Introduction

The aim of this book is to explain in full detail the proofs of Uhlenbeck's weak and strong compactness theorems. They were proven by Karen Uhlenbeck in 1982, c.f. [U1, U2]. Another textbook reference is [DK] by Donaldson and Kronheimer. Uhlenbeck's compactness results play a fundamental role in gauge theory. The strong and weak compactness theorems both concern sequences of connections on principal bundles with compact structure groups. The strong compactness theorem deals with Yang-Mills connections whereas in the weak compactness theorem the connections are not required to satisfy any equation.

An elementary observation in gauge theory is that the moduli space of flat connections over a compact manifold with a compact structure group is compact in the $\mathcal{C}^\infty$-topology. This is obvious from the fact that the gauge equivalence classes of flat connections are in one-to-one correspondence with conjugacy classes of representations of the fundamental group. (Here the bundle is not fixed but rather is also determined by the representation.) The weak Uhlenbeck compactness theorem is a remarkable generalization of this result. It asserts, in particular, that every sequence of connections with uniformly bounded curvature is gauge equivalent to a sequence, which has a weakly $W^{1,p}$-convergent subsequence (for any fixed p). In the case of abelian groups the proof reduces to Hodge theory, but in the nonabelian case it is highly nontrivial. This theorem lies at the heart of the compactness results for many equations in nonabelian gauge theory, such as the Yang-Mills equations, the vortex equations, or the rank two Seiberg-Witten monopole equations.

Uhlenbeck's strong compactness theorem asserts that every sequence of Yang-Mills connections with uniformly bounded curvature is gauge equivalent to a sequence, which has a $\mathcal{C}^\infty$-convergent subsequence. This result can be reduced to the weak compactness theorem as follows: The weak limit is again a Yang-Mills connection and hence is gauge equivalent to a smooth connection. Now one can put the sequence into relative Coulomb gauge with respect to the limit connection. Then $\mathcal{C}^\infty$-convergence follows from the fact that the Yang-Mills equation together with the gauge fixing condition form an elliptic system. By the same argument one obtains similar compactness results for all gauge theoretic equations that together with the relative Coulomb gauge form an elliptic system.

An important application of Uhlenbeck's theorem is the compactification of the moduli space of anti-self-dual instantons over a four-manifold. These compactified moduli spaces are the central ingredients in the construction of the Donaldson invariants of smooth four-manifolds [D2] and of the instanton Floer homology groups of three-manifolds [F]. Anti-self-dual instantons are special first order solutions of the Yang-Mills equation. Uhlenbeck's theorem asserts that noncompactness can only occur in sequences with unbounded curvature. In this case a conformal rescaling argument shows that instantons on the four-sphere bubble off. For a suitably chosen subsequence bubbling only occurs at finitely many points, and on the complement one has $\mathcal{C}^\infty$-convergence. Now Uhlenbeck's removable singularity theorem [U1] guarantees that the limit connection extends over the four-manifold. In the case of simply connected four-manifolds with negative definite intersection forms Donaldson used these compactified moduli spaces to prove his famous theorem about the diagonizability of intersection forms [D1].

Next we shall discuss the results proved in this book in more detail. A gauge invariant measure for the curvature is the L^p-energy of a connection,

$$\mathcal{E}(A) = \int |F_A|^p.$$

This energy is conformally invariant for $p = \frac{n}{2}$ on an n-manifold. As a consequence, for $p \leq \frac{n}{2}$ the moduli spaces of connections with bounded energy are not even compact in the L^1-topology.[1] For $p > \frac{n}{2}$, however, the weak Uhlenbeck compactness theorem asserts the compactness of these moduli spaces in the weak $W^{1,p}$-topology. The strong Uhlenbeck compactness theorem asserts the $\mathcal{C}^\infty$-compactness of the moduli spaces of Yang-Mills connections with bounded L^p-energy for $p > \frac{n}{2}$. Again this fails for $p = \frac{n}{2}$. Explicit examples in the case $n = 4$ follow from the ADHM construction [ADHM] of anti-self-dual instantons on the four-sphere with fixed L^2-energy. For example, the gauge equivalence classes of anti-self-dual $SU(2)$-connections over S^4 with L^2-energy $8\pi^2$ are parametrized by $\mathbb{R}^4 \times \mathbb{R}^+$ [DK, 3.4.1].

The weak and strong Uhlenbeck compactness theorems were originally stated for closed base manifolds with a fixed metric, but they generalize to several other situations. In this book we directly prove the Uhlenbeck compactness theorems for compact manifolds with boundary, as stated in theorems A and E. For the strong Uhlenbeck compactness this means that we consider the Yang-Mills equation with boundary condition $*F_A|_{\partial M} = 0$. Furthermore, there are generalizations to varying metrics and to manifolds that are exhausted by compact deformation retracts, as stated precisely in theorems A' and E'.

These generalizations are needed in the following applications: The proof of the metric independence of the Donaldson invariants requires the compactification of parametrized moduli spaces. These contain pairs consisting of a metric (in a

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[1]Let A be a connection of finite energy on the trivial bundle over $S^4 \cong \mathbb{R}^4 \cup \{\infty\}$ and consider the rescaled connections $A_\sigma(x) = \sigma A(\sigma x)$, then $\mathcal{E}(A_\sigma) = \sigma^{2p-n}\mathcal{E}(A)$ is bounded as $\sigma \to \infty$ but the pointwise norm of A_σ converges to a multiple of the Dirac distribution (by theorem C.5).

fixed path between two metrics) and a Yang-Mills connection with respect to that metric. So one has to allow for the metric to vary along with the sequence of connections in the strong Uhlenbeck compactness theorem. One can, however, a priori choose the sequence such that the metrics converge. The strong Uhlenbeck compactness also is frequently used for two types of noncompact base manifolds. Firstly, manifolds with finitely many punctures arise from the bubbling off analysis in two ways: One has convergence on a compact base manifold with finitely many punctures, and the bubbling off analysis near every puncture yields a sequence of connections on larger and larger balls exhausting $\mathbb{R}^4$. Secondly, one considers manifolds with cylindrical ends, for example, in order to define Donaldson invariants for manifolds with boundary. In that case one glues a cylindrical end to each boundary component. In Floer theory the product of the real line with a three-manifold occurs naturally as the domain of the gradient flow lines of the Chern–Simons functional.

Both generalizations, theorems A$'$ and E$'$, are proven by an extension argument of Donaldson and Kronheimer. This requires the restriction to manifolds that are exhausted by compact deformation retracts to ensure that gauge transformations on the compact sets can be extended to the whole manifold. Note that this covers both the case of manifolds with punctures and of manifolds with cylindrical ends. Hence these generalizations suffice for all the applications mentioned above.

For the weak compactness theorem A we will essentially follow Uhlenbeck's original proof. For the strong compactness theorem E, however, we use an alternative approach by Salamon. This reduces the strong compactness to the weak compactness with the help of a subtle local slice theorem F. It can be used to put the connections into relative Coulomb gauge with respect to the limit connection that is provided by the weak Uhlenbeck compactness theorem A. Then the strong Uhlenbeck compactness theorem E is a consequence of elliptic estimates for the connections. This approach circumvents a further patching argument. It is also useful for the generalization to manifolds with boundary: In the local slice theorem F we establish the relative Coulomb gauge with a suitable boundary condition. This complements the Yang-Mills equation with boundary condition $*F_A|_{\partial M} = 0$ to an elliptic boundary value problem. Furthermore, this line of argument is also suitable for the study of boundary value problems with nonlocal boundary conditions such as described in [Sa, W].

The 'standard' proof of the strong Uhlenbeck compactness theorem E essentially follows the same line of argument as the proof of the weak Uhlenbeck compactness theorem A: One first finds local Coulomb gauges in which one has convergent subsequences and then obtains global gauges from a patching construction. We slightly simplified the patching construction in the proof of theorem A, and also provide the generalization that allows to use this construction for a proof of theorem E.

The local Uhlenbeck gauges are provided by Uhlenbeck's local gauge theorem B for connections with sufficiently small L^q-energy. Here one can use the conformally invariant energy, that is $q = \frac{n}{2}$ (if we assume $n > 2$). That this energy is locally

small is ensured by a global bound on the L^p-energy for $p > \frac{n}{2}$. For the proof of the weak compactness theorem A, it would actually suffice to construct these gauges on Euclidean balls. However, if one wants to obtain the stronger convergence in theorem E, then the local gauges have to augment the Yang-Mills equation to an elliptic boundary value problem. This requires the more general form of theorem B, that constructs the Coulomb gauges with respect to a fixed metric on the manifold.

One of the main motivations for this book was to clarify the proof of this local gauge theorem B. It boils down to solving the boundary value problem posed by the Coulomb gauge for the gauge transformation. Then an a priori estimate provides the further estimates involved in the Uhlenbeck gauge. This a priori estimate is based on the L^p-estimate for the operator $\mathrm{d} \oplus \mathrm{d}^*$ on the space of 1-forms that satisfy the boundary condition from the Coulomb gauge (see theorem D).

Uhlenbeck's approach to solving the boundary value problem for the gauge transformation is to first construct a gauge transformation that solves the boundary condition and then use the implicit function theorem to solve the differential equation with homogenous boundary condition. The solution of the boundary condition requires the seemingly obvious theorem C. It asserts that the space of $W^{1,p}$-functions restricted to the boundary of a compact manifold is identical to the space of normal derivatives of $W^{2,p}$-functions. This was proven in higher generality by Agmon, Douglis, Nirenberg, [ADN], but even in this most basic case the proof requires the explicit solution of the Neumann boundary value problem on the half space with inhomogenous boundary conditions.

In this book we pursue the alternative approach suggested by Uhlenbeck: The boundary value problem for the gauge transformation can be directly solved with the implicit function theorem. This involves inhomogenous boundary conditions, so one has to work with boundary value spaces. Moreover, the surjectivity of the linearized operator requires the existence theorem for the Neumann boundary value problem with inhomogenous boundary conditions on L^p-spaces, which brings us back to the work of Agmon, Douglis, Nirenberg.

The L^p-theory for the Neumann boundary value problem with inhomogenous boundary conditions also enters in the elliptic estimates for the strong Uhlenbeck compactness theorem on manifolds with boundary and on manifolds exhausted by compact sets (which necessarily have nonempty boundaries). Thus it seemed appropriate to include an exposition of the Neumann problem in part I. This covers all the results that are required in this book.

Another result that fits well into this book but has no application within it is the local slice theorem F′ that provides a weak relative Coulomb gauge for L^p-connections. This gauge is used in [W] to generalize the compacntess of the moduli space of flat connections to (weakly) flat L^p-connections. This in turn is needed to deal with Lagrangian boundary conditions for anti-self-dual instantons.

The main results

The weak and strong Uhlenbeck compactness theorems deal with sequences of G-connections for compact Lie groups G. More precisely, let $P \to M$ be a principal G-bundle. Throughout this book the base manifold M is a smooth n-manifold with (possibly empty) boundary. If M is compact we will consider sequences in the Sobolev space $\mathcal{A}^{1,p}(P)$ of $W^{1,p}$-connections on P. The group $\mathcal{G}^{2,p}(P)$ of $W^{2,p}$-gauge transformations acts continuously on $\mathcal{A}^{1,p}(P)$. In the case of a noncompact base manifold M we consider the space $\mathcal{A}^{1,p}_{loc}(P)$ with the action of the gauge group $\mathcal{G}^{2,p}_{loc}(P)$.

These Sobolev spaces and actions are carefully defined in the appendices A and B; they are well defined for $p > \frac{n}{2}$. The action of the gauge group is in fact smooth, which leads to a Banach manifold structure of the moduli space $W^{1,p}(P)/\mathcal{G}^{2,p}(P)$ (away from the singularities which actually turn this into an orbifold). This becomes important in the study of moduli spaces of Yang-Mills connections, e.g. in Donaldson theory and Floer homology. However, for the compactness results which we fokus on here, it is enough to know that the gauge action is continuous.

The weak Uhlenbeck compactness theorem (for compact base manifolds) asserts that every subset of the quotient $\mathcal{A}^{1,p}(P)/\mathcal{G}^{2,p}(P)$ that satisfies an L^p-bound on the curvature is weakly compact. This theorem holds for compact manifolds with boundary as well as for closed manifolds.

Theorem A (Weak Uhlenbeck Compactness)
Assume M is a compact Riemannian n-manifold and let $1 < p < \infty$ be such that $p > \frac{n}{2}$. Let $(A^\nu)_{\nu \in \mathbb{N}} \subset \mathcal{A}^{1,p}(P)$ be a sequence of connections and suppose that $\|F_{A^\nu}\|_p$ is uniformly bounded. Then there exists a subsequence (again denoted $(A^\nu)_{\nu \in \mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}(P)$ such that $u^\nu {}^* A^\nu$ converges weakly in $\mathcal{A}^{1,p}(P)$.

Here the compactness of M is crucial. There also is a version of weak Uhlenbeck compactness for manifolds $M = \bigcup_{k \in \mathbb{N}} M_k$ that are exhausted by an increasing sequence of compact submanifolds, i.e. each compact submanifold M_k is contained in the interior of the next, M_{k+1}. In order to extend the weak Uhlenbeck compactness theorem to this situation we shall also assume that each submanifold M_k is a deformation retract of M.[2] This includes for example manifolds with finitely many punctures and cylindrical ends. The following theorem gives the precise formulation of weak Uhlenbeck compactness in this situation. It is a slight generalization of theorem A. Its proof uses an extension argument of Donaldson and Kronheimer [DK].

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[2]This condition ensures that every gauge transformation on M_k extends to M. It is an open question whether theorem A$'$ holds for more general manifolds.

Theorem A$'$
Assume that $M = \bigcup_{k \in \mathbb{N}} M_k$ is a Riemannian n-manifold exhausted by an increasing sequence of compact submanifolds M_k that are deformation retracts of M. Let $1 < p < \infty$ be such that $p > \frac{n}{2}$. Let $(A^\nu)_{\nu \in \mathbb{N}} \subset \mathcal{A}^{1,p}_{loc}(P)$ be a sequence of connections and for all $k \in \mathbb{N}$ suppose that $\|F_{A^\nu}\|_{L^p(M_k)}$ is uniformly bounded.

Then there exists a subsequence (again denoted $(A^\nu)_{\nu \in \mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}_{loc}(P)$ such that $u^\nu {}^* A^\nu|_{M_k}$ converges weakly in $\mathcal{A}^{1,p}(P|_{M_k})$ for all $k \in \mathbb{N}$.

The first step towards the proof of these weak compactness results is to establish the existence of a Coulomb type gauge over small trivializing neighbourhoods $U \subset M$. In a fixed trivialization of $P|_U$ connections are represented by elements of $\mathcal{A}^{1,p}(U)$, the $W^{1,p}$-space of 1-forms with values in $\mathfrak{g}$ (the Lie algebra of G). Gauge transformations are represented by elements of $\mathcal{G}^{2,p}(U)$, the $W^{2,p}$-space of G-valued functions. Now for $A \in \mathcal{A}^{1,q}(U)$ the L^q-energy is defined by

$$\mathcal{E}(A) := \int_U |F_A|^q.$$

Theorem B   (Uhlenbeck Gauge)
Fix a Riemannian n-manifold M, a compact Lie group G, and let $1 < q \leq p < \infty$ such that $q \geq \frac{n}{2}$ and $p > \frac{n}{2}$. In case $q < n$ assume in addition $p \leq \frac{nq}{n-q}$. Then there exist constants C_{Uh} and $\varepsilon_{Uh} > 0$ such that the following holds:

Every point in M has a neighbourhood $U \subset M$ with smooth boundary such that for every connection $A \in \mathcal{A}^{1,p}(U)$ with $\mathcal{E}(A) \leq \varepsilon_{Uh}$, there exists a gauge transformation $u \in \mathcal{G}^{2,p}(U)$ such that

> (i) $\mathrm{d}^*(u^*A) = 0,$ (iii) $\|u^*A\|_{W^{1,q}} \leq C_{Uh}\|F_A\|_q,$
>
> (ii) $*(u^*A)|_{\partial U} = 0,$ (iv) $\|u^*A\|_{W^{1,p}} \leq C_{Uh}\|F_A\|_p.$

The domains $U \subset M$ here will be diffeomorphic to the n-ball. For the proof of theorem A it would suffice to establish (i) and (ii) with respect to a metric that is pulled back from the Euclidean metric (on a ball in $\mathbb{R}^n$ whose diameter is comparable to the diameter of U in the given metric on M). However, if one wants to use theorem B for the 'standard' proof of the strong Uhlenbeck compactness theorem E below, then it is important to establish the gauge conditions (i) and (ii) with respect to the fixed metric on the manifold.

The proof of theorem B boils down to solving the boundary value problem posed by (i) and (ii) for the gauge transformation. In Uhlenbeck's original proof she first finds a gauge transformation that meets the boundary condition (ii) and then solves the homogeneous boundary value problem for (i). In this book we will solve the inhomogeneous boundary value problem right away using boundary

value spaces (as was suggested by Uhlenbeck in [U2]). That way one needs an existence theorem for the Neumann problem with inhomogeneous boundary conditions. This will be provided in the preliminary part I. The relevant estimate was proven in [ADN] in high generality. In our case it suffices to establish the following fact that also implies the existence of a gauge transformation that satisfies (ii). So this is an alternative proof of [U2, Lemma 2.6].

Theorem C (Agmon, Douglis, Nirenberg)
Let M be a compact Riemannian manifold, let $k \in \mathbb{N}_0$, and let $1 < p < \infty$. Then there is a constant C such that for every $f \in W^{1,p}(M)$ there exists a $u \in W^{2,p}(M)$ that satisfies

$$\frac{\partial u}{\partial \nu} = f\big|_{\partial M}, \qquad \|u\|_{W^{2,p}} \leq C\|f\|_{W^{1,p}}.$$

This will be proven along the lines of [ADN] using the Calderon-Zygmund inequality for the Poisson kernel. The Calderon-Zygmund inequality also lies at the heart of theorem D below. Assertion b) was stated by Uhlenbeck [U2] for the unit ball and was proven there in the case $p = 2$.

Theorem D
Let M be a compact Riemannian manifold and let $1 < p < \infty$. Then the following holds.

a) *There is a constant C such that for all $A \in W^{1,p}(M, \mathrm{T}^*M)$ with $*A|_{\partial M} = 0$*

$$\|A\|_{W^{1,p}} \leq C\big(\|\mathrm{d}A\|_p + \|\mathrm{d}^*A\|_p + \|A\|_p\big).$$

b) *Suppose in addition $H^1(M; \mathbb{R}) = 0$. Then there exists a constant C such that for all $A \in W^{1,p}(M, \mathrm{T}^*M)$ with $*A|_{\partial M} = 0$*

$$\|A\|_{W^{1,p}} \leq C\big(\|\mathrm{d}A\|_p + \|\mathrm{d}^*A\|_p\big).$$

Note that assertion b) provides an a priori estimate for connections that satisfy the Uhlenbeck gauge conditions (i) and (ii). This estimate will be used to establish (iii) and (iv) and thus to prove theorem B.

The second step in the proof of the weak Uhlenbeck compactness is a patching argument. One has to patch together the local gauge transformations obtained from theorem B to construct a sequence of global gauge transformations. In the case of a compact base manifold one can use the exponential map and cutoff functions in the Lie algebra. Topological obstructions only arise in the continuation of gauge transformations from a compact subset to the whole exhausted manifold. An argument of Donaldson and Kronheimer [DK, Lemma 4.4.5] proves theorem A′, but this requires the restriction to manifolds exhausted by deformation retracts.

The strong Uhlenbeck compactness theorem concerns sequences of connections that satisfy the Yang-Mills equation

$$\begin{cases} \mathrm{d}_A^* F_A = 0, \\ *F_A|_{\partial M} = 0. \end{cases} \tag{YM}$$

Note that our Yang-Mills equation incorporates a boundary condition. This is the natural boundary condition arising from the variational principle for the Yang-Mills functional on manifolds with boundary. Extrema of the Yang-Mills functional

$$\mathcal{YM}(A) = \int_M |F_A|^2$$

solve the Yang-Mills equation in its weak form: For every $\beta \in \Omega^1(P; \mathfrak{g})$ with compact support (but not necessarily $\beta = 0$ on ∂M)

$$\int_M \langle\, F_A \,,\, \mathrm{d}_A\beta \,\rangle = 0. \tag{wYM}$$

The solutions $A \in \mathcal{A}_{loc}^{1,p}(P)$ of this weak equation will be called weak Yang-Mills connections. In order for this equation to make sense the Sobolev exponent p has to be sufficiently large depending on the dimension $\dim M = n$ of the base manifold. [3] In the case of smooth connections this weak Yang-Mills equation (wYM) is equivalent to the boundary value problem (YM), i.e. the strong Yang-Mills equation. If moreover the base manifold M has no boundary then the weak Yang-Mills equation for smooth connections is equivalent to the usual Yang-Mills equation $\mathrm{d}_A^* F_A = 0$ without boundary condition.

The strong Uhlenbeck compactness theorem for G-bundles over manifolds with (possibly empty) boundary uses above definition of weak Yang-Mills connections (including the boundary condition).

Theorem E   (Strong Uhlenbeck Compactness)
Assume M is a compact Riemannian n-manifold. Let $1 < p < \infty$ be such that $p > \frac{n}{2}$ and in case $n = 2$ assume $p > \frac{4}{3}$. Let $(A^\nu)_{\nu \in \mathbb{N}} \subset \mathcal{A}^{1,p}(P)$ be a sequence of weak Yang-Mills connections and suppose that $\|F_{A^\nu}\|_p$ is uniformly bounded.

Then there exists a subsequence (again denoted $(A^\nu)_{\nu \in \mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}^{2,p}(P)$ such that $u^\nu {}^* A^\nu$ converges uniformly with all derivatives to a smooth connection $A \in \mathcal{A}(P)$.

Again, this theorem generalizes to manifolds which can be exhausted by compact deformation retracts. Moreover, one can perturb the Yang-Mills equation by considering a $\mathcal{C}^\infty$-convergent sequence of metrics g_ν and weak Yang-Mills connections A^ν with respect to the metrics g_ν.

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[3]If we assume $1 < p < \infty$ and $p > \frac{n}{2}$, then this is enough for $n = 1$ and $n \geq 3$. Only for $n = 2$ we need the additional condition $p \geq \frac{4}{3}$, see chapter 9.

Theorem E′

Assume $M = \bigcup_{k \in \mathbb{N}} M_k$ is a Riemannian n-manifold exhausted by an increasing sequence of compact submanifolds M_k that are deformation retracts of M. Let $1 < p < \infty$ be such that $p > \frac{n}{2}$ and in case $n = 2$ assume $p > \frac{4}{3}$. Let $(g_\nu)_{\nu \in \mathbb{N}}$ be a sequence of metrics on M that converges uniformly with all derivatives on every compact set. For all $\nu \in \mathbb{N}$ let $A^\nu \in \mathcal{A}_{loc}^{1,p}(P)$ be a weak Yang-Mills connection with respect to g_ν and suppose that for all $k \in \mathbb{N}$

$$\sup_{\nu \in \mathbb{N}} \|F_{A^\nu}\|_{L^p(M_k)} < \infty.$$

Then there exists a subsequence (again denoted $(A^\nu)_{\nu \in \mathbb{N}}$) and a sequence of gauge transformations $u^\nu \in \mathcal{G}_{loc}^{2,p}(P)$ such that $u^\nu{}^* A^\nu$ converges uniformly with all derivatives on every compact set to a smooth connection $A \in \mathcal{A}(P)$.

The key to the proofs of these two theorems is the existence of a global relative Coulomb gauge ensured by the local slice theorem F below. The weak Uhlenbeck compactness theorem provides a $W^{1,p}$-weakly convergent subsequence and a limit connection. After a common gauge transformation the limit connection is smooth. Now a further subsequence can be put into relative Coulomb gauge with respect to that limit connection. The strong Uhlenbeck compactness then follows from elliptic estimates for the operator

$$\alpha \mapsto \left(d_{\tilde{A}+\alpha}^* F_{\tilde{A}+\alpha}, \, *F_{\tilde{A}+\alpha}\big|_{\partial M}, \, d_{\tilde{A}}^* \alpha, \, *\alpha|_{\partial M} \right),$$

where $\tilde{A} \in \mathcal{A}(P)$ is a smooth connection. This approach by Salamon is different from the proofs of Uhlenbeck [U2] and Donaldson-Kronheimer [DK] (whose 'standard' proof will also be explained in chapter 10). It reduces the strong Uhlenbeck compactness theorem E to the weak compactness theorem A without using a further patching argument (in the case of a compact base manifold). The proof of theorem E′ moreover uses the same extension argument as the proof of theorem A′.

Theorem F   (Local Slice Theorem)

Let M be a compact Riemannian n-manifold with smooth boundary (that might be empty). Let $1 < p \le q < \infty$ such that

$$p > \frac{n}{2} \qquad and \qquad \frac{1}{n} > \frac{1}{q} > \frac{1}{p} - \frac{1}{n}.$$

Fix a reference connection $\hat{A} \in \mathcal{A}^{1,p}(P)$ and a constant $c_0 > 0$. Then there exist constants $\delta > 0$ and C such that the following holds: For every $A \in \mathcal{A}^{1,p}(P)$ with

$$\|A - \hat{A}\|_q \le \delta \qquad and \qquad \|A - \hat{A}\|_{W^{1,p}} \le c_0$$

there exists a gauge transformation $u \in \mathcal{G}^{2,p}(P)$ such that

(i) $d_{\hat{A}}^*(u^*A - \hat{A}) = 0$,   *(iii)* $\|u^*A - \hat{A}\|_q \le C\|A - \hat{A}\|_q$,

(ii) $*(u^*A - \hat{A})|_{\partial M} = 0$,   *(iv)* $\|u^*A - \hat{A}\|_{W^{1,p}} \le C\|A - \hat{A}\|_{W^{1,p}}$.

This theorem asserts the existence of a local slice through $\hat{A}$ that is transversal to the orbits of the gauge action. (This is because the infinitesimal action of the gauge group at $\hat{A}$ is given by $\mathrm{d}_{\hat{A}}$.) For every L^q-close connection A one then finds a gauge equivalent connection in the local slice, but can keep control of the $W^{1,p}$-norm. This goes beyond a simple application of the implicit function theorem since it only requires a $W^{1,p}$-bound on A, not $W^{1,p}$-closeness to $\hat{A}$.

There also is an L^p-version of the local slice theorem that will be proven in this book. In order to state the weak Coulomb equation involved we use the notation $\mathfrak{g}_P = P \times_{\mathrm{Ad}} \mathfrak{g}$ for the associated bundle arising from the adjoint representation of G on its Lie algebra $\mathfrak{g}$. Then the difference of any two smooth connections is an element of $\Gamma(\mathrm{T}^*M \otimes \mathfrak{g}_P)$.

Theorem F′   (Lᵖ-Local Slice Theorem)
Let M be a compact Riemannian n-manifold with smooth, possibly empty boundary. Let $2 \leq p < \infty$ be such that $p > n$ and fix a reference connection $\hat{A} \in \mathcal{A}^{0,p}(P)$. Then there exist constants $\delta > 0$ and C such that the following holds.

For every $A \in \mathcal{A}^{0,p}(P)$ with $\|A - \hat{A}\|_p \leq \delta$ there exists a gauge transformation $u \in \mathcal{G}^{1,p}(P)$ such that

$$\int_M \langle u^*A - \hat{A}, \mathrm{d}_{\hat{A}}\eta \rangle = 0 \qquad \forall \eta \in \Gamma(\mathfrak{g}_P)$$

and $\quad \|u^*A - \hat{A}\|_p \leq C\|A - \hat{A}\|_p.$

Following [CGMS] both local slice theorems F and F′ will be proven by Newton's iteration method. In fact, theorem F′ could also be proven by the implicit function theorem and one extra estimate. However, we use this easier case to illustrate the iteration method. For theorem F this iteration is considerably more complicated due to the boundary terms and the fact that the $W^{1,p}$-norm of the connection is only assumed to be bounded, not small. This, however, is just the setting that is obtained from the weak Uhlenbeck compactness theorem A: The connections converge in the weak $W^{1,p}$-topology, so they are $W^{1,p}$-bounded and they converge strongly only with respect to an L^q-norm.

Theorem F′ is used in [W] to generalize the regularity of flat connections to L^p-connections: A connection of class $W^{1,p}$ is called flat if its curvature vanishes, which is a partial differential equation of first order. One can use theorem F to prove that every flat connection is gauge equivalent to a smooth connection. For connections of class L^p one can introduce the notion of 'weak flatness' by a weak equation. The combination of the weak flatness and the weak relative Coulomb gauge provided by theorem F′ then constitutes an elliptic system whose L^p-solutions are in fact smooth. This proves that every weakly flat connection is gauge equivalent to a smooth connection.

Outline

This book is organized in four parts. Part I is of preliminary nature. It gives an exposition of the Neumann problem in chapters 1 to 3. We give proofs of a number of results that are wellknown but for which there seem to be no explicit proofs in the standard textbooks. For example, the L^2-regularity theorem 1.3 for weak solutions of the Neumann problem requires no minimum regularity of the solution. We also prove the regularity and existence results for L^p-spaces with general $p > 1$. Moreover, theorem C is of central importance for the Neumann problem with inhomogeneous boundary conditions, but some textbooks simply omit it. Here we give a proof that uses the methods of [ADN] but is considerably easier than their treatment of general boundary value problems, see theorem 3.4.

In chapter 4 some results on the Neumann problem are generalized to sections of vector bundles with nonsmooth connections. These are used in the Newton iteration for the local slice theorems F and F′.

In part II we prove the weak Uhlenbeck compactness theorems A and A′, see theorems 7.1 and 7.5. Firstly, chapter 5 provides regularity results for 1-forms which correspond to Hodge theory on manifolds with boundary. These are used to prove the L^p-estimates of theorem D for the operator $\mathrm{d} \oplus \mathrm{d}^*$, restated in theorem 5.1. Moreover, the regularity theory for Yang-Mills connections will again make use of these results.

In chapter 6 we then prove the Uhlenbeck local gauge theorem B, restated as theorem 6.1. We use boundary value spaces instead of Uhlenbeck's explicit construction of a gauge transformation that meets the boundary condition. We also filled in a lot of technicalities: Uhlenbeck proves the theorem for one model domain, that is the Euclidean unit ball [U2, Thm.2.1]. We show in theorem 6.3 that the theorem on the unit ball in fact holds for all metrics that are sufficiently $\mathcal{C}^2$-close to the Euclidean metric. Moreover, in order to generalize the theorem to manifolds with boundary and a fixed metric, we prove the same result for a second model domain, the "egg squeezed to the boundary". We furthermore explain the rescaling trick that proves the existence of the Uhlenbeck gauge in sufficiently small neighbourhoods on general manifolds.

Chapter 7 provides the patching constructions that complete the proof of the weak Uhlenbeck compactness theorems. In the case of a compact base manifold we have slightly modified Uhlenbeck's patching argument. The basic result is that every set of transition functions posesses a $\mathcal{C}^0$-neighbourhood of sets of transition functions that all define the same bundle. In [U2, Prop.3.2] the underlying cover has to be finite and the radius of the $\mathcal{C}^0$-neighbourhood depends on the number of covering patches and the set of transition functions. Our patching lemma 7.2 works for all countable covers of manifolds and the radius of the $\mathcal{C}^0$-neighbourhood simply is the radius of a convex geodesic ball in the Lie group. The proof of theorem A (restated as theorem 7.1) is based on this patching lemma.

The generalization of weak Uhlenbeck compactness to noncompact manifolds uses the extension argument of Donaldson and Kronheimer [DK, Lemma 4.4.5].

We explain that argument in lemma 7.8 and use it to prove proposition 7.6, which is a general tool for extending compactness results for moduli spaces over compact manifolds to base manifolds that are exhausted by compact deformation retracts. This result is then used to prove theorem A′, restated as theorem 7.5. It can again be used for the proof of theorem E′. It should be stressed that the extension argument requires that every gauge transformation on one of the exhausting compact submanifolds extends to the whole manifold. This is ensured by our assumption that the exhausting submanifolds are deformation retracts of the manifold.

Part III concerns the strong Uhlenbeck compactness theorems E and E′ (theorems 10.1 and 10.3). Here the generalization to manifolds with boundary requires to supplement the Yang-Mills equation with a boundary condition. This boundary value problem also occurs in the generalization to manifolds that are exhausted by compact sets, since these compact submanifolds necessarily have boundaries.

In chapter 8 we give proofs of the local slice theorem F and its L^p-version, theorem F′ (see theorem 8.1 and 8.3). These are adaptations of the Newton iteration method used for [CGMS, Thm.A.1] to manifolds with boundary and to L^p-connections. Moreover, we construct the relative Coulomb gauge with respect to different metrics under the same assumptions on the connection. This is needed for the case of varying metrics in theorem E′.

Chapter 9 introduces the Yang-Mills equation with boundary condition. We prove the smoothness of Yang-Mills connections up to a gauge transformation both on compact manifolds and on manifolds that are exhausted by compact deformation retracts, see theorem 9.4. This is done by the iteration of two regularity results. These already include estimates for the proof of the strong Uhlenbeck compactness.

Chapter 10 is devoted to the proofs of theorem E (theorem 10.1) and theorem E′ (theorem 10.3). Unlike to the 'standard' proof (which we also explain) this approach by Salamon requires no further patching construction for the proof of theorem E due to the use of the global relative Coulomb gauge provided by the local slice theorem F. For theorem E′ one again uses proposition 7.6, which relies on the extension argument of Donaldson and Kronheimer.

These last two chapters and chapter 5 contain careful details of the bootstrapping analysis. These are standard procedures but they did not seem entirely obvious, especially not for manifolds with boundary, and they have to be adapted separately to the case of noncompact manifolds exhausted by compact sets. Moreover, in order to obtain the compactness results for varying metrics, one has to obtain uniform constants for a small neighbourhood of metrics in all these estimates. This does not require a lot of extra work, but one always has to take care of what the constants depend on.

Part IV consists of a number of appendices that are designed to make this book as selfcontained as possible. Appendix A gives a brief introduction to gauge theory. This is not meant as an exposition but rather sets up the notation that is used throughout the book. Moreover, we prove some fundamental estimates on the energy and gauge transformations which will be needed frequently.

In appendix B one can find the definition of Sobolev spaces of sections of vector bundles and – more generally – fibre bundles. This leads to the definition of the Sobolev spaces of connections and gauge transformations. We also state all Sobolev embedding results that will be needed in the book and we give a coordinate free proof of a trace theorem (concerning the restriction of Sobolev functions to the boundary of a compact manifold).

Theorem C.2 in appendix C states a criterion for L^p-multipliers due to Mihlin [M]. It is much easier to check than the usual criteria e.g. in [St1]. We show how the criterion of Mihlin implies the standard criterion. Then we use this criterion to prove the Calderon-Zygmund inequality (theorem C.3). Furthermore, we use techniques of [ADN] to give a proof of the estimates on the Poisson kernel (theorem C.4) that are needed for theorem C. Similar techniques are used to prove a version of the mollifier theorem C.5 that again is more general than the one in most standard textbooks: We do not require compact support of the functions that are proven to converge to the Dirac distribution.

Appendix D states (without proofs) the main results on the Dirichlet problem. Appendix E states some results from Functional Analysis that are used at crucial places in this book.