Preface

Between April and July of 2001, I gave the Nachdiplom lecture series at ETH in Zurich. The lectures concerned the study of some non-linear partial differential equations related to curvature invariants in conformal geometry. A classic example of such a differential equation on a compact surface is the Gaussian curvature equation under conformal change of metrics. On manifolds of dimension four, an analogue of the Gaussian curvature is the Pfaffian integrand in the Gauss-Bonnet formula: on a Riemannian manifold (M, g) of dimension four, denote the Weyl–Schouten tensor A as

$$A_{ij} = R_{ij} - \frac{R}{6} g_{ij}$$

where R_{ij} is the Ricci tensor and R is the scalar curvature of the Riemannian metric g; denote the second elementary symmetric function of A as

$$\sigma_2(A) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}[(TrA)^2 - |A|^2],$$

where λ_i $(1 \leq i \leq 4)$ are the eigenvalues of A; then one has the Gauss Bonnet formula

$$8\pi^2(\chi M) = \int (\frac{1}{4}|W|^2 + \sigma_2(A))dv,$$

where W denotes the Weyl tensor. Under conformal change of metrics, $|W|^2 dv$ is point-wisely conformally invariant, thus$\int \sigma_2(A)dv$ is conformally invariant. The main focus of these lecture notes is the study of the partial differential equation describing the curvature polynomial $\sigma_2(A)$ under conformal change of metrics.

The notes are organized as follows: In Chapters 1 and 2, I discuss the equation prescribing Gaussian curvature on compact surface, provide background, and describe the main analytic tool, Moser–Trudinger inequalities, in the study. In Chapter 3, I describe the connection between Moser–Trudinger inequality to the Polyakov formula for the functional determinant of the Laplacian operator on compact surfaces. In Chapters 4 to 6, I discuss general conformal invariants, the connection of conformal invariants to conformal covariant operators on manifolds of dimension three and higher, with emphasis on a special 4-th operator (called the Paneitz operator) on manifolds of dimension 4. Finally in Chapters 7–10, I study the connection of the Paneitz operator to the curvature polynomial $\sigma_2(A)$ described above. I also report the work of Chang–Gursky–Yang [23] on the existence on manifolds (M^4, g) of solutions with $\sigma_2(A) > 0$ under the assumptions that $\int \sigma_(A) > 0$ and g be of positive Yamabe class.

The lectures were given at an early stage, when the study of the fully non-linear PDEs like that of $\sigma_2(A)$ were first developed. Since then, there has been much progress both in the form of existence and regularity results on such equations. Readers are referred to the article by Gursky–Viaclovsky [56], where a simpler proof, from a somewhat different perspective, of the main result in [23] discussed in these notes is given. There have also been important results on the existence of general conformal invariants by Graham–Zworski [50] and Fefferman–Graham [44]. There is also a more recent survey article [20] for recent developments in this research field.

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