is only some indication of how we might start to use expansion conditions to connect up paths into cycles, but the general principle is that if the expansion conditions are strong enough, then it might be possible even to find these connections iteratively in this fashion.

However, Conjecture 1.1 applies to all graphs, not just all graphs which conveniently satisfy some expansion condition! Conlon, Fox and Sudakov's first step, then, was to partition a graph into sub-

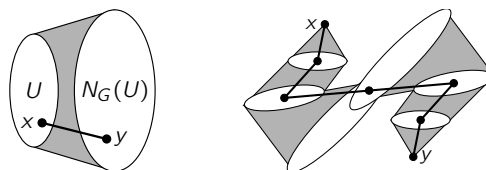


Figure 3. On the left, the neighbourhood $N_G(U)$ of a set U , the set of vertices y not in U for which there is some x in U such that xy is an edge. On the right, iteratively expanding about vertices x and y until a path (here of length six) between them is found.

graphs which satisfied some expansion conditions, before carrying out the outline above. This allowed them to show that, for any d , if an n -vertex graph has average degree d (and thus $dn/2$ edges in total), then $O(n)$ edge-disjoint cycles can be removed from it to leave a graph with average degree at most $d^{1-\varepsilon}$, for some small fixed $\varepsilon > 0$. Iterating this in $O(\log \log n)$ steps then gave a decomposition into at most $O(n \log \log n)$ cycles and edges.

As so often, there is a payoff: the stronger the expansion conditions we use, the easier it will be to connect up the paths into cycles, but we will need more edges remaining at each iteration to guarantee such strong conditions. Conlon, Fox and Sudakov used a strong expansion condition, where the ingenuity of their proof overall allowed them to work in subgraphs where, when they had t vertices remaining, say, the expansion condition corresponds very roughly to the above condition with $\alpha = t^{1-\mu}$ (though the precise condition used involves the number of edges leaving the set rather than the size of its neighbourhood).

In order to make much of an improvement to this, we need to use expansion conditions that can be found in an n -vertex graph with only $\log^{O(1)} n$ edges. To do this we must allow α to be much smaller. As α will need to be substantially less than 1, this is known as *sublinear expansion*, which was introduced in the 1990s by Komlós and Szemerédi [24, 25]. For illustration, a basic example of the type of expansion we might have here in an n -vertex graph G is that $|N_G(U)| \geq \frac{1}{\log^2 n} |U|$ for each vertex set U with at most $n/2$ vertices.

As this condition is so much weaker, whether or not we can work with it to some desired end often rests delicately on the details around the exact sublinear expansion conditions that can be achieved, but for simplicity here we will avoid these technicalities (for more details, and many applications and methods, see the recent comprehensive survey by Letzter [30]). Instead, let us emphasise a key point. These conditions may be weak, but in compensation essentially every graph has some subgraph satisfying some variation of these conditions. Indeed, as Komlós and Szemerédi showed, any graph contains within it a subgraph which has average degree not much smaller but which has some sublinear expansion properties.

For Theorem 1.2, Bucić and I followed the broad outline of Conlon, Fox and Sudakov [9] sketched above but using sublinear expansion conditions in place of the much stronger expansion conditions. To try this is a simple idea, the challenge is how hard it is to work with these much weaker conditions and this is what required considerable novelties and further work. It is particularly hard to show that, in very sparse sublinear expanders, a randomly chosen vertex set is likely to retain some expansion properties. However, we were able to show that if an n -vertex graph has average degree d , then we can remove $O(n)$ cycles to leave a graph with average degree $\log^{O(1)} d$. Thus, iterating this produces a graph with $O(n)$ edges in only $O(\log^* n)$ rounds, and therefore uses only $O(n \log^* n)$ cycles and edges in total.

While the iterated logarithm function grows extremely slowly with n , so that Theorem 1.2 comes within a hair's breadth of proving Conjecture 1.1, of course we should not be satisfied. The fundamental issue we currently have in our techniques is the iteration used, and more ideas appear needed, perhaps to decompose the graph more efficiently when this iteration is required by retrospectively lengthening the cycles found in previous rounds. Conjecture 1.1 is very simple to state, but its truth or falsity appears to reflect something deep about the structure of graphs, and its resolution may need to take into account different structural extremes that we do not yet understand.

2 Cycle lengths and the Erdős–Hajnal odd cycle problem

Any n -vertex graph with at least n edges has at least one cycle, but what can we say about its cycles? For any given k and n , how many edges in an n -vertex graph are sufficient to guarantee a cycle with length k , that is, with k edges? This (in wider generality) is the fundamental *Turán question* and a central part of extremal graph theory.

The case for odd cycles can be answered well without much trouble. An n -vertex graph can have a great many edges – $\lfloor n^2/4 \rfloor$ – yet contain no odd cycle (see Figure 6), and a single edge more immediately gives any odd cycle with length less than around $n/2$. Far fewer edges are needed to guarantee any specific even cycle. Though, in this sense, it is easier to find an even cycle, getting a satisfactory answer to the Turán question is much harder. It has been known since the 1970s, due to Bondy and Simonovits [5], that, for each fixed k , a bound of the form $O(n^{1+1/k})$ on the number of edges can be sufficient to guarantee a cycle with length $2k$ in an n -vertex graph. However, while this is widely expected to be tight up to the implicit constant multiple, this is only known for $k = 2, 3$ and 5.

In n -vertex graphs with $n^{1+o(1)}$ edges, we cannot guarantee any particular cycle length k . Could we instead guarantee a graph has some cycle whose length lies within some sequence k_1, k_2, k_3, \dots ? Answering a question of Erdős, in 2005 Verstraëte [40] showed that there is some such increasing sequence with limiting density 0 for which there is some C such that, for any graph G with average degree at least C , G contains a cycle of length k_i for some $i \geq 1$. This proof was non-constructive and thus did not determine any particular sequence of lengths k_1, k_2, \dots which has this property. Erdős [15] asked in particular whether the powers of 2 might satisfy this property. In 2008, Sudakov and Verstraëte [37] were able to show that any n -vertex graph with no cycle whose length is a power of 2 must have average degree at most $e^{O(\log^* n)}$, where, again, $\log^* n$ is the iterated logarithm function. The powers of 2 in this result is only an example sequence: the proof works for any sequence k_1, k_2, \dots of even numbers in which each term is at most C times the previous term, for any fixed $C > 0$ (i.e., $k_{i+1} \leq C k_i$ for each $i \geq 1$).

In the last few years, and using a new and fundamentally different approach, Liu and I [31] were able to improve this to answer Erdős's question, as follows.

Theorem 2.1. *There is some $d > 0$ such that every graph with average degree at least d has a cycle whose length is a power of 2.*

My methods with Liu apply to an even wider selection of sequences than those of Verstraëte and Sudakov, requiring only the sequence k_1, k_2, \dots consist of even numbers such that $k_{i+1} \leq \exp(k_i^{1/10})$ for each $i \geq 1$. The wide applicability of this result is perhaps its strongest quality, as for the powers of 2 the result is likely to hold for a much smaller constant than could be deduced from our methods. Indeed, Gyárfás and Erdős [16] conjectured that any graph with minimum degree at least 3 has a cycle whose length is a power of 2, as follows.

Conjecture 2.2. *Any graph with minimum degree at least 3 has a cycle whose length is a power of 2.*

Rather than only one cycle length from a sequence, we might also be able to say something about the set of cycles more generally. If a graph has average degree d and n vertices, then it may have no cycle shorter than $\Omega_d(\log n)$. However, if it has no such short cycles, then perhaps it correspondingly has many long cycles. Erdős and Hajnal suggested the harmonic sum of the cycle lengths as a measure of the density of the cycle lengths of a graph. Specifically, in 1966, they asked whether imposing a condition on the chromatic number³ of a graph G is sufficient to force $\sum_{\ell \in C(G)} \frac{1}{\ell}$ to be large, where $C(G)$ is the set of integers ℓ for which there is a cycle of length ℓ in G .

Erdős later wrote that they felt a much weaker condition, one only on the average degree, should actually be sufficient. In 1984, this was confirmed by Gyárfás, Komlós, and Szemerédi [22], who moreover showed that any graph G with average degree d satisfies $\sum_{\ell \in C(G)} \frac{1}{\ell} \geq c \log d$ for some small constant $c > 0$. This is tight up to the value of c , as shown by the example of the complete bipartite graph with d vertices in each class (see Figure 6 on the left for a similar graph). This showed that c cannot be taken to be larger than $\frac{1}{2}$ here, and led Erdős [12] to suggest in 1975 that this should be the best possible asymptotically. Using our methods behind Theorem 2.1, Liu and I confirmed that this is the right bound and c can be taken to be arbitrarily close to $\frac{1}{2}$ for sufficiently large d . That is, the following is true.

Theorem 2.3. *Every graph G with average degree d satisfies $\sum_{\ell \in C(G)} \frac{1}{\ell} \geq (\frac{1}{2} - o_d(1)) \log d$.*

As noted above, there is a sharp distinction between odd and even cycles with regard to the Turán question, and graphs may have a very high average degree indeed and yet no odd cycle. Average degree in a graph G is therefore not the right parameter to determine the appearance of cycle lengths from a sequence of odd numbers. Here the original suggestion by Erdős and Hajnal from 1966 to consider the chromatic number $\chi(G)$, as mentioned above, is more promising. Let $C_{\text{odd}}(G)$ be the set of odd numbers appearing in $C(G)$. In 1981, Erdős and Hajnal [14] asked whether $\sum_{\ell \in C_{\text{odd}}(G)} \frac{1}{\ell} \rightarrow \infty$ if $\chi(G) \rightarrow \infty$. This is a more difficult problem than the corresponding question for average degree and all cycle lengths and as such was widely open, though in 2011 Sudakov and Verstraëte [38] showed that it is true under an additional condition imposed on the 'independence ratio' of G . Using additional ideas on top of the methods behind Theorems 2.1 and 2.3, Liu and I were able to build on our techniques to answer this, giving the following asymptotically-tight lower bound.

Theorem 2.4. *Every graph G with chromatic number at least k satisfies $\sum_{\ell \in C_{\text{odd}}(G)} \frac{1}{\ell} \geq (\frac{1}{2} - o(1)) \log k$.*

If G is a graph with k vertices and every possible edge, then G has chromatic number k and $\sum_{\ell \in C_{\text{odd}}(G)} \frac{1}{\ell}$ is the sum of the odd numbers in the interval $[3, k]$, and, thus, is equal to $(\frac{1}{2} - o(1)) \log k$. Therefore, the constant $\frac{1}{2}$ in Theorem 2.4 cannot be improved.

Unlike all the other results mentioned on and towards these problems, the progress made for Theorems 2.1, 2.3 and 2.4 uses sublinear expansion. Building on our previous work – both together and apart – Liu and I showed that, for any sublinear expander H with at least a large constant average degree, there is a long interval in which $C(H)$ contains every even number. As every graph G with at least a large constant average degree contains a sublinear expander with almost the same average degree, $C(G)$ thus also contains every even number from a long interval. These intervals are long enough (relative to their start) that they will always catch, for example, some power of 2. Moreover, summing the reciprocals of the even numbers in any such interval will lead to a proof of Theorem 2.3.

Again, the weak properties of sublinear expanders make them very difficult to work with, and the methods introduced by Liu and myself here required several key ideas that subsequently led to other work (see, for example, [19], and again the survey by Letzter [30]). While Conjecture 2.2 appears currently far beyond our current techniques, progress continues to be made. Indeed, in forthcoming work, Milojević, Pokrovskiy, Sudakov and I have been able to answer, for large d , a question of Erdős [13] by giving an exact extremal result corresponding to Theorem 2.3. That is, there is some d_0 such that for every $d \geq d_0$, among the n -vertex graphs G with at least $d(n-d)$ edges, $\sum_{\ell \in C(G)} \frac{1}{\ell}$ is minimised exactly by the n -vertex graph with every possible edge between a set of d vertices and a set of $n-d$ vertices, and no other edges.

³The minimum number of colours required to colour the vertices so that no edge lies between two vertices of the same colour.

3 Hamilton cycles in expanders

A *Hamilton cycle* in a graph is a cycle which passes through every vertex exactly once. It is named for the Irish astronomer and mathematician William Rowan Hamilton, who considered the following question in 1857. Can you walk along some of the edges of a dodecahedron, pass through all the corners exactly once, and finish where you started? Equivalently, does the graph formed from the corners and edges of the dodecahedron contain a Hamilton cycle (see Figure 4)?

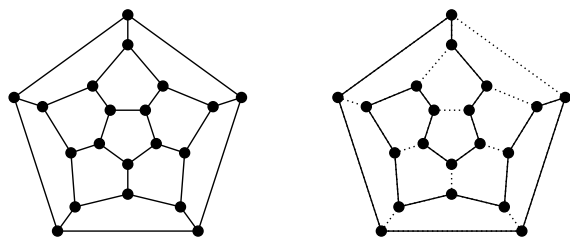


Figure 4. The graph of a dodecahedron, and a Hamilton cycle in it.

Hamilton was sufficiently enamoured with this question to sell the idea as a physical puzzle, which was sold in the UK and more widely in Europe as the ‘Icosian game’ and challenged the player to find such a cycle on a dodecahedron. Determining whether a graph contains a Hamilton cycle is a computationally difficult task. Indeed, this is one of Karp’s original examples of an NP-complete problem. It is perhaps then ironic that Hamilton’s Icosian game was a commercial failure in large part because it was too easy to solve! Hamilton may wish to be remembered more, then, for his invention of quaternions and his long period as the Royal Astronomer of Ireland.

Hamilton cycles in polyhedra had been considered a little earlier by the clergyman and amateur mathematician, T. P. Kirkman. Much earlier still, Hamilton cycles arose in the knight’s tour problem, whose history dates back at least to the 9th century in India. In this problem, the knight is to make a sequence of legal moves around the chessboard so that it occupies every square exactly once (see Figure 5). This gives a path through every vertex (a *Hamilton path*) in the graph corresponding to its possible moves, and, if this can be done so that it can return immediately to its starting square, then it gives a Hamilton cycle. The knight’s tour problem perhaps led to the first examples of Hamilton cycles in modern mathematics, with solutions for example given by Euler [18]. In the 20th century, Hamilton cycles have been an increasingly important object of study, with relevance for example to the travelling salesman problem.

The general difficulty of determining whether a graph has a Hamilton cycle has led to a lot of attention on proving simple conditions that imply a graph is Hamiltonian. For example, a stalwart

of many a first course in graph theory is Dirac’s theorem from 1953 that any graph with $n \geq 3$ vertices and minimum degree at least $n/2$ contains a Hamilton cycle. That this is tight is seen by two different *extremal examples* – an unbalanced complete bipartite graph and the disjoint union of two large complete graphs (see Figure 6). It is easy to determine whether Dirac’s *minimum degree* condition for Hamiltonicity holds, or not, but it will only be satisfied by graphs with very many edges. This condition has been generalised to others concerning the degrees of the graph (for example, Ore’s theorem), but all such conditions for Hamiltonicity require many edges if they are to be satisfied.

A famous condition for Hamiltonicity that can apply to sparser graphs is the Chvátal–Erdős condition from 1972. The corresponding result states that if the connectivity⁴ of a graph G is at least as large as the independence number⁵ of G , then it is Hamiltonian. Any n -vertex graph with average degree d is known to have an independent set with at least $n/(d+1)$ vertices, and the connectivity can easily be seen to be at most the average degree d . Thus, the Chvátal–Erdős condition can only hold in n -vertex graphs with average degree $\Omega(\sqrt{n})$.

Looking for conditions implying Hamiltonicity that apply to sparser graphs still, it is natural to consider two difficult and widely open conjectures from the 1970s. The first of these is by Chvátal and suggests a link between the *toughness* of a graph and its Hamiltonicity. A graph G is said to be t -*tough* if, for any s , the deletion of any set of s vertices from G gives either a connected graph, or one with at most s/t connected components. Any Hamiltonian graph, then, can be seen to be 1-tough. In 1973, Chvátal suggested that any sufficiently tough graph should be Hamiltonian, as follows.

Conjecture 3.1 (Chvátal). *There is some t such that any t -tough graph is Hamiltonian.*

In 2000, Bauer, Broersma and Veldman [3] showed that the stronger conjecture of Chvátal [8] that 2-toughness implies Hamiltonicity

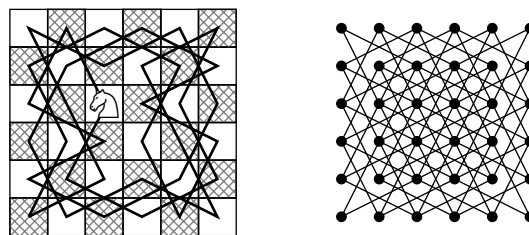


Figure 5. On the left, a closed knight’s tour on a 6×6 chess board, which corresponds to a Hamilton cycle in the graph of legal moves on the right.

⁴The minimum number of vertices whose removal disconnects G .

⁵The maximum number of vertices with no edges between them.

is false, and indeed if Conjecture 3.1 is true, then t must be at least $9/4$. Conjecture 3.1 remains wide open.

Perhaps even more difficult is an elegant conjecture due to Lovász [33] from the 1960s, and a variant later given by Thomassen (see, e.g., [21]). Lovász conjectured that any vertex-transitive graph⁶ which is connected contains a path that goes through every vertex exactly once. Thomassen suggested that only finitely many such graphs lack a cycle that goes through every vertex exactly once, as follows.

Conjecture 3.2 (Thomassen). *All but finitely many connected vertex-transitive graphs have a Hamilton cycle.*

There are only five known connected vertex-transitive non-Hamiltonian graphs with more than two vertices, all of which do not have many vertices. Note that the class of graphs which Conjecture 3.2 applies to contains the graph of the dodecahedron shown in Figure 4. Thus, in his very wide generalisation of the ‘Icosian game,’ Thomassen asks so much more of us than Hamilton ever did!

That Conjecture 3.2 is very challenging can be seen by our progress on the following question. Given any n -vertex vertex-transitive connected graph, how long a path or cycle can we find? (Hoping to eventually find an n -vertex path or cycle.) The best bound on this question for more than 40 years was that of Babai [2], who showed that in such graphs a cycle with length $\Omega(\sqrt{n})$ always exists. Only in 2023 was this improved, to $\Omega(n^{3/5})$ in a breakthrough by DeVos [10]. The current state of the art is that any n -vertex vertex-transitive connected graph has a cycle with length $\Omega(n^{13/21})$, due to Groenland, Longbrake, Steiner, Turcotte, and Yepremyan [21], and a path with length $\Omega(n^{9/14})$, due to Norin, Steiner, Thomassé, and Wollan [35].

Our discussion so far reflects the difficulty of finding sufficient conditions for Hamiltonicity applicable to sparse graphs. The sublin-

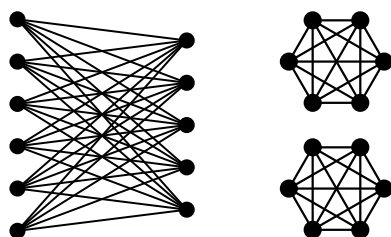


Figure 6. Extremal graphs with high minimum degree yet no Hamilton cycle. On the left, the graph with 11 vertices has no Hamilton cycle, as it cannot alternate between the right and left due to the imbalance in the number of vertices. On the right, the graph with 12 vertices has no Hamilton cycle, as it is not connected.

ear expansion conditions we considered previously are not strong enough. In particular, if a graph G is Hamiltonian, then any vertex set A in G which contains no edges must satisfy $|N(A)| > |A|$. That is to say, we could have a local problem at any scale that looks like the first of the extremal examples in Figure 6 and prevents the existence of any Hamilton cycle. It is natural, then, to consider expansion conditions which might imply Hamiltonicity, as has been done since the pioneering work of Pósa [36] on Hamilton cycles in random graphs.

Building on Pósa’s work, whether a Hamilton cycle is likely to appear or not in the most studied random graph models is very well understood, due to work by Bollobás [4] and by Ajtai, Komlós, and Szemerédi [1]. In all the corresponding methods, expansion conditions (as are likely to occur in the random graphs) are used in conjunction with random techniques. It is desirable then to find simple properties likely to hold in random graphs which will imply Hamiltonicity and avoid studying the random graph model directly. That is, which *pseudorandom* conditions in sparse graphs are sufficient to imply Hamiltonicity?

The study of pseudorandom graphs was begun by Thomason [39] in the 1980s, creating an active and influential area of research (for more on which, see, for example, the survey of Krivelevich and Sudakov [28]). A major class of graphs known to exhibit pseudorandom properties are (n, d, λ) -graphs: n -vertex d -regular⁷ graphs satisfying a certain condition (governed by λ) on the eigenvalues of their adjacency matrices. In their foundational study of (n, d, λ) -graphs in 2003, Krivelevich and Sudakov [27] conjectured that if $d/\lambda \geq C$ (for some universal constant $C > 0$), then any (n, d, λ) -graph is Hamiltonian. As evidence, they showed that, if $d/\lambda \geq \log n$ (for large n), then any (n, d, λ) -graph is Hamiltonian. For more details on this, and (n, d, λ) -graphs, see [27]. For our purposes now, however, we will consider the following conjecture (appearing, for example, in [6]), which implies the conjecture of Krivelevich and Sudakov on the Hamiltonicity of (n, d, λ) -graphs, and suggests pseudorandom expansion conditions for Hamiltonicity.

Conjecture 3.3. *There exists $C > 0$ such that any n -vertex graph satisfying the following two conditions is Hamiltonian.*

1. $|N(A)| \geq C|A|$ for any vertex set A of at most $n/2C$ vertices.
2. For any disjoint vertex sets A, B of at least n/C vertices each, there is an edge between A and B in G .

On a personal note, Conjecture 3.3 is a problem I first studied when writing my PhD thesis in 2015. There, to aid a long argument embedding certain graphs known as spanning trees into random graphs [34], an answer to this question would have been very convenient! In an example of how it is often easier to study random graphs rather than pseudorandom graphs, I was able to avoid the

⁶I.e., a graph G in which, for any pair of vertices, there is an isomorphism of G mapping one to the other.

⁷I.e., every vertex has degree d .

difficulty of Conjecture 3.3 by working in random graphs directly using some methods of Sudakov and Lee [29].

In 2009, Hefetz, Krivelevich, and Szabó [23] made progress towards Conjecture 3.3, essentially showing this is true if the constant expansion coefficient C is replaced by $\log n$. In 2024, Glock, Munhá Correia, and Sudakov [20] gave an improved bound on the conjecture of Krivelevich and Sudakov on the Hamiltonicity of (n, d, λ) -graphs, improving the condition $d/\lambda \geq \log n$ to $d/\lambda \geq (\log n)^{1/3}$. Very recently, I was able to confirm the conjecture in full with Draganić, Munhá Correia, Pokrovskiy, and Sudakov [11], which followed from our positive resolution of Conjecture 3.3. This confirms the most general, natural, conditions for Hamiltonicity which apply to even very sparse graphs, but the openness of Conjectures 3.1 and 3.2 shows that much remains to be done in this fascinating area.

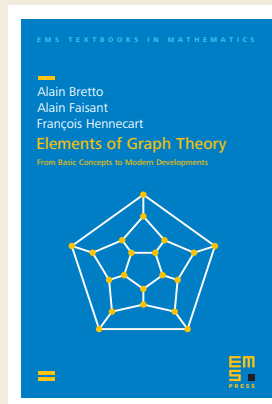
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Richard Montgomery is a professor of mathematics at the University of Warwick, working in extremal and probabilistic combinatorics. He received his PhD in 2015 at the University of Cambridge under the supervision of Andrew Thomason, after which he spent time at Trinity College, Cambridge, and then the University of Birmingham. Recognition of his work includes a 2024 EMS Prize and a 2025 LMS Whitehead Prize. richard.montgomery@warwick.ac.uk

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Interview with Sir John Kingman

Ana Isabel Mendes and Martin Raussen

Sir John Kingman, Fellow of the Royal Society, was the 4th president of the European Mathematical Society during the years 2003–2006. From his career as a professor of probability theory and statistics to holding top positions as a science administrator and leader, his professional journey continued until his retirement in 2006. Now, at age 86, he does not do contemporary mathematics any longer, but he still likes to think about and solve non-trivial puzzles of a mathematical flavour.

In January 2025, Ana Mendes and Martin Raussen had the opportunity to ask Sir John about his life within and around mathematics. Among many other things, we learned that mathematics is a fuzzy and messy subject, nevertheless connected, and quite often very useful!

A mathematical education

Ana Isabel Mendes/Martin Raussen: *To start this conversation, we would like to ask how you became a mathematician and a statistician. Can you please tell us what and perhaps who stimulated your interest in the first place?*

Sir John Kingman: Well, I went to an ordinary state grammar school in the north of London. In general, the teaching was good, and I enjoyed most of the subjects. I didn't enjoy subjects like art or woodwork or experimental science where I needed to use my hands because I wasn't very good at that. I famously dropped the apparatus that made hydrogen sulphide, which made me rather unpopular. But I found that I could do mathematics, and I enjoyed doing mathematics. As time went on, I found I enjoyed it even more. And of course you know, when you can do something, you are never bored.

So, I quite naturally thought, I should try and study mathematics at university. I was fortunate to get into Cambridge. Well, in Cambridge you choose a college, and my college did not have much in the way of mathematics. The person in charge was a very old-fashioned applied mathematician. He wouldn't have us use vectors or anything modern like that. Everything had to be written out. I can still write out Stokes's theorem in coordinates. And of



Sir John Kingman.

course, xyz -coordinates, not $x_1x_2x_3$. But I was very fortunate, because in my second year I was taught by a young mathematician who was still making his way, called M. F. Atiyah. That, as you can imagine, was quite an experience. I learned a great deal from Michael in that year.

But for reasons I'll explain if you'd like, I developed an interest in probability. In Cambridge, probability was neither pure mathematics nor applied mathematics. And there was a strict division between pure and applied, so it fell through the middle. But I was again fortunate because they fixed it up, so I could have, I think, four one-hour sessions with Dennis Lindley. He was one of the most brilliant statistical teachers that I have ever come across, and that was also a formative experience.

[AM/MR]: *Studying in the late 1950s must have been very different from what it was later on and how it is nowadays. Can you tell us a little more?*

JK: It was certainly different. Thinking about Cambridge, it was very old-fashioned. The way it was organised made it very difficult to bring in any new ideas. For instance, they never talked about probability theory. We always talked about random variables. One day I asked: *"What is a random variable?" "It's something that varies randomly"*. Oh, but what is it? And there was no answer.

It was also very strictly divided between pure maths and applied maths, and you had exactly half of one and half of the other. You went to lectures, some of the lectures were good and some were awful. Some of the lecturers were brilliant mathematicians, world-class, and some were completely unknown. And the correlation between those two classifications was not great: I'm not saying that the great mathematicians were the great lecturers or vice versa. In fact, the two of the most distinguished mathematicians, I will not give you their names, but you would certainly know them, they were dreadful lecturers. And there were other people who had not done any mathematics since their PhD, but who were very good at putting it across.

And then you had supervision. Normally, two students were meeting a senior person. During my first year, we had this old-fashioned applied mathematician, so we did very old-fashioned applied mathematics. But then in my second year, Atiyah took up supervision, and wow, I used to come out just wiping my brow. It was a quick fire! I remember one day I asked him a question about compound matrices. And I got 20 minutes of exterior algebra, which I of course never had met before. It was a wonderful experience, and I was very lucky to have it.

In the third year, you could choose what you wanted to do. And you just went to the lectures you felt like going to, and you could mix and match. So, I went to lectures on probability and statistics, and I went to lectures on genetics, which later had an effect. But I also went to lectures about fluid dynamics, differential geometry and general relativity, all sorts of things. And that was rather good because the lecturers were typically talking about what really enthused them. But some of them were not enthusiastic at all.

Perhaps the highlight of that third year was going to lectures on quantum theory by Dirac. He would, of course, talk about the things that he had invented himself. They were rather boring lectures; he read sections from his book on the subject. But one Saturday morning he decided to tell us how he discovered the spin on the electron. By pure mathematics, essentially by non-commutative algebra. And because it was non-commutative, something had to be spinning, and there was only one electron; the electron had a spin of $\frac{1}{2}$. That was rather remarkable, because what Dirac was saying was that if the creator had not given the electron a spin of $\frac{1}{2}$, he wouldn't have been doing as good a job as Dirac would have done.

[AM/MR]: *You must have studied very intensively. How was social life as a student?*

JK: I was what we call a swot or a nerd these days. That is to say, I spent my time reading and doing all the exam papers, and so on. I probably wasted many of the opportunities of social life.

I had quite a lot of friends. I went to concerts and there was a lot of music, things like that. But I spent an awful lot of time doing mathematics. I slightly regret that there was not so much else, but I was that sort of person.

[AM/MR]: *How did you then get into probability and into mathematical research?*

JK: The reason why I became interested in probability came from a quite different source. We had these long summer vacations, and I thought I'd better try and earn some money because we weren't very well off. I signed up with the post office engineering research station which was nearby.

A friend of my father worked there. I didn't realise that he was actually a very great man. He was Tommy Flowers who had been at Bletchley during the war. He was the person who built the Colossus computer, which was really the first electronic computer. He was a very quiet chap. It is famous that at Bletchley, the mathematicians and the theorists imagined what one could do with a machine that could do this and that, but no one had ever built such a machine. But Tommy Flowers, who was a telephone engineer said, *oh yes, we do this when we do telephone exchanges. I can do it*, and he did!

Anyway, he arranged for me to have this job, and I was given work which involved learning a bit about queueing theory. Queueing theory, as you know, originated in the Copenhagen telephone company with the work of Erlang, and it was very much a part of the telephone engineering world. I found it very interesting, and I wanted to try and do research in that direction.

Working as a university mathematician

[AM/MR]: *You obtained your first academic appointment, I believe, as a reader and then as a professor at the newly founded University of Sussex in the South of England. How was that experience in contrast with the traditional universities?*

JK: In fact, that wasn't my first job. I started doing research under the applied probabilist Peter Whittle. He was marvellous, but after a year, he said: *You had better go and work under David Kendall*. Kendall was the person who introduced the Kolmogorov type of probability to the UK. We had quite a tradition of statistics in England with people like Bartlett, Daniels, Lindley, and so on. But it was very much applied probability, and they really hadn't heard about the mathematical basis established by Kolmogorov before Kendall came along.

In my second year at work, I went to Oxford where Kendall was working, and I spent a year there. During that year, Kendall

was appointed to a professorship in Cambridge. Cambridge had never had a professor of statistics, and they decided that they wanted one that was mathematical. Hence, they went for the most mathematical person they could find in the subject, and that was Kendall. One day when I went to see him in Oxford, he said: “*I’m going to Cambridge.*” And he said that he needed an assistant lecturer to do the work. I didn’t apply, I was just appointed, that happened in those days, and I went back with Kendall to Cambridge. Indeed, I had to do the work! I had to give 1st year lectures and so on, which was quite an education!

But then I was offered a job at Sussex. Well, that was an attractive offer because the lady I was courting had also been offered, quite independently, a job at Sussex. We both accepted these offers and settled into married life in Brighton.

Sussex, as you were saying, was a new university. The difference was that in Cambridge, you do what you did last year, unless you fight very hard for a change. But in Sussex everything was new, so everything was being thought through from first principles. There wasn’t any last year. You must decide yourself. They tried all sorts of things. Some of them worked and some of them didn’t. I had quite an interesting time because I was the first statistician they appointed. One of the things they decided was that all the students of social science, economics and sociology, and so on, had to pass a compulsory course in statistics. And I was told that I was to teach this course. That was very different from teaching Cambridge mathematical undergraduates. One of the students had found the course very difficult because she had been away from school when they were taught decimals. She couldn’t do decimals, and that made statistics a little bit tricky!

[AM/MR]: *In 1969, you moved from the University of Sussex to Oxford University, which at that time hardly had highbrow expertise in statistics. How could you and statistics develop and thrive in Oxford at the time?*

JK: Oxford University did have a thing called the Institute of Economics and Statistics. It hosted a lot of economists, and John Hammersley, who was a mathematician who had managed to get a job there, so he didn’t have to teach any undergraduates and just could get on with doing mathematics. There was also a department of biomathematics which was by then headed by Maurice Bartlett, a renowned statistician. He was developing it as a sort of statistics department with a biological flavour. But there was no statistics in the mathematics faculty, and they did not want to introduce statistics because it was being done by Maurice Bartlett’s department.

But they thought they might need some probability because they had lost David Kendall. Well, Kendall never had a university job, he had a college job at Oxford. Anyway, they set up a chair in probability and I didn’t know that. I hadn’t seen an advertisement or anything, and I just got a letter from the Vice Chancellor saying we

have elected you to the professorship. And they just assumed that I would immediately pack up and go to Oxford, which I did! I had happy years there trying to introduce probability and then some statistics, and I tried to bring together the different statisticians in the university. Maurice Bartlett was very helpful, and a very good colleague, and we did in fact make a certain amount of progress. Later, they set up a proper department of statistics, but that was after my time.

Heavy traffic and the coalescent

[AM/MR]: *I hope it’s correct to say that your scientific work centred around random processes, random walks, queuing theory, and then also mathematical genetics. You are, among many other things, particularly known, at least to Wikipedia, for Kingman’s formula in queuing theory published when you were only 21 years old. Moreover, for the Kingman coalescent in mathematical genetics from the early 1980. Can you please explain what these are all about?*

JK: I find it one of the ironies with mathematics that people are not necessarily known for their best achievements. The things in mathematics which I am proud to have done are not the ones you mention; neither of them contains particularly deep mathematics. The heavy traffic came out of my work in the post office. Actually, for the telephone engineers we were talking about, ‘queues in heavy traffic’ was a telephone engineering term. They were interested because, if you have too many people coming into a queue, it becomes unstable, and the queue gets longer and longer. But when you’re just below that difficult point, the actual assumptions that you make are not very relevant. There are robust approximations which are very easy to handle.

I don’t think I invented that formula. I think it was pretty well-known to people in the telephone engineering world. What I did was to popularise it among the mathematicians, the applied probabilists who were working on traffic theory. They had not been interested in approximation. If you look at the classical work, including Kendall’s two papers in 1951 and 1952, they were all about exact solutions under very strong assumptions. But the people in the practical world were interested in approximate results under very loose assumptions. Of course, things don’t come in as Poisson processes, there are all sorts of irregularities which are difficult to describe, so you want robust results.

And then I published a theorem that you could prove. But I don’t think it was very new. I think it encouraged people to look at robust approximations, and from the point of view of the operational research people, who were using queuing theory, that’s exactly what they wanted. They didn’t want very long formulae with lots of Laplace transforms and things in it. They wanted some nice simple things that gave them some insight into what was

actually going on. So much about the heavy traffic approximation, the work that I have done as an undergraduate.

The coalescent was quite different. I have written about the way the coalescent came about.¹ As I already mentioned, I had been to this course of lectures on genetics in my third year in Cambridge. The lecturer had inherited the notes from R. A. Fisher, the great statistician, who had given that course for many years. When he retired, he handed his lecture notes, which he had used year after year, to this other lecturer who just leafed through and read us bits of Fisher's work. Well, one of them was a rather nice inequality that Fisher had conjectured. He had proved it for a locus with two alleles and there was increasing interest in situations with more than two alleles. Could you extend the result to a general case? And people had proved that you could. But the proofs were very long and involved, and it was clear that we were going to be asked to reproduce the proof in the exam. The lecturer dropped a heavy hint that one of the questions would be proving this inequality. This is why I thought I had better find a proof that I could remember, which I was able to do.

But that brought me into contact with the people working in mathematical genetics, particularly in Australia. In 1963, I visited Australia and met Moran, Ewens and Watterson, and we had lots of fruitful discussions. We went on over the years corresponding about the problem. Genetics was developing in a very interesting way. In classical genetics, you talk about genes, but you don't know what they are. They were simply abstract entities that could exist in different forms, and their patterns of inheritance could be studied. But of course, with the discoveries by Crick and Watson, people knew what genes actually were, these strings and double spirals and things like that. That raised quite new mathematical problems that people were grappling with. But I think there is a connection to what I felt when I was looking at queuing theory. What was needed was approximation and robust results. We didn't want to have to make very strong assumptions; they are not realistic!

I had produced some elaborate formulae that were pretty useless. But then I realised that, if you work *backwards* in time, then it can actually get much simpler. Particularly if there is not a strong selection operating, as is often the case at molecular level. You just look back.

Suppose you and I trace back our ancestry in the maternal line. We start with our mothers and then their mothers and so on. Maybe we find that once there was somewhere up there a Viking maiden who is an ancestor to both of us, Martin. Even for all three of us, we could still do this. You'd have to go further back. If you think about it mathematically, you've got the coalescent. It's not a deep thing at all, a straightforward Markov chain. The important thing is it just happened to be the tool that the people in genomics

wanted. That's marvellous! I am very pleased that this simple idea has proved to be very useful. But it's not hard maths.

Difficult mathematical results

[AM/MR]: *That allows us to ask you what piece of mathematical work you actually are most proud of.*

JK: I am afraid it is very old-fashioned. When I worked with Kendall, I got very interested in Markov processes in continuous time with only countably many states. That was very fashionable at the time. There is a marvellous book by my friend Kai Lai Chung, all about it, and there were all sorts of difficult problems. Kolmogorov had asked David Kendall, they met at a conference, if he could characterise the functions of time that can arise as transition probabilities. They are continuously differentiable, they satisfy various inequalities and so on. He said, could you actually characterise them? Kendall then, rather cruelly, passed the problem on to me. I was able to solve it, but that took me a decade, and that I am very proud of.

After I ceased to be a full-time mathematician, I went on doing some research rather as a hobby. I stayed in that field and over the years I have been able, from time to time, to prove some theorems which I like. But they are of very little interest because the subject has moved on. Young people now have different interests, and that is fair enough, that is as it should be. But it still gives me great pleasure that I was able to crack some, I think, quite difficult problems. But they will not get into Wikipedia.

[AM/MR]: *But they are published in the research literature, obviously.*

Applications

[AM/MR]: *Are you aware of other applications of your work outside mathematics?*

JK: Well, some of the results in queuing theory have been of interest in Operational Research and I think the coalescent has some other applications. I'm slowing down because I know that some of the applications are in the secret world, in cryptography and so on. There is for instance an interesting connection with large prime divisors of large numbers. That is apparently of interest to cryptographers, and that all links with the coalescent and a distribution which I have found and named after Poisson and Dirichlet, now called $PD(\theta)$. I know that $PD(\theta)$ is one of the things people talk about over coffee in those secret places. But I am not in that world, so I don't really know. I do not think that there are many more applications of the work I have done. But who knows?

¹ J. F. C. Kingman, [Origins of the coalescent: 1974–1982](#). *Genetics* 156, 1461–1463 (2000)

I did get involved during the Covid epidemic because I got very cross. I don't know whether it was like that where you are, but certainly in the UK, there was an awful lot of talk about the R -number. You know the R -number in models of an epidemic? It is essentially the average number of people you give the disease to. And if it is greater than 1, the epidemic will grow, and if it is less, it will die out. The government actually published these numbers every week, they published R -numbers for the United Kingdom and other local R -numbers for regions of the UK.

That was bad science, for a number of reasons, of which the strongest was that it ignored the heterogeneity of the epidemic. Even within a single country, the disease affected different regions, different age groups, different social classes, very differently. This means that R is really a matrix, with rows and columns corresponding to different subgroups of the population. The critical parameter is the largest eigenvalue of this matrix. Unfortunately, few politicians are familiar with the Perron–Frobenius theorem, and they want simpler tools. The local R -numbers are essentially row sums, which do not determine the spectral radius. This is, I think, now understood by many epidemiologists, and I hope that, when we next have a pandemic, we will not hear much about the R -number. Much more useful is the exponential rate of increase of the epidemic.

Mathematics is a mess, but this mess is connected!

[AM/MR]: *Can we ask you about the tensions between applied mathematicians and statisticians and pure mathematicians, from a historical perspective, but also how do you perceive the development nowadays?*

JK: You ask me about the unity of mathematics? I think that this is not a very useful concept because it suggests that mathematics is a nice neat compact discipline. But mathematics is really a great sprawling organism, that finds its way into all sorts of corner of life. It's a sort of fractal, since if you look at a bit of it in more detail it is still messy. No one can define mathematics. Maths is what mathematicians do, and mathematicians are people who do maths. But if you regard maths as a complicated topological space, the important thing is that it is a *connected* space.

Alright, there are pure mathematicians and applied mathematicians, and there are statisticians and there are computer scientists and so on. But when you have a practical problem which looks as though mathematics might help, someone will try and do it, and then they will realise that they do not really have the mathematics to do it. They go and talk to someone who knows a bit more mathematics and says: Yes, I know I could do some calculations on that. And then this person realises that she or he does not quite understand some of the bits and pieces and asks someone else, and this someone else is not really interested in the application but finds

the mathematics interesting. And then someone else comes along and constructs some vast abstract theory, and a new branch of mathematics suddenly starts to appear. And then someone spots a link with other applications, and we realise that some of the problems have already been solved.

That is why I strongly disagree with universities that divide between pure and applied mathematics or send the statisticians off into a corner somewhere. When I was at Stanford in the 1960s, I was in the mathematics department. The statisticians were in Sequoia Hall, a few yards away. If you were seen sneaking off to a seminar in Sequoia Hall, you were told in the coffee room that you were letting the side down. That is not my idea of maths! I was very shocked when I went back to Cambridge in 2001 to find that the divide between pure and applied mathematics, which I had seen in the 1960s, was still there. I had assumed that when the old men died off, common sense would prevail. But the trouble was the old men had successors, and they were just as keen to keep the two apart. That is ridiculous.

I am sorry, I am getting passionate, but I do feel passionate. Kai Lai Chung says in one of his prefaces that mathematicians have more interest in building fire stations than in putting out fires. Well, some mathematicians are! They have an abstract theory of fire stations which is formulated within category theory or something. But some other mathematicians like holding the hose, squirting water on the flames. And the important thing is that they keep talking to one another!

[AM/MR]: *In Portugal, for example, this is still a problem. Often, they don't speak with each other. But it begins changing.*

A second career: science administration and leadership

[AM/MR]: *In parallel with your scientific work, you moved on to an amazing series of occupations in science, administration and leadership. You headed the British Science and Engineering Research Council, you were the vice-chancellor of the University of Bristol for 16 years before you chaired the fairly new Isaac Newton Institute in Cambridge. You have also been president of the Royal Statistical Society, the London Mathematical Society, the European Mathematical Society, the British Statistics Commission, and we could move on mentioning top positions in private companies, and others.*

What did propel you into science administration and leadership? Did you enjoy this work as much as or perhaps even more than your work in the mathematical arena? Could you combine both a bit?

JK: It happened incrementally. I was a harmless professor of mathematics in Oxford doing what professors of mathematics do, when I was asked to sit on the mathematics committee of what was then the Science Research Council. That council gave grants to mathem-



Wills Memorial Building, Bristol University.

aticians and spent its time trying to persuade the real scientists, the experimentalists, that a little bit of money spent on mathematics would be a good investment.

Historically, very little grant money went into mathematics. The person who changed that was Christopher Zeeman. He had been chairman of this maths committee, and he made a real nuisance of himself. On one occasion, at the committee when people were arguing for lots of money to be spent on producing neutrons, he proposed the abolition of the neutron! Anyway, we were trying to build up grant income for mathematicians, so that we could go to conferences and publish, have decent libraries and even a work computer. We are talking now about what happened in the 1970s, when computers were coming in.

I was made chairman of this committee after a couple of years, and I sat on the next committee up, that was called the Science Board. And then I was made chairman of the Science Board. That meant I sat on the Council itself, which was bringing engineering into its title. And then I was asked to be chairman. Well, the previous jobs had all been, of course, part-time, and I was just doing them along with my ordinary work in Oxford. But the chairmanship of the Council itself, which meant that you were in charge of the whole thing, of spending several million pounds a year, was a full-time job. From then on, I took to doing mathematics really as a hobby, as an aid to sanity. When there was a little time, I could think about mathematics, and I was able to publish a little. But basically, I had given up being a full-time mathematician.



Isaac Newton Institute, Cambridge.

I did enjoy it; it was very hard work. It was very stimulating, and it brought me into contact with an enormous variety of scientific and engineering activity. I learned an awful lot of all sorts of physics, chemistry and biology and engineering, and I came to admire the wonderful things that were being done in some of these areas.

Then I had assumed that I would just go back to Oxford. But I was asked to be vice-chancellor of Bristol, which was the university where my father and my brother had started. Of course, that was not a very easy job either. Altogether, I had 20 years in which I was an amateur mathematician. Then, in 2001, I was asked to be director of the Isaac Newton Institute. I thought, that is fine, I can go back to mathematics.

But what I discovered was that the subject had moved on so much. Perhaps I should have worked harder to keep up with all the new developments, but I had not. I could please myself by working on the old-fashioned problems. But I really could not contribute to the way the subject would develop in the meantime, and that's good! I'm very much in favour of that. Subjects that stay the same, stagnate, and mathematics does not stagnate. But there were times when I felt like someone on a railway journey who decides to get off at an intermediate station to stretch his legs and then realises that the train has gone off without him and there is not much that can be done. I really have not been a full-time mathematician since 1981, when I went to the Research Council.

[AM/MR]: *You have already mentioned that not every brilliant mathematician is a good academic teacher as well. In the same spirit, not every brilliant mathematician is good as a science leader and promoter. From your perspective, which properties, which qualifications are essential as head of a scientific organisation or as a science politician?*

JK: That is a very difficult question. I have not thought about it thoroughly and in general. I am just considering and watching what has happened to the Research Council and to the University of Bristol since I left. My successors have been very different people

with very different qualities. I think it is actually quite difficult to list the qualities, to give a job specification, as it were, because different people bring different things to these jobs.

Obviously, a depth of vision and ability to appreciate. A broader point of view and the ability to put that across to the people who provide the money. When I was at the Research Council, I spent a lot of time talking to Mrs. Thatcher, the Prime Minister, and other influential people, trying to get them to put more support into science, and you have got to be able to do that. You have got to be organised to be able to chair a meeting and that sort of thing. But when you put all those qualities together, you already have something. The people who really make an impact have something which is quite difficult to define, and it is different for different people.

Also, often the job makes the person! You challenge someone, and they say, well, I would never do that. I just say, put their mind to it, and they may discover qualities in themselves that they did not suspect. Of course, it goes the other way, too. There are even historical examples for that: Someone who everyone thought would make a marvellous Roman emperor until he became emperor, he was not marvellous at all.

You asked about enjoying it. I found that I have enjoyed more or less each phase of my career. They have been very different. But I have never wished I was back in the previous job. I always found that the job I was doing was interesting and enjoyable even if sometimes a bit hairy.

The European Mathematical Society

[AM/MR]: *You were elected president of the EMS when it was still fairly new, 12 years after its foundation. What or perhaps who brought you into the EMS? What were the most important items on your agenda at the time?*

JK: I had not had anything to do with the EMS before. I think I was an individual member, but I had not taken part in any duty. It was quite a surprise when I was asked to be president. I think it was partly because I was about to take over the directorship of the Isaac Newton Institute. And of course, the research institutes are an important aspect to the EMS. The society had formed a sort of umbrella, called ERCOM (European Research Centres on Mathematics), for those institutes, like IHÉS in Paris, Oberwolfach, and so on. I was obviously very honoured to be asked.

But I knew very little about what was going on, so I investigated. I learned about the way it had been founded as the result of a considerable argument between the French and the British, with the French wanting a society of individual members and the British thinking more of a federation of national societies. I think it is one of the strengths of the EMS that it managed to be both of them. It gives it a very strong structure, relying on both individual members and corporate members. It means, of course, that it has



EMS Executive Committee meeting, Prague 2004.
(Photograph by David Salinger)

to add value to both aspects of the mathematical world in Europe. It should not be in a competition with the mathematical societies in the countries, or indeed with the International Mathematical Union, which does a very good job. But in between those two levels, the global and the national, there is room for a European dimension.

There have proved to be many useful things that the EMS could do. I have always thought that complementarity is the important thing. What can we deal with at the European level, because there are things which are specifically European.

When the European Research Council was set up, for instance, the EMS lobbied very hard to have a mathematician on it, and we got Pavel Exner, who was then the vice-president, on to the council, and later the inimitable Jean-Pierre Bourguignon took it over; that was splendid.

It was a great experience to work with colleagues from all over Europe on the Executive Committee, and I am most grateful to people like Helge Holden and Tuulikki Mäkeläinen. We found plenty to do, and I believe we did it well.

[AM/MR]: *How did you tackle the tensions between applied and pure mathematicians within the EMS?*

JK: I was lucky, because after the presidencies of Fritz Hirzebruch and then Jean-Pierre Bourguignon, Rolf Jeltsch came in as very much of an applied mathematician, and he really strengthened that aspect of the EMS. I tried to do the same on the statistical front.

Mathematics throughout the world

[AM/MR]: *When you were president of the EMS, the UK was still a solid part of the European Union. What effect has Brexit had on science cooperation in Europe?*

JK: I haven't been involved in the administration of any of this since Brexit, so my information is second hand. Scientists have tried very hard to find ways to maintain the collaboration that was set up under the European Union and with variable success. Of course, not all international collaboration is based on the European Union. CERN, for example, was an international organisation long before any of these. You can do that in a narrow area like in particle physics or some of the astronomy, you can set up a close collaboration. There is so much involved in building telescopes in the Canaries or in Hawaii. These international collaborations don't need something like the European Union. There are just a number of countries that have agreed to collaborate. And when scientists get the opportunity to have some wonderful new equipment, they are willing to work across the national boundaries.

[AM/MR]: *Do you have any clues about how the new US government will influence scientific cooperation more broadly?*

JK: I think we have to wait and see. Anyone who tries to predict what President Trump is going to do next week has my admiration [laughs]. But I think it's worth stressing that whatever the politicians decide, science is an international activity. You saw that when Europe was divided by the Iron Curtain. Despite all the obstacles, people managed to continue to collaborate somehow. For example, the way that Hungarian and Polish mathematicians maintained contact with Western Europe and other countries. They've got to find ways of following their passion, and if that means establishing international links, they find ways of doing it. They don't necessarily wait for the government to catch up.

Mathematical practice – earlier and now

[AM/MR]: *The way mathematicians work on a research topic, and how they publish, has changed a lot while you were active. Would you comment, please?*

JK: That's one of the things which has changed enormously over my career. When I was a young mathematician, we went to conferences from time to time, and we wrote letters and sent them out. If you published a paper, you got 50 offprints that you sent off to anyone who you thought might be interested. If you thought that someone was doing interesting work, you wrote them a letter and asked them for offprints. You might or might not get them! It was very sedate and rather slow work.

And now, mathematics, like the rest of science, moves incredibly fast. If you're working on a particular problem, if you have a contact via e-mail or on social media or whatever is your taste, with people working on it in different parts of the world, if you have an idea, you send it down to your friends in Japan or America

or whatever, and in a couple of hours later, someone comes up with a counter example.

That means that the subject is moving very fast. It also means that it's quite difficult for young people too, because you've got to get into these informal means of communication. I don't quite know how that works now. I think some young mathematicians find it quite difficult unless they have some strong links, for instance through a department or a senior collaborator. A present-day Ramanujan might have a hard time.

Preparing young mathematicians

[AM/MR]: *There are many indications, for example in the European PISA records, that we are not preparing our students in a very satisfactory way for their professional life in modern society. What should we do to prepare them better?*

JK: I think it has always been like that. When I think about the old-fashioned mathematics course I learned from Cambridge, I wasn't being prepared for the modern world. You weren't even allowed to learn functional analysis in those days. That was regarded as very unsound mathematics.

And we're never really preparing our young people. They have to find their own way, and they do. The nice thing about mathematics is that we are all being challenged by young people who come up. Perhaps you have been working on a problem all your career, and then some young woman or some young man comes along and proves the theorem!

I have always liked the story behind elliptic functions. People had been working on elliptic integrals for years and years. Then, Jacobi came along and said, *Let's look at them backwards*. It is as if we had invented trigonometric functions by studying the indefinite integral $\int \frac{1}{\sqrt{1-x^2}} dx$, and then someone comes along and says, *But you should take the inverse function!* And then trigonometry, sines and cosines, and all sorts of clever things like Fourier series arise...

Elegant mathematics

[AM/MR]: *Are there other particularly elegant or beautiful mathematical concepts or ideas that still fascinate you?*

JK: A lot of mathematics is incredibly exciting, even some very old mathematics. I still like Euclid's proof that there are infinitely many prime numbers. I knew a very distinguished biologist who didn't believe you could prove that, and he thought that there were only finitely many prime numbers, and he went to his grave believing that!

And you know, in probability there are wonderful things. I like even simple things like the central limit theorem or the law of the

iterated logarithm. Why is it $\log \log n$? It is very odd, but easy when you understand it.

Very often, something that looks ugly, when looked up at in the right way becomes beautiful and elegant. I am still very cross about my proof of the subadditive ergodic theorem. I gave the first proof of that theorem, but it's the most awful proof. I am ashamed to look back at the paper in which I published that proof. But since then, people have come along, and given much more natural elegant proofs, and I can now look at their proofs and admire what they've done. Einstein once said, "*leave elegance to the tailor.*" But that was a physicist's remark. For a mathematician, elegance is actually very important, because it is a sign that you are going with the grain of the subject. You are looking at the thing in the right way. If it is all very ugly and very hard, you are probably not looking at it in the right way.

I gave an example much earlier, going with the problem about determinants with Michael Atiyah, and being told about the exterior algebra. That was a case in point. I had been reading in a book about determinants and a theorem. It was incredibly difficult to prove just because the notation was so bad. Michael just drew some arrows and a commutative diagram, and push, push, and there you are! Because he was looking at it in the right way!

That, of course, is part of teaching mathematics. You try to teach people to think the way a mathematician would think. When I was at school, I had a very good and experienced maths teacher. One day, when I produced a very elaborate piece of algebra, he said to me: "*Kingman, let me give you a word of advice. Mathematicians are lazy.*" What he meant was: "*You don't fill pages of algebra if you can do it in a much slicker way. And the slicker way will lead to better understanding. Your pages of algebra don't help me to understand it, even if they were a valid proof.*"

Speculations about current and future mathematics

[AM/MR]: *In which mathematical areas do you observe or expect the most exciting developments, right now, and in the years to come?*

JK: That we can't say, can we? Because the exciting thing is that next week someone will get up and say, "*Look, let's do it this way*" and suddenly the whole area will take off.

That's what makes it such fun. We're not just doing things the way our fathers and grandfathers did it. The nice thing about mathematics is that young people can do this, and we have to take them seriously.

For instance, my wife was a historian. To be a professional historian, you have to spend years in archives and so on. And nowadays, doing interviews and all sorts of things. Gathering evi-

ence, you can't suddenly say "*This is the truth about Napoleon.*" You have to do lots and lots of work and go over the evidence. And then when you are about 55, you write a great book showing that Napoleon was actually a quite nice man after all.

Whatever your theory is within mathematics, a young person can come along and say "*Well, I just proved the Goldbach Conjecture.*" Unless you've made a mistake, quite possible. Even Andrew Wiles has made a mistake. But unless you've made a mistake, they have to give you the prize, because you've done it.

And everyone, all these grand professors in the grand universities, suddenly have to say "*Yes, she has done it, or he's done it. We didn't think of it that way, because we've been working in a particular way.*" When you have done that for many years, it's very difficult suddenly to say: "*Oh no, we'll do it in a different way.*" You need a fresh person coming on board.

Max Planck said that science progresses from funeral to funeral. It's not that the old men change their mind, it's that they die, and new people come with new ideas. You don't have to kill them off; you got to bring in the new ideas!

I don't know what areas are going to be particularly fruitful. I can see applications like artificial intelligence, for instance. Something struck me: When software people started producing browsers or search machine engines, at some account they had to use matrix algebra. That was a surprise. Undoubtedly, artificial intelligence will require new maths, perhaps some novel infinite-dimensional geometry?

[AM/MR]: *Now that artificial intelligence is on everybody's mind, it will certainly have an impact on how mathematics will be done in the future and new types of applications will arise.*

JK: Yes, it's amazing to think that AI might produce a proof of the Riemann hypothesis. But it raises questions, doesn't it? I remember when people started serious work on the four-colour problem, they realised that they needed computers to look at all the different cases. And for a long time, no human being had actually worked through the entire proof. So, we were trusting computers. Well, that is a sort of artificial intelligence. If you have got a computer proof that you really can't work through yourself line by line, you are relying on a sort of artificial intelligence. Examples like this made people think more deeply about the nature of mathematical proof. We never give a proof which is in formal symbolic logic. Bertrand Russell tried to do it, but he couldn't get very far, just well enough to get to quite elementary mathematics. Over the centuries we have worked out all sorts of shortcuts. If you read a mathematical paper, it has a proof. But that proof is really a rhetorical proof. We like to call it a proof when it is convincing the reader that all possibilities have been thought through and that the result must be true. It is not that there is a series of rigorous steps with every little bit filled in. A mathematician would often say: *It is easily seen that.* You know, that's rhetoric. That's not logic.

[AM/MR]: *But by now quite a lot of at least semi-elementary mathematics has been transformed into logic, into LEAN software, for example. Apparently even quite complicated proofs have been checked in the meanwhile. It's quite amazing.*

JK: Yes. But another thought. Mathematicians used to produce beautiful asymptotic expansions and other devices to derive accurate numerical results. Then computers made these redundant by churning out numbers to high orders of accuracy. Technology changed the nature of mathematical practice. This will happen again.

Young and experienced mathematicians

[AM/MR]: *A question of a different type at the end. When we last saw each other many years ago, I remember having asked you what you wanted to do after you retired. You answered at that time, tongue in cheek, as little as possible. I don't know if you remember. Have you really retired completely from mathematics and statistics? Or are you at heart still busy?*

JK: I'd stick to my original answer. It was complicated, because I retired in 2006. My wife and I had bought a house, this house in Bristol, where I still live, and we set up a retired life. We travelled. We had social activities. But then in 2010, she had a very bad stroke. For the next seven and a half years, I was really looking after her, and that was my major activity. I couldn't do mathematics. If I first started thinking about a mathematical problem, then we wouldn't get anything for lunch. I really had to push mathematics away. She died in 2018, and then I had to decide what kind of life to lead moving forward.

I didn't think that there was really any chance of my catching up with serious mathematics. I enjoy doing puzzles and things like that. And sometimes, you know some of the puzzles that you read in the newspaper actually have a mathematical structure. It is quite interesting to look at them. Not from the point of view of just filling in the numbers, but thinking what an algorithm would look like, that sort of thing. I have never published anything about that.

Just occasionally I return to real maths. My friend Persi Diaconis, famous for instance for proving that a pack of cards must be shuffled seven times, came to lunch in Bristol. He set me a problem in advance, about large permutations. I thought I'd solved it, but it turned out that I had misunderstood and solved a different problem. Still, it was a not quite trivial result, and it gained an honourable mention in one of Persi's papers.

I spend a lot of my time reading, listening to music, I go out, but only in Bristol, visiting friends. I have a son and a daughter and three very interesting teenage granddaughters.

I greatly admire mathematicians who can go on into their 80s and 90s, producing non-trivial mathematics. I remember when I was, many years ago, an editor of the Journal of the London Mathematical Society, we got papers submitted by Littlewood and Besicovitch; they were both very old. I had to find referees, and the only thing I could think of doing was sending Littlewood's paper to Besicovitch and Besicovitch's paper to Littlewood. We received some interesting comments. Some mathematicians just go on and on and on, and I greatly admire that.

[AM/MR]: *In a way, you have already commented that there is some truth in Hardy's dictum that mathematics is a young person's game.*

JK: I think it's true in the sense that there are sorts of handbrake turns when mathematics takes a quite new direction, and this is almost always caused by a young person, and that has always been the case. Think of Galois or Abel attacking the old question of solving equations by radicals. They brought a completely new dimension to the field, introducing ideas like group theory and Galois theory. But older mathematicians develop a low cunning which can often complement the ideas of the young.

I always like it when someone writes to say they have found a mistake in my 1971 paper.

[AM/MR]: *Really?*

JK: Yes.

[AM/MR]: *Thank you very much for an interesting conversation covering many insights and many years!*

Ana Isabel Mendes is a professor at the Polytechnic University of Leiria, School of Technology and Management (Portugal). She holds a PhD in pure mathematics and is currently conducting research on skin lesions classification problems. She is a guest researcher at LABI – Bioinformatics Applications Laboratory at Western Paraná State University, Brazil.

aimendes@ipleiria.pt

Martin Raussen is an emeritus professor at Aalborg University, Denmark. In recent years, his research has focused on applying methods from directed algebraic topology to geometric and combinatorial models in concurrency theory within computer science. He served as editor-in-chief of the EMS Newsletter from 2003 to 2008 and as a vice-president of the EMS from 2011 to 2016. An interview with him appeared in Issue 131 of the EMS Magazine.

raussen@math.aau.dk

Evolution of ChatGPT in Mathematica: a user experience

Galina Filipuk and Andrzej Kozłowski

This paper explores the impact of artificial intelligence on research and education, using ChatGPT and Mathematica as case studies to illustrate AI's influence across both fields.

When ChatGPT by OpenAI¹ [2] debuted in 2023, it almost overnight became a worldwide sensation. It seemed to be able to engage in remarkably human-like conversations on a wide range of topics, about which it seemed to have extensive knowledge. However, concerns soon emerged about disconcerting reports of it making up facts, false quotations and inaccurate bibliographic references as well as “lying.” This unexpected phenomenon was given the name “hallucinations.” The idea of testing ChatGPT’s skills in mathematics occurred to many, but it became clear that the results would be underwhelming, since chatbots are not designed to perform any computations, even basic arithmetic ones.

Given how Large Language Models (LLMs) operate, this outcome is unsurprising. LLMs are trained on huge collections of examples and generate text by predicting the “most likely word” that continues a given phrase. They cannot execute loops or branching, which are essential in most computational algorithms. Nevertheless, there have been reports of ChatGPT-4 writing adequate programs in certain programming languages. So it seemed interesting to determine whether it could be useful to mathematics students who wish to use Mathematica,² but are not skilled programmers. It turned out that a similar idea was explored earlier to Stephen Wolfram. First, ChatGPT gained the capability to access Wolfram|Alpha³ [5], enabling it to return reliable answers to purely computational questions. More impressively, from version 13 ChatGPT became integrated directly within Mathematica, serving two roles: as “Wolfram Language Assistant” and as “Code Writer.” It now seems possible that ChatGPT with this new ability could help lower Mathematica’s learning curve, and perhaps even to solve mathematical problems, at least those that can be solved with the help of Mathematica.

This note aims to describe the findings of our investigation into the potential of combining ChatGPT with Mathematica for teaching analysis to students with little or no prior knowledge of Mathematica. We believe that some of our conclusions may be both surprising and intriguing. However, a degree of caution appears advisable. GPT and other AI chatbots are evolving extremely fast. Even during our early experiments with ChatGPT, we noticed changes in ChatGPT’s behavior, often improvements, but occasionally regressions. With the proliferation of GPT models and versions, keeping track of their features and quirks has become increasingly difficult. In addition, many chatbots beyond GPT have appeared, which were added to later versions of Mathematica. More recently, Wolfram developed its own model, which is now the default in any “chat-enabled” notebook. However, using these models from within Mathematica is not free. When ChatGPT was the sole option available, we obtained a grant to carry out this investigation. For this reason we have not experimented other models for this investigation, although we believe the conclusions would largely hold [1, 3, 4, 6]. Wolfram’s own model looks particularly promising, as it was designed specifically for Mathematica.

Another natural question is what would happen if we replace Mathematica with other similar programs. We have not investigated this, because we specifically focused on using Mathematica for teaching analysis at the University of Warsaw. Also, one of our key conclusions is the importance of chatbots being able to perform their own evaluations, which allows them to identify and attempt to correct their own errors. When we asked GPT about this, it acknowledged having some familiarity with other programming languages, but explained that it cannot carry out its own evaluations in them. However, for those interested in using AI with other systems there are various resources available online.⁴

Our approach involved asking ChatGPT to solve, with the help of Mathematica, undergraduate analysis problems, mostly taken from classes we have taught. In our early experiments, we were often confused by the fact that ChatGPT’s behavior varied over

¹ <https://openai.com>

² <https://www.wolfram.com>

³ <https://www.wolframalpha.com>

⁴ For instance, https://www.maplesoft.com/products/maple/new_features/ or <https://www.mathworks.com/discovery/chatgpt>.

time, as well as between different models and “personas” that one could choose in a Mathematica notebook. One problem was that sometimes ChatGPT was able to evaluate the Mathematica code it wrote, while other times it could not. Additionally, it occasionally provided incorrect answers to a computational problem, claiming they were produced using Mathematica. When we pointed out discrepancies between ChatGPT’s responses and Mathematica’s actual output, it would apologize but fail to explain why it had not properly used Mathematica. When ChatGPT was unable to evaluate Mathematica code, its abilities were seriously reduced. We eventually discovered that ChatGPT’s ability to perform its own evaluations could be enabled or disabled in the “Add and Manage LLM Tools” menu that appears in the top right corner of a chat-enabled Mathematica notebook. After disabling this feature, specifically the WolframLanguageEvaluator, we repeated some of our earlier queries and observed notable differences in the responses. Wolfram Language, the programming language of Mathematica, is built for computation. Like other languages it is made up of “words,” but each word corresponds to a specific computational task (detailed in the documentation). By combining these words, one can, in principle, carry out every computation, subject to the limits imposed by computational complexity. Programs written in ordinary programming languages have to follow strict syntax. Mathematica supports “free-form” input. This allows users to express queries in natural language, which Mathematica then attempts to interpret and convert into a computationally meaningful form. This means that in principle one can simply ask Mathematica to perform a computation without needing to learn the intricacies of Wolfram Language.

1 Curvature of a surface

When one asks Mathematica, via free-form input, to compute the curvature of a surface such as a sphere, it fails to return an answer. This is because Mathematica does not include a built-in function for this specific task, and free-form input is limited to referencing existing built-in functions. While there are several ways to compute surface curvature in Mathematica without additional programming, these methods are often difficult to discover for inexperienced users. This is precisely the kind of situation where one might hope ChatGPT could offer meaningful assistance. Asking ChatGPT to compute the curvature of a sphere using Mathematica leads to a surprising outcome. Although this is a relatively simple problem that ChatGPT can solve on its own, ChatGPT “invents” a function, sometimes called “Curvature,” other times “GaussianCurvature,” which it incorrectly claims is built in Mathematica. It then writes code that could indeed accomplish the task, if such a function actually existed. Needless to say, its code does not work.

Is there a way to ask Mathematica to compute the Gaussian curvature of a surface? For example, a sphere of radius r ?

Yes, you can compute the Gaussian curvature of a surface using the Wolfram Language. For a parametric surface, you can use the [GaussianCurvature](#) function.

Here is how you can compute the Gaussian curvature of a sphere of radius r :

1. Define the parametric equations of the sphere.
2. Use the `GaussianCurvature` function to compute the curvature.

For a sphere of radius r , the parametric equations are:
 $r(\theta, \phi) = (r \sin(\theta) \cos(\phi), r \sin(\theta) \sin(\phi), r \cos(\theta))$,
 where θ ranges from 0 to π and ϕ ranges from 0 to 2π .

Here’s how you can do it in the Wolfram Language:

```
r = r; (* Radius of the sphere *)
parametricSphere = {r Sin[θ] Cos[φ],
                    r Sin[θ] Sin[φ], r Cos[θ]};

GaussianCurvature[parametricSphere, {θ, φ}]
```

This behavior typically occurred when ChatGPT lacked the ability to evaluate its own code. In such cases, it was necessary to inform ChatGPT that its code did not work. With some guidance or carefully phrased hints, it could sometimes be steered toward a valid method for performing the computation. This approach required the users to already know the solution themselves in order to prompt ChatGPT correctly. In contrast, when ChatGPT was able to evaluate its code, it would initially follow the same path by generating a non-functional solution. But then, upon attempting to execute the code, it would recognize that the result was invalid. Remarkably, it could then consult the documentation and revise its approach, often coming up to a correct and working solution on its own.

2 Painlevé transcendents

A similar issue happened when we asked ChatGPT to compute Painlevé transcendents, which are not implemented in the current version of Mathematica. Each time, ChatGPT confidently asserted that these functions were available and even generated plausible sounding names for them, often different with each attempt.

Can one compute Painlevé transcendents with Mathematica?

Yes, you can compute Painlevé transcendents using Mathematica (Wolfram Language). The Painlevé transcendents are solutions to the Painlevé equations, which are certain nonlinear ordinary differential equations.


In Wolfram Language, the function `PainleveP` can be used to compute these transcendents. Here is an example how to use it:

```
PainleveP[1,x]
```

This function computes the first Painlevé transcendent P_1 .

To see the full details and options for this function, let me get the documentation for you.

It seems there was an issue with the lookup. Let me try again.

► Used Documentation Searcher 

It seems that the search returned some general information about the Painlevé transcendents rather than specific functions in Wolfram Language. However, I can confirm that the function to compute Painlevé transcendents in Wolfram Language is indeed `PainleveP`.

Here's an example of how to use it:

```
(* Compute the first Painlevé transcendent P1 *)  
PainleveP[1,x]
```

Would you like to see computations or plots of specific Painlevé transcendents? Let me know how I can assist further!

Once again, when it was able to evaluate its code, ChatGPT recognized the error and identified an alternative way to compute the values.




So does Mathematica have a built-in Painlevé function or not?

The Wolfram Language does not have a built-in function that directly computes Painlevé transcendents in a closed form. However, you can solve the differential equations numerically using `NDSolve` as demonstrated.

It seems that there was an error in the replacement part of the code. Let me correct it and provide the accurate result for the value at $x = 1$:

Correcting the computation of the value at $x = 1$

```
(* Define the first Painlevé equation  
p1Equation = y'[x] == 6y[x]^2+x;  
  
(* Initial conditions *)  
initialConditions = {y[0] == 0, y'[0] == 1};  
  
(* Solve the differential equation numerically *)  
p1Solution = NDSolve[{p1Equation,initialConditions},  
y, {x, 0, 1}];  
  
(* Compute the value of y at x = 1 *)  
yValueAt1 = y[1] /. p1Solution[[1]]
```

 Insert and evaluate  Insert  Copy

Let me evaluate this for you.

3 Computation of a limit

There were several other instances where ChatGPT's performance noticeably improved once it was granted the ability to evaluate its own code. We asked it to compute the limit of a sequence of real numbers defined by a simple nonlinear recurrence relation. Although ChatGPT was aware that Mathematica includes a function (`RSolve`) for solving recurrence equations, it noticed that the equation was nonlinear and assumed `RSolve` would not be applicable. As a result, it recommended a numerical approach instead. However, while `RSolve` cannot generally solve nonlinear recurrence equations, it is capable of handling this particular case. When ChatGPT was able to perform evaluations, it correctly used the built-in functions `RSolve` and `Limit` to get the correct answer. In fact, for the example we provided, the limit of the sequence can be determined without solving the recurrence relation, through a simple mathematical argument. ChatGPT did not initially recognize this. However, after computing the limit and observing that it was 0, it was then able to formulate a valid mathematical proof. This behavior is fairly typical for ChatGPT, and as human reasoning: once it knows the correct answer, it can often construct a plausible, and sometimes even correct, justification.

Consider the system of recurrence equations: $a[1] == x$, $a[n+1] = a[n](99/100 + 10/n)$, where x is some real number. Can you use Wolfram Language to find the limit of the sequence $a[n]$?

To define the recursive equation, we can use the `RSolve` function. However, since `RSolve` cannot handle recursive equations with a variable coefficient, we'll need to use an alternative approach.

We conclude that ChatGPT, when used within Mathematica, can serve as a valuable tool for users who are not well acquainted with the Wolfram Language, helping them navigate and use it more effectively, provided, of course, that its outputs are approached with a critical mindset. For experienced mathematicians, this should not be a problem. However, the situation is quite different for students whose primary aim is to learn mathematics. In such cases, reliance on ChatGPT can be problematic, as it is not always a reliable mathematician. ChatGPT is quite good at stating general theorems and even outlining entire theories, but it is prone to errors in reasoning, often the kind that may appear convincing to a less careful student. For instance, we were able to persuade ChatGPT that “all positive integers are equal” using a well-known fallacious inductive proof. Although it initially accepted the argument, once we pointed out the obvious contradiction, ChatGPT acknowledged the mistake and was able to identify and explain the flaw in the reasoning. That said, ChatGPT's ability to use Mathematica for performing computations, writing simple code, and even constructing illustrative examples for mathematical concepts and theorems can still be of significant value to students.

Much like a diligent student, ChatGPT can often find a solution after receiving a hint and can then apply the same method to solve similar problems. Unfortunately, all of this progress is lost once the chat session ends or the allocated memory is exhausted. If ChatGPT could retain knowledge from these conversations, it could be trained on a wide range of mathematical problems. Combined with Mathematica's computational power, it might evolve into a powerful tool for mathematical education. We should note that ChatGPT has recently introduced a memory feature, allowing it to retain information from user interactions and apply it in future chats. Unfortunately, this functionality is currently limited to paying subscribers of ChatGPT Plus and does not support sharing learned information between users.

Generative AI models like GPT are evolving rapidly, and their performance is improving. Our recent experiments on GPT-4 Omni show that it can now produce correct, rigorous solutions to certain first-year undergraduate problems while generating and explaining appropriate Mathematica code. However, several persistent limitations remain: inventing plausible but non-existent Mathematica functions (such as a fictitious `RadiusOfConvergence` function for power series); sometimes showing an inexplicable reluctance to use Mathematica, resulting in incorrect answers to computations that Mathematica can solve accurately; and, on the mathematical side, favoring vague generalities over precise, detailed explanations and

engaging in “information dumping,” i.e., providing unnecessary and unsolicited information.


Despite these limitations, we believe ChatGPT, and similar AI models, can be genuinely useful in mathematics education if integrated with a reliable computational platform. Notably, its ability to generate code, and more importantly, to clearly explain complex code, combined with its ability to rapidly produce computational examples that illustrate mathematical concepts, could substantially enhance both teaching and learning experiences. Since our initial study, ChatGPT has shown marked improvement. For instance, ChatGPT now correctly solves recurrence relation problems that previously stumped it.

Consider the system of recurrence equations: $a[1] == x$, $a[n+1] = a[n](99/100 + 10/n)$, where x is some real number. Can you use Wolfram Language to find the limit of the sequence $a[n]$?

To find the limit of the sequence defined by the recurrence $a[1] = x$ and $a[n+1] = a[n](\frac{99}{100} + \frac{10}{n})$, we can analyze the behavior of the sequence as n approaches infinity.

We can use `DiscreteLimit` function to compute the limit of the sequence. Here is how you can set it up in Wolfram Language:

```
DiscreteLimit[
  RSolveValue[{a[1] == x,
    a[n+1] == a[n] * (99/100 + 10/n)}, a[n], n],
  n -> ∞
]
```

 Insert and evaluate  Insert  Copy

This will give us the limit of the sequence $a[n]$ as n approaches infinity. Let's evaluate this expression.

 Used Wolfram Language Evaluator +

The limit of the sequence $a[n]$ as n approaches infinity is 0. This means that regardless of the initial value x , the sequence converges to 0.

The answer to the same question given by Wolfram's new LLM kit is almost identical.

4 Computing with OpenAI O1 model

The evolution of ChatGPT (and other LLM chatbots) is proceeding at a great pace. Only after completing the first version of this article we discovered that a new version of GPT, named OpenAI O1 model,

appears to represent a quantum leap in LLM model's mathematical capabilities. The new version not only was able to almost flawlessly deal with all the computational problems which proved too demanding for previous versions but is often able to give rigorous mathematical proofs, which may be challenging even for advanced undergraduate students. Here is an example from complex analysis, which illustrates the computational and theoretical capabilities of this version of GPT.

We want to find a Möbius transformation f of the complex plane, which is not the identity itself and which preserves the real axis (takes real numbers to real numbers) and is such that $f(f(f(f(z))))=z$, that is, if we iterate f five times we get the identity transformation. How can we use Wolfram Language to do this?

Below is one illustrative way to have the Wolfram Language search systematically for real-coefficient Möbius transformations of exact order 5. The idea is:

1. Represent the Möbius transformation by a 2×2 real matrix M with $\det(M) = 1$.
2. Having $f^5 = \text{Identity}$ as a projective transformation means M^5 is (up to a nonzero scalar) the 2×2 identity matrix. In $SL(2, \mathbb{R})$, $M^5 = I$ exactly.
3. Solve for real (a, b, c, d) subject to $M^5 = I$ and $ad - bc = 1$.
4. Exclude the trivial identity solution $M = I$ (which would give $f = \text{identity}$).

A suitable code outline is:

This is followed by Wolfram language code, which indeed finds all the solutions asked for. Still, even this version is not ready to replace human mathematicians. For some reason it seems unable to examine the result of evaluating its own code and so cannot see that it returns a number of non-trivial solutions. In fact, the existence of such solutions is obvious when one views Möbius transformations as projective transformations of the Riemann sphere: they are given by rotations by $2\pi/5$ along the great circle corresponding to the real line. But even so, its performance already far exceeds the expectations we had when we embarked on this project.

Because real coefficients mean that the real axis is mapped to itself, any nontrivial real solution here will give you a Möbius transformation preserving the real axis and satisfying $f^5 = \text{identity}$. In many cases, you will find that the only real solutions are the \pm identity matrices (which correspond to the identity

projective transformation), demonstrating that there is no nontrivial real-axis-preserving Möbius map of exact order 5.

Why might there be no nontrivial solution? In the group $PSL(2, \mathbb{R})$, elements of finite order >2 are "elliptic" transformations that typically do not leave the entire real axis invariant as a set. (They have a single fixed point in the upper or lower half-plane instead, and so their action on the real boundary is not an order-5 cycle.) Hence the system of equations can force the matrix to collapse to ± 1 .

If you run the code above and do indeed get only the identity/negative identity solutions, that confirms there is no Möbius transformation (with real coefficients) of order 5 other than the identity itself.

► Key Points

- A Möbius transformation $f(z) = (az + b)/(cz + d)$ with real a, b, c, d preserves the real axis.
- Requiring $f^5 = \text{identity}$ means the associated 2×2 matrix M must satisfy $M^5 = \text{scalar} \times I$. In $SL(2, \mathbb{R})$, that means $M^5 = I$.
- For finite orders >2 , real-coefficient 2×2 matrices in $SL(2, \mathbb{R})$ typically fail to fix the whole real axis, so the only solutions end up being ± 1 .

You can experiment with similar code (perhaps relaxing the real-coefficient assumption) to discover Möbius transformations of higher finite order that do preserve some other curve or circle in the complex plane, but necessarily the real axis.

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Galina Filipuk and Andrzej Kozłowski are the authors of a three-volume book on mathematical analysis with Mathematica published by de Gruyter. The second edition now in print contains a chapter on using ChatGPT with Mathematica.

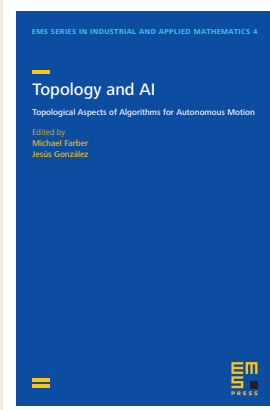
Galina Filipuk is an associate professor at the Institute of Mathematics, University of Warsaw. She received her PhD in mathematics from the Belarussian State University in 2002. She worked as a research associate at Loughborough University (UK) and as a research assistant professor at the Polish Academy of Sciences. She held Humboldt Research Fellowship for Experienced Researchers at TU Dresden and KU Eichstätt-Ingolstadt (Germany). She published more than 100 papers in special functions.

filipuk@mimuw.edu.pl

Andrzej Kozłowski is an adjunct professor at the Institute of Applied Mathematics and Mechanics, University of Warsaw. In 1968 he emigrated at the age of fifteen from Poland to the UK, where he completed his high school education and received his BA in mathematics from Warwick University and DPhil in algebraic topology from the University of Oxford. In 1980 he became a Japan Society for the Promotion of Science Postdoctoral Fellow at Tsukuba University (Japan). He worked as an assistant professor of mathematics at Wayne State University (USA) and as a professor at Toyama University of International Studies (Japan), and also taught at Tokyo Denki University (Japan). In 1993, he was a visiting professor at the University of Warsaw. He lived and worked in Japan for 25 years before returning to Poland in 2010. He has published over 30 research articles in algebraic topology and several in computer algebra and mathematics education. For several years he was a consultant for Wolfram Research, for whom he produced a number of Mathematica demonstrations.

akoz@mimuw.edu.pl

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Topology and AI
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Edited by
Michael Farber
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Jesús González (Center for
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Side by side: interview with Cristiana De Filippis and Tuomo Kuusi

Raffaella Mulas

In this new interview series for the EMS Magazine, each article brings together two mathematicians who share something – such as a research area, a passion, or a professional context – but differ in other ways, as for instance age, gender identity, background, or geographical location. Sitting “side by side,” they reflect on their personal and professional journeys, their views, and the worlds they live in.

The series is conducted by Raffaella Mulas and was born also thanks to the ideas and support of Apostolos Damialis (editorial director of EMS Press) and Donatella Donatelli (editor-in-chief of the EMS Magazine). By bringing together both common ground and contrast, side by side aims to reveal the richness of mathematical lives.

In this first article, Raffaella Mulas interviews Cristiana De Filippis (University of Parma) and Tuomo Kuusi (University of Helsinki).

Cristiana, Tuomo, and I are about to meet on Zoom. For the first few minutes, it’s just Cristiana and me. I’ve known her for a while, as we are both members of EMYA, and I have interviewed her before, on the occasion of her EMS Prize [1].

I know that she is a delight to interview, as she is a mathematical superstar who is also warm and easygoing. We chat about our summer plans and, of course, about Anakin – her cat, of whom I am a big fan.

We are waiting for Tuomo, and I’m not sure what to expect. Cristiana suggested him as her interview partner, and my pre-meeting search only turned up an intimidating list of publications and the fact that he is one of the most cited mathematicians. Impressive, yes – but will he be boring? Then Tuomo pops into the Zoom room, and my doubts vanish instantly. He radiates a spark of energy that’s immediately contagious.

Raffaella Mulas: Thank you both so much for being part of this! I’m really looking forward to seeing how this first double-interview takes shape. To get us started, could you each briefly share your journey, from when you first discovered your passion for mathematics to where you are today?

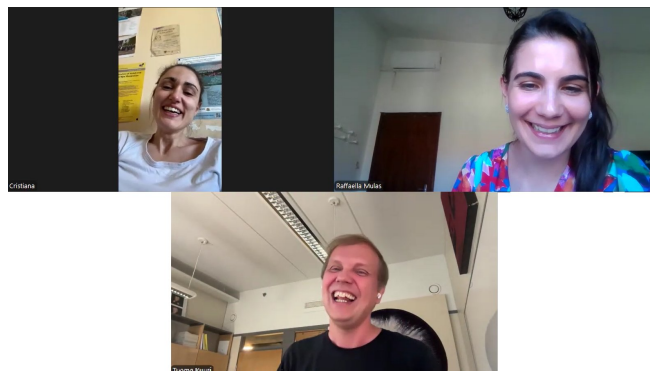


Figure 1. Cristiana, Raffaella and Tuomo on Zoom.

After a few “you go first” and “no, you go first,” Cristiana begins.

Cristiana De Filippis: My passion for mathematics probably began between primary and secondary school. Back then, I started doing a bit of extra work on my own, like trying to go beyond what we were learning in class and studying the subject more independently.

When it came time to choose a degree, I was actually surprised to discover that mathematics existed as a degree in its own right! Until then, it had been presented to me as something auxiliary – something you study only as a tool for subjects like engineering or physics. When I discovered that it can be studied at the university, that was it.

I did my bachelor’s in Turin, my master’s in Milan, then spent a few months in an internship at Inria Centre at Université Côte d’Azur, before moving to Oxford for my PhD. After that, I did a postdoc in Turin, and now I am a professor in Parma.

Tuomo Kuusi: For me, it happened later. I first went to an engineering-focused university. I had always been good at math, but I wasn’t especially fond of it at that stage.

Then, in my first year, I took a course called “The basics of modern analysis,” which, in reality, went much further. We were constructing Lebesgue measure, learning basic topology, and so on.



Figure 2. Giuseppe Mingione, Cristiana De Filippis and Tuomo Kuusi. (Picture courtesy of Cristiana De Filippis)

That's when I had my "wow" moment and decided to graduate in mathematics instead. A couple of postdocs and positions later, I am now a professor at the University of Helsinki.

[RM]: *I rarely hear stories like yours, Tuomo! I feel that many mathematicians, like Cristiana and me, choose mathematics at a young age.*

TK: For sure. I think a lot of it comes down to luck, as having a teacher who is motivating and can explain math well makes a huge difference. That's why, at our university, one focus is on training future teachers, starting from primary school. It's a really important part of our math department's mission.

[RM]: *That's very interesting! Now, the two of you have collaboration distance 2. How close are your research areas?*

TK: Well, you know, what I was working on before is actually quite close to Cristiana's area, which is regularity theory for elliptic and parabolic partial differential equations. Over time, I've moved more toward mathematical physics questions, but the techniques I learned back then are still extremely important in what I do now.

In fact, the last couple of times we've talked about problems, Cristiana has been fantastic. I'll have this very complicated proof – ten pages of horrible details – and she will say, "Why don't you just do it like this?" Suddenly my monster proof turns into two pages. It's pretty nuts!



Figure 3. Giuseppe Mingione, Giampiero Palatucci, Tuomo Kuusi and Cristiana De Filippis. (Picture courtesy of Giuseppe Mingione)

[RM]: *Haha, that's amazing! Does this mean you are going to become coauthors?*

CDF: Hopefully yes!

TK: What?! Cristiana, of course we are!

At this point, Cristiana and Tuomo dive deep into a discussion about techniques in their fields. I have no idea what they are talking about, but their excitement is so infectious that I smile and listen as if I do. The one thing I do understand is that they both love working with Rosario [Giuseppe Mingione] (Figures 2 and 3). When there is a pause, I jump in with my next question.

[RM]: *What were your PhD years like, and how do you think that period shaped you, both mathematically and personally?*

CDF: I did my PhD at Oxford, and it was a fantastic experience. I had a lot of freedom to work on what I found interesting, thanks to my advisor, Jan Kristensen, who gave me guidance without forcing me to follow trends. I could travel, meet people, and build collaborations both inside and outside my group. It was a fundamental time for me, both mathematically and personally.

TK: During my PhD I jumped into a completely new topic with my advisor, Juha Kinnunen. Very quickly, I found an open problem that had been around for decades, and I got obsessed with it. Eventually, my group and another group solved it independently, almost at the same time.

CDF: Not just "another group" – the *masters* of parabolic PDEs!

TK: Exactly! It was led by [Emmanuele] DiBenedetto. When I saw their paper, I thought, “Okay, fair enough, they found the same trick I did.”

CDF: You took it with so much fair play! I would have eaten the paper, like Scrooge McDuck’s competitor does with his bowler hat!

TK: I sort of did! But in the end, Rosario read my version, he liked it, and he suggested that we work together – and that was the beginning of our collaboration.

[RM]: *Sounds like a happy ending! Is there anything about the academic system that you wish you could change?*

TK: Funding could be structured better. Instead of just a few huge grants, I’d love to see more medium-sized ones that support more researchers.

CDF: Yes, and funding should also be adapted to each field. For some applied researchers, 1.5 million can vanish into two pieces of machinery; for mathematicians, the same amount could build an entire research group. But even then, local laws and bureaucracy can make it hard to use the funds in a way that keeps you competitive.

They also reflect on the difficulty of finding permanent positions. Then, the conversation takes an unexpected turn to artificial intelligence.

TK: The progress in just the last few years has been incredible. Having mathematical conversations with ChatGPT is now totally relevant.

CDF: Yes. It’s like knocking on a colleague’s door – not necessarily an expert in exactly what you’re doing, but still someone who can give feedback that helps you move forward.

TK: It’s not going to prove theorems for you, but it can suggest directions. You play around, and suddenly you think, “Oh, wow, maybe this is the right track.”

[RM]: *It’s indeed great for brainstorming. Now, would you like to share a short story that had a big impact on you or on the way you work? It can be something uplifting or challenging... anything that has stayed with you.*

TK: Well, my collaboration with Rosario had a miserable start. In our first paper, I wrote the fateful words, “by a simple covering argument.” Rosario went through the paper line by line, got to that point, and asked, “Okay, what is this simple covering argument?” Of course, it didn’t exist. That taught me a big lesson: you have

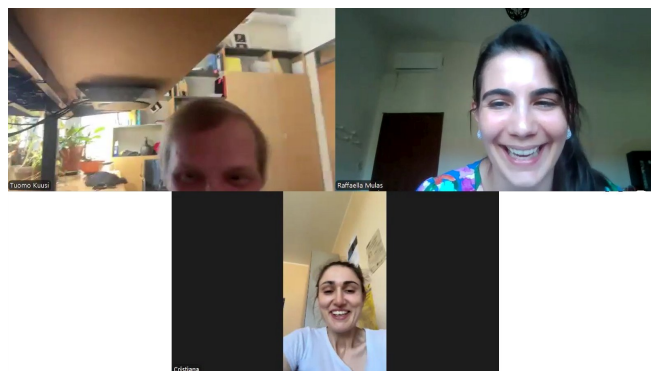


Figure 4. Cristiana, Raffaella and Tuomo on Zoom.

to be humble and check every detail, even when you think you understand it. Rosario once put it perfectly: “I saw this beautiful bird flying free in the sky – it was your professorship escaping!”

CDF: Haha! *Mutatis mutandis*, I’ve had the same experience: something that looked “standard” turned out to be completely wrong in the simplest case. I had to rethink the proof entirely to fix it.

TK: And what’s the moral of your story?

I love that the interviewees are interviewing each other.

CDF: The moral is that you have to be extremely careful, even with things that you believe are trivial, and it is helpful to check the most basic cases.

TK: So true! Here’s how I see it: when you are coming up with an argument, you should think like an artist: with big brushstrokes, no boundaries, trying ideas just to see where they might lead. But once the painting is done, you have to switch roles completely, and become an accountant, checking every single detail as if the person who wrote it (even if it’s you) had no idea what they were doing. It’s a painful process, and it’s not nearly as fun as the creative stage, but it’s absolutely essential.

CDF: Yes, right. You need to be able to look at your research from outside, from above, from far away, pretending that it was not yours.

Cristiana is joining the Zoom meeting from her phone. When the battery runs low, she disappears from the screen and then reappears under her table, plugging it in.

TK: Why did you go under the table now, Cristiana?

CDF: I just wanted to give you a different perspective!



Figure 5. Cristiana and Anakin. (Picture courtesy of Cristiana De Filippis)

TK: Yeah, should I go as well?

Then Tuomo slips under his table too (Figure 4). We all burst out laughing.

CDF: Don't worry, Raffaella, this is normal! That's how we are all the time!

[RM]: *I'm amused!*

They remind me of the time I saw Bernd Sturmfels lying on the floor during a keynote talk at a conference. He is also one of the most cited mathematicians. I'm starting to see a pattern.

[RM]: *Thank you for these beautiful images: the painter, and the mathematicians under the tables. I'll wrap up with one last*

question. Outside of mathematics, what do you enjoy doing that recharges you?

CDF: I stay charged by running every morning from Monday to Friday. I also love riding horses and spending time with my horse and my cat (Figure 5).

[RM]: *My friend Anakin! How about you, Tuomo?*

TK: I love dancing on roller skates! I even made Cristiana try it once.

CDF: And he brings his roller skates with him when he travels. He has the suitcase of a rock star!

TK: Well, you know, nothing lasts forever!

I wish this interview did.

[RM]: *Thank you both very much, this was fun!*

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Raffaella Mulas is a discrete mathematician but not a discreet one, as she loves listening to and sharing people's stories. She is an assistant professor at VU Amsterdam, where her work is supported by a VENI grant from the Dutch Research Council (NWO). She is also an elected member of EMYA and the founder of *Nora – Center for Science Communication*. r.mulas@vu.nl

Fraudulent publishing in mathematics: a European call to action and how information infrastructure can help

Moritz Schubotz and Jan Philip Solovej

The IMU–ICIAM working group’s new report on fraudulent publishing in the mathematical sciences documents how gaming of bibliometrics, predatory outlets and paper-mill activity are eroding trust in research, mathematics included. This short EMS note brings that analysis home to Europe. We urge readers to recognise the warning signs of fraudulent publishing, to report serious irregularities so that they can be investigated and sanctioned, and to reflect critically on their own editorial and reviewing practices. We then sketch why Europe is well placed to lead a structural response: a decade of policy development on open science; mature infrastructures for data, software and scholarly communication; and new capacity for community-led diamond open access. Finally, we outline developments towards non-print contributions across member countries, including the growth of formal proofs (e.g., with Lean and Isabelle) and we highlight the role of zbMATH Open as a European quality signal that can help editors, reviewers and authors steer clear of problematic venues.

1 A call to action

The core message of the IMU–ICIAM report [1, 2] is sobering: when numbers become targets, they invite manipulation. In fields like mathematics, with fewer papers and lower citation volumes, a handful of strategic behaviours (excessive self-citation, cartelised referencing, “special-issue mills”) can strongly skew metrics, rankings, and careers. Europe is not immune. The business shift to APC-funded mega-journals and the normalisation of quantitative research assessment have created perverse incentives. We now see: *Volume-over-value publishing*: large fleets of themed special issues; light or inconsistent peer review; guest-editor networks oriented around throughput.

Citation gaming: editorial pressure to add citations post-acceptance; reciprocal referencing; opaque “recommended citations.”

Paper-mill leakage: templated articles with “tortured phrases,” fabricated references, or recycled figures that sometimes pierce editorial checks.

Downstream effects: hiring and promotion tilted toward countable surrogates; university rank chasing; loss of trust among collaborators and early-career colleagues.

This is fixable, but only if we act where Europe has natural leverage: community-owned infrastructures, curated indexing, open-science standards, and editorial leadership. Fraudulent publishing damages the literature, wastes researchers’ time, and distorts incentives. While most mathematicians act in good faith, experience shows that small distortions of bibliometric targets can escalate into serious manipulation. The first responsibility is therefore individual [2]: read the papers you evaluate; check venues before submitting or accepting an editorial role; and, as a reviewer or editor, insist on transparent authorship and sound referencing practices. When you encounter serious irregularities (plagiarism, paper-mill products, manufactured citations, or identity misuse), report them through journal procedures; if necessary, raise concerns with institutions or recognised community channels.

Editors and editorial board members have special duties. If commercial pressure, guest-editor overload, or systematic citation manipulation erodes standards, resigning (and explaining why) is sometimes the only way to protect the record. This is uncomfortable, but it is often decisive. Ultimately, the community will only repair incentives if we act early and visibly. The aim is not policing for its own sake; it is to preserve trust in our literature and to protect younger colleagues from short-term temptations that can harm careers and institutions in the long run.

Within the Society, the EMS Committee for Publications and Electronic Dissemination (PED) has issued guidance on sound publishing practice and predatory outlets, which complements the IMU Committee on Publishing (CoP) recommendations [11, 12].

2 Why Europe can lead

Europe has aligned policy and infrastructure that make community-governed publishing feasible at scale. In May 2023 the Council of the European Union called for immediate and unrestricted open access to publicly funded research, for transparency in publishing contracts and costs, and for robust quality and integrity safeguards [6, 7]. These conclusions recognise the importance of non-profit, scholar-led models (often called *diamond* open access) as part of a sustainable ecosystem.

The European Open Science Cloud (EOSC) provides a federated fabric for data, software and services; importantly for our community, the European Mathematical Society (EMS) participates in the EOSC Association as an *observer*, ensuring that mathematical perspectives inform EOSC's evolution [9]. For journals and platforms, Europe's *European Diamond Capacity Hub* (EDCH) has emerged as a coordination point: it gathers funders, infrastructures and communities to produce shared guidance, a common access point to services, and a living map of the diamond landscape [13]. The Horizon Europe project DIAMAS underpins this work with practical recommendations for institutional publishing in Europe and with an inventory of services and requirements [10, 18].

These pan-European efforts dovetail with national capacity. In Germany, SeDOA (the Servicestelle Diamond Open Access) acts as a national node that curates information, offers guidance to journals, and connects institutions and funders with the European conversation [17]. In the Netherlands, *Openjournals.nl*, provides an integrated diamond platform used by many scholarly journals across disciplines [16]. Such initiatives make it realistic for editorial boards to migrate away from risky models without sacrificing visibility or service quality.

EMS itself has taken a leading role in shaping open access publishing. In 2019, the Society transformed its publishing house into *EMS Press*, today operating under the subscribe-to-open (S2O) model for its journals [4]. While EMS Press S2O publications are only made freely available to readers if sufficient subscriptions are secured, the model is free for authors, applies an open access-compliant licence, and lets the EMS or other journal-owning societies of institutions claim the publication rights. EMS Press is thus organised as a professional publisher, but with EMS as its sole shareholder its orientation is not towards maximising profit but towards serving the scientific community.

This approach resonates with the definition of *diamond open access* articulated by the German Servicestelle Diamond Open Access (SeDOA): "In diamond open access journals, quality-assured articles are published free of charge for authors and readers under an open access-compliant licence. The journal or title rights belong to the scholarly community. Diamond open access journals serve academic communication and non-commercial purposes [17]." In this sense, EMS Press is a distinctive hybrid: while operating commercially in form, its governance and S2O model place it firmly in the landscape of community-oriented publishing, ensuring that quality and openness remain at the centre.

3 Member-country developments: data, software and formal proofs

Open science in mathematics is not only about access to articles. Reproducibility depends on sharing *data*, *software* and, increasingly, *formal proofs*. Several member countries are investing in field-

specific infrastructures. In Germany, the Mathematical Research Data Initiative (MaRDI) within the National Research Data Infrastructure (NFDI) develops standards for mathematical research data, verified workflows, and services, including a portal that interlinks objects, algorithms, software and literature while implementing the FAIR principles [15].

On the software side, *swMATH* (now integrated into *zbMATH Open*) identifies and interlinks mathematical software with the literature, and is being connected to persistent-identifier graphs used in EOSC to improve discovery and credit [3, 21]. This makes it easier for editors and referees to check that computational claims are supported by citable software artefacts.

Formal proofs are also gaining traction. The COST action *Euro-ProofNet* brought together more than 40 countries to improve the interoperability and usability of proof assistants and libraries across Europe [5]. Meanwhile, communities around *Lean/mathlib* and *Isabelle* continue to expand, with training schools and research programmes across the continent [14, 19]. For editors and referees, this might matter in practice: when feasible, formally verified arguments may reduce the attack surface for paper-mill text, fabricated claims, and accidental error.

4 European quality signals: the role of *zbMATH Open*

Located in Europe and co-edited by the EMS, *zbMATH Open* curates the mathematical literature with post-publication reviews and rich interlinking to data and software. Crucially for quality assurance, indexing may be *cover-to-cover* or *in parts*, and indexing decisions are reviewed over time [20]. In consequence, journals whose practices deteriorate can be excluded from further coverage, and in multidisciplinary venues only mathematically relevant items are indexed. For editors, authors and evaluators, this provides a pragmatic signal: if a venue is no longer covered or is only partially indexed, caution is warranted. In parallel, recent developments at *zbMATH Open*, for example the display of OA and licensing information at article level and tighter integration of software metadata, help the community to verify provenance, reuse conditions and computational support [8, 21].

5 What EMS members can do now

Mathematicians are central stewards of the literature. In Europe, we benefit from aligned policy and strong community infrastructures, but their effect depends on our everyday choices. First, when you act as an editor or reviewer, hold the line on quality and transparency, and use community services like *zbMATH Open* and *swMATH* to check provenance and software. Second, when a venue's behaviour raises concerns, *say so*: report, withdraw support, and encourage colleagues to move to safer homes; European diamond

platforms now make this transition viable. Third, invest in openness by sharing data, code and, when appropriate, formal proofs; this narrows the space for low-quality and fraudulent work to hide. Europe has the policy framework and the infrastructure; if we align our publishing choices with these tools, we can rebuild incentives that reward *mathematics*, not metrics.

Acknowledgements. This paper summarises the companion report *Fraudulent publishing in the mathematical sciences* [1, 2] and aligns its recommendations with current European open-science infrastructures. The authors thank colleagues, in EMS in particular the EMS Committee for Publications and Electronic Dissemination (PED), IMU, in particular the Committee on Publishing (CoP)¹ and ICIAM for comments and discussions. The text of this article was improved using advanced grammar correction and large language models.

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Moritz Schubotz studied physics and computer science at the University of Wuppertal and TU Berlin. After his PhD, he was a postdoc at the University of Konstanz and the National Institute of Informatics in Tokyo before joining FIZ Karlsruhe in 2019. At FIZ Karlsruhe, he researches methods to improve zbMATH Open and related research infrastructure. moritz.schubotz@fiz-karlsruhe.de

Jan Philip Solovej is the president of the European Mathematical Society and a professor of mathematics at the University of Copenhagen, where he leads the Centre for the Mathematics of Quantum Theory (QMATH) at the University of Copenhagen. He is also the head of studies for the MSc programme in Quantum Information Science offered jointly by the University of Copenhagen and the Technical University of Denmark. He is the editor-in-chief of the *Journal of Mathematical Physics* and a member of the Royal Danish Academy of Sciences and Letters, Academia Europaea, and the European Academy of Sciences.

solvej@math.ku.dk

¹ <https://www.mathunion.org/cop>

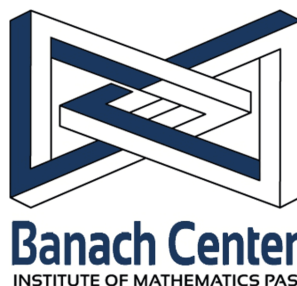
Banach Center and its mother institution, IMPAN (Institute of Mathematics of the Polish Academy of Sciences)

Adam Skalski

Stefan Banach International Mathematical Center, traditionally known as the Banach Center, forms an integral part of the Institute of Mathematics of the Polish Academy of Sciences (IMPAN). The institute was established in 1948, as the independent State Mathematical Institute, and in fact predates its umbrella institution, the Polish Academy of Sciences, created in 1952. IMPAN has originated in plans put forward by the leading figures of the world-famous pre-war Polish school of mathematics, such as Kazimierz Kuratowski, who was in fact the first director of the institute. Since its inception, IMPAN has continuously remained a top mathematical institution in Poland and one of the most important research centers in Central and Eastern Europe. Currently, it employs over 20 permanent research faculty, around 20 mathematicians on long term (tenure-track type) contracts, and almost 40 short-term researchers, mostly postdoctoral students, but also visiting professors. It continually hosts several national and international grants; notably, the only three individual ERC grants in mathematics in Poland were awarded to researchers based at IMPAN. The institute is also a partner in two doctoral schools, one joint with the University of Warsaw, the other, smaller and focused on applied mathematics, with the Institute of Geophysics of the Polish Academy of Sciences. The Warsaw headquarters of IMPAN is located in a historical building in the city center, which pre-1939 used to belong to the Warsaw Scientific Society, and hosted the Radiological Laboratory formally led by Maria Skłodowska-Curie, and the laboratory of Edward Flatau, the father of Polish neurology. Here also one can find the Central Mathematical Library, one of the biggest mathematical libraries in Eastern Europe, and the Publishing House of IMPAN, issuing nine international journals, including the 'golden three' of the Polish mathematical tradition, all established in the 1920s: "Fundamenta Mathematica," "Studia Mathematica" and "Acta Arithmetica," but also the series Banach Center Publications devoted to research produced by the visitors of the Banach Center.

History of the Banach Center and its Scientific Council

Since 1972 a particular role in the institute's activities has been played by the Stefan Banach International Mathematical Center,



founded within IMPAN for the purpose of promotion and intensification of international cooperation. It was a way of realizing a dream of Polish mathematicians concerning the building of a bridge between the scientists working in Central and Eastern Europe and the rest of the world. As one can imagine, obtaining the green light from the communist government for such an enterprise was not easy, but in the end, mostly due to efforts of the then director of IMPAN, Bogdan Bojarski, in 1972 the founding agreement was signed by the academies of sciences of Poland, Bulgaria, Czechoslovakia, GDR, Hungary, Romania and the Soviet Union. These partners were later joined by the academies of Vietnam, Cuba and North Korea. During the years 1972–1992 the Center gained a global reputation as the actual meeting point of mathematicians coming from all over the world. In that period, it hosted over 40 focused semesters, concerning practically all fields of mathematics.

In 1993 the original setup run its course, and it was replaced by a new agreement with the European Mathematical Society (the EMS itself having its roots at a meeting which took place in 1990 in Mađralin, near Warsaw), guaranteeing new perspectives of development of the Center. In particular, it was decided that the reinvigorated institution will organize intensive workshops, conferences and smaller research groups. The agreement with the EMS included a formation (or rather re-formation) of the Scientific Council of the Banach Center, which meets annually in Warsaw. The council includes nine members, which serve at most two four-year terms. Three of them are chosen by the EMS, three represent the original founding countries, and the remaining ones come from Poland. The chair of the council is elected by the members; for

many years this role was played by the President of the EMS. Here is the full list of the mathematicians who acted as chairs of the council:

- Lubomir Iliev (Bulgaria) 1972–1976
- Ákos Császár (Hungary) 1977–1980
- Klaus Matthes (East Germany) 1981–1982
- Kazimierz Urbanik (Poland) 1983–1986
- Romulus Cristescu (Romania) 1987–1989
- Sergeĭ M. Nikol'skiĭ (USSR) 1990–1992
- Friedrich Hirzebruch (Germany) 1993–2001
- Rolf Jeltsch (Switzerland) 2002–2005
- Ari Laptev (Sweden) 2006–2013
- Marta Sanz-Sole (Spain) 2013–2014
- Pavel Exner (Czech Republic) 2014–2021
- Gert-Martin Greuel (Germany) 2022–2023
- Šárka Nečasová (Czech Republic) 2023–

The daily activities of the Center are supervised by the director of IMPAN or one of his deputies; apart from the team working in the Będlewo center, which will be introduced below, the Center employs two secretaries located in Warsaw.

Locations

For many years, the main building of the Banach Center was a 19th century palace (Pałac Cukrowników) in Warsaw, situated not far from the IMPAN headquarters mentioned above. When in the 1990s it had to be returned to the pre-World War II owners, the efforts of the then directors of IMPAN led to acquiring from the Polish Academy of Sciences another 19th century palace, this time based in Będlewo, a small village near Poznań in western Poland. The Mathematical Research and Conference Center in Będlewo, as it is now called, includes a neo-Gothic palace, service buildings, a hotel and a large (with an area of almost nine hectares) park. It can host scientific meetings of up to 120 participants as well as support research in smaller groups. In 2021, it gained a separate modern, purpose-built lecture hall with 120 places.

Some of the Banach Center events take place in the Warsaw headquarters of IMPAN, which includes several lecture halls. The main auditorium there has 100 seats and recently installed high-quality audio-video equipment, allowing for efficient live streaming of lectures. The Warsaw building has also 20 guest rooms, in which the participants in the activities can be conveniently accommodated.

Current activities, collaborations

In recent years (apart from the pandemic slowdown) the Banach Center has been organizing annually around 50 events, with approximately 1000 Polish and 1500 foreign participants. As



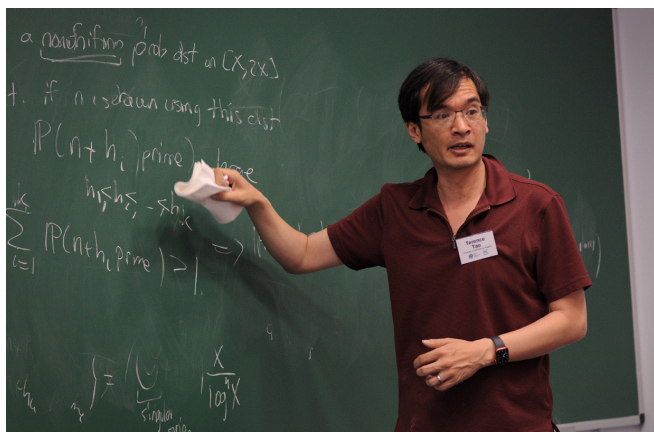
Będlewo Center. (© IMPAN)



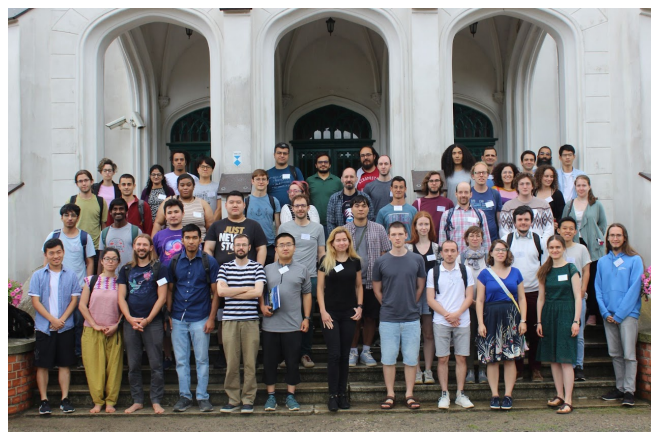
The main lecture hall. (© IMPAN)

mentioned above, some of the conference proceedings are published in the series Banach Center Publications, which already counts over 130 volumes. Typical events range from big, general mathematics conferences, through smaller workshops and graduate schools, to working groups based on the “research in pairs” model. A sample list of recently organized events is as follows.

- “Numbers in the Universe,” a number theory conference organized in summer 2023 jointly in Warsaw and Kyiv by the Banach Center and the International Centre for Mathematics in Ukraine (ICMU), with Maryna Viazovska, Vitaly Bergelson and Terence Tao as main speakers.
- Regular nationwide meetings, involving international contributors such as “Polish Combinatorics Meeting,” “Polish Probability Conference” or “Forum of the PDEs.”
- Graduate schools: for example, the Summer School in Algebraic, Asymptotic and Enumerative Combinatorics, organized in July 2023 in Będlewo.



Terence Tao at the conference "Numbers in the Universe," Warsaw.
(Photo © by Michał Kotowski)



Participants of the Summer School in Algebraic, Asymptotic and Enumerative Combinatorics, Będlewo, July 2023. (© IMPAN)

The Banach Center is one of the founding members of ERCOM (European Research Centres on Mathematics), a committee of the EMS gathering the top mathematical research institutes and conference centers in Europe. It is also involved in formal collaborations with several other mathematical institutions. Since 2015, it has been organizing Simons Semesters in Banach Center, a program of focused longer activities, supported by the Simons Foundation within the Targeted Grants for Institutes framework. This year marks the beginning of the already third edition of the program, called "Simons Semesters in Banach Center – New Energies."

Since 2019, the Będlewo Center hosts also each autumn a graduate school co-organized with the Mathematisches Forschungsinstitut Oberwolfach, within the scheme called MFO-Banach Center Graduate Seminars. Some other joint activities, such as events organized together with the International Centre for Mathematics in Ukraine, were already mentioned above.

Procedures and invitation to apply for organizing events

Despite running a very successful program of events for more than 50 years now, the Banach Center does not have an independent, stable long-term funding, and it is supported mostly from the budget of IMPAN, and occasionally from some short-term grants from the Ministry of Science and Higher Education. Due to these constraints, it only offers partial support to its workshops. In practice the procedure is as follows: a call is made at the beginning of the year X (usually with the deadline in early May), for events to take place in autumn of the year X and in the year $X + 1$. The applications are discussed at a meeting of the Scientific Council, where the organizers are given a chance to present their proposals. The council decides about the acceptance of the events and proposes a ranking, which is then converted by the board of IMPAN directors

to the actual support level, which traditionally, depending on the ranking, reaches between 20% and 50% of local costs. The results are announced by the end of June. Potential organizers may also apply outside the dates of the annual competition directly to the Banach Center office; in this case, however, the support offered is significantly smaller.

The applicants are supposed to propose some dates for their meeting and are encouraged to discuss it in advance with the Banach Center office (at office@impan.pl). As expected, the summer months are most popular, and the Center is especially keen to receive more proposals for events taking place in the period between November and March – this is not quite the right time for swimming in the lakes near Będlewo, but the park looks beautiful also in winter. Although the organizers have to find a way to complement the support offered by the Center, using conference fees or personal/institutional grants, the full cost remains rather low, with roughly 500 Euro per person being at the time of the writing an approximate price for a five-day meeting in Będlewo (including accommodation, food, access to all facilities and transport from/to Poznań).

You are very strongly encouraged to apply; more information can be found at <https://www.impan.pl/en/activities/banach-center>.

Adam Skalski completed his PhD in Nottingham in 2006. Later he worked in the UK, Japan and France, before returning to his native Poland in 2010. He has been the deputy director of IMPAN, responsible for the Banach Center, since 2015. During the term 2022–2024 he was the chair of ERCOM, and since 2025 has served on the Executive Committee of the EMS.

a.skalski@impan.pl

Traces and trajectories: in memory of Edi Zehnder (November 10, 1940–November 22, 2024)

Paul Biran and Felix Schlenk

Edi Zehnder began as a physicist, then turned to analytical dynamics studying small denominators, and eventually became one of the founding figures of (global) symplectic geometry and topology. He was born in Leuggern, a Swiss village in the canton of Aargau, where the Aare River merges with the Rhine. His father decided that the best education for him would be in a convent, so he attended high school at the boarding school of the Benedictine convent Einsiedeln. The education focused on ancient languages (Greek and Latin), and the only way to leave the convent during the week was to play in a local football club, which Edi did with enthusiasm. It was also at Einsiedeln that Edi met Jeannette (he 18 and she 15 at the time), who came with her class for a visit to the convent and was very impressed by Edi's pencil drawings and the fact that he smoked a pipe against the rules. From then on, they were together for 66 years, until Jeannette died three months before Edi.

Edi studied mathematics and physics from 1960 until 1965 at ETH Zürich. The curricula of these two disciplines were nearly identical at that time. Heinz Hopf was no longer accepting PhD students, so Edi became a PhD student in theoretical physics, working under the supervision of Res Jost.

Jürg Fröhlich, Edi's close friend and colleague at ETH, recalls:

I became a PhD student in theoretical physics in the autumn of 1969. At the time, the theory institute of ETH was located in a very large apartment of an old villa on Hochstrasse, above the main building of ETH. As the pictures show, Edi enjoyed smoking the pipe.

Since this was not appreciated by everyone, he was exiled into the bathroom of that apartment, not a very friendly place. I was a cigarette smoker and I did not mind the smell of pipe smoke. It was therefore decided that Edi and I should share a large, comfortable room of the institute. Edi was in the process of completing his PhD thesis on a subject in celestial mechanics: *On the restricted three-body problem*. The aim of his work was to understand the stability of orbits of the asteroids in the Hilda group. Res Jost, who was quite a heavy smoker too, often came to our office to discuss various questions related to Edi's thesis. He would argue that



Edi, 1985. (© Gerd Fischer, München; source: Archives of the Mathematisches Forschungsinstitut Oberwolfach)

the orbits of the Hilda group must be stable, because his wife Hilde was a stable person.

Res Jost admired Jürgen Moser's work on Hamiltonian mechanics, in particular his contribution to the KAM theorem, and he passed his admiration on to Edi. The topic of Edi's PhD thesis and his erudition in celestial mechanics made it natural for him to apply for a postdoctoral position in Moser's group at the Courant Institute. In 1971, the Zehnders (and their dog) moved to New York and, after a year, on to the Institute for Advanced Study (IAS), where Freeman Dyson was Edi's host.

In these years, Edi worked on small divisor problems (Nash–Moser method and hard implicit function theorems), clarifying and

simplifying the proof of the KAM theorem.¹ His stay at IAS led to a lifelong friendship with Freeman Dyson (Dyson invited Edi again for the academic year 1979/80).

In 1974, Edi moved to Germany and obtained his habilitation at the University of Erlangen–Nuremberg. He became a full professor at the University of Bochum in 1976.

In 1977 one of the great mathematical stories begins to develop, which ultimately culminates in Floer theory—it is a story of random interactions and mathematical curiosity across fields; a story of opportunities one needs to sense and grab, even when mathematical orthodoxy is dismissive. Edi plays a pivotal role in this story.

The context was as follows: In an important paper, Jürgen Moser proves an existence result about periodic orbits of Hamiltonian systems near an equilibrium point. For this, he uses the Hamiltonian action principle, which was generally considered useless for mathematical purposes. He carries out a local finite-dimensional reduction, so that the periodic orbits of interest can be found as critical points of a function defined locally on a finite-dimensional space by critical point theory (reminiscent of ideas from Morse theory). Moser, very knowledgeable about variational problems, also gives a short explanation of why the Hamiltonian action principle is useless for global questions—his local use of it is a breakthrough, but there is no way to use it globally. Paraphrased from today's perspective, he just said (pre-Floer) that there will never be a Floer theory. The error is short-lived. His former student Paul Rabinowitz shortly afterward submits a paper to the *Communications of Pure and Applied Mathematics (CPAM)* in which he finds global periodic orbits of an infinite-dimensional Hamiltonian system, a wave equation problem. Moser, who is an editor of CPAM, immediately sees his mistake and asks Rabinowitz if he can adapt his method to prove a global version of his local result, and Rabinowitz indeed succeeds in doing so.

Edi recognizes at once that Rabinowitz's ideas have broader ramifications, which he then explores with Herbert Amann.² While previous finite-dimensional reductions only approximated solutions to Hamilton's equation, the Amann–Zehnder reduction produces exact solutions, making it possible to count them. In the same year, Charles Conley publishes his influential book, which can be viewed as an extension of Morse theory to topological flows (Conley index theory). Together with Conley, Edi develops the idea of generalizing



Edi and Jeannette, around 1970. (Courtesy: Jürg Fröhlich)

the two-dimensional Poincaré–Birkhoff fixed point theorem to higher dimensions. They introduce the so-called Conley–Zehnder index and use Conley's index theorem in the study of asymptotically linear Hamiltonian systems.³

While writing this paper, Conley is on a sabbatical at Bochum. At the beginning of December 1982, on an invitation by Jürgen Moser they visit the *Forschungsinstitut für Mathematik (FIM)* of the ETH Zürich. An unexpected twist happens, which is summarized by Helmut Hofer:

John Mather from Princeton arrived in early December at the FIM to spend part of his sabbatical. He had just visited Michel Herman in Paris. According to Edi, Herman was one of the sharpest minds in the theory of dynamical systems, and Conley and Zehnder queried Mather about what Herman was up to. They learned that Herman and other members of the so-called Orsay topology group had tried to read a paper by the then-unknown Russian mathematician Yakov Eliashberg but struggled to understand it. The paper contained a complicated proof of the Arnold conjecture on symplectic fixed points in the case of surfaces.⁴ To be polite, Conley and Zehnder ask Mather what the Arnold conjecture is all about. Mather explains the conjecture, and more or

¹ E. Zehnder, Generalized implicit function theorems with applications to some small divisor problems. I. *Comm. Pure Appl. Math.* **28**, 91–140 (1975), and E. Zehnder, Generalized implicit function theorems with applications to some small divisor problems. II. *Comm. Pure Appl. Math.* **29**, 49–111 (1976).

² H. Amann and E. Zehnder, Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **7**, 539–603 (1980), and H. Amann and E. Zehnder, Periodic solutions of asymptotically linear Hamiltonian systems. *Manuscripta Math.* **32**, 149–189 (1980).

³ C. Conley and E. Zehnder, Morse-type index theory for flows and periodic solutions for Hamiltonian equations. *Comm. Pure Appl. Math.* **37**, 207–253 (1984).

⁴ The interesting story of Eliashberg's paper on the Arnold conjecture and the way it arrived to the "west" is retold in detail in the forthcoming book by Siobhan Roberts and Helmut Hofer.

less instantaneously, after looking at each other, Conley and Zehnder pronounce: “We can do this, give us a couple of days.” Indeed, three days later they explain their proof to Mather. And two weeks later Edi gave the last colloquium talk of the year at the University of Zürich, describing the proof of the Arnold conjecture for tori.

It turned out that the Conley–Zehnder ideas to prove an extension of the Poincaré–Birkhoff theorem were more than what was actually needed to prove the Arnold conjecture for the standard $2n$ -dimensional tori:

Theorem.⁵ *Every Hamiltonian diffeomorphism on the torus \mathbb{T}^{2n} must have at least as many fixed points as a smooth function must have, namely $2n + 1$. If the fixed points are non-degenerate there must be at least 2^{2n} .*

This was the first higher-dimensional global theorem in symplectic geometry. It showed that Hamiltonian flows are fundamentally different from volume-preserving flows. The method of proof shocked the specialists in the field, since it had little to do with symplectic geometry as understood at the time. Arnold invited Edi to Moscow, Gromov invited him and Conley for a talk in Paris, and Edi became an invited speaker at the 1986 ICM.

The novel use of Morse theory in their proof of the Arnold conjecture opened up a new field of research at the intersection of dynamical systems, geometry, and topology. The finite-dimensional reduction of the action functional inspired a great deal of work, in particular from the French school by Chaperon, Laudenbach, Sikorav, and Viterbo on generating functions. In a different direction, Edi’s student Andreas Floer intertwined the variational methods of Conley and Zehnder with Gromov’s pseudo-holomorphic curves to what is now called Floer homology, a key tool in symplectic geometry and topology, that has many applications to Hamiltonian systems, mirror symmetry, and low-dimensional topology. We again refer to the forthcoming book by Hofer and Roberts for more on Edi’s role in the creation of Floer homology. After proving the Arnold conjecture for tori, Conley and Zehnder tried to prove it for more general symplectic manifolds, but were not successful. However, in their proof of the torus case, which used a global finite-dimensional reduction, one could take a formal limit to infinite dimensions to obtain a perturbed version of Gromov’s pseudo-holomorphic curve equation about two years before Gromov introduced that equation. Edi had a gut feeling that one should be able to study this partial differential equation directly, but had no idea how to do this. However, his student Andreas Floer succeeded. After his undergraduate studies at Bochum, Floer began graduate studies at Berkeley in fall 1982.

Edi had never consciously met him before the following moment in early 1983.

One day, a student knocked on my office door on the 7th floor. I had never seen him before in my analysis lectures. He did not introduce himself but just asked, “Do you have an interesting topic for my thesis?”

Edi explained the Arnold conjecture and described his proof with Conley. For Floer’s thesis, he proposed: Analyze the structure of bounded solutions of the classical gradient flow, using Conley’s index theory.

Floer was immediately attracted to the challenging problem. I suggested that he begin with the Arnold conjectures for surfaces of higher genus, in order to verify the claims of Eliashberg.

While Alan Weinstein thought that Floer was a student of Cliff Taubes and vice versa, Floer was in regular communication with Edi, usually calling to his dismay at midnight in Bochum. Eventually, Floer received a PhD from Bochum. While still a graduate student, he started working in the direction of Edi’s suggestion, leading to his fundamental work during the period 1986–1989.

After ten years at Bochum, Edi moved to Aachen in 1986. Claude Viterbo recalls:

I vividly remember the first time I met Edi. He invited Andreas Floer and me—both of us postdocs at Courant at the time—for a week in Aachen. One morning, I went to knock on Edi’s office door. Inside a room that felt like a tennis court, I saw a dense cloud. Through the cloud, I could barely see Edi sitting at his desk, smoking his pipe, and greeting us in his cheerful, unmistakable manner. During our stay, we were invited to his home. I was struck by Jeannette’s remarkable ability to put two socially awkward mathematicians at ease—but we were likely not the first ones!

Edi returned to ETH Zürich in 1987. The same year, he discovered with Hofer⁶ the following phenomenon: *the C^0 -size of a Hamiltonian function (with support in a given domain) will force the existence of non-constant periodic orbits.* From this viewpoint the authors then define the Hofer–Zehnder capacity, prove that it is finite in a number of cases (e.g., bounded domains in \mathbb{R}^{2n}) and then promote as an important question the proof of

⁵ H. Hofer and E. Zehnder, Periodic solutions on hypersurfaces and a result by C. Viterbo. *Invent. Math.* **90**, 1–9 (1987), and H. Hofer and E. Zehnder, A new capacity for symplectic manifolds. In *Analysis, et cetera: Research papers published in honor of Jürgen Moser’s 60th birthday*, pp. 405–427, Academic Press, Boston (1990).

⁶ C. Conley and E. Zehnder, The Birkhoff–Lewis fixed point theorem and a conjecture of V. I. Arnold. *Invent. Math.* **73**, 33–49 (1983).

its finiteness on other manifolds. (This has inspired a lot of both old and recent work, see “Hofer–Zehnder capacity” on Zentralblatt or MathSciNet.) The Hofer–Zehnder capacity is a so-called symplectic capacity. The existence of this capacity immediately implies Gromov’s famous non-squeezing theorem as well as the C^0 -rigidity of symplectomorphisms discovered by Eliashberg and Gromov.

Edi continued to produce brilliant mathematics up to and beyond his retirement from ETH in 2005. With Dietmar Salamon⁷ he worked on the Conley conjecture, and they proved that every Hamiltonian diffeomorphism on the torus having only non-degenerate periodic points has infinitely many of them, paving the way for the resolution of the Conley conjecture by Hingston, later generalized by Ginzburg. With Helmut Hofer and Kris Wysocki, Edi systematically studied punctured holomorphic curves in noncompact symplectic manifolds, with the aim of capturing dynamical properties of Reeb flows (a special class of Hamiltonian flows).⁸ They used these curves to construct global sections for these flows (in the sense of Poincaré). As a result, they proved that on every convex energy level of an autonomous Hamiltonian flow on \mathbb{R}^4 there must be either exactly two closed orbits or infinitely many.⁹ Only recently, several groups of researchers have proved this dichotomy for any Reeb flow on a closed three-manifold. At the same time, the work of Hofer–Wysocki–Zehnder on punctured holomorphic curves builds the foundations of symplectic field theory, whose analytic underpinnings have been developed by the same authors during Edi’s last 15 research years.¹⁰

After their return to Switzerland, the Zehnders continued to enjoy the company of friends. Claude Viterbo remembers:

Edi and Jeannette’s hospitality was boundless—first on Gloriestrasse, and later in Greifensee. There was always wine, delicious food, and plenty of sharp humor. Their deep and enduring love was immediately apparent and deeply moving. On one of my last visits, Jeannette sweetly recalled the first time she saw Edi, dressed like a “young monk.” Jeannette,

a psychologist who had worked for many years in clinical practice, was deeply engaged in Edi’s mathematical world (for example, she translated Constance Reid’s book about Courant into German) and often joined Edi to conferences and extended visits.

Working out and formulating his mathematical findings “with loving care” gave Edi a lot of joy. His textbooks, which are striking in their clarity and elegance, testify to this. They teach valuable methods using well-chosen problems. In Jürg Fröhlich’s words:

Edi was a very hard-working researcher with an excellent taste for important problems and a clear focus. His contributions to Hamiltonian dynamics and symplectic topology have an everlasting impact on these fields and beyond. He was a very devoted teacher and advisor to students. Edi had strong opinions and was willing to defend them very forcefully if the situation demanded it. He always remained modest and did not mind acting, sometimes decisively, in the background, letting other colleagues occupy the limelight. He was a wonderful colleague and friend.—I will miss him.

And Claude Viterbo writes:

Edi was very close to Jürgen Moser, and both were strongly skeptical of overly formal mathematics. They believed that mathematics is not about simply generalizing known results. One important principle that Edi taught me is: “*Mathematics is about discovering new phenomena.*” He was undoubtedly a master at that. I will miss him and Jeannette terribly—personally, and like everyone else, mathematically.

Paul Biran studied mathematics at Tel Aviv University as an undergraduate, and then did his PhD there with Leonid Polterovich from 1994 to 1997, working on symplectic packing problems. Through regular visits to ETH he got to know Edi in person and received from him a lot of inspiration for mathematics and also in other directions. Since 2009, he is a professor of mathematics at ETH.

biran@math.ethz.ch

Felix Schlenk studied at ETH Zürich and did his PhD there with Edi Zehnder from 1996 to 2001, working on symplectic embedding problems. Ever since, he has learned a great deal from Edi—not only about mathematics. Since 2008, he has held the Chair of Dynamical Systems at the University of Neuchâtel.

schlenk@unine.ch

⁷ D. Salamon and E. Zehnder, Morse theory for periodic solutions of Hamiltonian systems and the Maslov index. *Comm. Pure Appl. Math.* **45**, 1303–1360 (1992).

⁸ Among this long series of fundamental works are F. Bourgeois, Ya. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder, Compactness results in symplectic field theory. *Geom. Topol.* **7**, 799–888 (2003), and H. Hofer, K. Wysocki and E. Zehnder, Finite energy foliations of tight three-spheres and Hamiltonian dynamics. *Ann. of Math.* **157**, 125–255 (2003).

⁹ H. Hofer, K. Wysocki and E. Zehnder, The dynamics on three-dimensional strictly convex energy surfaces. *Ann. of Math.* **148**, 197–289 (1998).

¹⁰ H. Hofer, K. Wysocki and E. Zehnder, *Polyfold and Fredholm theory*. *Erb. Math. Grenzgeb.* (3) **72**, Springer, Cham (2021).

Sergey P. Novikov (1938–2024)

Iskander Asanovich Taimanov

A great Soviet and Russian mathematician, Sergey Petrovich Novikov, passed away a year ago, on June 6, 2024, in Moscow.

He was born in a family of outstanding mathematicians on March 20, 1938, in Nizhnii Novgorod, called at that time Gorky, the city to which his parents moved for a short period from Moscow.

His father, Pyotr Sergeyevich Novikov (1901–1975) is well known for his proof of the existence of a finitely presented group for which the word problem is undecidable, and for the negative solution (jointly with his former student S. Adian) of the Burnside problem, which in turn follows from the proof that the free Burnside groups $B(m, n)$ are infinite for sufficiently large odd exponents n . In 1960 P. Novikov was elected full member of the Academy of Sciences of the Soviet Union.

His mother, Lyudmila Keldysh (1904–1976) worked in the Steklov Mathematical Institute and authored outstanding papers on set theory and set-theoretical topology.

In 1955, Sergey Novikov entered the Moscow State University. After a couple of years, Sergey chose as scientific advisor Mikhail Postnikov, whose impressive course on Galois theory he attended as a second-year student. This is how Sergey Novikov became a student in the Department of Higher Algebra.

At the beginning, Postnikov gave Novikov a preprint of the article “On the structure and applications of the Steenrod algebra” by J. F. Adams, and left for a year to teach in China.

Novikov always told that unlike other branches of mathematics, algebraic and differential topology have to be studied not from textbooks, but from the original articles, whose expository level is hardly achieved by graduate texts. In the 2000s, when he together with the author of this text organised the publication of the three-volume set of such articles, titled “Topological Library,” we included Adams’ article in which the Adams spectral sequence was introduced.

In 1959, Novikov published his first paper, “Cohomology of the Steenrod algebra,” *Dokl. Akad. Nauk SSSR* **128**, 893–895, in which he generalized certain results from Adams’ article and, in particular, showed that there are arbitrarily long compositions of maps that realize nontrivial elements of stable homotopy groups of spheres:

$$S^{n_1} \rightarrow S^{n_2} \rightarrow \dots \rightarrow S^{n_k}.$$



Sergey P. Novikov. (© Steklov Mathematical Institute, 2005)

The basic tool came from his observation that the analogue of the Steenrod operation Sq^0 , which acts on the cohomology of the Steenrod algebra, is not the identity map, but is given by the Frobenius automorphism. Novikov liked his first article very much, he wrote it completely by himself and thanked his advisor for the interest in the work. This was his start.

The next year, again by using the Adams spectral sequence, Novikov computed the ring Ω^U of U -bordisms (the Milnor–Novikov theorem) and obtained particular results on the rings of SO -, SU -, and Sp -bordisms.

Later, in 1966–67, Novikov returned to the Adams spectral sequence, proposing to consider its analogues for extraordinary

cohomology theories, introducing the so-called Adams–Novikov spectral sequence. In particular, he considered in detail such a sequence corresponding to U -cobordisms and described for them the analogue of the Steenrod algebra, the Landweber–Novikov algebra. In Novikov’s fundamental article “Methods of algebraic topology from the point of view of the cobordism theory,” *Izv. Akad. Nauk SSSR Ser. Mat.* **31**, 855–951 (1967), he discovered that the analogue of the Adams spectral sequence for unitary cobordisms is applicable to the classical problem of computing stable homotopy groups of spheres

$$\pi_k^S = \pi_{n+k}(S^n), \quad n \geq k + 1$$

(for these values of n the homotopy groups $\pi_{n+k}(S^n)$ depend on k only).

We want to mention that Novikov stressed that whenever he is doing some work, he always wants to solve some explicit problem and never produce generalizations or new constructions just for their own sake. When listening to talks delivered at his seminars, he used to ask sometimes: “Please tell us which problem would you like to solve.”

The problem of determining the stable homotopy groups of spheres π_k^S is still unsolved in general – these groups are known only for $k \leq 90$. The Adams–Novikov spectral sequence is a strong and important tool not only in the study of stable homotopy groups, but also in other problems of algebraic topology.

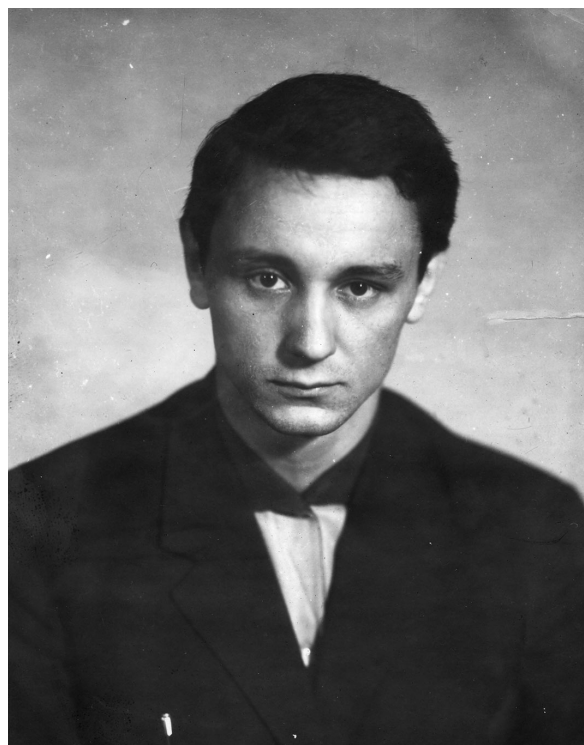
In “Methods of algebraic topology...,” formal groups appeared for the first time in algebraic topology via a theorem by Novikov’s student A. S. Mishchenko, who refused to publish it by himself and agreed that Novikov, who understood well the importance of this construction, will include it in an appendix to his article. Novikov had the intuition that the formal group of geometric cobordisms has rather special properties, and later Quillen established that this group is a universal group.

In 1960, Novikov graduated from Moscow University and became a PhD student at the Steklov Mathematical Institute.

In 1961, he solved the problem of the classification of simply-connected homotopically equivalent smooth manifolds of dimension > 4 up to diffeomorphisms. By a similar technique, in 1962 Browder characterized the homotopy types of smooth manifolds among simply-connected complexes. These results gave rise to the Browder–Novikov theory.

Once Novikov told that the great majority of results concern simply-connected manifolds, which form “the zero-measure set” and the wealth of interesting examples relate to nontrivial fundamental groups and their actions on higher homotopy groups.

In the mid-1960s, motivated by Grothendieck’s construction of étale cohomology, Novikov came to his famous toric trick for proving the topological invariance of rational Pontryagin classes. For its realization it was necessary to apply the technique of the Browder–Novikov theory to certain manifolds with free Abelian fundamental groups. At first there were doubts that this was possible, however,



Sergey P. Novikov in the middle of the 1960s.

the trick “came through” and the topological invariance of rational Pontryagin classes was established. Novikov considered this his best topological result. The methods and ideas he introduced in this proof were later widely used in topology.

His interest in non-simply-connected manifolds led to the formulation of the Novikov conjecture on higher signatures, in the study of which he involved Kasparov and Mishchenko. The conjecture is still open, though it is confirmed for many interesting cases.

We roughly attribute Novikov’s results obtained during the first ten years of his career (1959–1968) and mentioned above to two subjects: (1) the Adams spectral sequence and cobordism theory; (2) surgery theory and its applications. However, we have to speak about two more: (3) foliation theory, and (4) algorithmic problems in topology.

In the middle of the 1960s, Novikov became interested in the study of Haefliger’s work on foliations in the Anosov and Arnold’s seminars and proved the now-famous theorem on the existence of a compact leaf in every codimension-one foliation of the three-sphere.

Novikov’s theorem on the algorithmic unsolvability of the recognition problem for spheres of dimensions greater than four was proved in the early 1960s, but was never published by Novikov. It was presented much later in a special section of the paper: I. A. Volodin, V. E. Kuznetsov and A. T. Fomenko “The problem

of discriminating algorithmically the standard three-dimensional sphere," *Uspekhi Mat. Nauk* 29, 71–168 (1974). Now algorithmic and computational methods in algebraic and combinatorial topology are a very popular research subject.

These topological results brought Novikov worldwide fame.

In 1963, he finished his PhD study and became a research fellow in the Department of Algebra of the Steklov Institute. In 1964 Novikov received his PhD degree, and in 1966 he received the Doctor of Sciences degree and was elected a corresponding member of the Academy of Sciences of the Soviet Union. The next year Novikov received the Lenin Prize, the highest prize in the Soviet Union, and in 1970 he was awarded the Fields Medal, becoming the first mathematician from the Soviet Union who was awarded the medal.

In 1970–71, he did the well-known works on Hermitian K -theory and on symplectic cobordisms (joint with V. Buchstaber) where two-valued formal groups were introduced. However, at this time Novikov's interests moved in other directions. He also started to write articles with his students, many of whom later developed ideas from the joint texts.

At the beginning of 1971, Novikov left the Steklov Institute for the Landau Institute of Theoretical Physics.

He started to work in relativity theory, considering homogeneous cosmological models. In his joint article with O. Bogoyavlensky, for the purpose of studying the singularities of such models, a method of resolution of singularities motivated by ideas from algebraic geometry was introduced.

In 1973, Novikov became attracted by the spectacular progress in the theory of integrable systems. In this field he already applied his physical background to mathematical problems. In his famous paper on the periodic problem for the Korteweg–de Vries (KdV) equation

$$u_t = 6uu_x - u_{xxx},$$

Novikov introduced the so-called finite-zone potentials of the one-dimensional Schrödinger operator and proved that a potential has a finite number of zones in its spectrum if and only if it is a stationary solution of a higher Korteweg–de Vries equation, i.e.,

$$[L, c_{2n+1}A_{2n+1} + \dots + c_1A_1] = 0.$$

These equations are expressed as Lax pairs of the form

$$\frac{dL}{dt_{2n+1}} = [L, A_{2n+1}], \quad L = -\frac{d^2}{dx^2} + u(x),$$

where A_{2n+1} are ordinary differential operators of order $2n + 1$. Some results were shortly after and independently obtained by Lax, who worked with the classical eigenvalues of L .

As a crucial step, Novikov introduced an analytic continuation of the Bloch spectrum of the operator L with periodic coefficients. He considered the linear problem

$$L\psi = E\psi$$

for the Schrödinger operator $L = -\frac{d^2}{dx^2} + u(x)$ with a periodic potential $u(x) = u(x + T)$ for complex values of E , as well as the complex curve Γ which parameterizes all solutions of this linear problem with the property that $\psi(x + T) = \text{const} \cdot \psi(x)$ (called Floquet–Bloch functions), and he proved that Γ is a first integral of the KdV equations. Since the operator L is of second order, this complex curve Γ is a two-sheeted covering of the E -plane, i.e., it is a hyperelliptic Riemann surface, and it is of finite genus if and only if the potential $u(x)$ is finite-zone. This was the first appearance of spectral curves in the theory of integrable systems and one of the first applications of algebraic geometry in mathematical physics.

Later it was shown that the explicit finite-zone solutions of the KdV equations can be described in terms of the theta-function of Γ (by means of the Its–Matveev formula). The general method of finding algebraic-geometric solutions of soliton equations is now called the finite-gap integration method, and Novikov's former students B. Dubrovin and I. Krichever made crucial contributions to its development. However, Novikov did not agree with the term "finite gap" because, as he said, his original terminology came from solid state physics, where "zones of instability" in the Bloch spectrum have an important physical meaning.

In 1976, Krichever introduced the general Baker–Akhiezer functions and used them to construct algebraic-geometric solutions to the Kadomtsev–Petviashvili (KP) equation in the form

$$u(x, y, t) = -2\frac{d^2}{dx^2}\theta(Ux + Vy + Wt) + C,$$

where U, V, W are constant g -dimensional vectors, C is a constant, and θ is the theta function of the spectral curve, which is a Riemann surface of genus g . Novikov surprisingly proposed to use this formula for solving the Riemann–Schottky problem on characterizing Jacobi varieties. He conjectured that a principally polarized indecomposable Abelian variety is the Jacobi variety of a Riemann surface if and only if there are vectors U, V, W and a constant C such that such the above formula gives a solution to the KP equation. First, at the beginning of the 1980s, Dubrovin proved that the Novikov condition distinguishes the loci of Jacobian varieties up to irreducible components. Then in the middle of the 1980s Shiota proved the Novikov conjecture in full generality. Thus, a famous problem from algebraic geometry was solved by using an equation that originated in plasma physics.

Novikov made numerous other contributions to the "finite-zone theory," among which we mention here only two.

In 1976, he introduced in terms of algebraic-geometric spectral data the notion of a two-dimensional Schrödinger operator finite-zone on a single energy level (jointly with Dubrovin and Krichever) and later, in 1984, in joint articles with A. Veselov characterized among them the potential operators and introduced a new two-dimensional generalization of the KdV equation – the Novikov–Veselov equation – which describes isospectral deformations of such operators.

The theory of algebraic-geometric solutions to soliton equations revised the theory by Burchall and Chaundy of commuting ordinary differential operators of rank 1. The latter means that for generic constants λ and μ the space of solutions to the equations $L\psi = \lambda\psi$ and $M\psi = \mu\psi$ is one-dimensional, i.e., has rank 1. Here L and M are commuting ordinary differential operators. In 1978–79 Novikov jointly with Krichever proposed a method for studying commuting operators of higher rank based on the theory of holomorphic bundles over Riemann surfaces. They found the first explicit example of such operators of rank 2 and with an elliptic spectral curve.

In 2005, Novikov was awarded the Wolf Prize in Mathematics “for his fundamental and pioneering contributions to algebraic and differential topology, and to mathematical physics, notably the introduction of algebraic-geometric methods.”

Looking for problems in mathematical physics, Novikov was always interested in physically meaningful ones. In 1980, he published the monograph of “Theory of solitons. The Method of the Inverse Problem” in coauthorship with three physicists, Zakharov, Manakov, and Pitaevskii, one of the first books on this subject.

The stormy development of integrability theory in the 1970s–1990s involved both infinite-dimensional (solitons, etc.) and finite-dimensional systems, in particular, the Euler equations on Lie algebras. In 1980, Novikov became interested in the Kirchhoff equations which describe the motion of a body in an ideal fluid which is stationary at infinity. In the 1970s these equations were interpreted as the Euler equations on the Lie algebra of the group $E(3)$ of motions of the three-dimensional Euclidean space. For this Lie algebra the generic orbits of the coadjoint representation are four-dimensional manifolds, diffeomorphic to the tangent bundle of the two-sphere. Novikov and Schmelzer discovered that the reductions of the Euler equations to these orbits can mathematically be interpreted as the Euler–Lagrange equations for the motion of a charged particle on the sphere in a monopole-type magnetic field. The magnetic field on the two-sphere is given by a closed 2-form F , which in the case of a monopole is not exact — there is no globally defined vector-potential, i.e., no 1-form A such that $dA = F$. The periodic trajectories of a particle on a given energy level E satisfy the Euler–Lagrange equations for the multivalued Lagrange function

$$L^a(x, \dot{x}) = \sqrt{2E} \sqrt{g_{ik} \dot{x}^i \dot{x}^k} + A_k^a \dot{x}^k,$$

defined for $x \in U_a$, where $dA^a = F$. This immediately leads to multivalued functionals on spaces of closed forms, given by

$$S(\gamma) = \int_\gamma L^a(x, \dot{x}) dt;$$

here the functional is defined for curves lying in domains U such that the equation $dA = F$ is solvable in U , but the variational derivative δS is globally well defined.

Starting from that, he concluded the following.

1. Novikov initiated the study of periodic problem for such trajectories, now called magnetic geodesics. This problem cannot be approached by means of the classical Morse theory for geodesics, because even for exact magnetic fields the functional S is not necessarily bounded from below. He proposed several approaches to the periodic problem, including the principle of throwing out cycles; however, this is still not understood as well as its analogue for closed geodesics.
2. In a short note he considered multivalued functions on finite-dimensional manifolds, derived the Morse–Novikov inequalities for the critical points of such functions formulated in terms of the ranks of homologies with coefficients in what are presently called Novikov rings; these rings later found many applications, for instance, in quantum cohomology.
3. In the same note he formulated the general approach to multivalued functionals on spaces of maps not of closed curves, but of closed manifolds, and derived, as an example, the Wess–Zumino–Novikov–Witten (WZNW) functional.
4. Looking for physical examples of multivalued functionals, Novikov formulated the problem of the intersections of the Fermi surfaces $\{\varepsilon(p) = \text{const}\} \subset T^3 = \mathbb{R}^3/\mathbb{Z}^3$ in the space of quasimomenta, the three-torus T^3 , with the planes orthogonal to a constant vector in \mathbb{R}^3 . Such intersections are trajectories of Hamiltonian systems on Fermi surfaces. This led to many interesting mathematical results by Tsarev, Zorich, and Dynnikov, and even to the discovery by Novikov and Maltsev of experimentally-verifiable physical phenomena related to normal metals.

In the middle of the 1980s, returning to the Morse theory on manifolds that are not simply connected, Novikov together with his colleague M. Shubin introduced the well-known invariants that nowadays bear their names.

At the same time, Novikov jointly with Dubrovin introduced the so-called Hamiltonian systems of hydrodynamic type, which have the form

$$u_t^i = v_j^i(u) u_{\alpha}^j, \quad u_{\alpha}^j = \frac{\partial u^j}{\partial x^{\alpha}}$$

and are Hamiltonian with respect to the Poisson brackets of Dubrovin–Novikov type

$$\{u^q(x), u^p(y)\} = g^{qp}(u(x)) \delta'(x - y) + b_s^{qp}(u(x)) u_x^s \delta(x - y).$$

Such systems enjoy very rich geometrical properties. Their study led Dubrovin to introducing the Frobenius manifolds. The Novikov conjecture that the diagonalizable systems, i.e., those for which v_{α}^j is diagonal, are integrable was confirmed by Tsarev, who introduced the generalized hodograph method.

During the period 1996–2017 Novikov was a professor in the department of mathematics and Institute of Physical Science and Technology of University of Maryland, where in 1997 he was promoted to distinguished university professor.



Sergey P. Novikov at a workshop in Dubna.
(© Moscow Center for Continuous Mathematical Education)

In the 21st century, Novikov engaged in the development of discretization of complex analysis motivated by integrable systems (with Dynnikov) and the extension of the spectral theory of one-dimensional Schrödinger operators to potentials with special singularities (singular solitons) which appear in physics (jointly with Grinevich).

At that time Novikov returned to the Steklov Institute, where since 1982 he held the part-time position of the head of department of geometry and topology.

In addition to the already mentioned Fields Medal and the Lenin and Wolf Prizes, Novikov was awarded the Lobachevsky Prize of the Academy of Sciences of USSR and the Bogolyubov, Euler, and Lomonosov Gold Medals of the Russian Academy of Sciences. The last one is the top award of the Russian Academy.

Novikov was elected to the London Mathematical Society, the National Academy of Sciences (USA), Academia Europaea, the Accademia dei Lincei, the Pontificia Accademia delle Scienze, the Serbian Academy of Sciences and Arts, and the Montenegrin Academy of Sciences and Arts, and became doctor honoris causa of the University of Athens and of Tel Aviv University.

In 1978 Novikov gave a plenary talk at the ICM, and was an invited speaker at ICM-1966 and ICM-1970.

During the years 1985–96 he was president of Moscow Mathematical Society, and during the years 1988–2022, the chief editor of the journal *Uspekhi Matematicheskikh Nauk* (Russian Mathematical Surveys).

Since the middle of the 1960s, Novikov had been working at the Moscow University, where since 1983 he had been the head of the Chair of Higher Geometry and Topology. There he developed new courses, and that led to the book “Modern Geometry” by Dubrovin, Fomenko, and Novikov, published at the beginning of the 1980s in Russian and then as a three-volume set by Springer.

At the beginning of 2000s Novikov and the author of this text wrote a book for a graduate course on “Modern Geometrical Structures and Fields” which took into account the experience of the 1980–90s. It was published in Russian and then translated by the AMS.

The scientific school lead by Novikov was started very early, in 1960, when the third-year student V. Golo came to Novikov, who just became a PhD student at the Steklov Institute, and asked him to be his advisor. As Novikov told, that gave him a strong push. On his homepage he presented the names of his PhD students noting by double asterisks the ones who were invited speakers on ICM or plenary speakers on ECM or ICMP and by asterisks section speakers on ECM. We are reproducing this list (with later improvements) below.

The students who received ScD degrees:

V. Golo, V. Buchstaber,** A. Mishchenko,** G. Kasparov,** O. Bogoyavlensky,** F. Bogomolov,** S. Gusein-Zade, I. Krichever,** B. Dubrovin,** I. Taimanov,** A. Veselov,* I. Babenko, R. Nadiradze, V. Vedenyapin, M. Brodsky, S. Tsarev, O. Mokhov, R. Novikov, P. Grinevich, I. Dynnikov*, A. Maltsev, O. Musin*, D. Millionshchikov.

The students who received PhD degrees:

A. Brakhman, V. Peresetski, I. Volodin, A. Grigoryan, Th. Voronov, A. Zorich,** N. Panov, A. Lyskova, M. Pavlov, Thang T. Q. Le, L. Alania, S. Piunikhin, V. Sadov, A. Lazarev, R. De Leo, A. Giacobbe, K. Kaipa.

Sergey Novikov gave rise to many directions in modern mathematics and mathematical physics, and his ideas are developed by many researchers who, in particular, used his fundamental results in their studies.

Iskander Asanovich Taimanov works in geometry, topology, and integrable systems. Graduated in mathematics at Moscow State University in 1983 and defended his PhD there in 1987 (with S. P. Novikov as advisor). Since 1987 he has been working in the Sobolev Institute of Mathematics in Novosibirsk. In 2011, he was elected as a member of the Russian Academy of Sciences. Since 2024 he is the head of the Chair of Higher Geometry and Topology at Moscow State University.

iskander.taimanov@math.msu.ru

In2scienceUK and the University of Leeds: unlocking equitable opportunities in STEM

Ruth Holland and Hannah Newman

In2scienceUK and the University of Leeds are working together to deliver impactful STEM placements. By combining academic outreach with charitable expertise, they are opening doors for students from underserved backgrounds – offering practical experience, guidance, and a supportive community to help them navigate their path into science, technology, engineering, and mathematics.

In the UK, young people are encouraged to take part in work experience placements while at school or college as an opportunity to gain insight into a particular career.

Whilst schools help source placements where they can, provision varies across the UK and students and their parents are often involved in searching out opportunities themselves [3]. This can lead to students who are already connected to particular professions gaining high-quality experiences, while students with no relatable role models struggle to find relevant opportunities.

Lack of connections and access to work experience are particular challenges when it comes to educating students about potential careers in science, technology engineering, and mathematics (STEM). According to the latest Science Education Tracker [1], a survey conducted by the Royal Society in partnership with EngineeringUK, only 15% of young people surveyed had STEM-related work experience. A quarter of young people reported that they wanted to secure work experience in these areas but were unable to do so.

Improving representation in STEM

There are a number of groups who are currently underserved by the higher education system, including students from lower socioeconomic backgrounds. The additional expense of commuting to voluntary placements instead of earning money through a holiday job can be yet another barrier for these young people.

STEM subjects, in particular, have some additional challenges, for instance university physics, engineering and mathematics departments often have a low proportion of female colleagues and

these disparities begin earlier in the educational journey. How should we encourage more girls to consider continuing their study of these STEM subjects and perhaps to become researchers themselves?

Another challenge is that many higher-education STEM disciplines are new to students. For example, many haven't come across disciplines like engineering at school and so if they don't have a personal connection to an engineer, they may never think to consider this sector as a potential career.

As part of university access work, it is important for institutions to reach out to students who are underrepresented as we strive to increase diversity. In the case of STEM research at universities, how do we engage those students who are unaware of how universities work and what the world of research involves? How do we give them better insight into how subjects develop beyond school and where they might lead?

Our outreach approach at the University of Leeds

The STEM outreach team¹ at the University of Leeds has developed several programmes to provide university experiences and role models for students from communities or backgrounds which are underrepresented. These programmes introduce STEM subjects, including mathematics, in a different light, and include Leeds Futures – STEM, Medicine – not just medics,² the Materials Science Award,³ and our Royal Institution Mathematics Masterclass series.⁴

We also work with a number of external partners like IntoUniversity, With Insight Education, and the Advanced Mathematics Support Programme to enhance our in-house outreach offer. These partners provide additional resources and expertise, or a different channel to work with young people from underserved communities. They enable us to extend our reach, provide more opportunities to

¹ <https://accessandoutreach.leeds.ac.uk/stem/>

² <https://accessandoutreach.leeds.ac.uk/medicine-not-just-medics/>

³ <https://accessandoutreach.leeds.ac.uk/materials-science-award/>

⁴ <https://accessandoutreach.leeds.ac.uk/royal-institution-mathematics-masterclasses/>

involve colleagues and students within the University who want to share their enthusiasm for their disciplines, and allow us to welcome in more young people to give them a valuable experience of STEM at university.

Work placements at university

Whilst our extensive schools outreach programme is the main focus for our outreach team, we and colleagues across the University are often approached by teachers, students and their parents to offer STEM work experience opportunities. While colleagues have sometimes been able to host such placements, responding to these requests individually is inefficient and poses a number of challenges, including:

- substantial additional workloads;
- a lack of coordination which makes it difficult to respond in an equitable manner. Are those who reach out for placements already connected with universities and therefore receiving a disproportionate chance of gaining such placements at the expense of those who are already underserved?

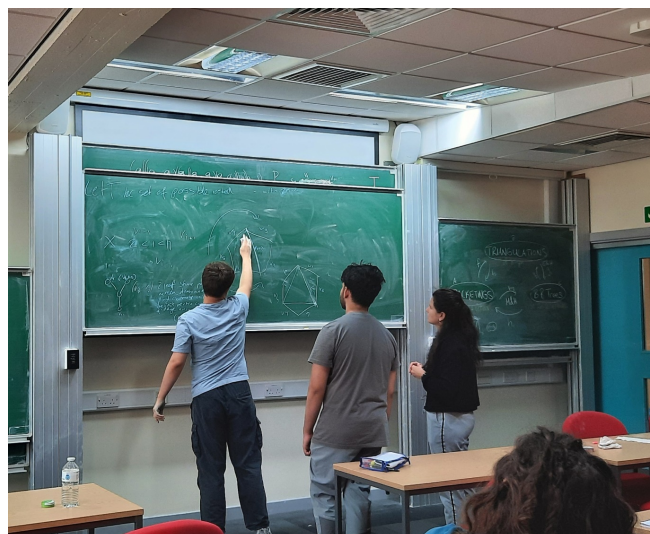
Ideally, we need:

- to provide enough support to make the process as straightforward as possible for colleagues, so that more choose to be involved, and many return to offer placements on a regular basis;
- a large enough scheme to provide effective matching which ensures we cater to and align with a range of young people's interests;
- a fair, objective way of allocating places to those who would most benefit due to limited resources.

Introducing In2scienceUK

This is where charitable organisations like In2scienceUK⁵ can provide valuable support. In2scienceUK is a charity which promotes social mobility by offering opportunities for young people from low socioeconomic backgrounds to explore education and careers across STEM-related fields. Established in 2010, the organisation works to break down barriers to STEM by offering a range of programmes⁶ to help equip beneficiaries with the knowledge, skills and confidence to navigate the decision-making process about their future.

Both the In2STEM⁷ and In2research⁸ programmes provide a mix of free in-person placements and events with online support and



In2STEM students discussing mathematics with their placement host.
(© In2scienceUK)

skills workshops. These interventions are designed to offer support at pivotal points in the participants' education when they are deciding on their next steps.

In2STEM is open to 16–19-year-old students who are deciding whether to pursue a STEM degree or apprenticeship, while In2research is designed to help explore postgraduate research. After attending these programmes, participants can sign up to further alumni support, where members can access career resources such as webinars, mentoring, networking events and other insights into the diverse range of career paths which exist in the STEM sphere. Participants can be a part of the alumni community for as long as they need after their placement has ended, so that resources are still available further on in their career journey.

For In2STEM, a wide range of STEM workplaces can sign up to host placements across both industry and academia. Applicants are matched by their subject area of interest and location, so they are within a commutable distance. Through the programme, charities subsidise expenses to ensure there are no financial barriers, neither for the hosts nor for the participants.

Year-on-year, In2STEM receives huge amounts of interest from young people all over the UK who are looking for support in kick-starting their STEM careers – the programme received 5000 applications in 2025 alone. This indicates that the shortage of skilled workers in a range of UK STEM industries isn't due to waning interest or engagement, but perhaps to the presence of persistent and systemic barriers to access.

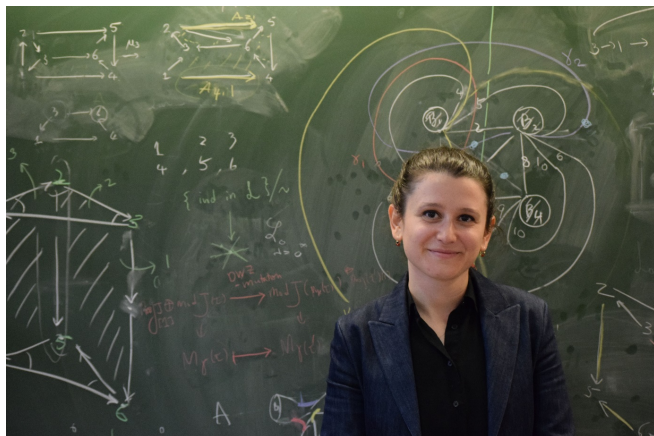
And when these barriers are removed for young people through programmes like In2STEM, the transformative effect is clear. Following the 2024 programme [2], 85% of participants said they

⁵ <https://in2scienceuk.org>

⁶ <https://in2scienceuk.org/our-programmes/>

⁷ <https://in2scienceuk.org/our-programmes/in2stem/>

⁸ <https://in2scienceuk.org/our-programmes/in2research/>



Dr. Emine Yildirim, one of the In2STEM hosts at the University of Leeds.
(© In2scienceUK)

understood the content and structure of a range of STEM degrees and apprenticeships, up from 53% before taking part. Out of the past participants surveyed, 97% who applied to university received offers, and 92% who accepted chose a STEM degree.

Bridging the experience gap through collaboration

Since 2020, the University of Leeds has partnered with In2scienceUK to provide In2STEM placements, which has resulted in a scheme which suits all our needs and now means we are able to provide around 45 placements each year. The charity is able to provide good advice and guidance to interested hosts and coordinates the application and matching processes so that young people are assigned appropriate placements.

The STEM outreach team actively encourages colleagues from PhD students to professors to be involved. Responding to feedback, In2scienceUK brought in the prospect of co-hosting placements and this has meant that more academics and research groups have felt able to be involved as it shares the time-commitment required both preparing for and during the placements.

Whilst the university advertises the scheme to schools and individuals, In2scienceUK are also active in recruiting young people to take part. This means that more young people are made aware of the possibility than if the university was working alone.

In addition to their placement on the scheme, young people are part of a larger cohort that spans the whole of the UK. This also enables the University of Leeds to be involved in additional workshops and online courses and engage with a greater number of young people from low socioeconomic backgrounds.

The benefits are not one-sided – with partners in academia and industry who support the programmes and provide volunteers,

it facilitates In2scienceUK's national mission to support as many young people as possible to pursue vibrant careers in STEM. With their generosity and dedication, the charity can deliver meaningful impact for beneficiaries across the UK.

One In2STEM participant from 2024 who took part in a placement at Leeds' School of Mathematics described the experience as "enlightening." They highlighted how it had been "insightful to learn how to think differently, to collaborate and work with people who think differently than you" as well having "such a positive impact" on their confidence.

By working together to offer transformative opportunities like these in mathematics, we can drive long-term change, starting with our own institutions.

Case study: Volunteering with In2STEM

Dr. Emine Yildirim, a research fellow based at the University of Leeds' School of Mathematics, shares her experiences as a recurring placement host for the In2STEM programme.

Why did you decide to take part in the In2STEM programme?

I enjoy volunteering during the summer for maths outreach events, and I find it very important for the community. I care about students, people who are enthusiastic about learning. It is a great joy for me to share my knowledge and experience, especially with the younger generation.

Tell us a bit about your placement – what did it involve?

It was an interactive placement – we shared our ways of thinking about maths problems and asked students to find their own way. Our project was based on different ways of thinking about mathematics. As a host, I was always with them, discussing and answering their questions, and also asking them questions. I think we tried to boost their curiosity about mathematics.

What have you learnt in your role as an In2STEM host?

Our research or work is meaningful when we share it with others. I learnt to be patient, and I saw how students are able to develop their skills once given the chance. It is a unique experience because as we teach, we also learn from students. Also, it is great to see young people who are passionate about mathematics!

Did you enjoy the experience? Would you recommend it to others?

I loved the experience, it was so much fun! These students are bright, smart, enthusiastic, curious and incredibly kind. If my time in Leeds permits, I would be interested in doing it again. If someone was interested in volunteering for the programme, I would tell them that it will be a priceless experience and to go ahead and host some of these amazing students.

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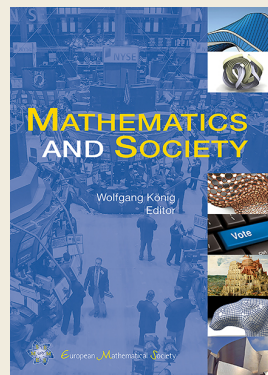
Ruth Holland is a mathematician working in STEM outreach at the University of Leeds. With experience teaching mathematics at both school and university levels, and nearly 20 years of involvement in outreach, she is committed to inspiring the next generation of scientists, engineers, and mathematicians. A strong advocate for inclusive STEM education, Ruth also serves as a Trustee of MathsWorldUK, a charity working to establish the UK's first Mathematics Discovery Centre. She is passionate about improving public understanding and enjoyment of mathematics.

r.m.holland@leeds.ac.uk

Hannah Newman is the Senior Communications Officer at social mobility charity In2scienceUK. Having prior experience in public engagement and science communication, she is passionate about breaking down barriers to equality in STEM.

h.newman@in2scienceuk.org

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The ubiquity and importance of mathematics in our complex society is generally not in doubt. However, even a scientifically interested layman would be hard pressed to point out aspects of our society where contemporary mathematical research is essential. Most popular examples are finance, engineering, weather and industry, but the way mathematics comes into play is widely unknown in the public. And who thinks of application fields like biology, encryption, architecture, or voting systems?

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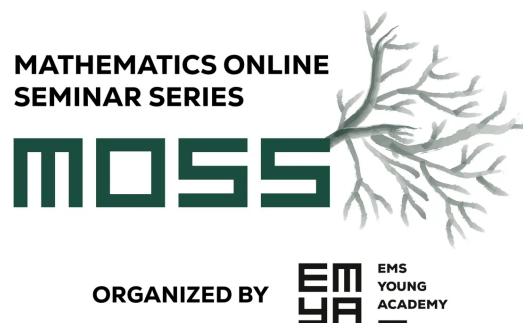
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MOSS: highlights from the first season

EMYA column regularly presented by Jesse Railo

Cristina Molero-Río and Belén Pulido



Can we bring together young mathematical talent so that everyone can benefit? MOSS makes it happen.

MOSS, the Mathematical Online Seminar Series, was created as a friendly and accessible space for young researchers to share their work, connect with others, and discover exciting areas of mathematics. What started as an idea quickly became a global seminar series with participants joining from all over the world.

EMYA,¹ the European Mathematical Society Young Academy, launched MOSS on February 2025.

MOSS includes presentations from leading young academics in mathematics from around the globe, which are accessible to a broad audience, no matter the mathematical subfield. This initiative is primarily intended to inspire the upcoming generation of young mathematicians, specially those who are working towards or have just completed a PhD in mathematics.

The origin of MOSS dates back to late 2024 during one of the EMYA monthly online meetings. The organizers by that time were the EMYA members Jelena Jankov Pavlović from the University of Osijek, Croatia, and Cristina Molero-Río from the University of Seville, Spain. Currently, Cristina continues and Belén Pulido, from the Universidad Nacional de Educación a Distancia (UNED), Spain, has taken over from Jelena. For more details, the reader is referred to [5].

The seminars are held entirely online every first Thursday of each month at 4 p.m. (CET), lasting approximately 45 minutes, followed by a 15-minute Q&A. They are recorded and are available at the EMS YouTube channel.² MOSS has also a mailing list³ for those who would like to receive news.

The first season of MOSS occurred between February and June 2025. In what follows, we recap the speakers and their talks, as well as some metrics that illustrate the reception of this initiative by the mathematical community.

Meet the speakers

In the first season of MOSS, all the speakers had obtained the EMS Prize of 2024,⁴ which is awarded to young mathematicians under 35 for exceptional contributions to mathematics.

Cristiana De Filippis⁵ is a professor at the University of Parma, Italy. She obtained her PhD in mathematics in 2020 at the University of Oxford, UK. She is an EMYA member of the first cohort (since 2023) and appeared at Forbes top 100 Italian women.⁶ She was awarded with the 2023 Bartolozzi Prize by the Italian Mathematical Union (UMI), the 2025 SIAM Early Career Prize in Partial Differential Equations and obtained an ERC Starting Grant in 2025. Her research interests focus on the regularity theory for elliptic and parabolic partial differential equations and on the calculus of variations.

Tom Hutchcroft⁷ is a professor at the California Institute of Technology, USA. He obtained his PhD in mathematics in 2017 at the University of British Columbia, Canada. He has received prestigious awards, including the Canadian Mathematical Society Doctoral Prize in 2018, the Rollo Davidson Prize in 2019, and the Whitehead Prize in 2025, as well as the competitive Packard Fellowship in 2024, among others. His research interests lie mostly in probability theory and mathematical physics, with additional focus on group theory, ergodic theory, and metric geometry.

Adam Kanigowski⁸ is a professor at the University of Maryland, USA. He obtained his PhD in mathematics in 2017 at the Institute of Mathematics of the Polish Academy of Sciences, Poland. He was honored with the Banach Prize in 2016 and the Polish Mathematical Society Prize for Young Mathematicians in 2016, among others, and has received competitive support including the Packard Fellowship in 2024. His research interests combine dynamical systems and ergodic theory, as well as their interactions with number theory, geometry and probability theory.

⁴ <https://euromathsoc.org/list-ems-prizes-history>

⁵ <https://sites.google.com/view/cristianadefilippis/home>

⁶ <https://forbes.it/2023/07/18/donne-successo-classifiche-forbes-2023>

⁷ <https://www.its.caltech.edu/~thutch/>

⁸ <https://akanigow.math.umd.edu/publications/>

¹ <https://euromathsoc.org/EMYA>

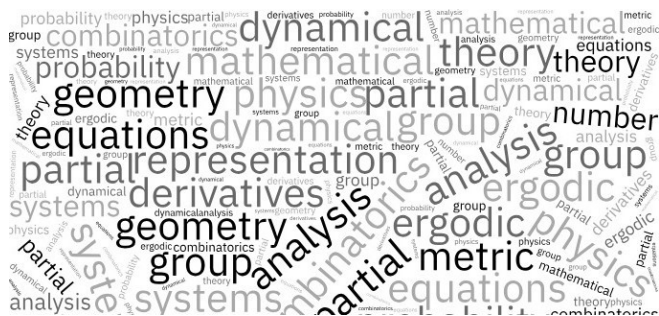
² youtube.com/playlist?list=PLNDz3HuMo23DREzRIWkRPF6oQoLo3-tmZ

³ docs.google.com/forms/d/e/1FAIpQLSfjXOc_H3IUqHTstGvEgYBYEWkqFd-UQVKgfv7mTzdshwckbw/viewform

Jessica Fintzen⁹ is a professor at the University of Bonn, Germany. She obtained her PhD in mathematics in 2016 at Harvard University. She was awarded with the Association for Women in Mathematics (AWM) Dissertation Prize in 2018, the Whitehead Prize of the London Mathematical Society in 2022 and the Frank Nelson Cole Prize in Algebra in 2024, among others. She also obtained different grants from the US National Science Foundation (NSF) and an ERC Starting Grant in 2022. Her research interests are related to number theory and representation theory, specially focused on p -adic groups and the Langlands program.

Richard Montgomery¹⁰ is a professor at the University of Warwick, UK. He obtained his PhD in mathematics in 2015 at the University of Cambridge. He received several awards such as the European Prize in Combinatorics in 2019, the Philip Leverhulme Prize in 2020 and the London Mathematical Society (LMS) Whitehead Prize in 2025. He also obtained an ERC Starting Grant in 2020. His research interests are mostly related to extremal and probabilistic combinatorics.

What we learned



The first talk of the season was given by *Cristiana De Filippis* on *'Nonuniformly elliptic Schauder estimates.'*

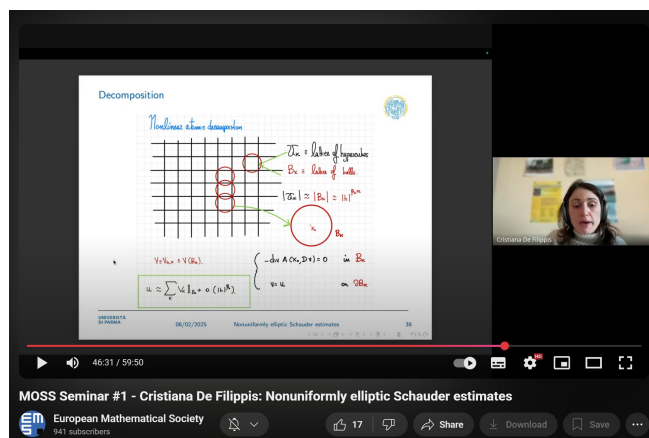
Schauder estimates are a fundamental tool in the study of elliptic and parabolic partial differential equations (PDEs), as they ensure that solutions inherit the regularity of the coefficients. They play a crucial role in various contexts, including establishing higher regularity of solutions to elliptic problems, handling free boundary issues, implementing bootstrap arguments, or proving existence theorems, among others.

In the linear case, Schauder estimates are classical, with foundational results dating back to the 1920s. Over the years, new proofs have been developed using different techniques, such as function space methods, convolution approaches, blow-up techniques, and nonlinear extensions.

All these classical results focus on uniformly elliptic operators and fundamentally rely on perturbation methods, which involve

⁹ <https://www.math.uni-bonn.de/people/fintzen/>

¹⁰ <https://rhmontgomery.warwick.ac.uk>



Nonuniformly elliptic Schauder estimates – Cristiana de Filippis, February 2025. (EMS YouTube channel)

freezing the coefficients and comparing the original solution to one with constant coefficients. However, these techniques fail in the nonuniformly elliptic setting, where homogeneous a priori estimates are lost and standard iteration arguments break down.

In this talk, Cristiana proposes a solution to the long-standing problem of extending Schauder estimates to nonlinear, nonuniformly elliptic problems. This approach introduces a novel method for obtaining a priori gradient bounds that does not rely on perturbation techniques, even in the presence of non-differentiable problems.

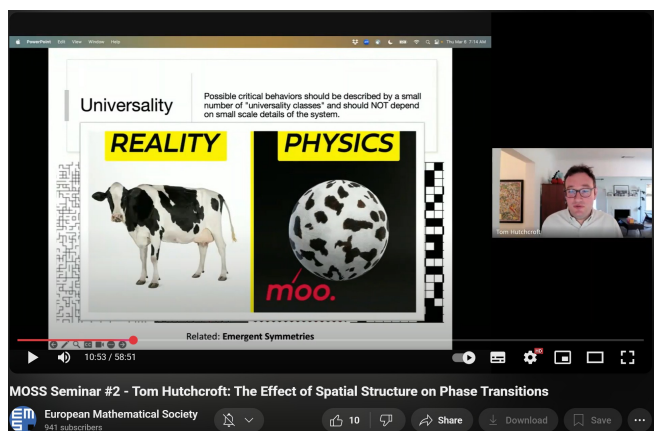
The reader is referred to [2], for instance.

The second talk of the season was given by *Tom Hutchcroft* on *'The effect of spatial structure on phase transitions.'*

An important and fascinating feature of many large, complex systems in mathematics, physics, and other fields is that they undergo phase transitions.

Mathematically, phase transitions usually appear in a specific way. Consider a system with many individual components, possibly arranged according to some spatial structure. Each component interacts with its neighbors in a manner governed by certain parameters, such as temperature, which we can vary continuously. A phase transition occurs when a small change in one of these parameters, crossing a special value called the critical value, causes a sudden, qualitative change in the system's behavior on a macroscopic scale.

We are all familiar with the solid/liquid/gas transitions from everyday life. However, phase transitions also appear in many other large systems unrelated to physics. Examples include the formation of traffic jams, the average-case computational complexity of optimization problems, and the spread of a novel disease through a population, a topic that has become particularly relevant recently. Understanding when, how, and why these systems undergo phase transitions is crucial for both theoretical insights



The effect of spatial structure on phase transitions – Tom Hutchcroft, March 2025. (EMS YouTube Channel)

and practical applications. Interestingly, the underlying mathematical principles are often very similar across these diverse situations, making the study of phase transitions a rich source of beautiful and deep mathematics beyond its original practical motivations.

The simplest models of phase transitions are called mean-field models, where every particle interacts equally with every other particle, ignoring any geometric arrangement. In this talk, Tom provides an introductory overview of what happens when geometry is introduced to these models, particularly how critical phenomena depend on spatial dimensions. He also briefly discusses some recent progress on these questions.

The reader is referred to [3], for instance.

The third talk of the season was given by Adam Kanigowski on ‘Ergodic theorems along sparse subsets of the integers.’

In the study of dynamical systems, a central question is how a system evolves over time. To capture this, mathematicians often consider ergodic averages, which measure the long-term average of a function along the orbit of a point under a transformation.

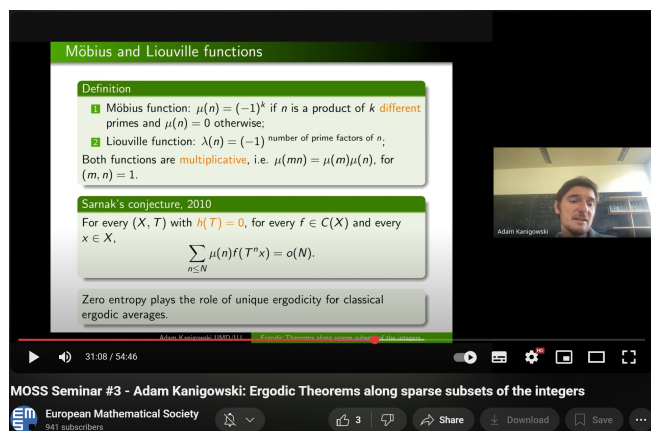
When the system is ergodic, these averages converge to a limit independent of the starting point, linking local orbit behavior to global properties of the system.

A more recent development studies ergodic averages along sparse subsets of the integers, such as the prime numbers or values of polynomials with integer coefficients. These sparse averages raise subtle questions about convergence and distribution, requiring a combination of ideas from number theory and ergodic theory.

In his talk, Adam focuses on precisely these kinds of sparse ergodic averages, and examines how they behave. He recalls some classical results, mentions some recent developments and highlights some interesting open problems.

The reader is referred to [4], for instance.

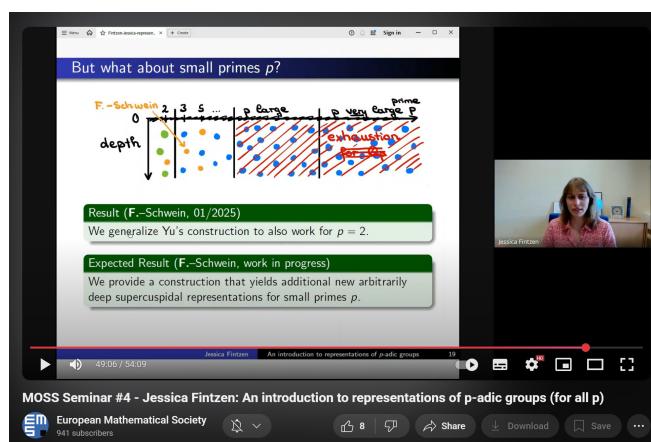
In May 2025, Jessica Fintzen delivered the talk titled ‘An introduction to representations of p -adic groups (for all p).’



Ergodic theorems along sparse subsets of the integers – Adam Kanigowski, April 2025. (EMS YouTube channel)

An explicit understanding of the category of all smooth, complex representations of p -adic groups provides an important tool not only within representation theory. It also has applications to number theory and other areas, and in particular enables progress on various different forms of the Langlands program.

The talk introduces p -adic groups and provides an overview of what is known about the representation theory of these groups, including developments from the last ten months. In particular, it surveys the construction of all so-called supercuspidal representations, which are the building blocks for all representations. For more than 20 years, it remained an open problem to extend a general construction to the case $p = 2$. The talk explains what makes this case so special and how the obstacles were finally overcome in recent joint work with David Schwein. See [6] for more details.



An introduction to representations of p -adic groups (for all p) – Jessica Fintzen, May 2025. (EMS YouTube channel)

- [5] E. Schlitzer, *MOSS: the mathematics online seminar series by EMYA*. European Mathematical Society News (2025) <https://euromathsoc.org/news/moss-the-mathematics-online-seminar-series-by-emya-133>
- [6] J. Fintzen and D. Schwein, Construction of tame supercuspidal representations in arbitrary residue characteristic. arXiv:2501.18553v2 (2025)

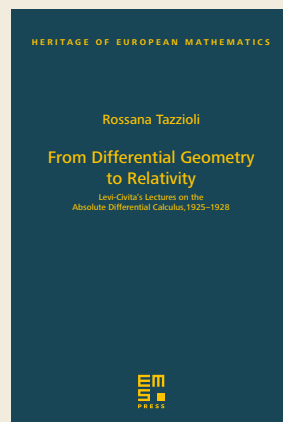
Cristina Molero-Río is a postdoctoral researcher at the University of Seville, Spain, where she earned her PhD in mathematics in 2022. Her research combines the disciplines of operations research and artificial intelligence, contributing to the fields of mathematical optimization and machine learning, respectively.

mmolero@us.es

Belén Pulido is an assistant professor at the Universidad Nacional de Educación a Distancia (UNED), Spain. She earned her PhD in mathematical engineering from the Universidad Carlos III de Madrid, Spain, in 2024. Her research focuses primarily on statistics, with particular interest in functional data analysis and its software implementation.

belen.pulido@ccia.uned.es

EMS Press title



From Differential Geometry to Relativity

Levi-Civita's Lectures on the Absolute Differential Calculus, 1925–1928

Rossana Tazzioli
(Université de Lille)

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2025. Hardcover. 535 pages.
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This book examines Levi-Civita's lectures on tensor calculus as a lens to illuminate key aspects of his scientific legacy. It highlights the deep interplay between his teaching and research, particularly in tensor calculus, differential geometry, and relativity, as well as his role as a mentor at the University of Rome. More broadly, it traces the history of Riemannian differential geometry from roughly 1870 to 1930.

Key themes emerge: the influence of the Italian mathematical tradition in Levi-Civita's work on tensor calculus, the intrinsic link between analysis, geometry, and relativity in his work, and his pedagogical approach, which incorporates physics and geometric intuition to extend mathematical results. The book also explores his collaborations with Enrico Fermi and Enrico Persico, shedding light on the Via Panisperna group during a pivotal period in theoretical physics.

Levi-Civita's treatise became a foundational text in absolute differential calculus, essential for physicists mastering tensor calculus in Einstein's theories.

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Thematic Working Group on the Teaching and Learning of Calculus, TWG25

ERME column regularly presented by Frode Rønning and Andreas Stylianides

In this issue presented by the group leader Laura Branchetti

CERME Thematic Working Groups

We continue the initiative of introducing the CERME Thematic Working Groups, which we began in the September 2017 issue, focusing on ways in which European research in the field of mathematics education may be of interest or relevance to people working in pure and applied mathematics. We aim to disseminate developments in mathematics education research discussed at CERMEs and enrich the ERME community with new participants, who may benefit from hearing about research methods and findings and contribute to future CERMEs.

Introduction of CERME's Thematic Working Group 25 – Teaching and Learning of Calculus

This paper reports on the current trends in research related to the teaching and learning of calculus within Thematic Working Group 25 (TWG25) at CERME14, as well as the development of research in this field, particularly in the context of interactions between mathematics and other disciplines. We explore the challenges related to conceptual understanding, formalization, institutional constraints, and interdisciplinary applications. We highlight innovative task designs, technological tools, and implications for teaching practices in diverse educational contexts.

1 Teaching and learning of calculus across grades and disciplines

The Thematic Working Group 25 (TWG25) is a new group proposed to create a new community that investigates the teaching and learning of calculus at secondary and tertiary levels and at the transition between secondary school and university. Calculus plays a key role in mathematics education, both for its interesting features *per se*—difficulties in learning calculus and infinitesimal processes have been investigated since the 1970s—and for its transversal role in society. Due to its powerful concepts, tools and theories developed to deal with graphs, variations, trends, and accumulation processes,

calculus is relevant for many other disciplines and professional fields (the so-called *non-mathematicians*). It is taught in most countries at secondary school, college and university first-year level. The undeniable value and flexibility of calculus in modeling processes and in the conceptualization and formalization of multiple real-world problems do not translate into recognition of its value by many teachers and students. Addressing the issue of teaching calculus to most students in several different contexts is challenging. Indeed, its intrinsic complexity—semiotic,¹ conceptual, and argumentative aspects—combined with lecturers' habit to formalize from the very beginning its notions, usually leads students to experience several difficulties [3, 18]. The difference between the conceptual and semiotic aspects of key notions like continuous function, derivative, integral, limit, and the formal and theoretical evolution of their definitions and properties is often stressed using two different terms—calculus and analysis—to refer to different mathematical practices and the related didactic issues, as highlighted by Bergé [2]. Separating calculus and analysis sharply and definitively is not easy and can be misleading, but a distinction is worthwhile. Transitioning from calculus to analysis, the students are asked to move from *doing* to *justifying* and *structuring*. Bergé [2] provided interesting examples of differences that appear in syllabi of courses addressed to undergraduate students. Calculus courses emphasize operational, representational and algorithmic tasks; analysis courses focus on theoretical, formal, axiomatic, and proof-oriented tasks. For instance, in calculus courses, continuity, derivative and integral are introduced via geometric intuition, computing with examples, graphs, and applications. Students work with tasks like “find where f is increasing,” “compute $\int f(x) dx$ from a to b .” In analysis, continuity and differentiability are explored in much greater generality: rigorous (ε, δ) -definitions, proofs of theorems, and often the role of completeness (or compactness) is used.

¹ *Semiotics*—the study of signs and meaning-making—has become a major focus in mathematics education over the past three decades. Research has shown that mathematical thinking and learning are mediated through symbols, gestures, diagrams, and language. Influential works highlighted the central role of semiotic processes in constructing and communicating mathematical meaning [14].

2 The main motivations to propose a new group on calculus

There are at least two motivations for making the effort to create a new community for the teaching and learning of calculus. First of all, it is necessary to provide new research—empirical, practice-based, and practice-oriented, and theoretical—to grasp the main variables affecting the conceptualization process, in particular the phase of representation and interpretation. This phase is necessary to ground the formal knowledge on strong and appropriate personal meanings. Unfortunately, as stressed in the pioneering work developed by David Tall and colleagues since the 1980s (see, for instance, [15–17]), often undergraduate students fail in constructing fruitful connection between their mental images, emerged after their learning experiences at the secondary level, and the formal definitions that are proposed suddenly at the undergraduate level. Tall and Vinner [17] introduced the terms *concept image* and *concept definition* to refer to these two aspects of the conceptualization, showing that many students struggle with the cognitive conflicts between them. Moreover, such conflicts often hinder their learning processes and cause ruptures between the manipulation of symbols and reasoning. Such undesired effects are partially due to promoting the learning of analysis without a meaning-making process and conceptual regulation processes behind, as documented by several researchers (e.g., [9]; for a systematic review, [5]).

The second motivation is that pre-calculus reasoning and notions and other calculus-related advanced mathematical practices (multivariable calculus, modeling processes based on rate of change and accumulation) have specificities that usually are lost while moving to the formal world of analysis and cannot be disregarded as informal knowledge and processed to be refined or formalized as soon as possible pursuing mainly the formal rigor [5, 13].

Several contributions to calculus had been proposed in other groups (in particular, TWG14 on University Mathematics Education), but some specificities were not addressed in a systematic and pertinent way. The main motivation for proposing a thematic group like TWG25 is precisely to clarify the specificities of the challenges related to teaching of calculus and to explore ways to support meaning-making processes. Taking this topic outside the university mathematics group opens a space to discuss it alongside high school calculus, thus bringing a richer perspective on pre-calculus and on the transition from secondary to tertiary.

The group inherits the work developed by researchers who worked on this topic for several years, including the calculus and analysis working groups attending the INDRUM conference [12, 18] and the calculus conferences (CalcConf1 in 2019 in Norway, CalcConf2 in 2023 in Norway, and CalcConf3 in 2025 in Italy) and new fresh perspectives by researchers of several countries with different theoretical perspectives and goals.

3 State of the art about the teaching and learning of calculus

Recent literature stressed that these issues, that mainly regard the epistemic and cognitive level, are intertwined with institutional, linguistic and affective variables that make the challenge even harder to address. The main results are summarized in a chapter of the first handbook about the INDRUM community (International Network on Didactics of University Mathematics), resuming the current challenging issues university mathematics education is facing [10]—in particular, [18]—and in the systematic review [4]. At the institutional level, two important phenomena have been stressed by researchers that have a strong impact on the teaching process and on the analysis of learning outcomes.

Artigue [1] emphasized that, among the various theoretical perspectives on investigating students' difficulties in mathematics during the transition from secondary school to university—as well as on innovations in university-level mathematics teaching—the institutional dimension has been less frequently addressed in the past. However, over the last two decades, there has been growing attention to the analysis of this aspect. In particular, the research community engaged in the research program developed by Chevallard since the late 1990s—adopting an anthropological perspective on the investigation of mathematical practices within institutional contexts (the anthropological theory of didactic, or ATD [6])—has made significant contributions to the characterization of mathematical practices at the secondary and tertiary levels across various fields (mathematics, engineering, physics, economics, etc.). A significant contribution is highlighting the mismatches between teaching and learning activities carried out in different contexts that can be a major source of difficulties for students and teachers. Among many contributions in this direction, the work by Gueudet et al. [11] is particularly important, since the authors highlighted relevant features of such a mismatch at the transition towards university and the related implications for research. A different use of ATD is the analysis of praxeologies across different tertiary institutions, in particular stressing the difference between the praxeologies that characterize the world of pure mathematics and the ones developed by other professional or scientific communities.

Biza et al. [3] presented a relevant issue that relates to the intention itself of teaching calculus to *non-mathematicians* in courses of the first year and on the assessment practices. Calculus is one of the main causes of dropout. Unfortunately, calculus is often intended, and used, as a filtering tool to test if students are good enough to attend the university courses [4]. Regardless of the features of mathematical practices related to a specific field, the students' interests and needs, calculus is taught in a “standard” way, without any attempt to fit the students' needs, or to take care of those

linguistic or affective needs of the students. The timely problem of inclusion and diversity in calculus teaching and learning, which deserves particular attention in courses for *non-mathematicians* [8], is still underexplored.

4 Overview of TWG25: research foci and theoretical perspectives

In this contribution, we summarize how the papers presented in TWG25 contributed to addressing these challenges, combining different theoretical lenses and addressing several of the aspects highlighted in the literature, and the new issues that emerged during the discussion groups that opened up new avenues for research. The group brought together a broad range of research contributions focused on the learning and teaching of calculus, bridging foundational secondary-level concepts and advanced topics in tertiary education. The authors addressed the development of students' understanding of key calculus ideas and examined instructional practices, theoretical frameworks, and methodologies that inform this field, building on research traditions spanning decades while responding to current curricular, technological, and interdisciplinary demands. Several contributors emphasized the importance of early conceptual readiness, focusing on variables, covariation, functional thinking, and graphical reasoning. These ideas form the cognitive substrate for understanding the rate of change and accumulation—central themes in calculus. Some papers deepened core topics such as the derivative as rate of change, the integral as accumulation, differentials, and the continuity and integrability of functions.

Among the diverse contributions to TWG25, we can identify three macro categories: students' conceptualization of properties of real numbers relevant to calculus, single variable calculus, and multivariable calculus.

A first area of inquiry concerns students' conceptualization of real numbers, particularly their understanding of the convergence of infinite series and the role of partial sums. Persistent challenges emerge here, including tensions between formal definitions and intuitive conceptions. Another crucial but often overlooked theme is the density of the rational numbers within the reals, which plays an essential role in grasping continuity. Classroom activities such as those involving continued fractions, together with analyses of student responses, highlight the variety and dynamism of students' reasoning processes.

A second major theme focuses on the notions of the derivative and integrals, explored through graphical and dynamic representations, technology-enhanced environments, and modeling tasks. Findings emphasize how different forms of representation shape students' evolving understanding and engagement with concepts of variation and rate of change. Research has also pointed to the importance of connecting procedural fluency with a deeper

conceptual awareness, especially regarding continuity and accumulation processes.

Another line of investigation examines the design of qualitative and graphical tasks. These activities have shown potential to foster flexible mathematical reasoning and to move learners beyond narrow algorithmic strategies. In this regard, extra-mathematical contexts appear particularly valuable: connecting calculus concepts to real-world situations allows students to anchor abstract notions in meaningful experiences, thereby strengthening their understanding of rates of change.

Teacher knowledge and classroom practices have also received attention. Studies reveal how pedagogical choices and epistemological perspectives strongly influence the way the derivative is taught, and how inconsistencies in the use of notation can hinder students' conceptual clarity. Some approaches propose rethinking traditional topics such as inverse proportion or substitution in integration, underlining the role of foundational reasoning structures in supporting students' learning trajectories.

Finally, research on multivariable calculus has highlighted the difficulties students face when dealing with double and triple integrals, multivariable limits, and functions of several variables. Digital tools and dynamic visualizations, such as interactive applets, have been shown to provide important scaffolds, supporting spatial reasoning and bridging the gap between symbolic manipulation and geometric intuition.

The contributions showcased a wide range of theoretical perspectives, including APOS theory, commognition, ATD, concept image and concept definition, and semiotic approaches. While this diversity enriched the dialogue, it also raised questions about coherence and integration. Participants debated whether future work should aim for theoretical unification or respect and encourage pluralism. Methodologies varied from thematic and content analysis to grounded theory and discourse analysis [7]. Participants emphasized the importance of aligning research methods with theoretical assumptions. Analysis of students' "scratch work" was highlighted as a valuable yet underutilized method for gaining insight into student thinking. Several recurring cross-cutting themes emerged, with particular attention to:

1. the tension between conceptual understanding and formal definitions;
2. the role of visual and embodied reasoning in fostering insight;
3. the influence of institutional and cultural contexts on curriculum;
4. the need to bridge the research-practice divide;
5. the foundational role of pre-calculus knowledge.

During the discussion session, we explored several important research questions and considerations that warrant further investigation. Examples include: How should various institutional conditions (e.g., cultural factors, prior knowledge, and experiences of both STEM and non-STEM students) be considered in the teaching and learning of calculus and multivariable calculus? How can

we enhance accessibility to calculus education while considering subject-specific norms? Do task design principles in calculus differ from those in other areas of mathematics? If so, what are the key design principles unique to calculus? For instance, what types of contextual tasks are most effective for teaching and learning calculus? Even within calculus, do these principles vary across topics—such as derivatives, limits, and integrals—or do they remain consistent?

5 The learning and teaching of calculus across disciplines

More recently, an explicitly interdisciplinary focus has gained attention due to the increasing awareness that the meanings and uses of calculus are strongly shaped by the disciplinary contexts in which it is applied. In this perspective, two conferences have been organized that focus on “The Learning and Teaching of Calculus Across Disciplines” (CalcConf2 in Bergen in 2023 and CalcConf3 in Milan in 2025).

Several contributions in the proceedings show how the teaching and learning of calculus cannot be separated from the practices of the sciences that rely on it. Several contributions addressed covariational reasoning as a cornerstone of students’ understanding of the derivative and accumulation. Others explored how basic mental models of the integral may support comprehension in thermodynamics and chemistry, or how conservation laws in physics can be interpreted through the fundamental theorem of calculus. A number of papers investigated the disciplinary specificity of calculus, for instance in economics, biology, chemistry and engineering, pointing out how meanings of differentials, partial derivatives or multivariable limits vary according to context.

The contributions also highlighted the increasing role of technology and innovation. Studies reported the use of eye-tracking to analyze reasoning with simulations, the integration of computational thinking in calculus teaching, or the affordances of digital platforms and AI tools, such as MAPLE Learn and large language models (LLMs), to create authentic problem situations. More creative approaches were presented as well, including the design of a calculus card game and the use of history-based artifacts as mediators of conceptual understanding.

Altogether, these contributions suggest that research on the teaching and learning of calculus is today both diversified and coherent: diversified, because of the variety of contexts, approaches and methods represented; coherent, because of the shared recognition that calculus learning should be meaningful, conceptually grounded, and responsive to the epistemological needs of the sciences.

From this perspective, the results and innovation proposed in CalcConf2 and CalcConf3 complement earlier discussions promoted in other international contexts, and reinforce the idea that future work should increasingly move toward concrete didactic

proposals and collaborative interdisciplinary designs. This appears to be a promising way forward to ensure that calculus teaching supports students not only in mastering procedures, but also in developing the conceptual and modeling resources needed for their scientific and professional trajectories. Future research should also consider the future potential of AI and LLMs in analyzing student responses and designing adaptive tasks. Cross-institutional collaboration and attention to diverse educational contexts will be key to addressing the evolving challenges in calculus education.

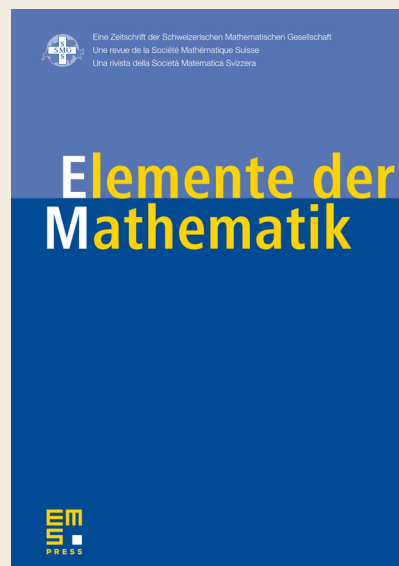
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Laura Branchetti is an associate professor of mathematics education at the Department of Mathematics “Federigo Enriques” of the University of Milan (Italy). She earned her PhD in history and didactics of mathematics, physics and chemistry (University of Palermo, 2016), and holds an MSc (2011) and BSc (2008) in mathematics from the University of Bologna. She obtained the Italian National Scientific Habilitation (ASN) for full professor in 2023. Her research lies in mathematics education and intersects with interdisciplinary studies between mathematics, physics and informatics. In the ERME community, she is the group leader of TWG25 “Teaching and Learning of Calculus” at CERME14 and CERME15, and she has served on INDRUM committees and related working groups. laura.branchetti@unimi.it

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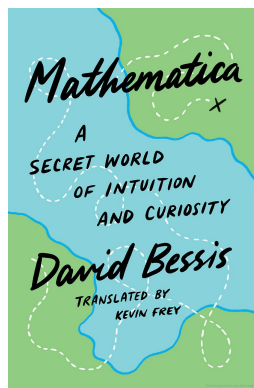
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Book review

Mathematica: A Secret World of Intuition and Curiosity

by David Bessis (translated by Kevin Frey)

Reviewed by Noah Giansiracusa



What happens inside the mind of a mathematician when they do math? This is in essence the question that drives Bessis throughout his sharply insightful and daringly original book *Mathematica* (published originally in French, then translated into English by Kevin Frey). This overarching question is explored not through the technical lens of neuroscience or cognitive science; it is explored through the informal lens of one mathematician's

introspection and personal journey. Bessis wants to know how the mathematical legends among us rose so high above the crowd, and how even ordinary mathematicians exhibit mathematical abilities that leave the non-mathematical public in awe.

Bessis looks for clues in a range of sources, from childhood games involving shapes and visualization, to graduate school experiences where mathematical concepts came more easily to some than others. Bessis also mines the more philosophical writings of giants such as Grothendieck and Thurston for kernels of wisdom and insight. The overarching answer Bessis finds to his overarching question of what transpires in the mind of a mathematician is, in a nutshell, that intuition plays a much bigger role in mathematical thinking than most of us realize.

The two deeply important corollaries of this finding that give the book a real *raison d'être* are that (1) mathematicians of all levels can up their game by learning how to do math more intuitively, and (2) everyone, even those who think of themselves as bad at math or math phobic, can do math—they just need to stop seeing math in terms of rigid rules and start seeing it as a world of imagination to explore and enjoy.

Early on the book offers a fantastic metaphor to convince readers that the enormous spread of mathematical abilities seen

in society cannot be a mere matter of intrinsic talent: “math is so unequal that it’s as if some people could run the one-hundred-meter dash in under a second, while the majority wouldn’t make it in a week.” In other words, there is surely some natural variance in math abilities just as there is in running—but the standard deviation in math is so absurdly large that something else must be going on. The explanation Bessis offers is that some people are inclined to think of mathematical objects intuitively and these are the people who hit no ceilings in their mathematical education, whereas others try to memorize definitions and formulas without wondering what they really mean and where they come from, an unsustainable (and unpleasant) way to learn and do math.

Those who do math more intuitively are not necessarily any more intelligent; I suspect Bessis would say that they are just lucky that the way their brain naturally operates happens to be the right way for doing math. But he sees this as a tremendous opportunity: if being better at math require more smarts, it only requires getting students to think differently about math, then we should help all students do this. Or, put another way, we should help everyone become privy to the “secret world” of intuitive mathematics. And as mentioned above, it is not just children who would benefit from this: working mathematicians of all levels could benefit by spending more time in mathematics’ secret world.

I will resist the urge to summarize what this secret world looks like and how to access it, for that is Bessis’s main task in *Mathematica* which he nicely develops in many directions over the book’s twenty chapters. Personally, I very much agree with Bessis’s arguments, and I am struck by the book’s novelty—among other things, it is the first book I have come across that really conveys to general readers what it is like to do research mathematics.

That said, I find it a confusing task to pin down the book’s readership. If the book is aimed at mathematicians, I do not know why it includes so many rudimentary details like what a conjecture is; but if it is aimed at non-mathematicians, then I am concerned that it spends all of its time explaining *how* mathematicians think without giving the non-mathematical reader any real motivation for *why* they should try to think like a mathematician. (Examples of a mathematical concept that a non-mathematical reader can grasp intuitively and then put to use practically in everyday life would

have gone a long way, in my view.) But while no single reader seems to be a perfect fit for the book, I nonetheless think that everyone will learn something important from this book and I have no hesitations about recommending it widely.

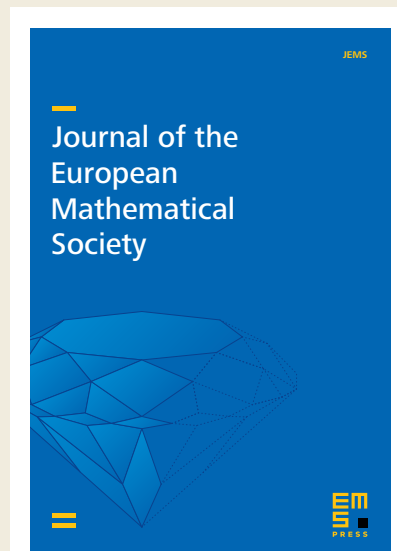
David Bessis (translated by Kevin Frey), *Mathematica: A Secret World of Intuition and Curiosity*. Yale University Press, 2024, 349 pages, Hardcover ISBN 978-0-300-27088-4, Softcover ISBN 978-0-300-28328-0, eBook ISBN 978-0-300-27714-2.

Noah Giansiracusa is an associate professor of mathematical sciences at Bentley University, a business-focused institution near Boston. He enjoys communicating math to the public and has written for a range of outlets including Washington Post, Wired, TIME, and Scientific American, and he's appeared on CNN live and BBC Radio. He is the author of the upcoming book "Robin Hood Math: Take Control of the Algorithms That Run Your Life."

ngiansiracusa@bentley.edu

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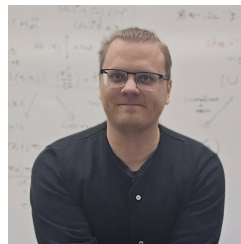


Boris Adamczewski is a French mathematician. He is a senior researcher at the CNRS and deputy director of the Centre International de Rencontres Mathématiques (CIRM) in Marseille. He currently conducts his research at the Camille Jordan Institute in Lyon. He received his PhD in mathematics from Aix-Marseille University

in 2002. His webpage is <https://boris-adamczewski.perso.math.cnrs.fr>.

He is a number theorist specializing in Diophantine approximation and transcendental number theory, working at the interface of number theory, theoretical computer science, and related fields such as combinatorics, dynamical systems, and the Galois theory of differential and difference equations. He is particularly known for contributions that combine finite automata with number theory.

He served as editor-in-chief of the *Gazette* of the French Mathematical Society from 2015 to 2020, held an ERC Consolidator Grant from 2015 to 2022, was awarded the Aisenstadt Chair at the Centre de recherches mathématiques (CRM) in Montreal in 2017, and received the Ernest Déchelle Prize from the French Academy of Sciences in 2022. He is currently head of the Combinatorics and Number Theory team (CTN) at the Camille Jordan Institute and director of the CNRS research network in number theory (RT2N).



Jesse Railo is an associate professor of applied mathematics at the Lappeenranta-Lahti University of Technology LUT in Finland. He received a PhD in mathematics at the University of Jyväskylä in 2019 and a MSc in mathematics at the University of Tampere in 2014. He held postdoctoral positions at the University of

Cambridge from 2021 to 2023 and at ETH Zürich from 2020 to 2021. Most recently, he was a Fulbright researcher at Stanford University in 2025. His webpage is <https://sites.google.com/view/jesserailo>.

His mathematical work focuses on inverse problems and imaging. His research uses techniques from partial differential equations, Riemannian geometry and statistics to provide rigorous answers to uniqueness, reconstruction and uncertainty related questions in mathematical models in imaging. In particular, he has made many contributions to Radon transforms and their generalizations, and Calderón-type inverse coefficient problems.

He has received the Finnish Inverse Prize of the Finnish Inverse Problems Society in 2020 as well as a distinction from the University of Jyväskylä for his doctoral dissertation in 2019. He has obtained research fellowships from the Finnish Academy of Science and Letters in 2020, Jenny and Antti Wihuri Foundation in 2023, and Emil Aaltonen Foundation in 2024. He was recently selected as a member of the EMS Young Academy (EMYA) for the term 2025–2028.

Announcement of the next meeting of the EMS Council

Lisbon, Portugal on June 27–28, 2026

The EMS Council meets every second year. The next meeting will be held in *Lisbon, Portugal on June 27–28, 2026* at the Academia das Ciências de Lisboa.¹ The registration starts on June 27 at 12:00 and ends at 13:45. The council meeting starts at 14:00 on June 27 and ends at 12:00 on June 28.

In the event that unexpected constraints prevent the meeting from being held fully in person, the Executive Committee may decide to hold the meeting in a hybrid format or entirely online.

Delegates

Delegates to the Council shall be elected for a period of four years. A delegate may be re-elected provided that consecutive service in the same capacity does not exceed eight years. *Re-election for the second term does not happen automatically.* Candidacies of delegates eligible for re-election for the second term must be submitted via the nomination form below if they are willing to serve. Delegates will be elected in the following categories.

(a) Full members

Full members are national mathematical societies, which elect 1, 2, 3, or 4 delegates according to their membership class. Council decides the membership class and societies are invited to apply for the class upgrade. However, the *current* membership class of the society determines the number of delegates for the 2026 council.

Each society is responsible for the election of its delegates. To be eligible to nominate its delegates the society must have paid the corporate membership fee for the year 2025 and/or 2026. It is not compulsory but is highly appreciated that the full member delegates join the EMS as individual members.

The link to the online nomination form for delegates of full members is below. The deadline for nominations for delegates of full members is *April 15, 2026*.



The venue of the 2026 council meeting: Salão Nobre of Biblioteca Academia das Ciências da Lisboa. (Public domain, source: acad-ciencias.pt)

(b) Associate members

Delegates representing associate members shall be elected by a ballot organized by the Executive Committee from a list of candidates who have been nominated and seconded by associate members and have agreed to serve. In November 2025, there were four associate members and, according to our statutes, (up to) one delegate may represent these members.

The associate member delegate who can be re-elected for the second term 2026–2029 is José Antonio Carrillo (European Society for Mathematical and Theoretical Biology).

The link to the online nomination form for delegates of associate members is below. The deadline for nominations for delegates of associate members is *March 9, 2026*.

¹ <https://www.acad-ciencias.pt/cedencia-de-espaco/>

(c) Institutional members

Delegates representing institutional members shall be elected by a ballot organized by the Executive Committee from a list of candidates who have been nominated and seconded by institutional members, and have agreed to serve. In November 2025, there were 57 institutional members and, according to our statutes, (up to) four delegates may represent these members.

The institutional member delegates whose term includes the 2026 council are Cristina Trombetti (Università di Napoli Federico II) and Jose A. Lozano (Basque Center for Applied Mathematics).

The following institutional member delegates can be re-elected for the second term 2026–2029:

José Francisco Rodrigues (University of Lisbon)
Adam Skalski (Institute of Mathematics of the Polish Academy of Sciences)

Nominations of these delegates must be submitted via the online form.

The link to the online nomination form for institutional member delegates is below. The deadline for nominations for delegates of institutional members is *March 9, 2026*.

(d) Individual members

Delegates representing individual members shall be elected by a ballot organized by the Executive Committee from a list of candidates who have been nominated and seconded, and have agreed to serve. These delegates must themselves be individual members of the European Mathematical Society and have paid the individual membership fees for the year 2025 and/or 2026.

In November 2025 there were 2663 individual members and, according to our statutes, these members may be represented by (up to) 27 delegates.

Here is a list of the delegates of individual members whose term includes 2026:

Lluís Alseda	Thierry Bouche
Fernando da Costa	Donatella Donatelli
Christian Kassel	Hrvoje Kraljević
Josef Málek	Pierangelo Marcati
Josep M. Miret	Piotr Oprocha
Joaquín Pérez	Carlo Petronio
Primož Potočnik	Giuseppe Savaré
Andreja Tepavčević	Enric Ventura Capell

The following delegates of individual members could be re-elected for the second term 2026–2029:

Krzysztof Ciesielski
Xavier Jarque

Pedro J. Miana
Barbara Nelli
Carola-Bibiane Schönlieb

Nominations of these delegates must be submitted via the online form.

The link to the online nomination form for delegates of individual members is below. The deadline for nominations for delegates of individual members is *March 9, 2026*.

We would like to thank Alice Fialowski and Muhammed Uludağ, whose second term has finished, for their excellent service. These delegates cannot participate as individual member delegates in the 2026 council meeting but can join in some other category (as a full, associate or institutional member delegate).

Agenda

The Executive Committee is responsible for preparing the matters to be discussed at council meetings. The proposed agenda items for this council meeting should be sent as soon as possible, but no later than *May 7, 2026*, to EMS Secretary Jiří Rákosník (rakosnik@math.cas.cz).

Executive Committee

The Council is responsible for electing the president, vice-presidents, secretary, treasurer and other members of the Executive Committee. The present membership of the Executive Committee, together with their individual terms of office, is as follows:

President: Jan Philip Solovej (2023–2026)
Vice-presidents: Jorge Buescu (2021–2026, 2019–2020 member-at-large)
Victoria Gould (2025–2028)
Secretary: Jiří Rákosník (2021–2028)
Treasurer: Samuli Siltanen (2023–2026)
Members: Barbara Kaltenbacher (2021–2028)
Susanna Terracini (2021–2028)
Adam Skalski (2025–2028)
Alain Valette (2025–2028)
María Ángeles García Ferrero (2025–2028)

Members of the Executive Committee are elected for a period of four years. The president can only serve one term. Committee members may be re-elected, provided that consecutive service shall not exceed eight years.

At the council meeting, elections will be held for the following opened positions in the Executive Committee:

- EMS president for the term 2027–2030
- vice-president for the term 2027–2030
- treasurer for the term 2027–2030

- second vice-president or member-at-large if one of the current EC members is elected as the new president or vice-president

The delegates must be technically prepared for voting electronically. Please bring your laptops, smartphones or tablets.

Based on Rule 18 of the By-laws, the EC will appoint a nomination committee whose task is to identify well-qualified candidates for the posts of the Executive Committee members. From the suggestions received, and any others of its own, the nomination committee shall select a short-list of top candidates and forward them to the Council for election.

Please send nominations for candidates for the Executive Committee to the ems-office@helsinki.fi before the deadline *February 9, 2026*. Nominations should include candidate's CV, recommendation letter, possibly a reference letter from the national society and any other supporting documents of choice.

All these arrangements are as required in the Statutes² and By-laws.³ See also the web page for the council meeting.⁴

The online nomination form for *full member delegates*:

<https://elomake.helsinki.fi/lomakkeet/137184/lomake.html>

The deadline for nominations is *April 15, 2026*.

The nomination form for *institutional, individual and associate member delegates*:

<https://elomake.helsinki.fi/lomakkeet/137183/lomake.html>

The deadline for nominations is *March 9, 2026*.

Secretary: Jiří Rákosník (rakosnik@math.cas.cz)

Secretariat: ems-office@helsinki.fi

² <https://euromathsoc.org/governance-statutes>

³ <https://euromathsoc.org/governance-by-laws>

⁴ <https://euromathsoc.org/Council2026>

European Mathematical Society

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solovej@math.ku.dk

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jbuescu@gmail.com

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rakosnik@math.cas.cz

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Instituto de Ciencias Matemáticas (ICMAT), Madrid, Spain
garciaferrero@icmat.es

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Universität Klagenfurt, Austria
barbara.kaltenbacher@aau.at

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Mathematical Institute of the Polish Academy of Sciences, Warsaw, Poland
skalski@impan.pl

Susanna Terracini (2025–2028)
Università di Torino, Italy
susanna.terracini@unito.it

Alain Valette (2025–2028)
University of Neuchâtel, Switzerland
alain.valette@unine.ch

Community Engagement Manager

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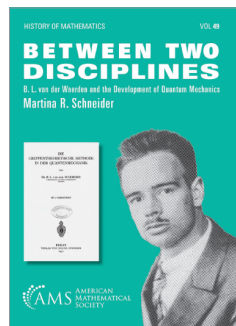
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* These discounts extend to members of national societies that are members of the EMS or with whom the EMS has a reciprocity agreement.

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- 40 € for persons belonging to a corporate EMS member society (full members and associate members)
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- Anyone who is a student at the time of becoming an individual EMS member, whether PhD or in a more junior category, shall enjoy a three-year introductory period with membership fees waived.
- Lifetime membership for the members over 60 years old.
- Option to join the EMS as reviewer of zbMATH Open.



BETWEEN TWO DISCIPLINES

B. L. van der Waerden and the Development of Quantum Mechanics

Martina R. Schneider, Wiesbaden
Translated by Alexander Blum

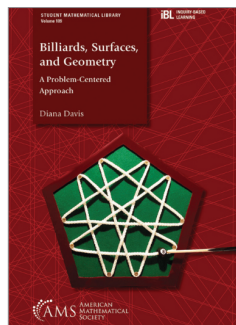
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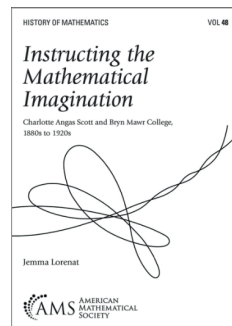
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Charlotte Angas Scott and Bryn Mawr College, 1880s to 1920s

Jemma Lorenat

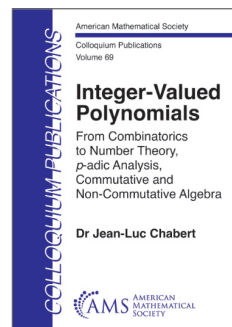
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Jean-Luc Chabert, University of Picardie

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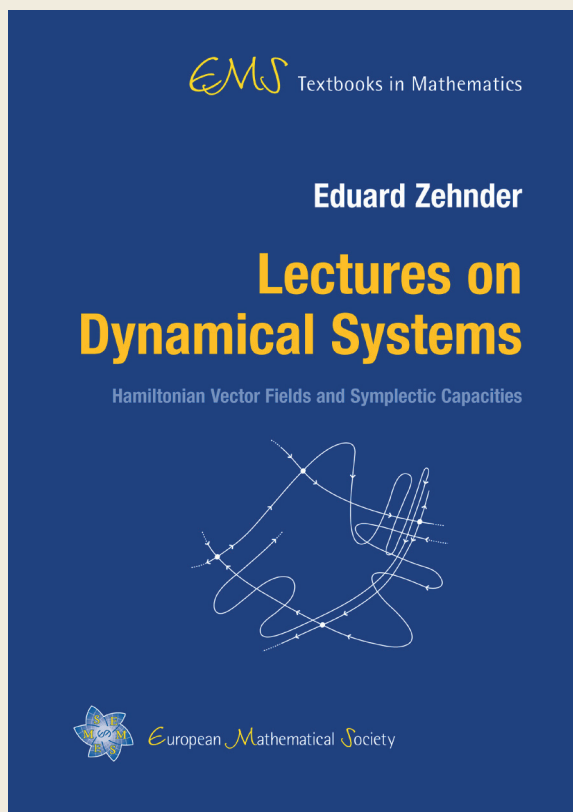
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Lectures on Dynamical Systems

Hamiltonian Vector Fields and Symplectic Capacities

EMS Textbooks in Mathematics

ISBN 978-3-03719-081-4. eISBN 978-3-03719-581-9

May 2010. Hardcover. 363 pages. € 59.00*

This book originated from an introductory lecture course on dynamical systems given by the author for advanced students in mathematics and physics at the ETH Zurich.

The first part centres around unstable and chaotic phenomena caused by the occurrence of homoclinic points. The existence of homoclinic points complicates the orbit structure considerably and gives rise to invariant hyperbolic sets nearby. The orbit structure in such sets is analyzed by means of the shadowing lemma, whose proof is based on the contraction principle. This lemma is also used to prove S. Smale's theorem about the embedding of Bernoulli systems near homoclinic orbits. The chaotic behavior is illustrated in the simple mechanical model of a periodically perturbed mathematical pendulum.

The second part of the book is devoted to Hamiltonian systems. The Hamiltonian formalism is developed in the elegant language of the exterior calculus. The theorem of V. Arnold and R. Jost shows that the solutions of Hamiltonian systems which possess sufficiently many integrals of motion can be written down explicitly and for all times. The existence proofs of global periodic orbits of Hamiltonian systems on symplectic manifolds are based on a variational principle for the old action functional of classical mechanics. The necessary tools from variational calculus are developed. There is an intimate relation between the periodic orbits of Hamiltonian systems and a class of symplectic invariants called symplectic capacities. From these symplectic invariants one derives surprising symplectic rigidity phenomena. This allows a first glimpse of the fast developing new field of symplectic topology.

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