Introduction

These notes reproduce the lectures which I gave in Masaryk University (Brno, Czech Republic) and in E. Schrödinger Institute for Mathematical Physics (Vienna, Austria) during the autumn semesters 2001 and 2002, respectively. The main goal was an exposition of the theory of finite dimensional representations of real semisimple Lie algebras.

The foundation of this theory was laid in É. Cartan [3]. Iwahori [13] gave an updated exposition of the É. Cartan's work (see also [10], Ch. 7). This theory reduces the classification of irreducible real representations of a real Lie algebra $\mathfrak{g}_0$ to description of the so-called *self-conjugate* irreducible complex representations of this algebra and to calculation of an invariant of such a representation (with values ± 1) which is called the *index*. No general method for solving any of these two problems were given in [10, 13], but they were reduced to the case when $\mathfrak{g}_0$ is simple and the highest weight of an irreducible complex representation is fundamental. A complete case-by-case classification for all simple real Lie algebras $\mathfrak{g}_0$ was given (without proof) in the tables of Tits [24].

The aim proclaimed by É. Cartan in [2, 3] was to find all irreducible subalgebras of the linear Lie algebras $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{gl}_n(\mathbb{R})$. As a continuation of this line, one should consider the works of Maltsev [17] and Dynkin [6] on semisimple subalgebras of simple complex Lie algebras, also based on the representation theory. A similar problem for real simple Lie algebras was studied in the paper of Karpelevich [15]. The results of this paper solve actually the problems on the self-conjugate complex representations and the index mentioned above. Note that the paper appeared before the publication of [13], but, unfortunately, still is not widely known. Our goal is to give a simplified (and somewhat extended and corrected) exposition of the main part of [15] and to relate it to the theory developped in [10, 13].

The first section contains some classical facts from the theory of Lie groups, Lie algebras and their representations, including the structure theory of complex semisimple Lie algebras, given without proofs. The necessary notation is also fixed there. In the main body of the lecture notes (§2 – §8) the exposition is given with detailed proofs.

§2 deals with general facts on complexification and realification, real and complex structures and real forms. In particular, a general description of simple real Lie algebras in terms of simple complex ones is given and two main examples of real forms (normal and compact) in a complex semisimple Lie algebra are constructed.

§3 is devoted to the correspondence between real forms of a complex semisimple Lie algebra $\mathfrak{g}$ and involutive automorphisms of $\mathfrak{g}$ discovered by É. Cartan [4] which is the main tool in the subsequent study of real Lie algebras and their representations. Instead of Riemannian geometry exploited in [4], we use the elementary techniques of Hermitian vector spaces, as in [11], Ch. III or [19], Ch. 5.

In §4, necessary facts about automorphisms of complex semisimple Lie algebras are given. As an example, we classify involutive automorphisms (and hence real forms) of the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. At the same time, we do not give such a classification for other simple Lie algebras, referring to [11] or [19]. We also construct here the so-called principal three-dimensional subalgebra of a complex semisimple Lie algebra $\mathfrak{g}$ and use it for studying the Weyl involution of $\mathfrak{g}$ which corresponds to the normal real form of this algebra.

In §5, we study the Cartan decompositions of a real semisimple Lie algebra $\mathfrak{g}_0$ and of the corresponding adjoint linear group $\operatorname{Int}\mathfrak{g}_0$. Then we prove the conjugacy of maximal compact subgroups of $\operatorname{Int}\mathfrak{g}_0$ (the elementary proof borrowed from [19] makes use of some properties of convex functions, instead of Riemannian geometry exploited in the original proof of É. Cartan).

§6 is devoted to the following problem. Suppose a homomorphism f of one complex semisimple Lie algebra $\mathfrak{g}$ into another $\mathfrak{h}$ be given. Choosing real forms $\mathfrak{g}_0 \subset \mathfrak{g}$ and $\mathfrak{h}_0 \subset \mathfrak{h}$, one asks, when $f(\mathfrak{g}_0) \subset \mathfrak{h}_0$. We prove the result of [15] which claims that this inclusion is equivalent to the relation $f\theta = \theta' f$ if the classes of conjugate real forms and conjugate automorphisms are considered. Thus, the problem is reduced to that of extension of involutive automorphisms by homomorphisms. To get this reduction, we use, as in §3, the elementary techniques of Hermitian vector spaces. A more precise result is got in the case when f is a so-called S-homomorphism (in [15], it was proved for an irreducible representation into a classical linear Lie algebra $\mathfrak{h}$).

In §7, we study the extension problem for involutive automorphisms in the main case, when $f = \rho$ is an irreducible representation $\mathfrak{g} \to \mathfrak{sl}_n(\mathbb{C})$. First we consider the relation $\rho\theta = \theta'\rho$ in the case when θ' is an outer automorphism of $\mathfrak{sl}_n(\mathbb{C})$, i.e.,

$$\theta'(X) = -BX^\top B^{-1}, \quad X \in \mathfrak{sl}_n(\mathbb{C}), \quad \text{where} \quad B^\top = \pm B.$$

A condition of existence of such an extension in terms of the highest weight of ρ is found; it is expressed as invariance of the highest weight of ρ under an involutive automorphism s_0 of the Dynkin diagram. An important problem is to calculate the sign $j = \pm 1$ in the above formula provided that this condition is satisfied. In [15], a simple explicit formula for this *Karpelevich index* j in the case when θ is an inner automorphism was proved. We generalize this formula to an arbitrary involutive θ. Then we consider the case of an inner θ', getting, in particular, two formulas from [15] which express the signature of the invariant Hermitian form through the character of ρ. We do not reproduce the explicit formulas expressing this signature through the highest weight of ρ which were deduced in [15], using cumbersome calculations. We also omit the study of the extension problem for reducible representations $\mathfrak{g} \to \mathfrak{sl}(V)$ and for representations $\mathfrak{g} \to \mathfrak{so}(V)$ and $\mathfrak{g} \to \mathfrak{sp}(V)$ which was made in [15], too.

In §8, we give a classification of irreducible real representations of real semisimple Lie algebras following the method exposed in [10, 13]. It turns out that an irreducible complex representation ρ_0 of a real semisimple Lie algebra $\mathfrak{g}_0$ is self-conjugate if and only if the corresponding involution θ of $\mathfrak{g}_0(\mathbb{C})$ extends to an outer involution of $\mathfrak{sl}_n(\mathbb{C})$ by the complexified representation $\rho_0(\mathbb{C})$. Moreover, the Cartan index of ρ_0 coincides with the Karpelevich index of $\rho_0(\mathbb{C})$. This allows to apply the results of §7 which leads to an explicit classification of irreducible real representations in terms of highest weights.

In §9, written by J. Šilhan, a description of the symmetry s_0, introduced in §7, in terms of the Satake diagram of a real semisimple Lie algebra is given. This allows to determine self-conjugate complex representations by means of the Satake diagram.

We conclude with some tables, giving in particular the indices of irreducible representations of simple complex Lie algebras.

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