Preface

In General Relativity spacetime is described mathematically by a Lorentzian manifold. Gravitation manifests itself as the curvature of this manifold. Physical fields, such as the electromagnetic field, are defined on this manifold and have to satisfy a wave equation. This book provides an introduction to the theory of linear wave equations on Lorentzian manifolds. In contrast to other texts on this topic, [Friedlander1975], [Günther1988], we develop the global theory. This means, we ask for existence and uniqueness of solutions which are defined on all of the underlying manifold. Such results are of great importance and are already used much in the literature despite the fact that published proofs are hard to find. Tracing back the references one typically ends at Leray's unpublished lecture notes [Leray1953] or their exposition [Choquet-Bruhat1968].

In this text we develop the global theory from scratch in a modern geometric language. In the first chapter we provide basic definitions and facts about distributions on manifolds, Lorentzian geometry, and normally hyperbolic operators. We study the building blocks for local solutions, the Riesz distributions, in some detail. In the second chapter we show how to solve wave equations locally. Using Riesz distributions and a formal recursive procedure one first constructs formal fundamental solutions. These are formal series solving the equations formally, but in general they do not converge. Using suitable cut-offs one gets "almost solutions" from these formal solutions. They are well-defined distributions but solve the equation only up to an error term. This is then corrected by some further analysis which yields true local fundamental solutions.

This procedure is similar to the construction of the heat kernel for a Laplace type operator on a compact Riemannian manifold. The analogy goes even further. Similar to the short-time asymptotics for the heat kernel, the formal fundamental solution turns out to be an asymptotic expansion of the true fundamental solution. Along the diagonal the coefficients of this asymptotic expansion are given by the same algebraic expression in the curvature of the manifold, the coefficients of the operator, and their derivatives as the heat kernel coefficients.

In the third chapter we use the local theory to study global solutions. This means we construct global fundamental solutions, Green's operators, and solutions to the Cauchy problem. This requires assumptions on the geometry of the underlying manifold. In Lorentzian geometry one has to deal with the problem that there is no good analog for the notion of completeness of Riemannian manifolds. In our context globally hyperbolic manifolds turn out to be the right class of manifolds to consider. Most basic models in General Relativity turn out to be globally hyperbolic but there are exceptions such as anti-deSitter spacetime. This is why we also include a section in which we study cases where one can guarantee existence (but not uniqueness) of global solutions on certain non-globally hyperbolic manifolds.

In the last chapter we apply the analytical results and describe the basic mathematical concepts behind field quantization. The aim of quantum field theory on curved spacetimes is to provide a partial unification of General Relativity with Quantum Physics where the gravitational field is left classical while the other fields are quantized. We develop the theory of C^*-algebras and CCR-representations in full detail to the extent that we need. Then we construct the quantization functors and check that the Haag–Kastler axioms of a local quantum field theory are satisfied. We also construct the Fock space and the quantum field.

From a physical perspective we just enter the door to quantum field theory but do not go very far. We do not discuss n-point functions, states, renormalization, nonlinear fields, nor physical applications such as Hawking radiation. For such topics we refer to the corresponding literature. However, this book should provide the reader with a firm mathematical basis to enter this fascinating branch of physics.

In the appendix we collect background material on category theory, functional analysis, differential geometry, and differential operators that is used throughout the text. This collection of material is included for the convenience of the reader but cannot replace a thorough introduction to these topics. The reader should have some experience with differential geometry. Despite the fact that normally hyperbolic operators on Lorentzian manifolds look formally exactly like Laplace type operators on Riemannian manifolds their analysis is completely different. The elliptic theory of Laplace type operators is not needed anywhere in this text. All results on hyperbolic equations which are relevant to the subject are developed in full detail. Therefore no prior knowledge of the theory of partial differential equations is needed.

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