Preface

These are notes of lectures on Kähler manifolds which I taught at the University of Bonn and, in reduced form, at the Erwin-Schrödinger Institute in Vienna. Besides giving a thorough introduction into Kähler geometry, my main aims were cohomology of Kähler manifolds, formality of Kähler manifolds after [DGMS], Calabi conjecture and some of its consequences, Gromov's Kähler hyperbolicity [Gr], and the Kodaira embedding theorem.

Let M be a complex manifold. A Riemannian metric on M is called Hermitian if it is compatible with the complex structure J of M,

$$\langle JX, JY\rangle = \langle X, Y\rangle.$$

Then the associated differential two-form ω defined by

$$\omega(X, Y) = \langle JX, Y\rangle$$

is called the Kähler form. It turns out that ω is closed if and only if J is parallel. Then M is called a Kähler manifold and the metric on M a Kähler metric. Kähler manifolds are modelled on complex Euclidean space. Except for the latter, the main example is complex projective space endowed with the Fubini–Study metric.

Let N be a complex submanifold of a Kähler manifold M. Since the restriction of the Riemannian metric of M to N is Hermitian and its Kähler form is the restriction of the Kähler form of M to N, N together with the induced Riemannian metric is a Kähler manifold as well. In particular, smooth complex projective varieties together with the Riemannian metric induced by the Fubini–Study metric are Kählerian. This explains the close connection of Kähler geometry with complex algebraic geometry.

I concentrate on the differential geometric side of Kähler geometry, except for a few remarks I do not say much about complex analysis and complex algebraic geometry. The contents of the notes is quite clear from the table below. Nevertheless, a few words seem to be in order. These concern mainly the prerequisites. I assume that the reader is familiar with basic concepts from differential geometry like vector bundles and connections, Riemannian and Hermitian metrics, curvature and holonomy. In analysis I assume the basic facts from the theory of elliptic partial differential operators, in particular regularity and Hodge theory. Good references for this are for example [LM, Section III.5] and [Wa, Chapter 6]. In Chapter 8, I discuss Gromov's Kähler hyperbolic spaces. Following the arguments in [Gr], the proof of the main result of this chapter is based on a somewhat generalized version of Atiyah's L^2-index theorem; for the version needed here, the best reference seems to be Chapter 13 in [Ro]. In Chapter 7, I discuss the proof of the Calabi conjecture. Without further reference I use Hölder spaces and Sobolev embedding theorems. This is standard material, and many textbooks on analysis provide these prerequisites. In addition, I need a result from the regularity theory of non-linear partial differential equations. For this, I refer to the lecture notes by Kazdan [Ka2] where the reader finds the necessary statements together with precise references for their proofs. I use some basic sheaf theory in the proof of the Kodaira embedding theorem in Chapter 9. What I need is again standard and can be found, for example, in [Hir, Section 1.2] or [Wa, Chapter 5]. For the convenience of the reader, I include appendices on characteristic classes, symmetric spaces, and differential operators.

The reader may miss historical comments. Although I spent quite some time on preparing my lectures and writing these notes, my ideas about the development of the field are still too vague for an adequate historical discussion.

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