Preface

The goal of this book is to explain some of the key ingredients of Grisha Perelman's first paper [28] on the Ricci flow, namely Li–Yau type differential Harnack inequalities, entropy formulas and space-time geodesics. We will only explain the main core of these three topics without too many applications. In particular, we will not consider any of the applications for three-dimensional manifolds towards the Poincaré conjecture or William Thurston's geometrization conjecture. The Ricci flow, introduced by Richard Hamilton [13], can be seen as a heat equation for the Riemannian metric on a manifold. Many of its properties are therefore connected with properties of heat equations and adjoint heat equations on manifolds. In this book, we show how certain properties of the heat equation lead to results about the Ricci flow and vice versa. As a general principle, we discuss each topic by first looking at its analog for the heat equation on a static manifold, since the computations are usually much easier for this case. Then we continue with the nonlinear case, i.e. with the Ricci flow equation or with the heat equation on an evolving manifold. We have chosen this method of presentation since it makes the theory more accessible to non-specialists.

After a short introduction, we set up notation and terminology and derive the most fundamental formulas for the Ricci flow in chapter one. In chapter two, we develop the theory of differential Harnack inequalities. We present the famous Li–Yau result [22], and continue with Hamilton's matrix version [17] and his Harnack estimates for the Ricci flow [18]. In chapter three, we will see all the different entropy formulas for Ricci solitons from the Perelman paper [28] and discuss their relation to Li–Yau type Harnack inequalities. We will also see Perelman's new Harnack inequality. Moreover, we explain the analogous entropy formulas for the heat equation on a static manifold, following Lei Ni [25], and we will see the analog of Perelman's Harnack inequality for the static case. Finally, in chapter four, we study Perelman's $\mathcal{L}$-functional, using the results we obtained in the two previous chapters. Again, we also present the analogs for the static case.

Our goal is not simply to explain a part of Perelman's paper – there are enough other good sources where this is done, see for example Kleiner and Lott [23] or Cao and Zhu [4]. We also do not want to take part in the discussion on whether Perelman's proofs of the two conjectures are correct or not. Most people we talked to agreed that at least the Poincaré conjecture seems to be solved. However, the analytic methods found in this book are of great interest in a much more general setting. We present here in detail these analytic methods to non-experts or students who are new to this subject, in the belief that that they can also be useful for the study of other geometric flows or other applications of the Ricci flow, for example in the Kähler case. Hence the book does not cover Perelman's paper [28] step by step. We rather picked out the part of the theory we find the most fundamental and the most interesting, and we connect this theory with Li–Yau's theorem, Hamilton's results for the Ricci flow and Lei Ni's recent work. Although some parts of the book are quite technical, we always make an attempt to give heuristic motivations and interpretations of the results.

We should like to mention that we do not claim any original work beside some simplifications of Perelman's results for the easier case of a heat equation on a static manifold. These results arise naturally from work done by Lei Ni.

As this book was written at a time when not many notes on Perelman's work had been published, it is independent of most of the recent publications on this topic. Of course there will be overlappings with other recently published texts discussing Perelman's results. However, most of these notes are written by experts for experts and contain a lot of details which are important for Perelman's proof but not for the general understanding of the theory of Li–Yau type Harnack inequalities, while the aim of this book is to present exactly this analytic core theory to non-experts.

As for prerequisites, we only assume that the reader is familiar with the basic concepts and notions of Riemannian geometry, but no preliminary knowledge of the Ricci flow, Harnack inequalities or Perelman's paper [28] is required.