Introduction

Calogero–Moser systems, which were originally discovered by specialists in integrable systems, are currently at the crossroads of many areas of mathematics and within the scope of interests of many mathematicians. More specifically, these systems and their generalizations turned out to have intrinsic connections with such fields as algebraic geometry (Hilbert schemes of surfaces), representation theory (double affine Hecke algebras, Lie groups, quantum groups), deformation theory (symplectic reflection algebras), homological algebra (Koszul algebras), Poisson geometry, etc. The goal of the present lecture notes is to give an introduction to the theory of Calogero–Moser systems, highlighting their interplay with these fields. Since these lectures are designed for non-experts, we give short introductions to each of the subjects involved, and provide a number of exercises.

We now describe the contents of the lectures in more detail.

In Lecture 1, we give an introduction to Poisson geometry and to the process of classical Hamiltonian reduction. More specifically, we define Poisson manifolds (smooth, analytic, and algebraic), momemt maps and their main properties, and then describe the procedure of (classical) Hamiltonian reduction. We give an example of computation of Hamiltonian reduction in algebraic geometry (the commuting variety). Finally, we define Hamiltonian reduction along a coadjoint orbit, and give the example which plays a central role in these lectures – the Calogero–Moser space of Kazhdan, Kostant, and Sternberg.

In Lecture 2, we give an introduction to classical Hamiltonian mechanics and the theory of integrable systems. Then we explain how integrable systems may sometimes be constructed using Hamiltonian reduction. After this we define the classical Calogero–Moser integrable system using Hamiltonian reduction along a coadjoint orbit (the Kazhdan–Kostant–Sternberg construction), and find its solutions. Then, by introducing coordinates on the Calogero–Moser space, we write both the system and the solutions explicitly, thus recovering the standard results about the Calogero–Moser system. Finally, we generalize these results to construct the trigonometric Calogero–Moser system.

Lecture 3 is an introduction to deformation theory. This lecture is designed, in particular, to enable us to discuss quantum-mechanical versions of the notions and results of Lectures 1 and 2 in a manner parallel to the classical case. Specifically, we develop the theory of formal and algebraic deformations of assosiative algebras, introduce Hochschild cohomology and discuss its role in studying deformations, and define universal deformations. Then we discuss the basics of the theory of deformation quantization of Poisson (in particular, symplectic) manifolds, and state the Kontsevich quantization theorem.

Lecture 4 is dedicated to the quantum-mechanical generalization of the material of Lecture 1. Specifically, we define the notions of quantum moment map and quantum Hamiltonian reduction. Then we give an example of computation of quantum reduction (the Levasseur–Stafford theorem), which is the quantum analog of the example of commuting variety given in Lecture 1. Finally, we define the notion of quantum reduction with respect to an ideal in the enveloping algebra, which is the quantum version of reduction along a coadjoint orbit, and give an example of this reduction, namely the construction of the spherical subalgebra of the rational Cherednik algebra. Being a quantization of the Calogero–Moser space, this algebra is to play a central role in subsequent lectures.

Lecture 5 contains the quantum-mechanical version of the material of Lecture 1. Namely, after recalling the basics of quantum Hamiltonian mechanics, we introduce the notion of a quantum integrable system. Then we explain how to construct quantum integrable systems by means of quantum reduction (with respect to an ideal), and give an example of this which is central to our exposition: the quantum Calogero–Moser system.

In Lecture 6, we define and study more general classical and quantum Calogero–Moser systems, which are associated to finite Coxeter groups (they were introduced by Olshanetsky and Perelomov). The systems defined in previous lectures correspond to the case of the symmetric group. In general, these integrable systems are not known (or expected) to have a simple construction using reduction; in their construction and study, Dunkl operators are an indispensible tool. We introduce the Dunkl operators (both classical and quantum), and explain how the Olshanetsky–Perelomov Hamiltonians are constructed from them.

Lecture 7 is dedicated to the study of the rational Cherednik algebra, which naturally arises from Dunkl operators (namely, it is generated by Dunkl operators, coordinates, and reflections). Using the Dunkl operator representation, we prove the Poincaré–Birkhoff–Witt theorem for this algebra, and study its spherical subalgebra and center.

In Lecture 8, we consider symplectic reflection algebras, associated to a finite group G of automorphisms of a symplectic vector space V. These algebras are natural generalizations of rational Cherednik algebras (although in general they are not related to any integrable system). It turns out that the PBW theorem does generalize to these algebras, but its proof does not, since Dunkl operators do not have a counterpart. Instead, the proof is based on the theory of deformations of Koszul algebras, due to Drinfeld, Braverman–Gaitsgory, Polishchuk–Positselski, and Beilinson–Ginzburg–Soergel. We also study the spherical subalgebra of the symplectic reflection algebra, and show by deformation-theoretic arguments that it is commutative if the Planck constant is equal to zero.

In Lecture 9, we describe the deformation-theoretic interpretation of symplectic reflection algebras. Namely, we show that they are universal deformations of semidirect products of G with the Weyl algebra of V.

In Lecture 10, we study the center of the symplectic reflection algebra in the case when the Planck constant equals zero. Namely, we consider the spectrum of the center, which is an algebraic variety analogous to the Calogero–Moser space, and show that the smooth locus of this variety is exactly the set of points where the symplectic reflection algebra is an Azumaya algebra; this requires some tools from homological algebra, such as the Cohen–Macaulay property and homological dimension, which we briefly introduce. We also study finite dimensional representations of symplectic reflection algebras with the zero value of the Planck constant. In particular, we show that for G being the symmetric group S_n (i.e. in the case of rational Cherednik algebras of type A), every irreducible representation has dimension $n!$, and irreducible representations are parametrized by the Calogero–Moser space defined in Lecture 1. A similar theorem is valid if $G = S_n \ltimes \Gamma^n$, where Γ is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$.

Lecture 11 is dedicated to representation theory of rational Cherednik algebras with a nonzero Planck constant. Namely, by analogy with semisimple Lie algebras, we develop the theory of category $\mathcal{O}$. In particular, we introduce Verma modules, irreducible highest weight modules, which are labeled by representations of G, and compute the characters of the Verma modules. The main challenge is to compute the characters of irreducible modules, and find out which of them are finite dimensional. We do some of this in the case when $G = S_n \ltimes \Gamma^n$, where Γ is a cyclic group. In particular, we construct and compute the characters of all the finite dimensional simple modules in the case $G = S_n$ (rational Cherednik algebra of type A). It turns out that a finite dimensional simple module exists if and only if the parameter k of the Cherednik algebra equals r/n, where r is an integer relatively prime to n. For such values of k, such representation is unique, its dimension is $|r|^{n-1}$, and it has no self-extensions.

At the end of each lecture, we provide remarks and references, designed to put the material of the lecture in a broader prospective, and link it with the existing literature. However, due to a limited size and scope of these lectures, we were, unfortunately, unable to give an exhaustive list of references on Calogero–Moser systems; such a list would have been truly enormous.

Acknowledgments. These lecture notes are dedicated to my mother Yelena Etingof on the occasion of her 75th birthday. Her continuous care from my early childhood to this day has shaped me both as a person and as a mathematician, and there are no words that are sufficient to express my gratitude and admiration.

I gave these lectures in the spring and summer of 2005 at ETH (Zurich). I am greatly indebted to Professor Giovanni Felder and the participants of his seminar for being active listeners; they are responsible for the very existence of these notes, as well as for improvement of their quality. I am also very grateful to ETH for its hospitality and financial support.

This work was partially supported by the NSF grant DMS-0504847 and the CRDF grant RM1-2545-MO-03.