Prologue and Summary

The equations describing the motion of a perfect fluid were first formulated by Euler in 1752 (see [Eu1], [Eu2]), based, in part, on the earlier work of D. Bernoulli [Be]. These equations were among the first partial differential equations to be written down, preceded, it seems, only by D'Alembert's 1749 formulation [DA] of the one-dimensional wave equation describing the motion of a vibrating string in the linear approximation. In contrast to D'Alembert's equation however, we are still, after the lapse of two and a half centuries, far from having achieved an adequate understanding of the observed phenomena which are supposed to lie within the domain of validity of the Euler equations.

The phenomena displayed in the interior of a fluid fall into two broad classes, the phenomena of sound, the linear theory of which is acoustics, and the phenomena of vortex motion. The sound phenomena depend on the compressibility of a fluid, while the vortex phenomena occur even in a regime where fluid may be considered to be incompressible. The formation of shocks, the subject of the present monograph, belongs to the class of sound phenomena, but lies in the nonlinear regime, beyond the range covered by linear acoustics. The phenomena of vortex motion include the chaotic form called turbulence, the understanding of which is one of the great challenges of science.

Let us make a short review of the history of the study of the phenomena of sound in fluids, in particular the phenomena of the formation and evolution of shocks in the non-linear regime. At the time when the equations of fluid mechanics were first formulated, thermodynamics was in its infancy, however it was already clear that the local state of a fluid as seen by a comoving observer is determined by two thermodynamic variables, say pressure and temperature. Of these, only pressure entered the equations of motion, while the equations involve also the density of the fluid. Density was already known to be a function of pressure and temperature for a given type of fluid. However in the absence of an additional equation, the system of equations at the time of Euler, which consisted of the momentum equations together with the equation of continuity, was underdetermined, except in the incompressible limit. The additional equation was supplied by Laplace in 1816 [La] in the form of what was later to be called the adiabatic condition, and allowed him to make the first correct calculation of the speed of sound.

The first work on the formation of shocks was done by Riemann in 1858 [Ri]. Riemann considered the case of isentropic flow with plane symmetry, where the equations of fluid mechanics reduce to a system of conservation laws for two unknowns and with two independent variables, a single space coordinate and time. He introduced for such

systems the so-called Riemann invariants, and with the help of these showed that solutions which arise from smooth initial conditions develop infinite gradients in finite time. Riemann also realized that the solutions can be continued further as discontinuous solutions, but here there was a problem. Up to this time the energy equation was considered to be simply a consequence of the laws of motion, not a fundamental law in its own right. On the other hand, the adiabatic condition was considered by Riemann to be part of the main framework. Now as long as the solutions remain smooth it does not matter which of the two equations we take to be the fundamental law, for each is a consequence of the other, modulo the remaining laws. However this is no longer the case once discontinuities develop, so one must make a choice as to which of the two equations to regard as fundamental and therefore remains valid thereafter.

Here Riemann made the wrong choice. For, only during the previous decade, in 1847, had the first law of thermodynamics been formulated by Helmholtz [He], based in part on the experimental work of Joule on the mechanical equivalence of heat, and the general validity of the energy principle had thereby been shown.

In 1865 the concept of entropy was introduced into theoretical physics by Clausius [Cl2], and the adiabatic condition was understood to be the requirement that the entropy of each fluid element remains constant during its evolution. The second law of thermodynamics, involving the increase of entropy in irreversible processes, had first been formulated in 1850 by Clausius [Cl1] without explicit reference to the entropy concept. After these developments the right choice in Riemann's dilemma became clear. The energy equation must remain at all times a fundamental law, but the entropy of a fluid element must jump upward when the element crosses a hypersurface of discontinuity. The formulation of the correct jump conditions that must be satisfied by the thermodynamic variables and the fluid velocity across a hypersurface of discontinuity was begun by Rankine in 1870 [Ra] and completed by Hugoniot in 1889 [Hu].

With Einstein's discovery of the special theory of relativity in 1905 [Ei], and its final formulation by Minkowski in 1908 [Mi] through the introduction of the concept of spacetime with its geometry, the domain of geometry being thereby extended to include time, a unity was revealed in physical concepts which had been hidden up to this point. In particular, the concepts of energy density, momentum density or energy flux, and stress, were unified into the concept of the energy-momentum-stress tensor and energy and momentum were likewise unified into a single concept, the energy-momentum vector. Thus, when the Euler equations were extended to become compatible with special relativity, it was obvious from the start that it made no sense to consider the momentum equations without considering also the energy equation, for these two were parts of a single tensorial law, the energy-momentum conservation law. This law together with the particle conservation law (the equation of continuity of the non-relativistic theory), constitute the laws of motion of a perfect fluid in the relativistic theory. The adiabatic condition is then a consequence for smooth solutions.

A new basic physical insight on the shock development problem was reached first, it seems, by Landau in 1944 [Ln]. This was the discovery that the condition that the entropy jump be positive as a hypersurface of discontinuity is traversed from the past to the future, should be equivalent to the condition that the flow is evolutionary, that is, that conditions

in the past determine the fluid state in the future. More precisely, what was shown by Landau was that the condition of determinism is equivalent, at the linearized level, to the condition that the tangent hyperplane at a point on the hypersurface of discontinuity, is on one hand contained in the exterior of the sound cone at this point corresponding to the state before the discontinuity, while on the other hand intersects the sound cone at the same point corresponding to the state after the discontinuity, and that this latter condition is equivalent to the positivity of the entropy jump.

This is interesting from a general philosophical point of view, because it shows that irreversibility can arise, even though the laws are all time-reversible, once the solution ceases to be regular. To a given state at a given time there always corresponds a unique state at any given later time. If the evolution is regular in the associated time interval, then the reverse is also true: to a given state at a later time there corresponds a unique state at any given earlier time, the laws being time-reversible. This reverse statement is however false if there is a shock during the time interval in question. Thus determinism in the presence of hypersurfaces of discontinuity selects a direction of time and the requirement of determinism coincides, modulo the other laws, with what is dictated by the second law of thermodynamics which is in its nature irreversible. This recalls the interpretation of entropy, first discovered by Boltzmann in 1877 [Bo], as a measure of disorder at the microscopic level. An increase of entropy was thus understood to be associated to an increase in disorder or to loss of information, and determinism can only be expected in the time direction in which information is lost, not gained.

An important mathematical development with direct application to the equations of fluid mechanics in the physical case of three space dimensions, was the introduction by Friedrichs of the concept of a symmetric hyperbolic system in 1954 [F] and his development of the theory of such systems. It is through this theory that the local existence and domain of dependence property of solutions of the initial value problem associated to the equations of fluid mechanics are established. Another development in connection to this was the general investigation by Friedrichs and Lax in 1971 [F-L] (see also [Lx1]) of nonlinear first order systems of conservation laws which for smooth solutions have as a consequence an additional conservation law. This is the case for the system of conservation laws of fluid mechanics, which consists of the particle and energy-momentum conservation laws, which for smooth solutions imply the conservation law associated to the entropy current. It was then shown that if the additional conserved quantity is a convex function of the original quantities, the original system can be put into symmetric hyperbolic form. Moreover, for discontinuous solutions satisfying the jump conditions implied by the integral form of the original conservation laws, an inequality for the generalized entropy was derived.

The problem of shock formation for the equations of fluid mechanics in one space dimension, and more generally for systems of conservation laws in one space dimension, was studied by Lax in 1964 [Lx2], and 1973 [Lx3], and John [J1] in 1974. The approach of these works was analytic, the strategy being to deduce an ordinary differential inequality for a quantity constructed from the first derivatives of the solution, which showed that this quantity must blow up in finite time, under a certain structural assumption on the system called genuine nonlinearity and suitable conditions on the initial data. The gen-

uine nonlinearity assumption is in particular satisfied by the non-relativistic compressible Euler equations in one space dimension provided that the pressure is a strictly convex function of the specific volume.

A more geometric approach in the case of systems with two unknowns was developed by Majda in 1984 [Ma1] based in part on ideas introduced by Keller and Ting in 1966 [K-T]. In this approach, which is closer in spirit to the present monograph, one considers the evolution of the inverse density of the characteristic curves of each family and shows that under appropriate conditions this inverse density must somewhere vanish within finite time. In this way, not only were the earlier blow-up results reproduced, but, more importantly, insight was gained into the nature of the breakdown. Moreover Majda's approach also covered the case where the genuine nonlinearity assumption does not hold, but we have linear degeneracy instead. He showed that in this case, global-in-time smooth solutions exist for any smooth initial data.

The problem of the global-in-time existence of solutions of the equations of fluid mechanics in one space dimension was treated by Glimm in 1965 [Gl] through an approximation scheme involving at each step the local solution of an initial value problem with piecewise constant initial data. The convergence of the approximation scheme then produced a solution in the class of functions of bounded variation. Now, by the previously established results on shock formation, a class of functions in which global existence holds must necessarily include functions with discontinuities, and the class of functions of bounded variation is the simplest class having this property. Thus, the treatment based on the total variation, the norm in this function space, in itself an admirable investigation, would be insuperable if the development of the one-dimensional theory was the goal of scientific effort in the field of fluid mechanics. However that goal can only be the mathematical description of phenomena in real three-dimensional space and one must ultimately face the fact that methods based on the total variation do not generalize to more than one space dimension. In fact it is clear from the study of the linearized theory, acoustics, which involves the wave equation, that in more than one space dimension only methods based on the energy concept are appropriate.

The first and thus far the only general result on the formation of shocks in three-dimensional fluids was obtained by Sideris in 1985 [S]. Sideris considered the compressible Euler equations in the case of a classical ideal gas with adiabatic index $\gamma > 1$ and with initial data which coincide with those of a constant state outside a ball. The assumptions of his theorem on the initial data were that there is an annular region bounded by the sphere outside which the constant state holds, and a concentric sphere in its interior, such that a certain integral in this annular region of $\rho - \rho_0$, the departure of the density ρ from its value ρ_0 in the constant state, is positive, while another integral in the same region of ρv^r, the radial momentum density, is non-negative. These integrals involve kernels which are functions of the distance from the center. It is also assumed that everywhere in the annular region the specific entropy s is not less than its value s_0 in the constant state. The conclusion of the theorem is that the maximal time interval of existence of a smooth solution is finite. The chief drawback of this theorem is that it tells us nothing about the nature of the breakdown. Also the method relies on the strict convexity of the pressure as

a function of the density displayed by the equation of state of an ideal gas, and does not extend to more general equations of state.

The other important work on shocks in three space dimensions was the 1983 work of Majda [Ma2], [Ma3], on what he calls the shock front problem. In this problem we are given initial data in $\Re^3$ which is smooth in the closure of each component of $\Re^3 \setminus S$, where S is a smooth surface in $\Re^3$. The data is to satisfy the condition that there exists a function σ on S such that the jumps of the data across S satisfy the Rankine–Hugoniot jump conditions as well as the entropy condition with σ in the role of the shock speed. The higher order compatibility conditions associated to an initial boundary value problem are also required to be satisfied. We are then required to find a time interval $[0, \tau]$, a smooth hypersurface K in the spacetime slab $[0, \tau] \times \Re^3$ and a solution of the compressible Euler equations which is smooth in the closure of each component of $[0, \tau] \times \Re^3 \setminus K$, and satisfies across K the Rankine–Hugoniot jump conditions as well as the entropy condition. We may think of this problem as the local-in-time shock continuation problem. Majda solved this problem under an additional condition on the initial data which seems to be necessary for the stability of the linearized problem. The additional condition follows from the other conditions in the case of a classical ideal gas, but it does not follow for a general equation of state.

The present monograph considers the relativistic Euler equations in three space dimensions for a perfect fluid with an arbitrary equation of state. We consider regular initial data on a space-like hyperplane Σ_0 in Minkowski spacetime which outside a sphere coincide with the data corresponding to a constant state. We consider the restriction of the initial data to the exterior of a concentric sphere in Σ_0 and we consider the maximal classical development of this data. Then, under a suitable restriction on the size of the departure of the initial data from those of the constant state, we prove certain theorems which give a complete description of the maximal classical development, which we call maximal solution. In particular, the theorems give a detailed description of the geometry of the boundary of the domain of the maximal solution and a detailed analysis of the behavior of the solution at this boundary. A complete picture of shock formation in three-dimensional fluids is thereby obtained. Also, sharp sufficient conditions on the initial data for the formation of a shock in the evolution are established and sharp lower and upper bounds for the temporal extent of the domain of the maximal solution are derived.

The reason why we consider only the maximal development of the restriction of the initial data to the exterior of a sphere is in order to avoid having to treat the long time evolution of the portion of the fluid which is initially contained in the interior of this sphere. For, we have no method at present to control the long time behavior of the pointwise magnitude of the vorticity of a fluid portion, the vorticity satisfying a transport equation along the fluid flow lines. Our approach to the general problem is the following. We show that given arbitrary regular initial data which coincide with the data of a constant state outside a sphere, if the size of the initial departure from the constant state is suitably small, we can control the solution for a time interval of order $1/\eta_0$, where η_0 is the sound speed in the surrounding constant state. We then show that at the end of this interval a thick annular region has formed, bounded by concentric spheres, where the flow is irrotational and isentropic, the constant state holding outside the outer sphere. We then

study the maximal classical development of the restriction of the data at this time to the exterior of the inner sphere. In the irrotational isentropic case there is a function ϕ which we call a wave function, the differential of which at a point determines the state of the fluid at that point, and the fluid equations reduce to a nonlinear wave equation for ϕ, as is shown in Chapter 1.

The order of presentation in this monograph is however the reverse of that just outlined. After the first four chapters which set up the general framework, we confine attention to the irrotational isentropic problem up to Chapter 13, where the main theorem, Theorem 13.1, is proved. We return to the general problem in Chapter 14, after establishing a theorem, Theorem 14.1, which, in the irrotational isentropic context, gives sharp sufficient conditions on the initial data for the formation of a shock in the evolution. It is at this point where our treatment of the general problem resumes, and we analyze the solution of the general problem during the initial time interval. In fact, our analysis allows us to find which conditions on the data at the beginning of the time interval result in data at the end of the time interval verifying the assumptions of Theorem 14.1. In this way we are able to establish a theorem, Theorem 14.2, which, in the general context of fluid mechanics, gives sharp sufficient conditions on the initial data for the formation of a shock in the evolution. We should emphasize at this point that if we were to restrict ourselves from the beginning to the irrotational isentropic case, we would have no problem extending the treatment to the interior region, thereby treating the maximal solution corresponding to the data on the complete initial hyperplane Σ_0. In fact, it is well known that sound waves decay in time faster in the interior region and our constructions can readily be extended to cover this region. It is only our present inability to achieve long time control of the magnitude of the vorticity along the flow lines of the fluid, that prevents us from treating the interior region in the general case.

The geometry of the boundary of the domain of the maximal solution is studied in Chapter 15, the main results being expressed by Theorem 15.1 and Propositions 15.1, 15.2, and 15.3. The boundary consists of a regular part and a singular part. Each component of the regular part $\underline{C}$ is an incoming characteristic hypersurface with a singular past boundary. The singular part of the boundary of the domain of the maximal solution is the locus of points where the inverse density of the wave fronts vanishes. It is the union $\partial_- H \bigcup H$, where each component of $\partial_- H$ is a smooth embedded surface in Minkowski spacetime, the tangent plane to which at each point is contained in the exterior of the sound cone at that point.

On the other hand each component of H is a smooth embedded hypersurface in Minkowski spacetime, the tangent hyperplane to which at each point is contained in the exterior of the sound cone at that point, with the exception of a single generator of the sound cone, which lies on the hyperplane itself. The past boundary of a component of H is the corresponding component of $\partial_- H$. The latter is at the same time the past boundary of a component of $\underline{C}$. As is explained in the Epilogue, the maximal classical solution is the physical solution of the problem up to $\underline{C} \bigcup \partial_- H$, but not up to H. The problem of the physical continuation of the solution is set up in the Epilogue as the shock development problem. This problem is associated to each component of $\partial_- H$ and its solution requires the construction of a hypersurface of discontinuity K, lying in the past of the cor-

responding component of H, but having the same past boundary as the latter, namely the given component of $\partial_- H$. Thus, although the notion of maximal classical solution is not physically appropriate up to H, it does provide the basis for constructing the physical continuation, the solution of the shock development problem, by providing not only the right boundary conditions at $\underline{C} \bigcup \partial_- H$, but also a barrier at H which is indispensable for controlling the physical continuation. The actual treatment of the shock development problem and the subsequent shock interactions shall be the subject of a subsequent monograph.

The present monograph concludes with a derivation of a formula for the jump in vorticity across K, which shows that while the flow is irrotational ahead of the shock, it acquires vorticity immediately behind, which is tangential to the shock front and is associated to the gradient along the shock front of the entropy jump.

We have chosen to work in this monograph with the relativistic Euler equations rather than confining ourselves to their non-relativistic limit, for three reasons. The first is the obvious reason that there is a class of natural phenomena, those of relativistic astrophysics, which lie beyond the domain of the non-relativistic equations. The second reason is that there is a substantial gain in geometric insight in considering the relativistic equations. At a fundamental level, the picture looks simpler from the relativistic perspective, because of the aforementioned unity of physical concepts brought about by the spacetime geometry viewpoint of special relativity. As an example we give the equation (1.51) of Chapter 1:

$$i_u \omega = -\theta ds. \tag{1}$$

Here ω is the vorticity 2-form. According to the definitions of Chapter 1,

$$\omega = d\beta \tag{2}$$

where β is the 1-form defined, relative to an arbitrary system of coordinates, by:

$$\beta_\mu = -\sqrt{\sigma} u_\mu, \qquad u_\mu = g_{\mu\nu} u^\nu, \tag{3}$$

$\sqrt{\sigma}$ being the relativistic enthalpy per particle, u^μ the fluid velocity and $g_{\mu\nu}$ the Minkowski metric. In (1), θ is the temperature and s the entropy per particle, while i_u denotes contraction on the left by the vectorfield u. Equation (1) is arguably the simplest explicit form of the energy-momentum equations. Our derivation in the Epilogue of the jump in vorticity behind a shock relies on this equation. The 1-form β plays a fundamental role in this monograph. In the irrotational isentropic case it is given by $\beta = d\phi$, where ϕ is the wave function.

The third reason why we have chosen to work with the relativistic equations is that no special care is needed to extract information on the non-relativistic limit. This is due to the fact that the non-relativistic limit is a regular limit, obtained by letting the speed of light in conventional units tend to infinity, while keeping the sound speed fixed. To allow the results in the non-relativistic limit to be extracted from our treatment in a straightforward manner, we have chosen to avoid summing quantities having different physical dimensions when such sums would make sense only when a unit of velocity has been chosen, even though we follow the natural choice within the framework of special

relativity of setting speed of light equal to unity in writing down the relativistic equations of motion. We shall presently give an example to illustrate what we mean. Consider the vectorfield K_0 defined by equation (5.15) of Chapter 5:

$$K_0 = (\eta_0^{-1} + \alpha^{-1}\kappa)L + \underline{L}, \qquad \underline{L} = \alpha^{-1}\kappa L + 2T. \tag{4}$$

The terms here have not yet been defined, but the reader may return to this example after assimilating the appropriate definitions. In any case, η_0 is, as mentioned above, the sound speed in the surrounding constant state. The function α is the inverse density of the hyperplanes Σ_t corresponding to the constant values of the time coordinate t, with respect to the acoustical metric $h_{\mu\nu}$:

$$h_{\mu\nu} = g_{\mu\nu} + (1 - \eta^2)u_\mu u_\nu, \qquad u_\mu = g_{\mu\nu}u^\nu, \tag{5}$$

$g_{\mu\nu}$ being again the Minkowski metric, η the sound speed, and u^μ the fluid velocity. This is a Lorentzian metric on spacetime, the null cones of which are the sound cones. The function α has the physical dimension of velocity. The function κ is the inverse spatial density of the wave fronts with respect to the acoustical metric, a dimensionless quantity. Thus in the sum $\eta_0^{-1} + \alpha^{-1}\kappa$, which is the coefficient of L in the first term of (4), each term has the physical dimension of inverse velocity. The vectorfield L is the tangent vectorfield of the bicharacteristic generators, parametrized by t, of a family of outgoing characteristic hypersurfaces C_u, the level sets of an acoustical function u. The wave fronts $S_{t,u}$ are the surfaces of intersection $C_u \bigcap \Sigma_t$. The physical dimension of L is inverse time. Thus the first term in (4) has the physical dimension of inverse length. The vectorfield T defines a flow on each of the Σ_t, taking each wave front onto another wave front, the normal, relative to the induced acoustical metric $\overline{h}$, the flow of the foliation of Σ_t by the surfaces $S_{t,u}$. It has the physical dimension of an inverse length. The first term in the second part of (4) also has the same physical dimension, hence the physical dimension of the vectorfield $\underline{L}$ is inverse length as well. We conclude that each term in (4) has the physical dimension of inverse length, thus the physical dimension of K_0 is inverse length.

Denoting, as above, by σ the square of the relativistic enthalpy per particle, we have:

$$\sqrt{\sigma} = e + pv \tag{6}$$

where e is the relativistic energy per particle, p the pressure and v is the volume per particle. Let H be the function defined by:

$$1 - \eta^2 = \sigma H. \tag{7}$$

The derivative of H with respect to σ at constant s plays a central role in shock theory . This quantity is expressed by (see equation (E.47) in the Epilogue):

$$\left(\frac{dH}{d\sigma}\right)_s = -a \left\{ \left(\frac{d^2v}{dp^2}\right)_s + \frac{3v}{\sqrt{\sigma}}\left(\frac{dv}{dp}\right)_s \right\} \tag{8}$$

where a is the positive function:

$$a = \frac{\eta^4}{2\sigma v^3}.$$

The sign of $(dH/d\sigma)_s$ in the state ahead of a shock determines, as is shown in the Epilogue, the sign of the jump in pressure in crossing the shock to the state behind. The jump in pressure is positive if this quantity is negative, the reverse otherwise. The value of $(dH/d\sigma)_s$ in the surrounding constant state is denoted by ℓ in this monograph. This constant determines the character of the shocks for small initial departures from the constant state. In particular when $\ell = 0$, no shocks form and the domain of the maximal classical solution is complete. Consider the function $(dH/d\sigma)_s$ as a function of the thermodynamic variables p and s. Suppose that we have an equation of state such that at some value s_0 of s we have $(dH/d\sigma)_{s=s_0} = 0$, that is, the function $(dH/d\sigma)_s$ vanishes everywhere along the adiabat $s = s_0$. We show in Chapter 1 that in this case the irrotational isentropic fluid equations corresponding to the value s_0 of the entropy are equivalent to the minimal surface equation, the wave function ϕ defining a minimal graph in a Minkowski spacetime of one more spatial dimension. Thus the minimal surface equation defines a dividing line between two different types of shock behavior.

Now, the relativistic enthalpy is dominated by the term mc^2, the contribution of the particle rest mass m, to the energy per particle, c being the speed of light. We note here that the particle rest mass may be taken to be unity, so that all quantities per particle are quantities per unit rest mass. Thus in the non-relativistic limit the second term in parenthesis in (8) vanishes and the expression in parenthesis reduces simply to: $(d^2v/dp^2)_s$.

Now, the case where $(d^2v/dp^2)_s > 0$, the adiabats being convex curves in the p, v plane, so that $(dH/d\sigma)_s < 0$, is the more commonly found in nature, however the reverse case does occur in the gaseous region near the critical point in the liquid-to-vapor phase transition and in similar transitions at higher temperatures associated to molecular dissociation and to ionization (see [Z-R]).

One of the basic concepts on which our approach relies is the general concept of variation, or variation through solutions, on which our treatment not only of the irrotational isentropic case but also of the general equations of motion is based. This concept has been discussed in the general context of Euler–Lagrange equations, that is, systems of partial differential equations arising from an action principle, in the monograph [Ch]. It was shown there that to a variation is associated a linearized Lagrangian, and it was also shown how energy currents are in general constructed on the basis of this linearized Lagrangian. It is through energy currents and their associated integral identities that the estimates essential to our approach are derived. Here the first order variations correspond to the one-parameter subgroups of the Poincaré group, the isometry group of Minkowski spacetime, extended by the one-parameter scaling or dilation group, which leave the surrounding constant state invariant. The higher order variations correspond to the one-parameter groups of diffeomorphisms generated by a set of vectorfields, the commutation fields, to be discussed below. The construction in [Ch] of an energy current requires a multiplier vectorfield which at each point belongs to the closure of the positive component of the inner characteristic core in the tangent space at that point. In the irrotational isentropic case the characteristic in the tangent space at a point consists only of the sound cone at that point and this requirement becomes the requirement that the multiplier vectorfield be non-space-like and future-directed with respect to the acoustical metric (5). We use two multiplier vectorfields in our analysis of the isentropic irrotational problem.

The first is the vectorfield K_0 defined by (4) and the second is the vectorfield K_1 defined by equation (5.16) of Chapter 5:

$$K_1 = (\omega/\nu)L. \tag{9}$$

Here ν is the mean curvature of the wave fronts $S_{t,u}$, sections of the outgoing characteristic hypersurfaces C_u, relative to their characteristic normal L, the tangent vectorfield to the bi-characteristic generators of C_u, parametrized by t. However ν is defined not relative to the acoustical metric $h_{\mu\nu}$ but rather relative to a conformally related metric $\tilde{h}_{\mu\nu}$:

$$\tilde{h}_{\mu\nu} = \Omega h_{\mu\nu}. \tag{10}$$

It turns out that there is a choice of conformal factor Ω such that in the isentropic irrotational case a first order variation $\dot{\phi}$ of the wave function ϕ satisfies the wave equation relative to the metric $\tilde{h}_{\mu\nu}$. This is shown in Chapter 1 and this choice defines Ω in the remainder of the monograph. The definition makes Ω the ratio of a function of σ to the value of this function in the surrounding constant state, thus Ω is equal to unity in the constant state. It turns out moreover that Ω is bounded above and below by positive constants. The function ω appearing in (9) is required to satisfy certain conditions (conditions **D1**–**D5** of Chapter 5) and it is shown in Proposition 13.4 that the function $\omega = 2\eta_0(1+t)$ does satisfy these requirements. A similar analysis to the one done above in the case of the multiplier field K_0 shows, taking into account the fact that the physical dimension of ν is inverse time, that the multiplier field K_1 has the physical dimension of length. The vectorfield K_1 corresponds to the generator of inverted time translations, which are proper conformal transformations of the Minkowski spacetime with its Minkowskian metric $g_{\mu\nu}$. The latter was first used by Morawetz [Mo] to study the decay of solutions of the initial boundary value problem for the classical wave equation outside an obstacle. The vectorfield K_1 is an analogue of the multiplier field of Morawetz for the acoustical spacetime which is the same underlying manifold but equipped with the acoustical metric $h_{\mu\nu}$. The energy currents associated to K_0 and K_1 are defined by equations (5.18) and (5.19) of Chapter 5, respectively. The energy current associated to K_1 contains certain additional lower order terms, defined through the function ω. Analogous terms were present in the work of Morawetz. The general structure of these terms has been investigated, in the general context of Euler–Lagrange equations, in [Ch].

To each variation ψ, of any order, there are energy currents associated to ψ and to K_0 and K_1 respectively. These currents define the energies $\mathcal{E}_0^u[\psi](t)$, $\mathcal{E}_1^{\prime u}[\psi](t)$, and fluxes $\mathcal{F}_0^t[\psi](u)$, $\mathcal{F}_1^{\prime t}[\psi](u)$. For given t and u the energies are integrals over the exterior of the surface $S_{t,u}$ in the hyperplane Σ_t, while the fluxes are integrals over the part of the outgoing characteristic hypersurface C_u between the hyperplanes Σ_0 and Σ_t. To obtain the energy and flux associated to K_1, certain integrations by parts are performed. This construction is presented in Chapter 5. The precise choice of the factor ω/ν in (9) is dictated by the need to eliminate certain error integrals which would otherwise be present. It is these energy and flux integrals, together with a spacetime integral $K[\psi](t,u)$ associated to K_1, to be discussed below, which are used to control the solution.

It is evident from the above that the means by which the solution is controlled depend on the choice of the acoustical function u, the level sets of which are the outgoing

characteristic hypersurfaces C_u. The function u is determined by its restriction to the initial hyperplane Σ_0. The divergence of the energy currents, which determines the growth of the energies and fluxes, itself depends on $^{(K_0)}\tilde{\pi}$, in the case of the energy current associated to K_0, and $^{(K_1)}\tilde{\pi}$, in the case of the energy current associated to K_1. Here for any vectorfield X in spacetime, we denote by $^{(X)}\tilde{\pi}$ the Lie derivative of the conformal acoustical metric $\tilde{h}$ with respect to X. We may call $^{(X)}\tilde{\pi}$ the deformation tensor corresponding to X. In the case of higher order variations, the divergences of the energy currents depend also on the $^{(Y)}\tilde{\pi}$, for each of the commutation fields Y to be discussed below.

All these deformation tensors ultimately depend on the acoustical function u, or, which is the same, on the geometry of the foliation of spacetime by the outgoing characteristic hypersurfaces C_u, the level sets of u. The most important geometric property of this foliation from the point of view of the study of shock formation is the density of the packing of its leaves C_u. One measure of this density is the inverse spatial density of the wave fronts, that is, the inverse density of the foliation of each spatial hyperplane Σ_t by the surfaces $S_{t,u}$. This is the function κ which appears in (4) and is given in arbitrary coordinates on Σ_t by:

$$\kappa^{-2} = (\overline{h}^{-1})^{ij}\,\partial_i u \partial_j u \tag{11}$$

where $\overline{h}_{ij}$ is the induced acoustical metric on Σ_t. Another measure is the inverse temporal density of the wave fronts, the function μ given in arbitrary coordinates in spacetime by:

$$\frac{1}{\mu} = -(h^{-1})^{\mu\nu}\partial_\mu t \partial_\nu u. \tag{12}$$

The two measures are related by:

$$\mu = \alpha\kappa \tag{13}$$

where α is the inverse density, with respect to the acoustical metric, of the foliation of spacetime by the hyperplanes Σ_t. The function α also appears in (4) and is given in arbitrary coordinates in spacetime by:

$$\alpha^{-2} = -(h^{-1})^{\mu\nu}\partial_\mu t \partial_\nu t. \tag{14}$$

It is expressed directly in terms of the 1-form β in the general case, or $d\phi$ in the irrotational isentropic case. It turns out moreover, that it is bounded above and below by positive constants. Consequently μ and κ are equivalent measures of the density of the packing of the leaves of the foliation of spacetime by the C_u. Shock formation is characterized by the blowup of this density or equivalently by the vanishing of κ or μ. The above and the basic geometric construction are discussed in Chapter 2.

The other entity, besides κ or μ which describes the geometry of the foliation by the C_u, is the second fundamental form of the C_u. Since the C_u are null hypersurfaces with respect to the acoustical metric h, their tangent hyperplane at a point is the set of all vectors at that point which are h-orthogonal to the generator L, and L itself belongs to the tangent hyperplane, being h-orthogonal to itself. Thus the second fundamental form χ of C_u is intrinsic to C_u and in terms of the metric $\hbar$ induced by the acoustical metric on the $S_{t,u}$ sections of C_u, it is given by:

$$\mathcal{L}_L \hbar = 2\chi \tag{15}$$

where $\mathcal{L}_X \vartheta$ for a covariant $S_{t,u}$ tensorfield ϑ denotes the restriction of $\mathcal{L}_X \vartheta$ to $T S_{t,u}$. The acoustical structure equations such as the propagation equation for χ along the generators of C_u, the Codazzi equation which expresses $d\!\!\!/\,\text{iv}\,\chi$, the divergence of χ intrinsic to $S_{t,u}$, in terms of $d\!\!\!/\,\text{tr}\chi$, the differential on $S_{t,u}$ of $\text{tr}\chi$, and a component of the acoustical curvature and of k, the second fundamental form of the Σ_t relative to h, are presented in Chapter 3. Also included in the acoustical structure equations is the Gauss equation which expresses the Gauss curvature of $(S_{t,u}, h\!\!\!/\,)$ in terms of χ and a component of the acoustical curvature and of k, and an equation which expresses $\mathcal{L}_T \chi$ in terms of the Hessian of the restriction of μ to $S_{t,u}$ and another component of the acoustical curvature and of k. In the same chapter the components of k are analyzed. These acoustical structure equations contain terms which blow up as κ or μ tend to zero. The regular form of these equations is given in the next chapter, Chapter 4, the subject of which is the analysis of the acoustical curvature. It is there shown that the terms which blow up as κ or μ tend to zero cancel.

The most important acoustical structure equation from the point of view of the formation of shocks is the propagation equation for μ along the generators of C_u, equation (3.96):

$$L\mu = m + \mu e. \tag{16}$$

An equivalent equation, (3.99), is satisfied by κ. This equation is derived in Chapter 3 only in the irrotational isentropic case, in contrast to the other structure equations which hold in general. The function m is in this case given by:

$$m = \frac{1}{2}(L\phi)^2 \frac{dH}{d\sigma}(T\sigma) \tag{17}$$

while the function e, given by (3.98), depends only on the derivatives along L of the ψ_α, $\alpha = 0, 1, 2, 3$, the first variations corresponding to the spacetime translations. A similar propagation equation holds in the general case with the function m given by:

$$m = \frac{1}{2}(\beta_L)^2 \left(\frac{dH}{d\sigma}\right)_s (T\sigma) \tag{18}$$

and the function e depending only on the derivatives of the β_α, the rectangular components of β, tangential to the C_u. We shall presently indicate why this has to be the case without performing the calculations, which in any case parallel those of Chapter 3, with the general formulas (3.4) and (3.10) for the vectorfield V^i and the metric $\overline{h}_{ij}$ induced on the Σ_t, in place of the corresponding formulas (3.6) and (3.11) in the irrotational isentropic case. The fluid velocity being transversal to C_u, by virtue of the adiabatic condition $u \cdot ds = 0$, any derivative of s can be expressed as a derivative tangential to C_u. On the other hand, according to (1) for any vector X we have:

$$\omega(u, X) = -\theta X \cdot ds.$$

It follows that the differential of the entropy and the vorticity do not contribute to the function m, hence an equation of the form (16) results with m given by (18). It is the function m which determines shock formation, when being negative, causing μ to decrease to zero.

The path we have followed in attacking the problem of shock formation in 3-dimensional fluids illustrates the following approach in regard to quasilinear hyperbolic systems of partial differential equations. That is, the quantities which are used to control the solution must be defined using the causal, or characteristic, structure of spacetime determined by the solution itself, not an artificial background structure. The original system of equations must then be considered in conjunction with the system of equations which this structure obeys, and it is only through the study of the interaction of the two systems that results are obtained. The work [C-K] on the stability of the Minkowski space in the framework of general relativity was the first illustration of this approach. In the present case however, the structure, which is here the acoustical structure, degenerates as shocks begin to form, and the precise way in which this degeneracy occurs must be guessed beforehand and established in the course of the argument of the mathematical proof. The fact that the underlying structure degenerates implies that our estimates are no longer even locally equivalent to standard energy estimates, which would of necessity have to fail when shocks appear.

Chapter 5 establishes the fundamental energy estimate, Theorem 5.1. This applies to a solution of the homogeneous wave equation in the acoustical spacetime, in particular to any first order variation. Now the higher order variations satisfy inhomogeneous wave equations in the acoustical spacetime, the source functions depending on the deformation tensors of the commutation fields. These source terms give rise to error integrals, that is to spacetime integrals of contributions to the divergence of the energy currents, which are written down but not estimated in Chapter 5. The remaining error integrals however, are all estimated in the proof of Theorem 5.1, and since these estimates apply to any variation of any order, Chapter 5 contributes in an essential way to the main theorem of Chapter 13. The proof of Theorem 5.1 relies on certain bootstrap assumptions on the acoustical entities. The most crucial of these assumptions concern the behavior of the function μ. These are the assumptions **C1**, **C2**, and **C3**, which are established in the first part of Chapter 13, by Propositions 13.1, 13.2, and 13.3, respectively, on the basis of the final set of bootstrap assumptions, which consists only of pointwise estimates for the variations up to certain order. To give an idea at this point of the nature of these assumptions, the assumption **C2** required in Chapter 5 to obtain the fundamental energy estimate up to time s is (modulo assumption **C1**):

$$\mu^{-1}(T\mu)_+ \leq B_s(t) \quad : \text{for all } t \in [0, s] \tag{19}$$

where $B_s(t)$ is a function such that:

$$\int_0^s (1+t)^{-2}[1+\log(1+t)]^4 B_s(t)dt \leq C \tag{20}$$

with C a constant independent of s. Here T is the vectorfield defined above and we denote by f_+ and f_-, respectively the positive and negative parts of an arbitrary function f. This assumption is then established by Proposition 13.2 with $B_s(t)$ the following function:

$$B_s(t) = C\sqrt{\delta_0}\frac{(1+\tau)}{\sqrt{\sigma - \tau}} + C\delta_0(1+\tau) \tag{21}$$

where $\tau = \log(1+t)$, $\sigma = \log(1+s)$, and δ_0 is a small positive constant appearing in the final set of bootstrap assumptions. Now, the spacetime integral $K[\psi](t, u)$ mentioned above, is defined by (5.169). It is essentially the integral of

$$-\frac{1}{2}(\omega/\nu)(L\mu)_-|d\!\!\!/\psi|^2$$

in the spacetime exterior to C_u and bounded by Σ_0 and Σ_t. Assumption **C3** on the other hand states that there is a positive constant C independent of s such that in the region below Σ_s where $\mu < \eta_0/4$ we have:

$$L\mu \leq -C^{-1}(1+t)^{-1}[1 + \log(1+t)]^{-1} \tag{22}$$

In view of this assumption, the integral $K[\psi](t, u)$ gives effective control of the derivatives of the variations tangential to the wave fronts in the region where shocks are to form. The same assumption, which is established by Proposition 13.3, also plays an essential role in the study of the singular boundary in Chapter 15. The final stage of the proof of Theorem 5.1 is the analysis of a system of integral inequalities in two variables t and u satisfied by the five quantities $\mathcal{E}_0^u[\psi](t)$, $\mathcal{E}_1'^u[\psi](t)$, $\mathcal{F}_0^t[\psi](u)$, $\mathcal{F}_1'^t[\psi](u)$, and $K[\psi](t, u)$. This analysis is reflected at analogous stages in the course of the proof of Theorem 13.1.

The commutation fields Y are defined in Chapter 6. They are five: the vectorfield T which is transversal to the C_u, the field $Q = (1+t)L$ along the generators of the C_u and the three rotation fields $R_i : i = 1, 2, 3$ which are tangential to the $S_{t,u}$ sections. The latter are defined to be $\Pi \stackrel{\circ}{R_i} : i = 1, 2, 3$, where the $\stackrel{\circ}{R_i}$ $i = 1, 2, 3$ are the generators of spatial rotations associated to the background Minkowskian structure, while Π is the h-orthogonal projection to the $S_{t,u}$. Note that the commutation field T has the physical dimension of inverse length while the other commutation fields are dimensionless. Expressions for the deformation tensors $^{(T)}\tilde{\pi}$, $^{(Q)}\tilde{\pi}$, and $^{(R_i)}\tilde{\pi} : i = 1, 2, 3$ are then derived, which show that these depend on the acoustical entities μ and χ. The last however depend in addition on the derivatives of the restrictions to the surfaces $S_{t,u}$ of the spatial rectangular coordinates $x^i : i = 1, 2, 3$, as well as on the derivatives of the x^i with respect to T and L, that is, on the rectangular components T^i and L^i of the vectorfields T and L (note that $L^0 = 1$, $T^0 = 0$). The estimates of these and their derivatives with respect to the commutation fields in terms of the acoustical entities occupies a major part of Chapters 10 and 11 as shall be discussed below.

In Chapter 7 a recursion formula is obtained for the source functions associated to the higher order variations, on the basis of which an explicit formula for these source functions is obtained in Chapter 13. Then the error integrals arising from the contributions to the source functions containing the top and next to the top order derivatives of the variations are estimated.

Chapters 8 and 9 are crucial for the entire work because it is here that the estimates for the top order derivatives of the acoustical entities are derived. The expressions for the source functions and the associated error integrals from Chapter 7 show that the error integrals corresponding to the energies of the $n+1$st order variations contain the nth order derivatives of the deformation tensors, which in turn contain the nth order derivatives of

χ and $n + 1$st order derivatives of μ. Thus to achieve closure, we must obtain estimates for the latter in terms of the energies of up to the $n + 1$st order variations. Now, the propagation equations of Chapter 3 for χ and μ, appearing in regular form in Chapter 4, give appropriate expressions for $\mathcal{L}_L \chi$ and $L\mu$. However, if these propagation equations, which may be thought of as ordinary differential equations along the generators of the C_u, are integrated with respect to t to obtain the acoustical entities χ and μ themselves, and their spatial derivatives are then taken, a loss of one degree of differentiability would result and closure would fail.

We overcome this difficulty in the case of χ in Chapter 8 by considering the propagation equation for $\mu \mathrm{tr}\chi$. We show that, by virtue of a wave equation for σ, which follows from the wave equations satisfied by the first variations corresponding to the spacetime translations, the principal part on the right-hand side of this propagation equation can be put into the form $-L\check{f}$ of a derivative of a function $-\check{f}$ with respect to L. This function is then brought to the left-hand side and we obtain a propagation equation for $\mu \mathrm{tr}\chi + \check{f}$. In this equation $\hat{\chi}$, the trace-free part of χ enters, but the propagation equation in question is considered in conjunction with the Codazzi equation, which constitutes an elliptic system on each $S_{t,u}$ for $\hat{\chi}$, given $\mathrm{tr}\chi$. We thus have an ordinary differential equation along the generators of C_u coupled to an elliptic system on the $S_{t,u}$ sections. More precisely, the propagation equation which is considered at the same level as the Codazzi equation is a propagation equation for the $S_{t,u}$ 1-form $\mu \dslash \mathrm{tr}\chi + \dslash \check{f}$, which is a consequence of the equation just discussed. To obtain estimates for the angular derivatives of χ of order l we similarly consider a propagation equation for the $S_{t,u}$ 1-form:

$$^{(i_1 \ldots i_l)} x_l = \mu \dslash (R_{i_l} \ldots R_{i_1} \mathrm{tr}\chi) + \dslash (R_{i_l} \ldots R_{i_1} \check{f})$$

In the case of μ the aforementioned difficulty is overcome in Chapter 9 by considering the propagation equation for $\mu \slashed{\Delta} \mu$, where $\slashed{\Delta} \mu$ is the Laplacian of the restriction of μ to the $S_{t,u}$. We show that by virtue of a wave equation for $T\sigma$, which is a differential consequence of the wave equation for σ, the principal part on the right-hand side of this propagation equation can again be put into the form $L\check{f}'$ of a derivative of a function $\check{f}'$ with respect to L. This function is then likewise brought to the left-hand side and we obtain a propagation equation for $\mu \slashed{\Delta} \mu - \check{f}'$. In this equation $\hat{\slashed{D}}^2 \mu$, the trace-free part of the Hessian of the restriction of μ to the $S_{t,u}$ enters, but the propagation equation in question is considered in conjunction with the elliptic equation on each $S_{t,u}$ for μ, which the specification of $\slashed{\Delta} \mu$ constitutes. Again we have an ordinary differential equation along the generators of C_u coupled to an elliptic equation on the $S_{t,u}$ sections. To obtain estimates of the spatial derivatives of μ of order $m + l + 2$ of which m are derivatives with respect to T, we similarly consider a propagation equation for the function:

$$^{(i_1 \ldots i_l)} x'_{m,l} = \mu R_{i_l} \ldots R_{i_1} (T)^m \slashed{\Delta} \mu - R_{i_l} \ldots R_{i_1} (T)^m \check{f}'.$$

This allows us to obtain estimates for the top order spatial derivatives of μ of which at least two are angular derivatives. A remarkable fact, which is shown in Chapter 13, is that the missing top order spatial derivatives do not enter the source functions, hence do not contribute to the error integrals. In fact it is shown that the only top order spatial

derivatives of the acoustical entities entering the source functions are those in the 1-forms $^{(i_1\dots i_l)}x_l$ and the functions $^{(i_1\dots i_l)}x'_{m,l}$.

The paradigm of an ordinary differential equation along the generators of a characteristic hypersurface coupled to an elliptic system on the sections of the hypersurface as the means to control the regularity of the entities describing the geometry of the characteristic hypersurface and the stacking of such hypersurfaces in a foliation, was first encountered in [C-K]. It is interesting to note that this paradigm does not appear in space dimension less than three. In the case of the work on the stability of the Minkowski space however, in contrast to the present case, the gain of regularity achieved in this treatment is not essential for obtaining closure, because there is room for one degree of differentiability. This is because of the fact that the Einstein equations arise from a Lagrangian which is quadratic in the canonical velocities, that is, in the derivatives of the unknown functions, in contrast to the equations of fluid mechanics, or more generally of continuum mechanics, which in the Lagrangian picture are equations for a mapping of spacetime into the material manifold, the Lagrangian not depending quadratically on the differential of this mapping (see [Ch]). As a consequence, the metric determining the causal structure depends in continuum mechanics on the derivatives of the unknowns, rather than only on the unknowns themselves.

In the present work, the appearance of the factor of μ, which vanishes where shocks originate, in front of $\displaystyle{\not{d}}R_{i_l}\dots R_{i_1}\operatorname{tr}\chi$ and $R_{i_l}\dots R_{i_1}(T)^m\!\!\not{\triangle}\mu$ in the definitions of $^{(i_1\dots i_l)}x_l$ and $^{(i_1\dots i_l)}x'_{m,l}$ above, makes the analysis far more delicate. This is compounded with the difficulty of the slower decay in time which the addition of the terms $-\!\!\not{d}R_{i_l}\dots R_{i_1}\check{f}$ and $R_{i_l}\dots R_{i_1}(T)^m\check{f}'$ forces. The analysis requires a precise description of the behavior of μ itself, given by Proposition 8.6, and a separate treatment of the condensation regions, where shocks are to form, from the rarefaction regions, the terms referring not to the fluid density but rather to the density of the stacking of the wave fronts. To overcome the difficulties the following weight function is introduced:

$$\overline{\mu}_{m,u}(t) = \min\left\{\frac{\mu_{m,u}(t)}{\eta_0}, 1\right\}, \qquad \mu_{m,u}(t) = \min_{\Sigma_t^u}\mu \tag{23}$$

where Σ_t^u is the exterior of $S_{t,u}$ in Σ_t, and, in Chapter 13, the quantities $\mathcal{E}_0^u[\psi](t)$, $\mathcal{E}_1'^u[\psi](t)$, $\mathcal{F}_0^t[\psi](u)$, $\mathcal{F}_1'^t[\psi](u)$, and $K[\psi](t,u)$ corresponding to the highest order variations are weighted with a power, $2a$, of this weight function. Lemma 8.11, then plays a crucial role in Chapters 8 and 9 as well as in Chapter 13 where everything comes together. We present this lemma here in an imprecise manner to indicate what is involved. Let:

$$M_u(t) = \max_{\Sigma_t^u}\left\{-\mu^{-1}(L\mu)_-\right\}, \qquad I_{a,u} = \int_0^t \overline{\mu}_{m,u}^{-a}(t')M_u(t')dt'. \tag{24}$$

Then under certain bootstrap assumptions in the past of Σ_s, for any constant $a \geq 2$, there is a positive constant C *independent of s, u and a* such that for all $t \in [0, s]$ we have:

$$I_{a,u}(t) \leq Ca^{-1}\overline{\mu}_{m,u}^{-a}(t). \tag{25}$$

As mentioned in the discussion of Chapter 6 above, estimates for the derivatives of the spatial rectangular coordinates x^i with respect to the commutation fields must also be obtained, the derivative of the x^i with respect to the vectorfields $\hat{T}$ and L being the spatial rectangular components $\hat{T}^i$ and L^i of these vectorfields. Here $\hat{T} = \kappa^{-1}T$ is the vectorfield of unit magnitude with respect to h corresponding to T. Thus, although the argument depends mainly on the causal structure of the acoustical spacetime, the underlying Minkowskian structure, to which the rectangular coordinates belong, has a role to play as well, and it is the estimates in question which analyze the mutual relationship of the two structures. The major part of Chapters 10 and 11 is the derivation of estimates for the spatial derivatives of the first derivatives of the x^i, in terms of the acoustical entities. In particular, Propositions 10.1 and 10.2 give estimates for the angular derivatives of the x^i and of the $\hat{T}^i = \hat{T}x^i$, that is, their derivatives with respect to the rotation fields R_j, while Propositions 11.1 and 11.2 give estimates for the spatial derivatives of the $\hat{T}^i$, that is the derivatives with respect to T and the R_j, of which at least one is a T-derivative. The corollaries of these propositions provide the remaining estimates, including the required estimates for the deformation tensors of the commutation fields in terms of the acoustical entities. In particular, Corollaries 10.1.e and 10.2.e give, through Lemma 10.6, estimates for the iterated commutators of the set of rotation fields, Corollaries 10.1.i and 10.2.i give estimates for the angular derivatives of the commutators $[L, R_i] = {}^{(R_i)}Z$ (see Lemma 8.2), while Lemma 10.24 gives estimates for the angular derivatives of the commutators $[T, R_i] = {}^{(R_i)}\Theta$ (see Lemma 10.22). On the other hand, Corollaries 11.1.c and 11.2.c give estimates for the spatial derivatives of the commutators $[L, T] = \Lambda$. The remainder of Chapters 10 and 11 deduce the required estimates for the quantities appearing in the final estimates of Chapters 8 and 9 for the 1-forms ${}^{(i_1 \ldots i_l)}x_l$ and the functions ${}^{(i_1 \ldots i_{l-m})}x'_{m, l-m}$, respectively.

Chapter 12 contains the recovery of the acoustical bootstrap assumptions used in the previous chapters, in particular in Chapters 10 and 11. That is, these acoustical assumptions are established, using the method of continuity, on the basis of the final set of bootstrap assumptions, which consists only of pointwise estimates for the variations up to certain order. In the same chapter the estimates for up to the next to the top order angular derivatives of χ and spatial derivatives of μ are derived. These, when substituted in the estimates of Chapters 10 and 11, give control of all quantities involved in terms of estimates for the variations. A fundamental role in Chapter 12 is played by Propositions 12.2, 12.4, 12.5, and 12.7 which establish the coercivity hypotheses **H0**, **H1**, **H2** and **H2**$'$ on which the previous chapters depend. These propositions roughly speaking show that for any covariant $S_{t,u}$ tensorfield ϑ, the sum $\sum_i |\mathcal{L}_{R_i}\vartheta|^2$ bounds pointwise $|D\!\!\!/\,\vartheta|^2$. Proposition 12.8, which shows that if X is any $S_{t,u}$-tangential vectorfield and ϑ any covariant $S_{t,u}$ tensorfield then we can bound pointwise $\mathcal{L}_X\vartheta$ in terms of the $\mathcal{L}_{R_i}\vartheta$ and the $\mathcal{L}_{R_i}X = [R_i, X]$, also plays an important role.

Chapter 13 begins by establishing the basic assumptions **C1**, **C2**, and **C3**, on the behavior of the function μ on which the energy estimates rely. We then formulate the final bootstrap assumption, to which all other assumptions have been reduced, and which consists only of pointwise estimates for the variations up to certain order. After that we deduce an explicit formula for the source functions, using the recursion formula derived

in Chapter 7, and analyze the structure of the terms containing the top order spatial derivatives of the acoustical entities, showing that these can be expressed in terms of the 1-forms $^{(i_1\ldots i_l)}x_l$ and the functions $^{(i_1\ldots i_{l-m})}x'_{m,l-m}$. These terms are shown to contribute *borderline error integrals*, the treatment of which is the main source of difficulties in the problem. These borderline integrals are all proportional to the constant ℓ mentioned above, the value of (8) in the surrounding constant state, hence are absent in the case $\ell = 0$.

We should make clear here that the only variations which are considered up to this point are the variations arising from the first order variations corresponding to the group of spacetime translations. In particular the final bootstrap assumption involves only variations of this type, and each of the five quantities $\mathcal{E}^u_{0,[n]}(t)$, $\mathcal{F}^t_{0,[n]}(u)$, $\mathcal{E}'^u_{1,[n]}(t)$, $\mathcal{F}'^t_{1,[n]}(u)$, and $K_{[n]}(t, u)$, which together control the solution, is defined to be the sum of the corresponding quantity $\mathcal{E}^u_0[\psi](t)$, $\mathcal{F}^t_0[\psi](u)$, $\mathcal{E}'^u_1[\psi](t)$, $\mathcal{F}'^t_1[\psi](u)$, and $K[\psi](t, u)$, over all variations ψ of this type, up to order n. To estimate the borderline integrals however, we introduce an additional assumption, assumption **J**, which concerns the first order variations corresponding to the scaling or dilation group and to the rotation group, and the second order variations arising from these by applying the commutation field T. This assumption is later established through energy estimates of order 4 arising from these first order variations and derived on the basis of the final bootstrap assumption, just before the recovery of the final bootstrap assumption itself. It turns out that the borderline integrals all contain the factor $T\psi_\alpha$, where $\psi_\alpha : \alpha = 0, 1, 2, 3$ are the first variations corresponding to spacetime translations and assumption **J** is used to obtain an estimate for $\sup_{\Sigma^u_t}\left(\mu^{-1}|T\psi_\alpha|\right)$ in terms of $\sup_{\Sigma^u_t}\left(\mu^{-1}|L\mu|\right)$, which involves on the right the factor $|\ell|^{-1}$ (see (13.198)). Upon substituting this estimate in the borderline integrals, the factors involving ℓ cancel, and the integrals are estimated using (25).

The above is an outline of the main steps in the estimation of the borderline integrals associated to the vectorfield K_0. The estimation of the borderline integrals associated to the vectorfield K_1, is however still more delicate. In this case we first perform an integration by parts on the outgoing characteristic hypersurfaces C_u, obtaining hypersurface integrals over Σ^u_t and Σ^u_0 and another spacetime volume integral. In this integration by parts the terms, including those of lower order, must be carefully chosen to obtain appropriate estimates, because here the long time behavior, as well as the behavior as μ tends to zero, is critical. Another integration by parts, this time on the surfaces $S_{t,u}$, is then performed to reduce these integrals to a form which can be estimated. The estimates of the hypersurface integrals over Σ^u_t are the most delicate (the hypersurface integrals over Σ^u_0 only involve the initial data) and require separate treatment of the condensation and rarefaction regions, in which the properties of the function μ, in particular those established by Proposition 8.6, come into play.

In proceeding to derive the energy estimates of top order, $n = l + 2$, the power $2a$ of the weight $\overline{\mu}_{m,u}(t)$ is chosen suitably large to allow us to transfer the terms contributed by the borderline integrals to the left-hand side of the inequalities resulting from the integral identities associated to the multiplier fields K_0 and K_1. The argument then proceeds along the lines of that of Chapter 5, but is more complex because account must be taken of the terms corresponding to the estimates of Chapter 7 and of all the other terms contributed by the source functions, and also because of the fact that here we are dealing with weighted

quantities. Once the top order energy estimates are established, we revisit the lower order energy estimates, using at each order the energy estimates of the next order in estimating the error integrals contributed by the highest spatial derivatives of the acoustical entities at that order. We then establish a descent scheme, which yields, after finitely many steps, estimates for the five quantities $\mathcal{E}^u_{0,[n]}(t)$, $\mathcal{F}^t_{0,[n]}(u)$, $\mathcal{E}'^u_{1,[n]}(t)$, $\mathcal{F}'^t_{1,[n]}(u)$, and $K_{[n]}(t,u)$, for $n = l + 1 - [a]$, where $[a]$ is the integral part of a, in which weights no longer appear.

It is these unweighted estimates which are used to close the bootstrap argument by recovering the final bootstrap assumption. This is accomplished by the method of continuity through the use of the isoperimetric inequality on the wave fronts $S_{t,u}$, and leads to the main theorem, Theorem 13.1. This theorem shows that there is another differential structure, that defined by the acoustical coordinates t, u, ϑ introduced in Chapter 2, such that relative to this structure the maximal classical solution extends smoothly to the boundary of its domain. This boundary contains however a singular part where the function μ vanishes, hence, in these coordinates, the acoustical metric h degenerates. With respect to the standard differential structure induced by the rectangular coordinates x^α : $\alpha = 0, 1, 2, 3$ in Minkowski spacetime, the solution is continuous but not differentiable on the singular part of the boundary, the derivative $\hat{T}^\mu \hat{T}^\nu \partial_\mu \beta_\nu$ blowing up as we approach the singular boundary. Thus, with respect to the standard differential structure, the acoustical metric h is everywhere in the closure of the domain of the maximal solution non-degenerate and continuous, but not differentiable on the singular part of the boundary of this domain, while with respect to the differential structure induced by the acoustical coordinates h is everywhere smooth, but degenerate on the singular part of the boundary. We have not sought to obtain an optimal lower bound for exponent a. This lower bound is significantly reduced by observing that only one among the first order variations actually contributes borderline integrals, namely the variation corresponding to time translations.

As has already been mentioned, the first part of Chapter 14 establishes a theorem, Theorem 14.1, which gives sharp sufficient conditions on the initial data for the formation of a shock in the evolution, in the case of irrotational isentropic initial data. The proof is through Proposition 8.6 and is based on the study of the evolution with respect to t of the mean value on the sections $S_{t,u}$ of each outgoing characteristic hypersurface C_u of the quantity:

$$\underline{\tau} = (1 - u + \eta_0 t)\underline{L}\psi_0 - (\psi_0 - k). \tag{26}$$

Here ψ_0 is the first variation corresponding to time translations and k is its value in the surrounding constant state. The proof of Theorem 14.1 uses the estimate provided by the spacetime integral $K[\psi_0](t,u)$ associated to ψ_0. Theorem 14.1 is followed by the analysis of the solution of the problem with general initial data during the initial time interval of order $1/\eta_0$, as has already been discussed above. The last part of Chapter 14 establishes a theorem, Theorem 14.2, which extends Theorem 14.1 to the general case, removing the irrotational and isentropic restrictions on the initial data. The proof of Theorem 14.2 is based on the 1-form:

$$\xi_\mu = \dot{\beta}_\mu + \theta \dot{s} u_\mu \tag{27}$$

corresponding to any first order variation $(\dot{p}, \dot{s}, \dot{u})$ of a general solution (p, s, u), through solutions of the general equations of motion, and the associated functions:

$$i = L^\mu \xi_\mu, \quad \underline{i} = \underline{L}^\mu \xi_\mu. \tag{28}$$

We then study the evolution of the mean value on the $S_{t,u}$ sections of each C_u of the quantity:

$$\underline{\tau}' = (1 - u + \eta_0 t)\underline{i} - v_0(p - p_0) \tag{29}$$

where v_0 and p_0 are respectively the volume per particle and pressure in the surrounding constant state. Here certain crucial integrations by parts on the $S_{t,u}$ sections as well as on C_u itself are performed, in which the structure of C_u as a characteristic hypersurface comes into play. We remark that Theorems 14.1 and 14.2 also give a sharp upper bound on the time interval required for the onset of shock formation.

The contents of Chapter 15 have already been briefly described above. Proposition 15.1 describes the singular part of the boundary of the domain of the maximal classical solution from the point of view of the acoustical spacetime. It shows that this singular part has the intrinsic geometry of a regular null hypersurface in a regular spacetime and, like the latter, is ruled by invariant curves of vanishing arc length. On the other hand, the extrinsic geometry of the singular boundary is that of a space-like hypersurface which becomes null at its past boundary. The invariant curves are then used to define canonical acoustical coordinates.

Theorem 15.1 is the main result of the chapter. This theorem shows that at each point q of the singular boundary, the past sound cone in the cotangent space at q degenerates into two hyperplanes intersecting in a 2-dimensional plane. We thus have a trichotomy of the bi-characteristics, or null geodesics of the acoustical metric, ending at q, into the set of outgoing null geodesics ending at q, which corresponds to one of the hyperplanes, the set of incoming null geodesics ending at q, which corresponds to the other hyperplane, and the set of the remaining null geodesics ending at q, which corresponds to the 2-dimensional plane. The intersection of the past characteristic cone of q with any Σ_t in the past of q similarly splits into three parts, the parts corresponding to the outgoing and to the incomings sets of null geodesics ending at q being embedded discs with a common boundary, an embedded circle, which corresponds to the set of the remaining null geodesics ending at q. *All outgoing null geodesics ending at q have the same tangent vector at q.* This vector is then an invariant characteristic vector associated to the singular point q.

This striking result is in fact the reason why the considerable freedom in the choice of the acoustical function does not matter in the end. For as is shown in Proposition 15.2, which considers the transformation from one acoustical function to another, the foliations corresponding to different families of outgoing characteristic hypersurfaces have equivalent geometric properties and degenerate in precisely the same way on the same singular boundary. Finally, Proposition 15.3 gives a detailed description of the boundary of the domain of the maximal classical solution from the point of view of Minkowski spacetime. The contents of the Epilogue have already been adequately described above.

In concluding this introduction we remark that shocks develop not only in the context of fluid mechanics but also in magnetohydrodynamics, that is, the mechanics of a perfectly electrically conducting fluid in the presence of a magnetic field, in the nonlinear regime of the theory of elasticity, that is, the mechanics of deformable solids, isotropic or crystalline, as well as in the electrodynamics of continuous nonlinear media, in particular in the propagation of electromagnetic waves in electrically insulating fluids or solids with a nonlinear relationship between the electromagnetic field and the electromagnetic displacement. The pioneering work on shock formation in the theory of elasticity, in the spherically symmetric case, has been done by John [J2]. It is hoped that the present monograph will provide a springboard for those wishing to attack the general problem of shock formation in any of these fields.

The present work relies more heavily on differential geometric concepts and methods than previous works on the same subject. For those prospective readers whose background is mainly in the fields of fluid mechanics or partial differential equations, but who may not have acquired an equally strong background in differential geometry, we recommend as a reference the book by Bishop and Crittenden [B-C] for the basic differential geometric concepts, and the book by Schoen and Yau [S-Y] as an excellent introduction to geometric analysis.

In regard to notational conventions, Latin indices take the values 1,2,3, while Greek indices take the values 0,1,2,3. Repeated indices are meant to be summed, unless otherwise specified.