1. Introduction

The goal of these lectures is to present a comprehensive exposition of modern partial hyperbolicity theory. They contain the core of the theory as well as outline some recent new achievements in this rapidly developing area. The material is accessible to students and non-experts who possess some basic knowledge in dynamical systems and wish to learn some new phenomena outside classical hyperbolicity. These lectures may also be of interest to experts as they provide a unified and systematic treatment of partial hyperbolicity and stable ergodicity and are unique in that.

Partial hyperbolicity is a relatively new field, just over 30 years old, but has proven to be rich in interesting ideas, sophisticated techniques and exciting applications. It appears naturally in some models in science. To illustrate this consider the FitzHugh-Nagumo partial differential equation which is used in neurobiology to model propagation of electrical impulse through the nerve membrane:

$$u_t(x,t) = \epsilon \Delta_x u(x,t) + h(u),$$

where $u(x,t) = (u_1(x,t), u_2(x,t))$ and

$$h(u_1, u_2) = (g(u_1) - bu_2, cu_1 - du_2)$$

is the *local map*. The function g introduces a cubic non-linearity

$$g(u_1) = -au_1(u_1 - \theta)(u_1 - 1).$$

We shall discuss traveling wave solutions of the FitzHugh-Nagumo equation. These are solutions of the form

$$\varphi(\xi) = \varphi(x - ct) = (\varphi_1(x - ct), \varphi_2(x - ct)),$$

where $c > 0$ is the velocity of the wave. The function $\varphi(\xi)$ satisfies the *traveling wave equation*

$$\epsilon\varphi''(\xi) + c\varphi'(\xi) + h(\varphi(\xi)) = 0.$$

Setting $\varphi' = v$ we obtain

$$\begin{cases} \varphi' = v \\ \epsilon v' = -cv - h(\varphi) \end{cases}$$

By changing the function $h(\varphi)$ outside a ball $B(0,R)$ of some large radius R, one can obtain that $(\varphi', v') \cdot (\varphi, v) < 0$. This modification of the original system guarantees that no solutions escape to infinity which is thus a repelling fixed point. This allows us to consider the equation (and the corresponding flow) on the two-dimensional sphere.

Following principles of singular perturbation theory let us change the time to "slow time" by substituting $t = \epsilon\,\tau$. Denote the slow time derivative by $\dot{\varphi}$. We have

$$\begin{cases} \dot{\varphi} = \epsilon v \\ \dot{v} = -cv - h(\varphi). \end{cases}$$

For $\epsilon = 0$, the manifold $\mathcal{C}$, defined by $v = -\frac{1}{c}h(\varphi)$, is a manifold of equilibrium points. Consider the expanded system

$$\begin{cases} \dot{\varphi} = \epsilon v \\ \dot{v} = -cv - h(\varphi) \\ \dot{\epsilon} = 0 \end{cases}$$

and linearize it at $\epsilon = 0$, $v = -\frac{1}{c}h(\varphi)$. The Jacobian matrix for the linearized system has eigenvalues $\lambda = -c, -c, 0, 0, 0$. It follows that for $\varepsilon = 0$ there exist a three-dimensional center manifold $\mathcal{C}^0 = \mathcal{C}$ and a two-dimensional stable manifold to it, i.e., $\mathcal{C}$ is *normally hyperbolic* (see Section 5.1 below). By the singular perturbation theory normal hyperbolicity survives: for any sufficiently small ϵ there exist a three-dimensional center manifold $\mathcal{C}^\epsilon$ and a two-dimensional stable manifold to it. One can show that the restriction of the dynamics to $\mathcal{C}^\epsilon$ is of a Morse-Smale type (see [35]).

In a more general setting one can observe partial hyperbolicity in systems described by partial differential equations possessing inertial manifolds. It often happens that the system acts as a contraction or/and expansion in directions transversal to the inertial manifold whose rates exceed the rates of contraction and expansion along this manifold. In this case the inertial manifold is normally hyperbolic.

Partial hyperbolicity can also occur when a periodic force acts on a dissipative system f possessing a "strange" attractor. The resulting system is the product $f \times \mathrm{Id}$. It acts on the phase space, that is the product of the phase space for f and the circle, and possesses a partially hyperbolic "strange" attractor. A small perturbation of this map often also possesses a partially hyperbolic "strange" attractor.

The structure of these lectures is as follows. In Chapter 2 we introduce the concept of partial hyperbolicity and also describe some basic examples of partially hyperbolic diffeomorphisms. In Chapter 3 we present the Mather spectrum theory for diffeomorphisms which allows one to characterize a partially hyperbolic map in terms of the spectrum of the linear operator generated by the map in the space of all continuous vector fields. Using this characterization we establish stability of partially hyperbolic maps.

In Chapters 4, 5 and 6 we discuss various aspects of stability theory for partially hyperbolic diffeomorphisms including: 1) constructions of invariant stable and unstable foliations (see Sections 4.2–4.7); 2) some criteria for integrability of the central distribution (see Section 5.3; in general, this distribution is not integrable, see Section 6.1 but it is often integrable in a weak sense, see Section 5.6); 3) stability of the central foliation under small perturbations (see Section 5.5), and 4) the branching phenomenon for intermediate foliations (see Section 6.3). We also introduce the concept of normal hyperbolicity which originated in works of Hirsch, Pugh and Shub [25, 26] and is closely related to partial hyperbolicity.

Our approach is based on an extension and adaptation to our case of a method which originated in the work of Perron [36] (see also [3], Chapter 4; the formal description of this method is given by Theorem 4.3). This method is quite powerful and can be used in various situations. We apply it to establish structural stability of Anosov maps (see Section 4.8) and to describe some interesting phenomena associated with insufficient smoothness of intermediate foliations (see Sections 6.2 and 6.3).

In Chapter 7 we discuss a crucial absolute continuity property of invariant foliations which provides a main technical tool in studying ergodic properties of partially hyperbolic systems with respect to smooth invariant probability measures. Chapter 8 is devoted to another crucial property of invariant foliations known as the accessibility property. It is necessary and in many cases sufficient to establish topological transitivity and ergodicity of the system.

In the last two chapters we outline basic elements and recent results in Pugh-Shub stable ergodicity theory with applications to skew products over Anosov maps, to Anosov flows (in particular, geodesic flows) and to frame flows on manifolds of negative curvature. In particular, we describe the surprising "Fubini's nightmare" phenomenon associated with non-absolutely continuous "pathological" foliations arising "typically" in partial hyperbolicity theory.

The majority of results presented in these lectures come with complete proofs. However, for some results, which require sophisticated techniques, we either just outline their proofs omitting technical details (but providing necessary references) or consider the proofs of some particular cases where the main idea can still be seen. For completeness of the exposition and to broaden applications we also included some results without proofs.

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