Introduction

In this book we compare the linear heat equation on a static manifold with the Ricci flow – which is a nonlinear heat equation for a Riemannian metric $g(t)$ on M – and with the heat equation on a manifold moving by Ricci flow. The aim of the book is to explain some of the theory of Perelman's first Ricci flow paper [28] as well as its thematic context. Concretely, we will discuss differential Harnack inequalities, entropy formulas and Perelman's $\mathcal{L}$-functional. In particular, we want to emphasize how these three topics are related with each other and with special solutions of the Ricci flow, so-called Ricci solitons. We will see that Li–Yau type Harnack inequalities are related with Perelman's entropy formulas and the $\mathcal{L}$-functional in a quite natural way. To motivate the results and to start with simpler computations, we will first consider the analogs of these topics for the heat equation on a static manifold. We will see that the results in the static and in the Ricci flow case are often very similar and sometimes coincide on special solutions.

This introduction contains a brief exposition of the historical background and summarizes the main content of the book.

Hamilton's Ricci flow and its early success

Nonlinear heat flows first appeared in Riemannian geometry in 1964, when James Eells and Joseph H. Sampson [10] introduced the harmonic map heat flow as a tool to deform given maps $u\colon M \to N$ between two manifolds into extremal maps (i.e. critical points in the sense of calculus of variations) for the energy functional

$$E(u) = \int_M |\nabla u|^2 \, dV. \tag{1}$$

In particular, these so-called harmonic maps include geodesics, harmonic functions and minimal surfaces. Eells and Sampson showed that the gradient flow of the energy functional (1) above – the harmonic map heat flow – will converge to a harmonic map if the target manifold N has negative sectional curvature. Since then, geometric heat flows have become an intensively studied topic in geometric analysis.

A fundamental problem in differential geometry is to find canonical metrics on Riemannian manifolds, i.e. metrics which are highly symmetrical, for example metrics with constant curvature in some sense. Using the idea of evolving an object to such an ideal state by a nonlinear heat flow, Richard Hamilton [13] invented the Ricci flow in 1981. Hamilton's idea was to smooth out irregularities of the curvature by evolving a given Riemannian metric on a manifold M with respect to the nonlinear weakly parabolic equation

$$\partial_t g_{ij}(t) = -2R_{ij}(t), \tag{2}$$

where g_{ij} denotes the Riemannian metric and R_{ij} its Ricci curvature. Hamilton showed that there exists a solution for a short time for any smooth initial metric g_0, see [13] (or also [9] for a simplified proof).

A solution metric $g(t)$ shrinks where its Ricci curvature is positive, while it expands in regions where the Ricci curvature is negative. For example a round sphere, which has positive constant Ricci curvature at time $t = 0$, will shrink – faster and faster – and collapse to a single point in finite time. In particular, the volume of the sphere is strictly decreasing along the flow. Hamilton therefore also considered the normalized (i.e. volume preserving) version of the Ricci flow, given by the equation

$$\partial_t g_{ij} = -2R_{ij} + \frac{2}{n} r g_{ij},$$

where n denotes the dimension of M and $r = \int_M R\, dV / \int_M dV$ is the average scalar curvature.

The stationary metrics under the Ricci flow (2) are Ricci flat metrics (i.e. metrics with $\mathrm{Ric} \equiv 0$). These are also the critical points of the Einstein–Hilbert functional

$$\mathcal{E}(g) = \int_M R\, dV, \tag{3}$$

a fact which we will prove in chapter one. However, we will see that the gradient flow of this functional differs from the Ricci flow by an additional term, which makes the equation impossible to solve in general. More precisely, the gradient flow of (3) is given by the equation

$$\partial_t g_{ij} = -2R_{ij} + R g_{ij}, \tag{4}$$

which does not have to be solvable (in either time direction) even for a short time, since $\partial_t g = -2\,\mathrm{Ric}$ is parabolic, while $\partial_t g = Rg$ is backwards parabolic and at the symbol level these two terms do not cancel. One can hence regard the Ricci flow as the part of the Einstein–Hilbert gradient flow that one gets by cancelling out the bad (i.e. backwards parabolic) term. Now the natural question arises whether the Ricci flow is the gradient flow of any functional at all, but unfortunately the answer is no. This seems to be a well-known proposition of Hamilton, but we have not yet seen any published proof in the literature. A proof can be found in chapter one.

If the Riemannian metric evolves with respect to a nonlinear heat equation, then so do its derivatives, in particular the different curvature tensors. All these equations will be presented in chapter one. For example the scalar curvature satisfies

$$\partial_t R = \triangle R + 2\,|\mathrm{Ric}|^2 \geq \triangle R + \frac{2}{n} R^2, \tag{5}$$

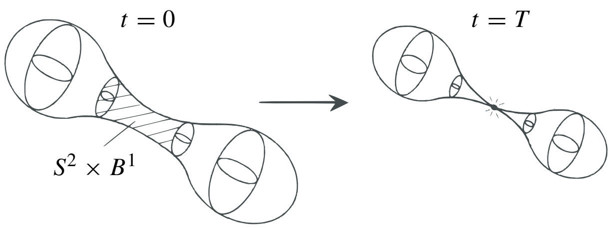
so by the maximum principle its minimum $R_{\min}(t) = \inf_M R(\cdot, t)$ is non-decreasing along the flow. Hamilton developed a maximum principle for tensors, with which he could find similar results for the Ricci and Riemannian curvature tensors. He proved that the Ricci flow preserves the positivity of the Ricci tensor in dimension three [13] and of the curvature operator in all dimensions [14]. Moreover, he also proved that the eigenvalues of the Ricci tensor in dimension three approach each other under the normalized flow. This allowed him to prove the following convergence result in his seminal paper.

Proposition 1 (Hamilton, [13]). *For a three-manifold with initial metric of strictly positive Ricci tensor a solution to the normalized Ricci flow will exist for all time and will converge exponentially fast to a metric of constant positive sectional curvature as time t tends to infinity.*

Such a three-manifold must hence be diffeomorphic to the three-sphere or a quotient of it by a finite group of isometries. Given a homotopy three-sphere, if one can show that it always admits a metric of positive Ricci curvature, then with Hamilton's result the Poincaré conjecture would follow. In [14], Hamilton proved a similar result for four-manifolds with initial metric of positive curvature operator.

However, such a simple result does not hold if one starts with an arbitrary metric without curvature assumptions. In the general case, the solution of the Ricci flow (2) may behave much more complicatedly and develop singularities in finite time, in particular the curvature may become arbitrarily large in some region while staying bounded in its complement. For example, if one starts with an almost round cylindrical neck, which looks like $S^2 \times B^1$ connecting two large pieces of low curvature, then the positive curvature in the S^2-direction will dominate the slightly negative curvature in the B^1-direction and therefore one expects the neck to shrink and pinch off.

An existence proof and detailed analysis of such neckpinches can be found in a recent book by Bennett Chow and Dan Knopf [8], the first rigorous examples have been constructed by Sigurd Angenent and Dan Knopf in [1].



Remark. The above picture is justified by Angenent's and Knopf's paper [2], where they proved that the diameter of the neck remains finite and that the singularity occurs solely on a hypersurface diffeomorphic to S^2 rather than along $S^2 \times [a, b]$, for instance.

In order to deal with such neckpinches, Hamilton [15] invented a topological surgery where one cuts the neck open and glues small caps to each of the boundaries in such a way that one can continue running the Ricci flow. He proposed a surgery procedure for four-manifolds that satisfy certain curvature assumptions and conjectured that a similar surgery would also work for three-manifolds with no a-priori assumptions at all. This led him to a program of attacking William Thurston's geometrization conjecture [29], which states that every closed three-manifold can be decomposed along spheres S^2 or tori T^2 into pieces that admit one of eight different geometric structures. In this context, neckpinch surgery corresponds to the topological decomposition along two-spheres into such pieces. However, neckpinches can also occur for purely PDE-related reasons, as in the picture above, where a sphere S^3 is decomposed into two spheres S^3. Note that attaching a three-sphere along a two-sphere to *any* manifold M does not change the topology of M. A good source for Hamilton's program is his survey [19] from 1995. To analyze singularities, one can use an analytic tool that allows to compare the curvatures of the solution at different points and different times. This tool is known as a Harnack type inequality, which we will now describe.

Differential Harnack inequalities

The classical Harnack inequality from parabolic PDE theory states that for $0 < t_1 < t_2 \leq T$ a non-negative smooth solution $u \in C^\infty(M \times [0, T])$ of the linear heat equation $\partial_t u = \triangle u$ on a closed, connected manifold M satisfies

$$\sup_M u(\,\cdot\,, t_1) \leq C \inf_M u(\,\cdot\,, t_2), \tag{6}$$

where C depends on t_1, t_2 and the geometry of M. However, in the classical presentation, the geometric dependency of C is not easy to analyze and the inequality does not provide the optimal comparison at points that are far from the infimum and the supremum.

In 1986, Peter Li and Shing Tung Yau [22] found a completely new Harnack type result, namely a pointwise gradient estimate (called a *differential* Harnack inequality) that can be integrated along a path to find a classical Harnack inequality of the form (6). They proved that on a manifold with $\mathrm{Ric} \geq 0$ and convex boundary, the differential Harnack expression

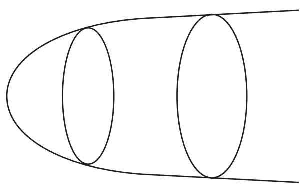
$$H(u, t) := \frac{\partial_t u}{u} - \frac{|\nabla u|^2}{u^2} + \frac{n}{2t} \tag{7}$$

is non-negative for any positive solution u of the linear heat equation. To prove this differential Harnack inequality one uses the maximum principle for parabolic equations. With his maximum principle for systems mentioned above, Hamilton [17] was able to prove a matrix version of the Li–Yau result. On the one hand Hamilton's result is more general since his matrix inequality contains the Li–Yau inequality as its trace, on the other hand the matrix inequality demands stronger curvature assumptions than Li–Yau's trace version. We will see in chapter two that the Li–Yau Harnack expression $H(u,t)$ as well as Hamilton's matrix version vanish identically on the heat kernel (which is the expanding self-similar solution of the heat equation which tends to a δ-function as $t \to 0$).

For the Ricci flow, the self-similar solutions are called Ricci solitons. Concretely, a Ricci soliton is a solution metric of the Ricci flow which moves only by diffeomorphisms and scaling, i.e. a solution $g(t) = a(t) \cdot \Phi_t^*(g(0))$ which is the pull-back of the initial metric $g(0)$ by a one-parameter family of diffeomorphisms $\Phi_t \colon M \to M$ multiplied with a scaling function $a(t)$. An easy example for a *shrinking* (i.e. $a(t)$ decreasing) soliton is the sphere S^n discussed above. If the scaling function $a(t)$ equals 1 for all time, then the soliton is called *steady*. The simplest nontrivial example here is the so-called *cigar soliton*, which is defined to be $\mathbb{R}^2$ equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2}. \tag{8}$$

It is called a cigar because it is asymptotic to a cylinder at infinity, has maximal Gauss curvature at the origin and burns away.



The cigar

If the diffeomorphisms in the definition of a soliton are generated by the gradient of some scalar function f on M – the so-called soliton potential – we call $g(t)$ a *gradient* soliton.

Hamilton's idea was to find a nonlinear analog to the Li–Yau Harnack inequality involving positive curvature in place of u. Motivated by the fact that Li Yau's Harnack expression and his own matrix version are zero on expanding self-similar solutions as mentioned above, he looked for curvature expressions which vanish on expanding Ricci solitons and then searched for a linear combination of these expressions and derivatives of it. With this idea he found a matrix and a trace Harnack inequality for the Ricci flow, cf. [18]. We will present them at the end of chapter two.

Using his Harnack inequalities, Hamilton [19] was able to prove the following classification of blow-ups of three-dimensional singularities – modulo the control of the injectivity radius.

Proposition 2 (Hamilton, Theorem 26.5 of [19]). *Let $(M, g(t))$ be a solution to the Ricci flow on a compact three-manifold where a singularity develops in finite time T. Then either the injectivity radius times the square root of the maximum curvature goes to zero, or else there exists a sequence of dilations of the solution which converges to a quotient by isometries of either S^3, $S^2 \times \mathbb{R}$ or $\Sigma \times \mathbb{R}$, where Σ is the cigar soliton.*

Notice that all three possible limits in the proposition are gradient solitons, the first two being shrinking and the last one being steady. A key question for Hamilton's program is: How can one prove that all singularities are modelled by self-similar solutions? This question was answered by Grisha Perelman [28], who introduced a new differential Harnack inequality for the Ricci flow, described below. In fact, Hamilton's and Perelman's Harnack estimates and gradient Ricci solitons are important for the study of the flow in arbitrary dimensions, not only in dimension three.

Perelman's gradient flow approach

Recall that it was Hamilton's idea to take linear combinations of expressions that vanish on soliton solutions in order to find interesting estimates. Therefore, we list various vanishing expressions of a soliton potential f for steady, shrinking and expanding gradient solitons in chapter one. Combining some of them, one finds the remarkable result that in the steady soliton case the function e^{-f} satisfies the adjoint heat equation on a manifold evolving by Ricci flow. Similar results hold for expanding and shrinking solitons. This connection between Ricci solitons and the adjoint heat equation is the starting point for a completely new approach to the subject, introduced by Perelman [28] in 2002.

Perelman presented a new functional, which may be regarded as an improved version of the Einstein–Hilbert functional (3), namely

$$\mathcal{F}(g, f) := \int_M \big(R + |\nabla f|^2\big) e^{-f} \, dV, \tag{9}$$

which has the gradient flow system

$$\left.\begin{aligned} \partial_t g_{ij} &= -2(R_{ij} + \nabla_i \nabla_j f), \\ \partial_t f &= -\triangle f - R. \end{aligned}\right\} \tag{10}$$

After a pull-back by the family of diffeomorphisms generated by ∇f, this system becomes

$$\left.\begin{aligned} \partial_t g_{ij} &= -2R_{ij} && \text{(Ricci flow)}, \\ \square^* e^{-f} &= 0 && \text{(adjoint heat equation)}, \end{aligned}\right\} \tag{11}$$

where $\square^* := -\partial_t - \triangle + R$ denotes the adjoint heat operator under the Ricci flow. So in this sense the Ricci flow may be regarded as a gradient flow up to a modification with a family of diffeomorphisms. Note that if we interpret the Ricci flow as a dynamical system on the space of Riemannian metrics modulo diffeomorphisms and scaling, the soliton solutions correspond to fixed points in this space.

In chapter three we will show that the entropy functional $\mathcal{F}(g, f)$ is non-decreasing along the flow and constant exactly on steady Ricci solitons with potential f. Entropy functionals similar to (9) have been found for shrinking solitons (by Perelman [28]) and later for expanding solitons (by Feldman, Ilmanen and Lei Ni [12]) as well. We will introduce them all in the third chapter. We will also explain the corresponding entropy functionals for the heat equation on a static manifold. This is taken from Lei Ni's papers [25] and [26].

In Perelman's paper [28] no clear motivation for the various entropy functionals can be found, so at first read it is almost impossible to understand where these functionals come from, even if one already knows the connection between solitons and the adjoint heat equation mentioned above. But using Lei Ni's result (for the heat equation on a fixed manifold), we will see that the entropy functionals are quite natural and are strongly connected to the Nash entropy and, surprisingly, also to the Li–Yau Harnack inequality. This relationship will be our main interest in chapter three.

In this chapter, we also show how one finds another differential Harnack inequality for the heat kernel via a local version of Ni's entropy formula (cf. [25]). In the Ricci flow case, this corresponds to Perelman's Harnack inequality for the adjoint heat kernel on a manifold evolving by Ricci flow (cf. [28], Section 9), which one gets via a local version of the shrinking soliton entropy. Note that while the classical Li–Yau result presented in chapter two holds for the linear heat equation on a static manifold and Hamilton's Harnack inequalities are results for the curvature under Ricci flow (i.e. a nonlinear heat equation on the space of Riemannian metrics), Perelman's Harnack inequality involves both heat equations together: it holds for a solution of the (adjoint) heat equation on a manifold that also evolves by a heat equation.

In the same way as one can integrate the Li–Yau Harnack estimate along a path to get a classical Harnack inequality, one can also integrate Perelman's Harnack inequality and the new Harnack inequality that corresponds to Lei Ni's entropy formula. One gets a lower bound for the adjoint heat kernel under Ricci flow in the Perelman case or for the heat kernel on a static manifold in the other case. This will lead us to Perelman's $\mathcal{L}$-length functional (cf. [28], Section 7), a functional defined on the space of all space-time paths. We will discuss this functional, the corresponding space-time geodesics and exponential map in chapter four. We will also introduce and analyze the corresponding "length" functional for the static case. The new results that we will get there follow as a natural continuation of Lei Ni's results. On flat $\mathbb{R}^n$ the two cases coincide modulo the necessary sign changes.

One discovers an interesting fact there: If one computes the first and second variation of Perelman's $\mathcal{L}$-functional with respect to variations of the space-time path one is led to Hamilton's matrix and trace Harnack expressions for the Ricci flow. So these two (at first glance completely different) Harnack expressions come together in this chapter on the $\mathcal{L}$-functional in a very natural way. Surprisingly, to draw conclusions, Hamilton's Harnack inequalities for the Ricci flow do not have to be satisfied, since the Harnack terms appear during the computations but cancel out in the final result! If one does the analogous computation for the static case, the Ricci curvature appears where Hamilton's Harnack expressions appeared in the Ricci flow case. However, the curvature terms do not cancel out and hence the conclusions one wants to make on the corresponding $\mathcal{L}$-length (as well as the monotonicity of Ni's entropy functionals mentioned above or the positivity of Li–Yau's Harnack expression) only hold if the Ricci curvature is non-negative. This shows that the results for the Ricci flow case – even so they are harder to compute – are in fact more natural since they hold without any a-priori curvature assumptions converse to the static case.

Here is a heuristic explanation why the results for the linear heat equation hold exactly in the case of $\mathrm{Ric} \geq 0$: In this case the constant metric $\tilde{g}(t) \equiv g(0)$ is a supersolution to the solution of the Ricci flow $\partial_t g = -2\,\mathrm{Ric} \leq 0$. Moreover a positive solution to the backwards heat equation $\partial_t u = -\triangle u$ is a subsolution to the adjoint heat equation under the Ricci flow $\partial_t u = -\triangle u + Ru$ (since $R \geq 0$ if $\mathrm{Ric} \geq 0$). Putting this together, the inequalities which hold for the Ricci flow and the adjoint heat equation will also hold for the backwards heat equation on a manifold with fixed metric. Changing the time direction will then lead to the results for the (forward) heat equation. With this explanation in mind, there are two possible orders of presentation. On the one hand, one can start with the static case and interpret the results in the Ricci flow case as more complicated analogs. This is how historically the Harnack inequalities were found. Since the computations in the case of a fixed metric are usually much easier, we have chosen this order to present the results. Alternatively one could also see the Ricci flow combined with the adjoint heat equation as the geometrically natural case and – in contrast to our order of presentation – interpret the formulas for the static case as a conversion of the corresponding formulas for the Ricci flow to super- (and sub-)solution inequalities in the special case of non-negative Ricci curvature.

For this book, we have condensed some of the already existing proofs or carried them out in more detail to make the topic more accessible to non-experts. However, it is always a good idea to look at the original sources, especially since some propositions are not presented in their most general version here, which would only distract from the main geometric ideas. In particular, we recommend Hamilton's original sources (especially [13], [17] and [18]) and Lei Ni's papers [25] and [26] on Perelman's ideas.

Not to get sidetracked too much, we only present a *compact core* of the subject without many applications. In particular, we will not discuss any of the applications to three-manifolds towards the Poincaré conjecture or Thurston's geometrization conjecture at all, but only present results which are valid for arbitrary dimensions. In addition there are various applications to general dimensions (such as non-collapsing, no breathers, ε-regularity, etc.) which we skip. Most of them can be found in Perelman's paper [28].

Outline of the book

The book is divided into four main chapters. In chapter one we explain general variations of Riemannian metrics and introduce the Ricci flow as the (weakly) parabolic part of the L^2-gradient flow of the Einstein–Hilbert functional. We then prove that there exists no functional which has the Ricci flow as its gradient flow (Proposition 1.7). After computing various evolution equations for the Ricci flow, we introduce gradient Ricci solitons and derive equations for the three cases of steady, shrinking and expanding solitons (Propositions 1.14, 1.15 and 1.16). A reader who is familiar with Hamilton's papers will already know most of the results in this chapter.

Chapter two is devoted to the study of differential Harnack inequalities. We start with the famous Li–Yau Harnack estimate, which we present in its original form (Proposition 2.5) as well as in a quadratic and an integrated version (Corollaries 2.6 and 2.7). We then explain the maximum principle for systems to prove Hamilton's matrix Harnack inequality for the heat equation (Proposition 2.11). Finally, we proceed with Hamilton's Harnack inequalities for the Ricci flow (Theorem 2.14), where we only give a heuristic motivation instead of rewriting the rather lengthy proof.

In the third chapter we present Lei Ni's entropy formulas for the heat equation on a static manifold. The main result (Theorem 3.8) will relate these entropy formulas with the Nash entropy and the Li–Yau Harnack inequality. Moreover, a local version of an entropy formula for the positive heat kernel (Proposition 3.6) will lead us to a new Li–Yau type differential Harnack inequality, which gives a lower bound for the heat kernel when being integrated (Corollary 3.10). We also discuss the entropy formulas for steady, shrinking and expanding solitons, the most important being the shrinking case in which we will again integrate a Harnack type inequality (Proposition 3.15) to find a lower bound for the adjoint heat kernel under Ricci flow (Corollary 3.16), as in Section 9 of Perelman's paper [28]. The quantities $\ell(q, \bar{\tau})$ and $\ell(q, T)$ which one finds in these two corollaries 3.16 and 3.10, respectively, will turn out to be Perelman's backwards reduced distance and its analog for the heat kernel on a static manifold.

This motivates our investigation of the two corresponding distance-functionals in chapter four. We first examine the static case, where the computations are much easier. This will lead to Proposition 4.2, a new result for the heat kernel on a manifold, which coincides with Perelman's result (Theorem 4.9) in the case of a Ricci flat manifold. The second part of chapter four provides a detailed exposition of Perelman's $\mathcal{L}$-length, $\mathcal{L}$-geodesics and the $\mathcal{L}$-exponential map. We finish with the monotonicity of Perelman's backwards reduced volume (Corollary 4.13).

Development of the book and acknowledgements. My first contact with this subject was in the winter semester 2003/2004, when I was writing a term paper in analysis under the guidance of Michael Struwe. The aim was to analyze the very first section of Perelman's paper from 2002, where Perelman introduces his famous steady soliton entropy functional.

In the following semester, Tom Ilmanen held an introductory lecture course on the Ricci flow at ETH Zürich that was inspired by Perelman's work and Ilmanen's discussions with Richard Hamilton, Mu-Tao Wang and Lei Ni. There were also invited guest speakers: Richard Hamilton, Dan Knopf and Natasa Sesum. This lecture course intensified my interests in the subject of geometric flows, entropy formulas and Harnack inequalities and I would like to thank Tom Ilmanen as well as the above mentioned guest speakers for it. Much of the first chapter as well as some parts of the third chapter are inspired by this lecture.

This book is a revised and extended version of my diploma thesis written in the winter semester 2004/2005 at ETH Zürich, again under the guidance of Michael Struwe. I wish to express my gratitude to Michael Struwe for many interesting and helpful discussions, comments and suggestions during this semester. I would also like to thank Daniel Perez for proof-reading parts of my diploma thesis.

In the winter term 2005/2006, I organized a student seminar on geometric heat flows together with Tom Ilmanen, where my thesis was used as a source which was often easier to understand for the students than the original papers. So we believe that this book can not only be used for an introductory lecture or seminar on this topic, but is also suitable for self-study.