1 Preliminaries

We want to study solutions to wave equations on Lorentzian manifolds. In this first chapter we develop the basic concepts needed for this task. In the appendix the reader will find the background material on differential geometry, functional analysis and other fields of mathematics that will be used throughout this text without further comment.

A wave equation is given by a certain differential operator of second order called a "normally hyperbolic operator". In general, these operators act on sections in vector bundles which is the geometric way of saying that we are dealing with systems of equations and not just with scalar equations. It is important to allow that the sections may have certain singularities. This is why we work with distributional sections rather than with smooth or continuous sections only.

The concept of distributions on manifolds is explained in the first section. One nice feature of distributions is the fact that one can apply differential operators to them and again obtain a distribution without any further regularity assumption.

The simplest example of a normally hyperbolic operator on a Lorentzian manifold is given by the d'Alembert operator on Minkowski space. Its fundamental solution, a concept to be explained later, can be described explicitly. This gives rise to a family of distributions on Minkowski space, the Riesz distributions, which will provide the building blocks for solutions in the general case later.

After explaining the relevant notions from Lorentzian geometry we will show how to "transplant" Riesz distributions from the tangent space into the Lorentzian manifold. We will also derive the most important properties of the Riesz distributions.

1.1 Distributions on manifolds

Let us start by giving some definitions and by fixing the terminology for distributions on manifolds. We will confine ourselves to those facts that we will actually need later on. A systematic and much more complete introduction may be found e.g. in [Friedlander1998].

1.1.1 Preliminaries on distributions. Let M be a manifold equipped with a smooth volume density dV. Later on we will use the volume density induced by a Lorentzian metric but this is irrelevant for now. We consider a real or complex vector bundle $E \to M$. We will always write $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex. The space of compactly supported smooth sections in E will be denoted by $\mathcal{D}(M, E)$. We equip E and T^*M with connections, both denoted by ∇. They induce connections on the tensor bundles $T^*M \otimes \cdots \otimes T^*M \otimes E$, again denoted by ∇. For a continuously differentiable section $\varphi \in C^1(M, E)$ the covariant derivative is a continuous section in $T^*M \otimes E$, $\nabla\varphi \in C^0(M, T^*M \otimes E)$. More generally, for $\varphi \in C^k(M, E)$ we get $\nabla^k\varphi \in C^0(M, \underbrace{T^*M \otimes \cdots \otimes T^*M}_{k \text{ factors}} \otimes E)$.

We choose a Riemannian metric on T^*M and a Riemannian or Hermitian metric on E depending on whether E is real or complex. This induces metrics on all bundles $T^*M \otimes \cdots \otimes T^*M \otimes E$. Hence the norm of $\nabla^k \varphi$ is defined at all points of M.

For a subset $A \subset M$ and $\varphi \in C^k(M, E)$ we define the *C^k-norm* by

$$\|\varphi\|_{C^k(A)} := \max_{j=0,\ldots,k} \sup_{x \in A} |\nabla^j \varphi(x)|. \tag{1.1}$$

If A is compact, then different choices of metric and connection yield equivalent norms $\|\cdot\|_{C^k(A)}$. For this reason there will usually be no need to explicitly specify the metrics and the connections.

The elements of $\mathcal{D}(M, E)$ are referred to as *test sections* in E. We define a notion of convergence of test sections.

Definition 1.1.1. Let $\varphi, \varphi_n \in \mathcal{D}(M, E)$. We say that the sequence $(\varphi_n)_n$ *converges to φ in $\mathcal{D}(M, E)$* if the following two conditions hold:

(1) There is a compact set $K \subset M$ such that the supports of all φ_n are contained in K, i.e., $\mathrm{supp}(\varphi_n) \subset K$ for all n.

(2) The sequence $(\varphi_n)_n$ converges to φ in all C^k-norms over K, i.e., for each $k \in \mathbb{N}$

$$\|\varphi - \varphi_n\|_{C^k(K)} \xrightarrow[n\to\infty]{} 0.$$

We fix a finite-dimensional $\mathbb{K}$-vector space W. Recall that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ depending on whether E is real or complex.

Definition 1.1.2. A $\mathbb{K}$-linear map $F : \mathcal{D}(M, E^*) \to W$ is called a *distribution in E with values in W* if it is continuous in the sense that for all convergent sequences $\varphi_n \to \varphi$ in $\mathcal{D}(M, E^*)$ one has $F[\varphi_n] \to F[\varphi]$. We write $\mathcal{D}'(M, E, W)$ for the space of all W-valued distributions in E.

Note that since W is finite-dimensional all norms $|\cdot|$ on W yield the same topology on W. Hence there is no need to specify a norm on W for Definition 1.1.2 to make sense. Note moreover, that distributions in E act on test sections in E^*.

Lemma 1.1.3. *Let F be a W-valued distribution in E and let $K \subset M$ be compact. Then there is a nonnegative integer k and a constant $C > 0$ such that for all $\varphi \in \mathcal{D}(M, E^*)$ with $\mathrm{supp}(\varphi) \subset K$ we have*

$$|F[\varphi]| \le C \cdot \|\varphi\|_{C^k(K)}. \tag{1.2}$$

The smallest k for which inequality (1.2) holds is called the *order* of F over K.

Proof. Assume (1.2) does not hold for any pair of C and k. Then for every positive integer k we can find a nontrivial section $\varphi_k \in \mathcal{D}(M, E^*)$ with $\mathrm{supp}(\varphi_k) \subset K$ and $|F[\varphi_k]| \ge k \cdot \|\varphi_k\|_{C^k}$. We define sections $\psi_k := \frac{1}{|F[\varphi_k]|}\varphi_k$. Obviously, these ψ_k satisfy $\mathrm{supp}(\psi_k) \subset K$ and

$$\|\psi_k\|_{C^k(K)} = \frac{1}{|F[\varphi_k]|}\|\varphi_k\|_{C^k(K)} \le \frac{1}{k}.$$

Hence for $k \geq j$

$$\|\psi_k\|_{C^j(K)} \leq \|\psi_k\|_{C^k(K)} \leq \tfrac{1}{k}.$$

Therefore the sequence $(\psi_k)_k$ converges to 0 in $\mathcal{D}(M, E^*)$. Since F is a distribution we get $F[\psi_k] \to F[0] = 0$ for $k \to \infty$. On the other hand, $|F[\psi_k]| = \left|\frac{1}{|F[\varphi_k]|} F[\varphi_k]\right| = 1$ for all k, which yields a contradiction. □

Lemma 1.1.3 states that the restriction of any distribution to a (relatively) compact set is of finite order. We say that a distribution F is of order m if m is the smallest integer such that for each compact subset $K \subset M$ there exists a constant C so that

$$|F[\varphi]| \leq C \cdot \|\varphi\|_{C^m(K)}$$

for all $\varphi \in \mathcal{D}(M, E^*)$ with $\mathrm{supp}(\varphi) \subset K$. Such a distribution extends uniquely to a continuous linear map on $\mathcal{D}^m(M, E^*)$, the space of C^m-sections in E^* with compact support. Convergence in $\mathcal{D}^m(M, E^*)$ is defined similarly to that of test sections. We say that φ_n converge to φ in $\mathcal{D}^m(M, E^*)$ if the supports of the φ_n and φ are contained in a common compact subset $K \subset M$ and $\|\varphi - \varphi_n\|_{C^m(K)} \to 0$ as $n \to \infty$.

Next we give two important examples of distributions.

Example 1.1.4. Pick a bundle $E \to M$ and a point $x \in M$. The *delta-distribution* δ_x is an E_x^*-valued distribution in E. For $\varphi \in \mathcal{D}(M, E^*)$ it is defined by

$$\delta_x[\varphi] = \varphi(x).$$

Clearly, δ_x is a distribution of order 0.

Example 1.1.5. Every locally integrable section $f \in L^1_{\mathrm{loc}}(M, E)$ can be interpreted as a $\mathbb{K}$-valued distribution in E by setting for any $\varphi \in \mathcal{D}(M, E^*)$

$$f[\varphi] := \int_M \varphi(f)\,\mathrm{dV}.$$

As a distribution f is of order 0.

Lemma 1.1.6. *Let M and N be differentiable manifolds equipped with smooth volume densities. Let $E \to M$ and $F \to N$ be vector bundles. Let $K \subset N$ be compact and let $\varphi \in C^k(M \times N, E \boxtimes F^*)$ be such that* $\mathrm{supp}(\varphi) \subset M \times K$. *Let $m \leq k$ and let $T \in \mathcal{D}'(N, F, \mathbb{K})$ be a distribution of order m. Then the map*

$$f : M \to E,$$
$$x \mapsto T[\varphi(x, \cdot)],$$

defines a C^{k-m}-section in E with support contained in the projection of $\mathrm{supp}(\varphi)$ *to the first factor, i.e.,* $\mathrm{supp}(f) \subset \{x \in M \mid$ *there exists* $y \in K$ *such that* $(x, y) \in \mathrm{supp}(\varphi)\}$. *In particular, if φ is smooth with compact support, and T is any distribution in F, then f is a smooth section in E with compact support.*

Moreover, x-derivatives up to order $k-m$ may be interchanged with T. More precisely, if P is a linear differential operator of order $\le k-m$ acting on sections in E, then

$$Pf = T[P_x\varphi(x,\cdot)].$$

Here $E \boxtimes F^*$ denotes the vector bundle over $M \times N$ whose fiber over $(x,y) \in M \times N$ is given by $E_x \otimes F^*_y$.

Proof. There is a canonical isomorphism

$$E_x \otimes \mathcal{D}^k(N, F^*) \to \mathcal{D}^k(N, E_x \otimes F^*),$$
$$v \otimes s \mapsto (y \mapsto v \otimes s(y)).$$

Thus we can apply $\mathrm{id}_{E_x} \otimes T$ to $\varphi(x,\cdot) \in \mathcal{D}^k(N, E_x \otimes F^*) \cong E_x \otimes \mathcal{D}^k(N, F^*)$ and we obtain $(\mathrm{id}_{E_x} \otimes T)[\varphi(x,\cdot)] \in E_x$. We briefly write $T[\varphi(x,\cdot)]$ instead of $(\mathrm{id}_{E_x} \otimes T)[\varphi(x,\cdot)]$.

To see that the section $x \mapsto T[\varphi(x,\cdot)]$ in E is of regularity C^{k-m} we may assume that M is an open ball in $\mathbb{R}^p$ and that the vector bundle $E \to M$ is trivialized over M, $E = M \times \mathbb{K}^n$, because differentiability and continuity are local properties.

For fixed $y \in N$ the map $x \mapsto \varphi(x,y)$ is a C^k-map $U \to \mathbb{K}^n \otimes F^*_y$. We perform a Taylor expansion at $x_0 \in U$, see [Friedlander1998, p. 38f]. For $x \in U$ we get

$$\begin{aligned}
\varphi(x,y) &= \sum_{|\alpha|\le k-m-1} \tfrac{1}{\alpha!} D^\alpha_x \varphi(x_0,y)(x-x_0)^\alpha \\
&\quad + \sum_{|\alpha|=k-m} \frac{k-m}{\alpha!} \int_0^1 (1-t)^{k-m-1} D^\alpha_x \varphi((1-t)x_0 + tx, y)(x-x_0)^\alpha \, dt \\
&= \sum_{|\alpha|\le k-m} \tfrac{1}{\alpha!} D^\alpha_x \varphi(x_0,y)(x-x_0)^\alpha \\
&\quad + \sum_{|\alpha|=k-m} \tfrac{k-m}{\alpha!} \int_0^1 (1-t)^{k-m-1} \big(D^\alpha_x \varphi((1-t)x_0 + tx, y) \\
&\qquad\qquad - D^\alpha_x \varphi(x_0,y)\,\big)\, dt \cdot (x-x_0)^\alpha .
\end{aligned}$$

Here we used the usual multi-index notation, $\alpha = (\alpha_1,\dots,\alpha_p) \in \mathbb{N}^p$, $|\alpha| = \alpha_1 + \dots + \alpha_p$, $D^\alpha_x = \frac{\partial^{|\alpha|}}{(\partial x^1)^{\alpha_1}\dots(\partial x^p)^{\alpha_p}}$, and $x^\alpha = x_1^{\alpha_1}\dots x_p^{\alpha_p}$. For $|\alpha| \le k-m$ we certainly have $D^\alpha_x\varphi(\cdot,\cdot) \in C^m(U \times N, \mathbb{K}^n \otimes F^*)$ and, in particular, $D^\alpha_x\varphi(x_0,\cdot) \in \mathcal{D}^m(N, \mathbb{K}^n \otimes F^*)$. We apply T to get

$$\begin{aligned}
T[\varphi(x,\cdot)] &= \sum_{|\alpha|\le k-m} \tfrac{1}{\alpha!} T[D^\alpha_x \varphi(x_0,\cdot)](x-x_0)^\alpha \\
&\quad + \sum_{|\alpha|=k-m} \tfrac{k-m}{\alpha!} T\Big[\int_0^1 (1-t)^{k-m-1} \big(D^\alpha_x \varphi((1-t)x_0 + tx, \cdot) \\
&\qquad\qquad - D^\alpha_x\, \varphi(x_0,\cdot)\big)\, dt\Big](x-x_0)^\alpha .
\end{aligned} \tag{1.3}$$

Restricting the x to a compact convex neighborhood $U' \subset U$ of x_0 the $D_x^\alpha \varphi(\cdot,\cdot)$ and all their y-derivatives up to order m are *uniformly* continuous on $U' \times K$. Given $\varepsilon > 0$ there exists $\delta > 0$ so that $|\nabla_y^j D_x^\alpha \varphi(\tilde{x}, y) - \nabla_y^j D_x^\alpha \varphi(x_0, y)| \le \frac{\varepsilon}{m+1}$ whenever $|\tilde{x} - x_0| \le \delta$, $j = 0, \dots, m$. Thus for x with $|x - x_0| \le \delta$

$$\begin{aligned}
&\left\| \int_0^1 (1-t)^{k-m-1} \left(D_x^\alpha \varphi((1-t)x_0 + tx, \cdot) - D_x^\alpha \varphi(x_0, \cdot) \right) dt \right\|_{C^m(M)} \\
&\quad = \left\| \int_0^1 (1-t)^{k-m-1} \left(D_x^\alpha \varphi((1-t)x_0 + tx, \cdot) - D_x^\alpha \varphi(x_0, \cdot) \right) dt \right\|_{C^m(K)} \\
&\quad \le \int_0^1 (1-t)^{k-m-1} \left\| D_x^\alpha \varphi((1-t)x_0 + tx, \cdot) - D_x^\alpha \varphi(x_0, \cdot) \right\|_{C^m(K)} dt \\
&\quad \le \int_0^1 (1-t)^{k-m-1} \varepsilon \, dt \\
&\quad = \frac{\varepsilon}{k-m}.
\end{aligned}$$

Since T is of order m this implies in (1.3) that $T[\int_0^1 \dots dt] \to 0$ as $x \to x_0$. Therefore the map $x \mapsto T[\varphi(x,\cdot)]$ is $k - m$ times differentiable with derivatives $D_x^\alpha|_{x=x_0} T[\varphi(x,\cdot)] = T[D_x^\alpha \varphi(x_0,\cdot)]$. The same argument also shows that these derivatives are continuous in x. □

1.1.2 Differential operators acting on distributions. Let E and F be two $\mathbb{K}$-vector bundles over the manifold M, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. Consider a linear differential operator $P : C^\infty(M, E) \to C^\infty(M, F)$. There is a unique linear differential operator $P^* : C^\infty(M, F^*) \to C^\infty(M, E^*)$ called the *formal adjoint* of P such that for any $\varphi \in \mathcal{D}(M, E)$ and $\psi \in \mathcal{D}(M, F^*)$

$$\int_M \psi(P\varphi) \, \mathrm{dV} = \int_M (P^*\psi)(\varphi) \, \mathrm{dV}. \tag{1.4}$$

If P is of order k, then so is P^* and (1.4) holds for all $\varphi \in C^k(M, E)$ and $\psi \in C^k(M, F^*)$ such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi)$ is compact. With respect to the canonical identification $E = (E^*)^*$ we have $(P^*)^* = P$.

Any linear differential operator $P : C^\infty(M, E) \to C^\infty(M, F)$ extends canonically to a linear operator $P : \mathcal{D}'(M, E, W) \to \mathcal{D}'(M, F, W)$ by

$$(PT)[\varphi] := T[P^*\varphi]$$

where $\varphi \in \mathcal{D}(M, F^*)$. If a sequence $(\varphi_n)_n$ converges in $\mathcal{D}(M, F^*)$ to 0, then the sequence $(P^*\varphi_n)_n$ converges to 0 as well because P^* is a differential operator. Hence $(PT)[\varphi_n] = T[P^*\varphi_n] \to 0$. Therefore PT is again a distribution.

The map $P : \mathcal{D}'(M, E, W) \to \mathcal{D}'(M, F, W)$ is $\mathbb{K}$-linear. If P is of order k and φ is a C^k-section in E, seen as a $\mathbb{K}$-valued distribution in E, then the distribution $P\varphi$ coincides with the continuous section obtained by applying P to φ classically.

The case when P is of order 0, i.e., $P \in C^\infty(M, \mathrm{Hom}(E, F))$, is of special importance. Then $P^* \in C^\infty(M, \mathrm{Hom}(F^*, E^*))$ is the pointwise adjoint. In particular, for a function $f \in C^\infty(M)$ we have

$$(fT)[\varphi] = T[f\varphi].$$

1.1.3 Supports

Definition 1.1.7. The *support* of a distribution $T \in \mathcal{D}'(M, E, W)$ is defined as the set

$$\begin{aligned}\mathrm{supp}(T) := \{x \in M \mid & \text{ for all neighborhoods } U \text{ of } x \text{ there exists} \\ & \varphi \in \mathcal{D}(M, E) \text{ with } \mathrm{supp}(\varphi) \subset U \text{ and } T[\varphi] \neq 0\}.\end{aligned}$$

It follows from the definition that the support of T is a closed subset of M. In case T is a L^1_{loc}-section this notion of support coincides with the usual one for sections.

If for $\varphi \in \mathcal{D}(M, E^*)$ the supports of φ and T are disjoint, then $T[\varphi] = 0$. Namely, for each $x \in \mathrm{supp}(\varphi)$ there is a neighborhood U of x such that $T[\psi] = 0$ whenever $\mathrm{supp}(\psi) \subset U$. Cover the compact set $\mathrm{supp}(\varphi)$ by finitely many such open sets $U_1, \dots, U_k$. Using a partition of unity one can write $\varphi = \psi_1 + \cdots + \psi_k$ with $\psi_j \in \mathcal{D}(M, E^*)$ and $\mathrm{supp}(\psi_j) \subset U_j$. Hence

$$T[\varphi] = T[\psi_1 + \cdots + \psi_k] = T[\psi_1] + \cdots + T[\psi_k] = 0.$$

Be aware that it is not sufficient to assume that φ vanishes on $\mathrm{supp}(T)$ in order to ensure $T[\varphi] = 0$. For example, if $M = \mathbb{R}$ and E is the trivial $\mathbb{K}$-line bundle let $T \in \mathcal{D}'(\mathbb{R}, \mathbb{K})$ be given by $T[\varphi] = \varphi'(0)$. Then $\mathrm{supp}(T) = \{0\}$ but $T[\varphi] = \varphi'(0)$ may well be nonzero while $\varphi(0) = 0$.

If $T \in \mathcal{D}'(M, E, W)$ and $\varphi \in C^\infty(M, E^*)$, then the evaluation $T[\varphi]$ can be defined if $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi)$ is compact even if the support of φ itself is noncompact. To do this pick a function $\sigma \in \mathcal{D}(M, \mathbb{R})$ that is constant 1 on a neighborhood of $\mathrm{supp}(T) \cap \mathrm{supp}(\varphi)$ and put

$$T[\varphi] := T[\sigma\varphi].$$

This definition is independent of the choice of σ since for another choice σ' we have

$$T[\sigma\varphi] - T[\sigma'\varphi] = T[(\sigma - \sigma')\varphi] = 0$$

because $\mathrm{supp}((\sigma - \sigma')\varphi)$ and $\mathrm{supp}(T)$ are disjoint.

Let $T \in \mathcal{D}'(M, E, W)$ and let $\Omega \subset M$ be an open subset. Each test section $\varphi \in \mathcal{D}(\Omega, E^*)$ can be extended by 0 and yields a test section $\varphi \in \mathcal{D}(M, E^*)$. This defines an embedding $\mathcal{D}(\Omega, E^*) \subset \mathcal{D}(M, E^*)$. By the restriction of T to Ω we mean its restriction from $\mathcal{D}(M, E^*)$ to $\mathcal{D}(\Omega, E^*)$.

Definition 1.1.8. The *singular support* $\mathrm{sing\,supp}(T)$ of a distribution $T \in \mathcal{D}'(M, E, W)$ is the set of points which do not have a neighborhood restricted to which T coincides with a smooth section.

The singular support is also closed and we always have $\mathrm{sing\,supp}(T) \subset \mathrm{supp}(T)$.

Example 1.1.9. For the delta-distribution δ_x we have $\mathrm{supp}(\delta_x) = \mathrm{sing\,supp}(\delta_x) = \{x\}$.

1.1.4 Convergence of distributions. The space $\mathcal{D}'(M, E)$ of distributions in E will always be given the *weak topology*. This means that $T_n \to T$ in $\mathcal{D}'(M, E, W)$ if and only if $T_n[\varphi] \to T[\varphi]$ for all $\varphi \in \mathcal{D}(M, E^*)$. Linear differential operators P are always continuous with respect to the weak topology. Namely, if $T_n \to T$, then we have for every $\varphi \in \mathcal{D}(M, E^*)$

$$PT_n[\varphi] = T_n[P^*\varphi] \to T[P^*\varphi] = PT[\varphi].$$

Hence

$$PT_n \to PT.$$

Lemma 1.1.10. *Let $T_n, T \in C^0(M, E)$ and suppose $\|T_n - T\|_{C^0(M)} \to 0$. Consider T_n and T as distributions.*

Then $T_n \to T$ in $\mathcal{D}'(M, E)$. In particular, for every linear differential operator P we have $PT_n \to PT$.

Proof. Let $\varphi \in \mathcal{D}(M, E)$. Since $\|T_n - T\|_{C^0(M)} \to 0$ and $\varphi \in L^1(M, E)$, it follows from Lebesgue's dominated convergence theorem that

$$\begin{aligned}
\lim_{n\to\infty} T_n[\varphi] &= \lim_{n\to\infty} \int_M T_n(x) \cdot \varphi(x)\, \mathrm{dV}(x) \\
&= \int_M \lim_{n\to\infty} (T_n(x) \cdot \varphi(x))\, \mathrm{dV}(x) \\
&= \int_M (\lim_{n\to\infty} T_n(x)) \cdot \varphi(x)\, \mathrm{dV}(x) \\
&= \int_M T(x) \cdot \varphi(x)\, \mathrm{dV}(x) \\
&= T[\varphi].
\end{aligned}$$

□

1.1.5 Two auxiliary lemmas. The following situation will arise frequently. Let E, F, and G be $\mathbb{K}$-vector bundles over M equipped with metrics and with connections which we all denote by ∇. We give $E \otimes F$ and $F^* \otimes G$ the induced metrics and connections. Here and henceforth F^* will denote the dual bundle to F. The natural pairing $F \otimes F^* \to \mathbb{K}$ given by evaluation of the second factor on the first yields a vector bundle homomorphism $E \otimes F \otimes F^* \otimes G \to E \otimes G$ which we write as $\varphi \otimes \psi \mapsto \varphi \cdot \psi$.[1]

Lemma 1.1.11. *For all C^k-sections φ in $E \otimes F$ and ψ in $F^* \otimes G$ and all $A \subset M$ we have*

$$\|\varphi \cdot \psi\|_{C^k(A)} \le 2^k \cdot \|\varphi\|_{C^k(A)} \cdot \|\psi\|_{C^k(A)}.$$

Proof. The case $k = 0$ follows from the Cauchy–Schwarz inequality. Namely, for fixed $x \in M$ we choose an orthonormal basis f_i, $i = 1, \dots, r$, for F_x. Let f_i^* be

[1] If one identifies $E \otimes F$ with $\mathrm{Hom}(E^*, F)$ and $F^* \otimes G$ with $\mathrm{Hom}(F, G)$, then $\varphi \cdot \psi$ corresponds to $\psi \circ \varphi$.

the basis of F_x^* dual to f_i. We write $\varphi(x) = \sum_{i=1}^r e_i \otimes f_i$ for suitable $e_i \in E_x$ and similarly $\psi(x) = \sum_{i=1}^r f_i^* \otimes g_i$, $g_i \in G_x$. Then $\varphi(x) \cdot \psi(x) = \sum_{i=1}^r e_i \otimes g_i$ and we see

$$\begin{aligned}
|\varphi(x) \cdot \psi(x)|^2 &= \Big|\sum_{i=1}^r e_i \otimes g_i\Big|^2 = \sum_{i,j=1}^r \langle e_i \otimes g_i, e_j \otimes g_j\rangle = \sum_{i,j=1}^r \langle e_i, e_j\rangle\langle g_i, g_j\rangle \\
&\le \sqrt{\sum_{i,j=1}^r \langle e_i, e_j\rangle^2} \cdot \sqrt{\sum_{i,j=1}^r \langle g_i, g_j\rangle^2} \\
&\le \sqrt{\sum_{i,j=1}^r |e_i|^2|e_j|^2} \cdot \sqrt{\sum_{i,j=1}^r |g_i|^2|g_j|^2} \\
&= \sqrt{\sum_{i=1}^r |e_i|^2 \sum_{j=1}^r |e_j|^2} \cdot \sqrt{\sum_{i=1}^r |g_i|^2 \sum_{j=1}^r |g_j|^2} \\
&= \sum_{i=1}^r |e_i|^2 \cdot \sum_{i=1}^r |g_i|^2 \\
&= |\varphi(x)|^2 \cdot |\psi(x)|^2.
\end{aligned}$$

Now we proceed by induction on k.

$$\begin{aligned}
\|\nabla^{k+1}(\varphi \cdot \psi)\|_{C^0(A)} &\le \|\nabla(\varphi \cdot \psi)\|_{C^k(A)} \\
&= \|(\nabla\varphi) \cdot \psi + \varphi \cdot \nabla\psi\|_{C^k(A)} \\
&\le \|(\nabla\varphi) \cdot \psi\|_{C^k(A)} + \|\varphi \cdot \nabla\psi\|_{C^k(A)} \\
&\le 2^k \cdot \|\nabla\varphi\|_{C^k(A)} \cdot \|\psi\|_{C^k(A)} + 2^k \cdot \|\varphi\|_{C^k(A)} \cdot \|\nabla\psi\|_{C^k(A)} \\
&\le 2^k \cdot \|\varphi\|_{C^{k+1}(A)} \cdot \|\psi\|_{C^{k+1}(A)} \\
&\qquad + 2^k \cdot \|\varphi\|_{C^{k+1}(A)} \cdot \|\psi\|_{C^{k+1}(A)} \\
&= 2^{k+1} \cdot \|\varphi\|_{C^{k+1}(A)} \cdot \|\psi\|_{C^{k+1}(A)}.
\end{aligned}$$

Thus

$$\begin{aligned}
\|\varphi \cdot \psi\|_{C^{k+1}(A)} &= \max\{\|\varphi \cdot \psi\|_{C^k(A)}, \|\nabla^{k+1}(\varphi \cdot \psi)\|_{C^0(A)}\} \\
&\le \max\{2^k \cdot \|\varphi\|_{C^k(A)} \cdot \|\psi\|_{C^k(A)}, 2^{k+1} \cdot \|\varphi\|_{C^{k+1}(A)} \cdot \|\psi\|_{C^{k+1}(A)}\} \\
&= 2^{k+1} \cdot \|\varphi\|_{C^{k+1}(A)} \cdot |\psi\|_{C^{k+1}(A)}.
\end{aligned}$$ ☐

This lemma allows us to estimate the C^k-norm of products of sections in terms of the C^k-norms of the factors. The next lemma allows us to deal with compositions of functions. We recursively define the following universal constants:

$$\alpha(k, 0) := 1, \quad \alpha(k, j) := 0$$

for $j > k$ and for $j < 0$, and

$$\alpha(k+1, j) := \max\{\alpha(k, j),\ 2^k \cdot \alpha(k, j-1)\} \tag{1.5}$$

if $1 \le j \le k$. The precise values of the $\alpha(k, j)$ are not important. The definition was made in such a way that the following lemma holds.

Lemma 1.1.12. *Let Γ be a real valued C^k-function on a Lorentzian manifold M and let $\sigma : \mathbb{R} \to \mathbb{R}$ be a C^k-function. Then for all $A \subset M$ and $I \subset \mathbb{R}$ such that $\Gamma(A) \subset I$ we have*

$$\|\sigma \circ \Gamma\|_{C^k(A)} \le \|\sigma\|_{C^k(I)} \cdot \max_{j=0,\dots,k} \alpha(k, j) \|\Gamma\|^j_{C^k(A)}.$$

Proof. We again perform an induction on k. The case $k = 0$ is obvious. By Lemma 1.1.11

$$\begin{aligned}
\|\nabla^{k+1}(\sigma \circ \Gamma)\|_{C^0(A)} &= \|\nabla^k[(\sigma' \circ \Gamma) \cdot \nabla\Gamma]\|_{C^0(A)} \\
&\le \|(\sigma' \circ \Gamma) \cdot \nabla\Gamma\|_{C^k(A)} \\
&\le 2^k \cdot \|\sigma' \circ \Gamma\|_{C^k(A)} \cdot \|\nabla\Gamma\|_{C^k(A)} \\
&\le 2^k \cdot \|\sigma' \circ \Gamma\|_{C^k(A)} \cdot \|\Gamma\|_{C^{k+1}(A)} \\
&\le 2^k \cdot \|\sigma'\|_{C^k(I)} \cdot \max_{j=0,\dots,k} \alpha(k, j) \|\Gamma\|^j_{C^{k+1}(A)} \cdot \|\Gamma\|_{C^{k+1}(A)} \\
&\le 2^k \cdot \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=0,\dots,k} \alpha(k, j) \|\Gamma\|^{j+1}_{C^{k+1}(A)} \\
&= 2^k \cdot \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=1,\dots,k+1} \alpha(k, j-1) \|\Gamma\|^j_{C^{k+1}(A)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\|\sigma \circ \Gamma\|_{C^{k+1}(A)} &= \max\{\|\sigma \circ \Gamma\|_{C^k(A)}, \|\nabla^{k+1}(\sigma \circ \Gamma)\|_{C^0(A)}\} \\
&\le \max\{\|\sigma\|_{C^k(I)} \cdot \max_{j=0,\dots,k} \alpha(k, j) \|\Gamma\|^j_{C^k(A)}, \\
&\qquad 2^k \cdot \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=1,\dots,k+1} \alpha(k, j-1) \|\Gamma\|^j_{C^{k+1}(A)}\} \\
&\le \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=0,\dots,k+1} \max\{\alpha(k, j), 2^k \alpha(k, j-1)\} \|\Gamma\|^j_{C^{k+1}(A)} \\
&= \|\sigma\|_{C^{k+1}(I)} \cdot \max_{j=0,\dots,k+1} \alpha(k+1, j) \|\Gamma\|^j_{C^{k+1}(A)}.
\end{aligned}$$

□

1.2 Riesz distributions on Minkowski space

The distributions $R_+(\alpha)$ and $R_-(\alpha)$ to be defined below were introduced by M. Riesz in the first half of the 20th century in order to find solutions to certain differential equations. He collected his results in [Riesz1949]. We will derive all relevant facts in full detail.

Let V be an n-dimensional real vector space, let $\langle \cdot , \cdot \rangle$ be a nondegenerate symmetric bilinear form of index 1 on V. Hence $(V, \langle \cdot , \cdot \rangle)$ is isometric to n-dimensional Minkowski space $(\mathbb{R}^n, \langle \cdot , \cdot \rangle_0)$ where $\langle x, y \rangle_0 = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$. Set

$$\gamma : V \to \mathbb{R}, \quad \gamma(X) := -\langle X, X \rangle. \tag{1.6}$$

A nonzero vector $X \in V \setminus \{0\}$ is called *timelike* (or *lightlike* or *spacelike*) if and only if $\gamma(X) > 0$ (or $\gamma(X) = 0$ or $\gamma(X) < 0$ respectively). The zero vector $X = 0$ is considered as spacelike. The set $I(0)$ of timelike vectors consists of two connected components. We choose a *time-orientation* on V by picking one of these two connected components. Denote this component by $I_+(0)$ and call its elements *future directed*. Put $J_+(0) := \overline{I_+(0)}$, $C_+(0) := \partial I_+(0)$, $I_-(0) := -I_+(0)$, $J_-(0) := -J_+(0)$, and $C_-(0) := -C_+(0)$.

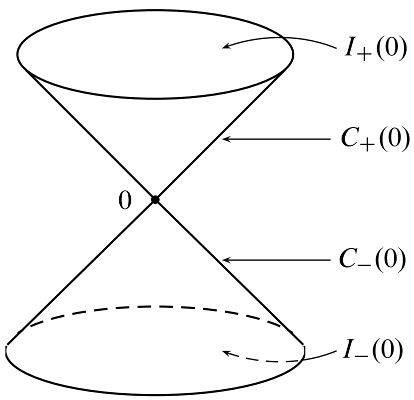


Figure 1. Light cone in Minkowski space.

Definition 1.2.1. For any complex number α with $\Re e(\alpha) > n$ let $R_+(\alpha)$ and $R_-(\alpha)$ be the complex-valued continuous functions on V defined by

$$R_\pm(\alpha)(X) := \begin{cases} C(\alpha, n)\gamma(X)^{\frac{\alpha-n}{2}}, & \text{if } X \in J_\pm(0), \\ 0, & \text{otherwise,} \end{cases}$$

where $C(\alpha, n) := \frac{2^{1-\alpha}\pi^{\frac{2-n}{2}}}{(\frac{\alpha}{2}-1)!(\frac{\alpha-n}{2})!}$ and $z \mapsto (z-1)!$ is the Gamma function.

For $\alpha \in \mathbb{C}$ with $\Re e(\alpha) \le n$ this definition no longer yields continuous functions due to the singularities along $C_\pm(0)$. This requires a more careful definition of $R_\pm(\alpha)$ as a distribution which we will give below. Even for $\Re e(\alpha) > n$ we will from now on consider the continuous functions $R_\pm(\alpha)$ as distributions as explained in Example 1.1.5.

Since the Gamma function has no zeros the map $\alpha \mapsto C(\alpha, n)$ is holomorphic on $\mathbb{C}$. Hence for each fixed testfunction $\varphi \in \mathcal{D}(V, \mathbb{C})$ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ yields a holomorphic function on $\{\Re e(\alpha) > n\}$.

There is a natural differential operator $\Box$ acting on functions on V, $\Box f := \partial_{e_1}\partial_{e_1} f - \partial_{e_2}\partial_{e_2} f - \cdots - \partial_{e_n}\partial_{e_n} f$ where $e_1, \ldots, e_n$ is any basis of V such that $-\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = \cdots = \langle e_n, e_n\rangle = 1$ and $\langle e_i, e_j\rangle = 0$ for $i \neq j$. Such a basis $e_1, \ldots, e_n$ is called *Lorentzian orthonormal*. The operator $\Box$ is called the *d'Alembert operator*. The formula in Minkowski space with respect to the standard basis may look more familiar to the reader,

$$\Box = \frac{\partial^2}{(\partial x^1)^2} - \frac{\partial^2}{(\partial x^2)^2} - \cdots - \frac{\partial^2}{(\partial x^n)^2}.$$

The definition of the d'Alembertian on general Lorentzian manifolds can be found in the next section. In the following lemma the application of differential operators such as $\Box$ to the $R_\pm(\alpha)$ is to be taken in the distributional sense.

Lemma 1.2.2. *For all $\alpha \in \mathbb{C}$ with $\Re e(\alpha) > n$ we have*

(1) $\gamma \cdot R_\pm(\alpha) = \alpha(\alpha - n + 2)R_\pm(\alpha + 2)$,

(2) $(\operatorname{grad} \gamma) \cdot R_\pm(\alpha) = 2\alpha \ \operatorname{grad}\ R_\pm(\alpha + 2)$,

(3) $\Box R_\pm(\alpha + 2) = R_\pm(\alpha)$.

(4) *The map $\alpha \mapsto R_\pm(\alpha)$ extends uniquely to $\mathbb{C}$ as a holomorphic family of distributions. In other words, for each $\alpha \in \mathbb{C}$ there exists a unique distribution $R_\pm(\alpha)$ on V such that for each testfunction φ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ is holomorphic.*

Proof. Identity (1) follows from

$$\frac{C(\alpha, n)}{C(\alpha + 2, n)} = \frac{2^{(1-\alpha)}\ (\frac{\alpha+2}{2} - 1)!\ (\frac{\alpha+2-n}{2})!}{2^{(1-\alpha-2)}\ (\frac{\alpha}{2} - 1)!\ (\frac{\alpha-n}{2})!} = \alpha\ (\alpha - n + 2).$$

To show (2) we choose a Lorentzian orthonormal basis $e_1, \ldots, e_n$ of V and we denote differentiation in direction e_i by ∂_i. We fix a testfunction φ and integrate by parts:

$$\begin{aligned}
\partial_i \gamma \cdot R_\pm(\alpha)[\varphi] &= C(\alpha, n) \int_{J_\pm(0)} \gamma(X)^{\frac{\alpha-n}{2}}\ \partial_i \gamma(X) \varphi(X)\, dX \\
&= \frac{2C(\alpha, n)}{\alpha + 2 - n} \int_{J_\pm(0)} \partial_i (\gamma(X)^{\frac{\alpha-n+2}{2}}) \varphi(X)\, dX \\
&= -2\alpha C(\alpha + 2, n) \int_{J_\pm(0)} \gamma(X)^{\frac{\alpha-n+2}{2}}\ \partial_i \varphi(X)\, dX \\
&= -2\alpha R_\pm(\alpha + 2)[\partial_i \varphi] \\
&= 2\alpha \partial_i R_\pm(\alpha + 2)[\varphi],
\end{aligned}$$

which proves (2). Furthermore, it follows from (2) that

$$\begin{aligned}
\partial_i^2 R_\pm(\alpha+2) &= \partial_i \left(\frac{1}{2\alpha} \partial_i \gamma \cdot R_\pm(\alpha) \right) \\
&= \frac{1}{2\alpha} \left(\partial_i^2 \gamma \cdot R_\pm(\alpha) + \partial_i \gamma \cdot \left(\frac{1}{2(\alpha-2)} \partial_i \gamma \cdot R_\pm(\alpha-2) \right) \right) \\
&= \frac{1}{2\alpha} \partial_i^2 \gamma \cdot R_\pm(\alpha) + \frac{1}{4\alpha(\alpha-2)} (\partial_i \gamma)^2 \frac{(\alpha-2)(\alpha-n)}{\gamma} \cdot R_\pm(\alpha) \\
&= \left(\frac{1}{2\alpha} \partial_i^2 \gamma + \frac{\alpha-n}{4\alpha} \cdot \frac{(\partial_i \gamma)^2}{\gamma} \right) \cdot R_\pm(\alpha),
\end{aligned}$$

so that

$$\begin{aligned}
\Box\, R_\pm(\alpha+2) &= \left(\frac{n}{\alpha} + \frac{\alpha-n}{4\alpha} \cdot \frac{4\gamma}{\gamma} \right) R_\pm(\alpha) \\
&= R_\pm(\alpha).
\end{aligned}$$

To show (4) we first note that for fixed $\varphi \in \mathcal{D}(V,\mathbb{C})$ the map $\{\Re e(\alpha) > n\} \to \mathbb{C}$, $\alpha \mapsto R_\pm(\alpha)[\varphi]$, is holomorphic. For $\Re e(\alpha) > n-2$ we set

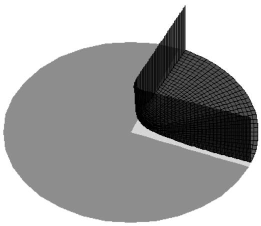
$$\widetilde{R}_\pm(\alpha) := \Box\, R_\pm(\alpha+2). \tag{1.7}$$

This defines a distribution on V. The map $\alpha \mapsto \widetilde{R}_\pm(\alpha)$ is then holomorphic on $\{\Re e(\alpha) > n-2\}$. By (3) we have $\widetilde{R}_\pm(\alpha) = R_\pm(\alpha)$ for $\Re e(\alpha) > n$, so that $\alpha \mapsto \widetilde{R}_\pm(\alpha)$ extends $\alpha \mapsto R_\pm(\alpha)$ holomorphically to $\{\Re e(\alpha) > n-2\}$. We proceed inductively and construct a holomorphic extension of $\alpha \mapsto R_\pm(\alpha)$ on $\{\Re e(\alpha) > n-2k\}$ (where $k \in \mathbb{N} \setminus \{0\}$) from that on $\{\Re e(\alpha) > n-2k+2\}$ just as above. Note that these extensions necessarily coincide on their common domain since they are holomorphic and they coincide on an open subset of $\mathbb{C}$. We therefore obtain a holomorphic extension of $\alpha \mapsto R_\pm(\alpha)$ to the whole of $\mathbb{C}$, which is necessarily unique. $\Box$

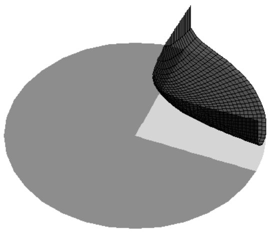
Lemma 1.2.2 (4) defines $R_\pm(\alpha)$ for all $\alpha \in \mathbb{C}$, not as functions but as distributions.

Definition 1.2.3. We call $R_+(\alpha)$ the *advanced Riesz distribution* and $R_-(\alpha)$ the *retarded Riesz distribution* on V for $\alpha \in \mathbb{C}$.

The following illustration shows the graphs of Riesz distributions $R_+(\alpha)$ for $n=2$ and various values of α. In particular, one sees the singularities along $C_+(0)$ for $\Re e(\alpha) \leq 2$.



$\alpha = 0.1$



$\alpha = 1$

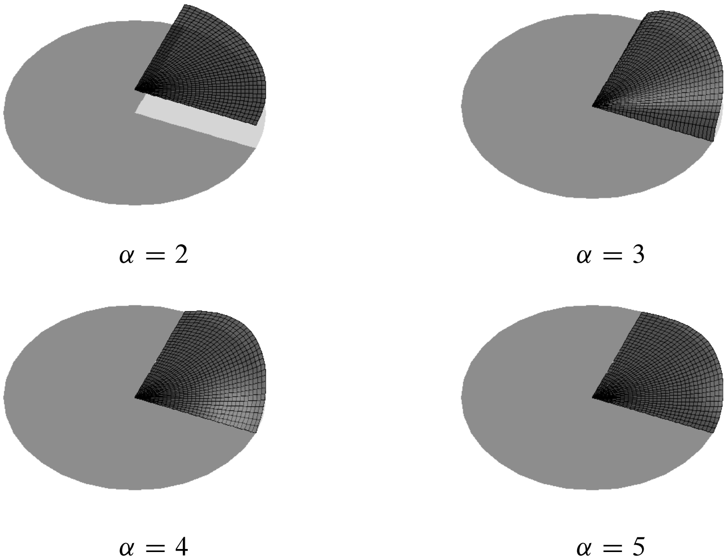


Figure 2. Graphs of Riesz distributions $R_+(\alpha)$ in two dimensions.

We now collect the important facts on Riesz distributions.

Proposition 1.2.4. *The following holds for all* $\alpha \in \mathbb{C}$:

(1) $\gamma \cdot R_\pm(\alpha) = \alpha(\alpha - n + 2)\, R_\pm(\alpha + 2)$.
(2) $(\operatorname{grad}\gamma) R_\pm(\alpha) = 2\alpha \operatorname{grad}(R_\pm(\alpha + 2))$.
(3) $\Box R_\pm(\alpha + 2) = R_\pm(\alpha)$.
(4) *For every* $\alpha \in \mathbb{C} \setminus (\{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\})$, *we have*
$$\operatorname{supp}(R_\pm(\alpha)) = J_\pm(0) \quad \textit{and} \quad \operatorname{sing\,supp}(R_\pm(\alpha)) \subset C_\pm(0).$$
(5) *For every* $\alpha \in \{0, -2, -4, \dots\} \cup \{n-2, n-4, \dots\}$, *we have* $\operatorname{supp}(R_\pm(\alpha)) = \operatorname{sing\,supp}(R_\pm(\alpha)) \subset C_\pm(0)$.
(6) *For* $n \geq 3$ *and* $\alpha = n-2, n-4, \dots, 1$ *or* 2 *respectively, we have* $\operatorname{supp}(R_\pm(\alpha)) = \operatorname{sing\,supp}(R_\pm(\alpha)) = C_\pm(0)$.
(7) $R_\pm(0) = \delta_0$.
(8) *For* $\Re e(\alpha) > 0$ *the order of* $R_\pm(\alpha)$ *is bounded from above by* $n + 1$.
(9) *If* $\alpha \in \mathbb{R}$, *then* $R_\pm(\alpha)$ *is real, i.e.,* $R_\pm(\alpha)[\varphi] \in \mathbb{R}$ *for all* $\varphi \in \mathcal{D}(V, \mathbb{R})$.

Proof. Assertions (1), (2), and (3) hold for $\Re e(\alpha) > n$ by Lemma 1.2.2. Since, after insertion of a fixed $\varphi \in \mathcal{D}(V, \mathbb{C})$, all expressions in these equations are holomorphic in α they hold for all α.

(4). Let $\varphi \in \mathcal{D}(V, \mathbb{C})$ with $\operatorname{supp}(\varphi) \cap J_\pm(0) = \emptyset$. Since $\operatorname{supp}(R_\pm(\alpha)) \subset J_\pm(0)$ for $\Re e(\alpha) > n$, it follows for those α that
$$R_\pm(\alpha)[\varphi] = 0,$$

and then for all α by Lemma 1.2.2 (4). Therefore $\operatorname{supp}(R_\pm(\alpha)) \subset J_\pm(0)$ for all α.

On the other hand, if $X \in I_\pm(0)$, then $\gamma(X) > 0$ and the map $\alpha \mapsto C(\alpha,n)\gamma(X)^{\frac{\alpha-n}{2}}$ is well defined and holomorphic on all of $\mathbb{C}$. By Lemma 1.2.2 (4) we have for $\varphi \in \mathcal{D}(V,\mathbb{C})$ with $\operatorname{supp}(\varphi) \subset I_\pm(0)$

$$R_\pm(\alpha)[\varphi] = \int_{\operatorname{supp}(\varphi)} C(\alpha,n)\gamma(X)^{\frac{\alpha-n}{2}}\varphi(X)\,dX$$

for *every* $\alpha \in \mathbb{C}$. Thus $R_\pm(\alpha)$ coincides on $I_\pm(0)$ with the smooth function $C(\alpha,n)\gamma(\cdot)^{\frac{\alpha-n}{2}}$ and therefore $\operatorname{sing\,supp}(R_\pm(\alpha)) \subset C_\pm(0)$. Since furthermore the function $\alpha \mapsto C(\alpha,n)$ vanishes only on $\{0,-2,-4,\dots\} \cup \{n-2,n-4,\dots\}$ (caused by the poles of the Gamma function), we have $I_\pm(0) \subset \operatorname{supp}(R_\pm(\alpha))$ for every $\alpha \in \mathbb{C} \setminus (\{0,-2,-4,\dots\} \cup \{n-2,n-4,\dots\})$. Thus $\operatorname{supp}(R_\pm(\alpha)) = J_\pm(0)$. This proves (4).

(5). For $\alpha \in \{0,-2,-4,\dots\} \cup \{n-2,n-4,\dots\}$ we have $C(\alpha,n) = 0$ and therefore $I_\pm(0) \cap \operatorname{supp}(R_\pm(\alpha)) = \emptyset$. Hence $\operatorname{sing\,supp}(R_\pm(\alpha)) \subset \operatorname{supp}(R_\pm(\alpha)) \subset C_\pm(0)$. It remains to show $\operatorname{supp}(R_\pm(\alpha)) \subset \operatorname{sing\,supp}(R_\pm(\alpha))$. Let $X \notin \operatorname{sing\,supp}(R_\pm(\alpha))$. Then $R_\pm(\alpha)$ coincides with a smooth function f on a neighborhood of X. Since $\operatorname{supp}(R_\pm(\alpha)) \subset C_\pm(0)$ and since $C_\pm(0)$ has a dense complement in V, we have $f \equiv 0$. Thus $X \notin \operatorname{supp}(R_\pm(\alpha))$. This proves (5).

Before we proceed to the next point we derive a more explicit formula for the Riesz distributions evaluated on testfunctions of a particular form. Introduce linear coordinates $x^1,\dots,x^n$ on V such that $\gamma(x) = -(x^1)^2 + (x^2)^2 + \dots + (x^n)^2$ and such that the x^1-axis is future directed. Let $f \in \mathcal{D}(\mathbb{R},\mathbb{C})$ and $\psi \in \mathcal{D}(\mathbb{R}^{n-1},\mathbb{C})$ and put $\varphi(x) := f(x^1)\psi(\hat{x})$ where $\hat{x} = (x^2,\dots,x^n)$. Choose the function ψ such that on $J_+(0)$ we have $\varphi(x) = f(x^1)$.

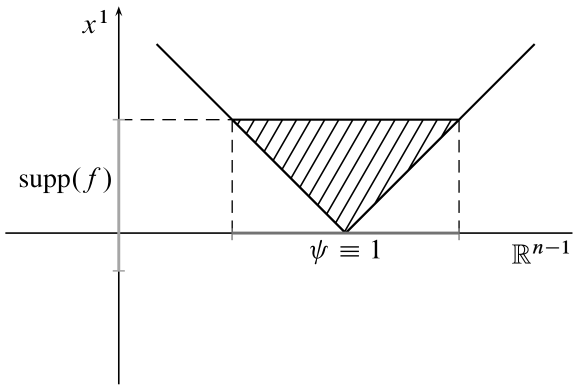


Figure 3. Support of φ.

Claim: If $\Re\mathrm{e}(\alpha) > 1$, then

$$R_+(\alpha)[\varphi] = \frac{1}{(\alpha-1)!}\int_0^\infty r^{\alpha-1} f(r)\,dr.$$

Proof of the claim. Since both sides of the equation are holomorphic in α for $\Re e(\alpha) > 1$ it suffices to show it for $\Re e(\alpha) > n$. In that case we have by the definition of $R_+(\alpha)$

$$\begin{aligned}
R_+(\alpha)[\varphi] &= C(\alpha,n)\int_{J_+(0)} \varphi(X)\gamma(X)^{\frac{\alpha-n}{2}}\,dX \\
&= C(\alpha,n)\int_0^\infty \int_{\{|\hat{x}|<x^1\}} \varphi(x^1,\hat{x})((x^1)^2-|\hat{x}|^2)^{\frac{\alpha-n}{2}}\,d\hat{x}\,dx^1 \\
&= C(\alpha,n)\int_0^\infty f(x^1)\int_{\{|\hat{x}|<x^1\}} ((x^1)^2-|\hat{x}|^2)^{\frac{\alpha-n}{2}}\,d\hat{x}\,dx^1 \\
&= C(\alpha,n)\int_0^\infty f(x^1)\int_0^{x^1}\int_{S^{n-2}} ((x^1)^2-t^2)^{\frac{\alpha-n}{2}}t^{n-2}d\omega\,dt\,dx^1,
\end{aligned}$$

where S^{n-2} is the $(n-2)$-dimensional round sphere and $d\omega$ its standard volume element. Renaming x^1 we get

$$R_+(\alpha)[\varphi] = \mathrm{vol}(S^{n-2})\,C(\alpha,n)\int_0^\infty f(r)\int_0^r (r^2-t^2)^{\frac{\alpha-n}{2}}t^{n-2}dt\,dr.$$

Using $\int_0^r (r^2-t^2)^{\frac{\alpha-n}{2}}t^{n-2}dt = \frac{1}{2}r^{\alpha-1}\frac{(\frac{\alpha-n}{2})!(\frac{n-3}{2})!}{(\frac{\alpha-1}{2})!}$ we obtain

$$\begin{aligned}
R_+(\alpha)[\varphi] &= \frac{\mathrm{vol}(S^{n-2})}{2}\,C(\alpha,n)\int_0^\infty f(r)r^{\alpha-1}\frac{(\frac{\alpha-n}{2})!(\frac{n-3}{2})!}{(\frac{\alpha-1}{2})!}dr \\
&= \frac{1}{2}\frac{2\pi^{(n-1)/2}}{(\frac{n-1}{2}-1)!}\cdot\frac{2^{1-\alpha}\pi^{1-n/2}}{(\alpha/2-1)!(\frac{\alpha-n}{2})!}\cdot\frac{(\frac{\alpha-n}{2})!(\frac{n-3}{2})!}{(\frac{\alpha-1}{2})!}\cdot\int_0^\infty f(r)r^{\alpha-1}dr \\
&= \frac{\sqrt{\pi}\cdot 2^{1-\alpha}}{(\alpha/2-1)!(\frac{\alpha-1}{2})!}\cdot\int_0^\infty f(r)r^{\alpha-1}dr.
\end{aligned}$$

Legendre's duplication formula (see [Jeffrey1995, p. 218])

$$\left(\frac{\alpha}{2}-1\right)!\left(\frac{\alpha+1}{2}-1\right)! = 2^{1-\alpha}\sqrt{\pi}\,(\alpha-1)! \tag{1.8}$$

yields the claim.

To show (6) recall first from (5) that we know already

$$\mathrm{sing\,supp}(R_\pm(\alpha)) = \mathrm{supp}(R_\pm(\alpha)) \subset C_\pm(0)$$

for $\alpha = n-2, n-4, \ldots, 2$ or 1 respectively. Note also that the distribution $R_\pm(\alpha)$ is invariant under time-orientation preserving Lorentz transformations, that is, for any such transformation A of V we have

$$R_\pm(\alpha)[\varphi\circ A] = R_\pm(\alpha)[\varphi]$$

for every testfunction φ. Hence $\operatorname{supp}(R_\pm(\alpha))$ as well as $\operatorname{sing\,supp}(R_\pm(\alpha))$ are also invariant under the group of those transformations. Under the action of this group the orbit decomposition of $C_\pm(0)$ is given by

$$C_\pm(0) = \{0\} \cup (C_\pm(0) \setminus \{0\}).$$

Thus $\operatorname{supp}(R_\pm(\alpha)) = \operatorname{sing\,supp}(R_\pm(\alpha))$ coincides either with $\{0\}$ or with $C_\pm(0)$.

The claim shows for the testfunctions φ considered there

$$R_+(2)[\varphi] = \int_0^\infty r f(r)\, dr.$$

Hence the support of $R_+(2)$ cannot be contained in $\{0\}$. If n is even, we conclude $\operatorname{supp}(R_+(2)) = C_+(0)$ and then also $\operatorname{supp}(R_+(\alpha)) = C_+(0)$ for $\alpha = 2, 4, \ldots, n-2$.

Taking the limit $\alpha \searrow 1$ in the claim yields

$$R_+(1)[\varphi] = \int_0^\infty f(r)\, dr.$$

Now the same argument shows for odd n that $\operatorname{supp}(R_+(1)) = C_+(0)$ and then also $\operatorname{supp}(R_+(\alpha)) = C_+(0)$ for $\alpha = 1, 3, \ldots, n-2$. This concludes the proof of (6).

Proof of (7). Fix a compact subset $K \subset V$. Let $\sigma_K \in \mathcal{D}(V, \mathbb{R})$ be a function such that $\sigma_{|K} \equiv 1$. For any $\varphi \in \mathcal{D}(V, \mathbb{C})$ with $\operatorname{supp}(\varphi) \subset K$ write

$$\varphi(x) = \varphi(0) + \sum_{j=1}^n x^j \varphi_j(x)$$

with suitable smooth functions φ_j. Then

$$\begin{aligned} R_\pm(0)[\varphi] &= R_\pm(0)[\sigma_K \varphi] \\ &= R_\pm(0)[\varphi(0)\sigma_K + \sum_{j=1}^n x^j \sigma_K \varphi_j] \\ &= \varphi(0) \underbrace{R_\pm(0)[\sigma_K]}_{=:c_K} + \sum_{j=1}^n \underbrace{(x^j R_\pm(0))}_{=0 \text{ by } (2)}[\sigma_K \varphi_j] \\ &= c_K \varphi(0). \end{aligned}$$

The constant c_K actually does not depend on K since for $K' \supset K$ and $\operatorname{supp}(\varphi) \subset K$,

$$c_{K'}\varphi(0) = R_+(0)[\varphi] = c_K \varphi(0),$$

so that $c_K = c_{K'} =: c$. It remains to show $c = 1$.

We again look at testfunctions φ as in the claim and compute using (3)

$$\begin{aligned} c \cdot \varphi(0) &= R_+(0)[\varphi] \\ &= R_+(2)[\Box\varphi] \\ &= \int_0^\infty r f''(r)\, dr \\ &= -\int_0^\infty f'(r)\, dr \\ &= f(0) \\ &= \varphi(0). \end{aligned}$$

This concludes the proof of (7).

Proof of (8). By its definition, the distribution $R_\pm(\alpha)$ is a continuous function if $\mathfrak{Re}(\alpha) > n$, therefore it is of order 0. Since $\Box$ is a differential operator of order 2, the order of $\Box R_\pm(\alpha)$ is at most that of $R_\pm(\alpha)$ plus 2. It then follows from (3) that:

• If n is even: for every α with $\mathfrak{Re}(\alpha) > 0$ we have $\mathfrak{Re}(\alpha)+n = \mathfrak{Re}(\alpha)+2\cdot\frac{n}{2} > n$, so that the order of $R_\pm(\alpha)$ is not greater than n (and so $n+1$).

• If n is odd: for every α with $\mathfrak{Re}(\alpha) > 0$ we have $\mathfrak{Re}(\alpha) + n + 1 = \mathfrak{Re}(\alpha) + 2 \cdot \frac{n+1}{2} > n$, so that the order of $R_\pm(\alpha)$ is not greater than $n+1$.

This concludes the proof of (8).

Assertion (9) is clear by definition whenever $\alpha > n$. For general $\alpha \in \mathbb{R}$ choose $k \in \mathbb{N}$ so large that $\alpha + 2k > n$. Using (3) we get for any $\varphi \in \mathcal{D}(V, \mathbb{R})$

$$R_\pm(\alpha)[\varphi] = \Box^k R_\pm(\alpha + 2k)[\varphi] = R_\pm(\alpha + 2k)[\Box^k\varphi] \in \mathbb{R}$$

because $\Box^k\varphi \in \mathcal{D}(V, \mathbb{R})$ as well. $\Box$

In the following we will need a slight generalization of Lemma 1.2.2 (4):

Corollary 1.2.5. *For $\varphi \in \mathcal{D}^k(V, \mathbb{C})$ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ defines a holomorphic function on $\{\alpha \in \mathbb{C} \mid \mathfrak{Re}(\alpha) > n - 2[\frac{k}{2}]\}$.*

Proof. Let $\varphi \in \mathcal{D}^k(V, \mathbb{C})$. By the definition of $R_\pm(\alpha)$ the map $\alpha \mapsto R_\pm(\alpha)[\varphi]$ is clearly holomorphic on $\{\mathfrak{Re}(\alpha) > n\}$. Using (3) of Proposition 1.2.4 we get the holomorphic extension to the set $\{\mathfrak{Re}(\alpha) > n - 2[\frac{k}{2}]\}$. $\Box$

1.3 Lorentzian geometry

We now summarize basic concepts of Lorentzian geometry. We will assume familiarity with semi-Riemannian manifolds, geodesics, the Riemannian exponential map etc. A summary of basic notions in differential geometry can be found in Appendix A.3. A thorough introduction to Lorentzian geometry can e.g. be found in [Beem–Ehrlich–Easley1996] or in [O'Neill1983]. Further results of more technical nature which could distract the reader at a first reading but which will be needed later are collected in Appendix A.5.

Let M be a time-oriented Lorentzian manifold. A piecewise C^1-curve in M is called *timelike, lightlike, causal, spacelike, future directed*, or *past directed* if its tangent vectors are timelike, lightlike, causal, spacelike, future directed, or past directed respectively. A piecewise C^1-curve in M is called *inextendible*, if no piecewise C^1-reparametrization of the curve can be continuously extended to any of the end points of the parameter interval.

The *chronological future* $I_+^M(x)$ of a point $x \in M$ is the set of points that can be reached from x by future directed timelike curves. Similarly, the *causal future* $J_+^M(x)$ of a point $x \in M$ consists of those points that can be reached from x by causal curves and of x itself. In the following, the notation $x < y$ (or $x \leq y$) will mean $y \in I_+^M(x)$ (or $y \in J_+^M(x)$ respectively). The *chronological future* of a subset $A \subset M$ is defined to be $I_+^M(A) := \bigcup_{x \in A} I_+^M(x)$. Similarly, the *causal future* of A is $J_+^M(A) := \bigcup_{x \in A} J_+^M(x)$. The *chronological past* $I_-^M(A)$ and the *causal past* $J_-^M(A)$ are defined by replacing future directed curves by past directed curves. One has in general that $I_\pm^M(A)$ is the interior of $J_\pm^M(A)$ and that $J_\pm^M(A)$ is contained in the closure of $I_\pm^M(A)$. The chronological future and past are open subsets but the causal future and past are not always closed even if A is closed (see also Section A.5 in the appendix).

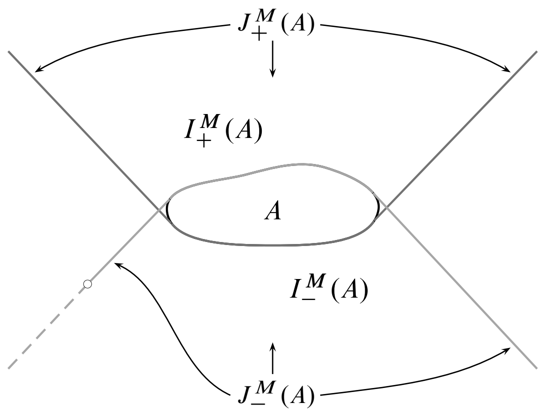


Figure 4. Causal and chronological future and past of subset A of Minkowski space with one point removed.

We will also use the notation $J^M(A) := J_-^M(A) \cup J_+^M(A)$. A subset $A \subset M$ is called *past compact* if $A \cap J_-^M(p)$ is compact for all $p \in M$. Similarly, one defines *future compact* subsets.

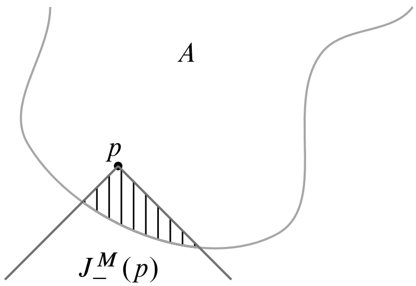


Figure 5. The subset A is past compact.

Definition 1.3.1. A subset $\Omega \subset M$ in a time-oriented Lorentzian manifold is called *causally compatible* if for all points $x \in \Omega$

$$J_\pm^\Omega(x) = J_\pm^M(x) \cap \Omega$$

holds.

Note that the inclusion "$\subset$" always holds. The condition of being causally compatible means that whenever two points in Ω can be joined by a causal curve in M this can also be done inside Ω.

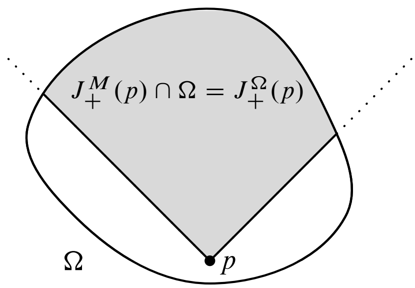


Figure 6. Causally compatible subset of Minkowski space.

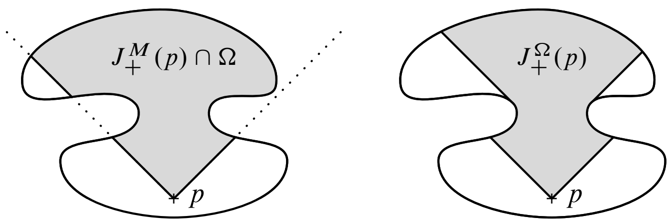


Figure 7. Domain which is not causally compatible in Minkowski space.

If $\Omega \subset M$ is a causally compatible domain in a time-oriented Lorentzian manifold, then we immediately see that for each subset $A \subset \Omega$ we have

$$J_\pm^\Omega(A) = J_\pm^M(A) \cap \Omega.$$

Note also that being causally compatible is transitive: If $\Omega \subset \Omega' \subset \Omega''$, if Ω is causally compatible in Ω', and if Ω' is causally compatible in Ω'', then so is Ω in Ω''.

Definition 1.3.2. A domain $\Omega \subset M$ in a Lorentzian manifold is called

- *geodesically starshaped* with respect to a fixed point $x \in \Omega$ if there exists an open subset $\Omega' \subset T_xM$, starshaped with respect to 0, such that the Riemannian exponential map $\exp_x$ maps Ω' diffeomorphically onto Ω;
- *geodesically convex* (or simply *convex*) if it is geodesically starshaped with respect to all of its points.

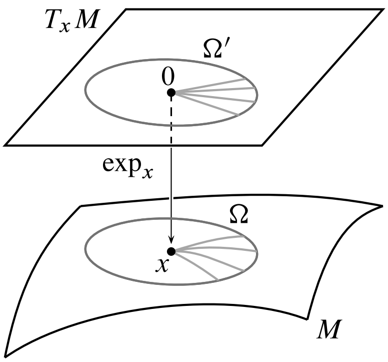


Figure 8. Ω is geodesically starshaped with respect to x.

If Ω is geodesically starshaped with respect to x, then $\exp_x(I_\pm(0) \cap \Omega') = I_\pm^\Omega(x)$ and $\exp_x(J_\pm(0) \cap \Omega') = J_\pm^\Omega(x)$. We put $C_\pm^\Omega(x) := \exp_x(C_\pm(0) \cap \Omega')$.

On a geodesically starshaped domain Ω we define the smooth positive function $\mu_x : \Omega \to \mathbb{R}$ by

$$\mathrm{dV} = \mu_x \cdot (\exp_x^{-1})^* (dz), \tag{1.9}$$

where dV is the Lorentzian volume density and dz is the standard volume density on $T_x\Omega$. In other words, $\mu_x = \det(d\exp_x) \circ \exp_x^{-1}$. In normal coordinates about x, $\mu_x = \sqrt{|\det(g_{ij})|}$.

For each open covering of a Lorentzian manifold there exists a refinement consisting of convex open subsets, see [O'Neill1983, Chap. 5, Lemma 10].

Definition 1.3.3. A domain Ω is called *causal* if $\overline{\Omega}$ is contained in a convex domain Ω' and if for any $p, q \in \overline{\Omega}$ the intersection $J_+^{\Omega'}(p) \cap J_-^{\Omega'}(q)$ is compact and contained in $\overline{\Omega}$.

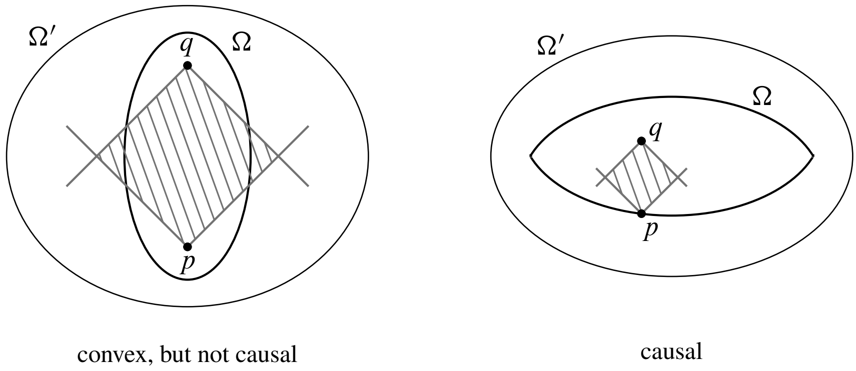


Figure 9. Convexity versus causality.

Definition 1.3.4. A subset S of a connected time-oriented Lorentzian manifold is called *achronal* (or *acausal*) if and only if each timelike (respectively causal) curve meets S at most once.

A subset S of a connected time-oriented Lorentzian manifold is a *Cauchy hypersurface* if each inextendible timelike curve in M meets S at exactly one point.

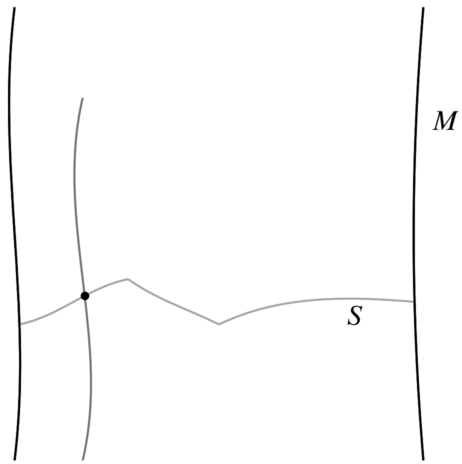


Figure 10. Cauchy hypersurface S met by a timelike curve.

Obviously every acausal subset is achronal, but the reverse is wrong. However, every achronal spacelike hypersurface is acausal (see Lemma 42 from Chap. 14 in [O'Neill1983]).

Any Cauchy hypersurface is achronal. Moreover, it is a closed topological hypersurface and it is hit by each inextendible causal curve in at least one point. Any two Cauchy hypersurfaces in M are homeomorphic. Furthermore, the causal future and past of a Cauchy hypersurface is past- and future-compact respectively. This is a consequence of e.g. [O'Neill1983, Ch. 14, Lemma 40].

Definition 1.3.5. The *Cauchy development* of a subset S of a time-oriented Lorentzian manifold M is the set $D(S)$ of points of M through which every inextendible causal curve in M meets S.

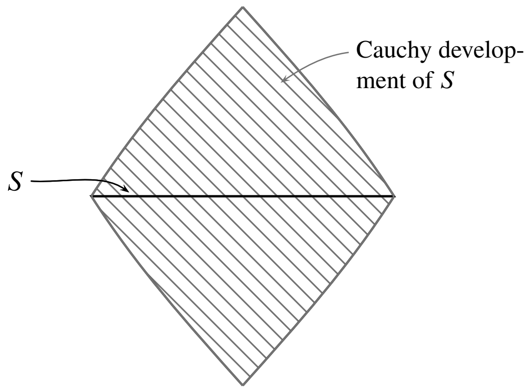


Figure 11. Cauchy development.

Remark 1.3.6. It follows from the definition that $D(D(S)) = D(S)$ for every subset $S \subset M$. Hence if $T \subset D(S)$, then $D(T) \subset D(D(S)) = D(S)$.

Of course, if S is achronal, then every inextendible causal curve in M meets S at most once. The Cauchy development $D(S)$ of every *acausal* hypersurface S is open, see [O'Neill1983, Chap. 14, Lemma 43].

Definition 1.3.7. A Lorentzian manifold is said to satisfy the *causality condition* if it does not contain any closed causal curve.

A Lorentzian manifold is said to satisfy the *strong causality condition* if there are no almost closed causal curves. More precisely, for each point $p \in M$ and for each open neighborhood U of p there exists an open neighborhood $V \subset U$ of p such that each causal curve in M starting and ending in V is entirely contained in U.

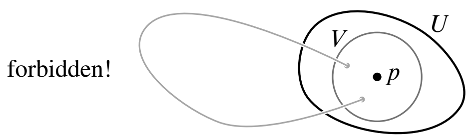


Figure 12. Strong causality condition.

Obviously, the strong causality condition implies the causality condition. Convex open subsets of a Lorentzian manifold satisfy the strong causality condition.

Definition 1.3.8. A connected time-oriented Lorentzian manifold is called *globally hyperbolic* if it satisfies the strong causality condition and if for all $p, q \in M$ the intersection $J_+^M(p) \cap J_-^M(q)$ is compact.

Remark 1.3.9. If M is a globally hyperbolic Lorentzian manifold, then a nonempty open subset $\Omega \subset M$ is itself globally hyperbolic if and only if for any $p, q \in \Omega$ the intersection $J_+^\Omega(p) \cap J_-^\Omega(q) \subset \Omega$ is compact. Indeed non-existence of almost closed causal curves in M directly implies non-existence of such curves in Ω.

We now state a very useful characterization of globally hyperbolic manifolds.

Theorem 1.3.10. *Let M be a connected time-oriented Lorentzian manifold. Then the following are equivalent:*

(1) *M is globally hyperbolic.*
(2) *There exists a Cauchy hypersurface in M.*
(3) *M is isometric to $\mathbb{R} \times S$ with metric $-\beta dt^2 + g_t$ where β is a smooth positive function, g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface in M.*

Proof. Using work of Geroch [Geroch1970, Thm. 11], it has been shown by Bernal and Sánchez in [Bernal–Sánchez2005, Thm. 1.1] that (1) implies (3). See also [Ellis–Hawking1973, Prop. 6.6.8] and [Wald1984, p. 209] for earlier mentionings of this fact. That (3) implies (2) is trivial and that (2) implies (1) is well-known, see e.g. [O'Neill1983, Cor. 39, p. 422]. □

Examples 1.3.11. Minkowski space is globally hyperbolic. Every spacelike hyperplane is a Cauchy hypersurface. One can write Minkowski space as $\mathbb{R} \times \mathbb{R}^{n-1}$ with the metric $-dt^2 + g_t$ where g_t is the Euclidean metric on $\mathbb{R}^{n-1}$ and does not depend on t.

Let (S, g_0) be a connected Riemannian manifold and $I \subset \mathbb{R}$ an interval. The manifold $M = I \times S$ with the metric $g = -dt^2 + g_0$ is globally hyperbolic if and only if (S, g_0) is complete. This applies in particular if S is compact.

More generally, if $f : I \to \mathbb{R}$ is a smooth positive function we may equip $M = I \times S$ with the metric $g = -dt^2 + f(t)^2 \cdot g_0$. Again, (M, g) is globally hyperbolic if and only if (S, g_0) is complete, see Lemma A.5.14. *Robertson–Walker spacetimes* and, in particular, *Friedmann cosmological models*, are of this type. They are used to discuss big bang, expansion of the universe, and cosmological redshift, compare [Wald1984, Ch. 5 and 6] or [O'Neill1983, Ch. 12]. Another example of this type is *deSitter spacetime*, where $I = \mathbb{R}$, $S = S^{n-1}$, g_0 is the canonical metric of S^{n-1} of constant sectional curvature 1, and $f(t) = \cosh(t)$. *Anti-deSitter spacetime* which we will discuss in more detail in Section 3.5 is not globally hyperbolic.

The interior and exterior *Schwarzschild spacetimes* are globally hyperbolic. They model the universe in the neighborhood of a massive static rotationally symmetric body such as a black hole. They are used to investigate perihelion advance of Mercury, the bending of light near the sun and other astronomical phenomena, see [Wald1984, Ch. 6] and [O'Neill1983, Ch. 13].

Corollary 1.3.12. *On every globally hyperbolic Lorentzian manifold M there exists a smooth function $h: M \to \mathbb{R}$ whose gradient is past directed timelike at every point and all of whose level-sets are spacelike Cauchy hypersurfaces.*

Proof. Define h to be the composition $t \circ \Phi$ where $\Phi: M \to \mathbb{R} \times S$ is the isometry given in Theorem 1.3.10 and $t: \mathbb{R} \times S \to \mathbb{R}$ is the projection onto the first factor. □

Such a function h on a globally hyperbolic Lorentzian manifold will be referred to as a *Cauchy time-function*. Note that a Cauchy time-function is strictly monotonically increasing along any future directed causal curve.

We quote an enhanced form of Theorem 1.3.10, due to A. Bernal and M. Sánchez (see [Bernal–Sánchez2006, Theorem 1.2]), which will be needed in Chapter 3.

Theorem 1.3.13. *Let M be a globally hyperbolic manifold and S be a spacelike smooth Cauchy hypersurface in M. Then there exists a Cauchy time-function $h: M \to \mathbb{R}$ such that $S = h^{-1}(\{0\})$.* □

Any given smooth spacelike Cauchy hypersurface in a (necessarily globally hyperbolic) Lorentzian manifold is therefore the leaf of a foliation by smooth spacelike Cauchy hypersurfaces.

Recall that the *length* $L[c]$ of a piecewise C^1-curve $c: [a,b] \to M$ on a Lorentzian manifold (M,g) is defined by

$$L[c] := \int_a^b \sqrt{|g(\dot{c}(t),\dot{c}(t))|}\,dt.$$

Definition 1.3.14. The *time-separation* on a Lorentzian manifold (M,g) is the function $\tau: M \times M \to \mathbb{R} \cup \{\infty\}$ defined by

$$\tau(p,q) := \begin{cases} \sup\{L[c] \mid c \text{ future directed causal curve from } p \text{ to } q, & \text{if } p < q \\ 0, & \text{otherwise,} \end{cases}$$

for all p, q in M.

The properties of τ which will be needed later are the following:

Proposition 1.3.15. *Let M be a time-oriented Lorentzian manifold. Let p, q, and $r \in M$. Then*

(1) $\tau(p,q) > 0$ *if and only if* $q \in I_+^M(p)$.

(2) *The function τ is lower semi-continuous on $M \times M$. If M is convex or globally hyperbolic, then τ is finite and continuous.*

(3) *The function* τ *satisfies the* inverse triangle inequality*: If* $p \leq q \leq r$*, then*

$$\tau(p,r) \geq \tau(p,q) + \tau(q,r). \tag{1.10}$$

See e.g. Lemmas 16, 17, and 21 from Chapter 14 in [O'Neill1983] for a proof. □

Now let M be a Lorentzian manifold. For a differentiable function $f : M \to \mathbb{R}$, the *gradient* of f is the vector field

$$\operatorname{grad} f := (df)^\sharp. \tag{1.11}$$

Here $\omega \mapsto \omega^\sharp$ denotes the canonical isomorphism $T^*M \to TM$ induced by the Lorentzian metric, i.e., for $\omega \in T_x^*M$ the vector $\omega^\sharp \in T_xM$ is characterized by the fact that $\omega(X) = \langle \omega^\sharp, X\rangle$ for all $X \in T_xM$. The inverse isomorphism $TM \to T^*M$ is denoted by $X \mapsto X^\flat$. One easily checks that for differentiable functions $f, g : M \to \mathbb{R}$

$$\operatorname{grad}(fg) = g \operatorname{grad} f + f \operatorname{grad} g. \tag{1.12}$$

Locally, the gradient of f can be written as

$$\operatorname{grad} f = \sum_{j=1}^{n} \varepsilon_j \, df(e_j)\, e_j$$

where $e_1, \ldots, e_n$ is a local Lorentz orthonormal frame of TM, $\varepsilon_j = \langle e_j, e_j\rangle = \pm 1$. For a differentiable vector field X on M the *divergence* is the function

$$\operatorname{div} X := \operatorname{tr}(\nabla X) = \sum_{j=1}^{n} \varepsilon_j \langle e_j, \nabla_{e_j} X\rangle.$$

If X is a differentiable vector field and f a differentiable function on M, then one immediately sees that

$$\operatorname{div}(fX) = f \operatorname{div} X + \langle \operatorname{grad} f, X\rangle. \tag{1.13}$$

There is another way to characterize the divergence. Let dV be the volume form induced by the Lorentzian metric. Inserting the vector field X yields an $(n-1)$-form $\mathrm{dV}(X, \cdot, \ldots, \cdot)$. Hence $d(\mathrm{dV}(X, \cdot, \ldots, \cdot))$ is an n-form and can therefore be written as a function times dV, namely

$$d(\mathrm{dV}(X, \cdot, \ldots, \cdot) = \operatorname{div} X \cdot \mathrm{dV}. \tag{1.14}$$

This shows that the divergence operator depends only mildly on the Lorentzian metric. If two Lorentzian (or more generally, semi-Riemannian) metrics have the same volume form, then they also have the same divergence operator. This is certainly not true for the gradient.

The divergence is important because of Gauss' divergence theorem:

Theorem 1.3.16. *Let M be a Lorentzian manifold and let $D \subset M$ be a domain with piecewise smooth boundary. We assume that the induced metric on the smooth part of the boundary is non-degenerate, i.e., it is either Riemannian or Lorentzian on each connected component. Let $\mathfrak{n}$ denote the exterior normal field along ∂D, normalized to $\langle \mathfrak{n}, \mathfrak{n} \rangle =: \varepsilon_{\mathfrak{n}} = \pm 1$.*

Then for every smooth vector field X on M such that $\mathrm{supp}(X) \cap \overline{D}$ *is compact we have*

$$\int_D \mathrm{div}(X)\ \mathrm{dV} = \int_{\partial D} \varepsilon_{\mathfrak{n}} \langle X, \mathfrak{n} \rangle\, \mathrm{dA} \qquad \square$$

where dA *is the induced volume element on ∂D.*

Let $e_1, \dots, e_n$ be a Lorentz orthonormal basis of $T_x M$. Then $(\xi^1, \dots, \xi^n) \mapsto \exp_x(\sum_j \xi^j e_j)$ is a local diffeomorphism of a neighborhood of 0 in $\mathbb{R}^n$ onto a neighborhood of x in M. This defines coordinates $\xi^1, \dots, \xi^n$ on any open neighborhood of x which is geodesically starshaped with respect to x. Such coordinates are called *normal coordinates* about the point x.

We express the vector X in normal coordinates about x and write $X = \sum_j \eta^j \frac{\partial}{\partial \xi^j}$. From (1.14) we conclude, using $\mathrm{dV} = \mu_x \cdot d\xi^1 \wedge \cdots \wedge d\xi^n$

$$\begin{aligned}
\mathrm{div}(\mu_x^{-1} X) \cdot \mathrm{dV} &= d(\, \mathrm{dV}(\mu_x^{-1} X, \cdot, \dots, \cdot)) \\
&= d\Big(\sum_j (-1)^{j-1} \eta^j\, d\xi^1 \wedge \cdots \wedge \widehat{d\xi^j} \wedge \cdots \wedge d\xi^n \Big) \\
&= \sum_j (-1)^{j-1}\, d\eta^j \wedge d\xi^1 \wedge \cdots \wedge \widehat{d\xi^j} \wedge \cdots \wedge d\xi^n \\
&= \sum_j \frac{\partial \eta^j}{\partial \xi^j}\, d\xi^1 \wedge \cdots \wedge d\xi^n \\
&= \sum_j \frac{\partial \eta^j}{\partial \xi^j}\, \mu_x^{-1}\ \mathrm{dV}.
\end{aligned}$$

Thus

$$\mu_x\ \mathrm{div}(\mu_x^{-1} X) = \sum_j \frac{\partial \eta^j}{\partial \xi^j}. \tag{1.15}$$

For a C^2-function f the *Hessian* at x is the symmetric bilinear form

$$\mathrm{Hess}(f)|_x \colon T_x M \times T_x M \to \mathbb{R}, \quad \mathrm{Hess}(f)|_x(X, Y) := \langle \nabla_X \operatorname{grad} f, Y \rangle.$$

The *d'Alembert operator* is defined by

$$\square f := -\,\mathrm{tr}(\mathrm{Hess}(f)) = -\operatorname{div}\operatorname{grad} f.$$

If $f \colon M \to \mathbb{R}$ and $F \colon \mathbb{R} \to \mathbb{R}$ are C^2 a straightforward computation yields

$$\square(F \circ f) = -(F'' \circ f)\langle df, df \rangle + (F' \circ f)\square f. \tag{1.16}$$

Lemma 1.3.17. *Let Ω be a domain in M, geodesically starshaped with respect to $x \in \Omega$. Then the function μ_x defined in (1.9) satisfies*

$$\mu_x(x) = 1, \quad d\mu_x|_x = 0, \quad \mathrm{Hess}(\mu_x)|_x = -\frac{1}{3}\,\mathrm{ric}_x, \quad (\Box\mu_x)(x) = \frac{1}{3}\mathrm{scal}(x),$$

where ric_x *denotes the Ricci curvature considered as a bilinear form on $T_x\Omega$ and* scal *is the scalar curvature.*

Proof. Let $X \in T_x\Omega$ be fixed. Let $e_1, \ldots, e_n$ be a Lorentz orthonormal basis of $T_x\Omega$. Denote by $J_1, \ldots, J_n$ the Jacobi fields along $c(t) = \exp_x(tX)$ satisfying $J_j(0) = 0$ and $\frac{\nabla J_j}{dt}(0) = e_j$ for every $1 \le j \le n$. The differential of $\exp_x$ at tX is, for every t for which it is defined, given by

$$d_{tX}\exp_x(e_j) = \frac{1}{t}J_j(t),$$

$j = 1, \ldots, n$. From the definition of μ_x we have

$$\begin{aligned}\mu_x(\exp_x(tX))e_1 \wedge \cdots \wedge e_n &= \det(d_{tX}\exp_x)e_1 \wedge \cdots \wedge e_n\\ &= (d_{tX}\exp_x(e_1)) \wedge \cdots \wedge (d_{tX}\exp_x(e_n))\\ &= \frac{1}{t}J_1(t) \wedge \cdots \wedge \frac{1}{t}J_n(t).\end{aligned}$$

Jacobi fields J along the geodesic $c(t) = \exp_x(tX)$ satisfy the Jacobi field equation $\frac{\nabla^2}{dt^2}J(t) = -R(J(t), \dot c(t))\dot c(t)$, where R denotes the curvature tensor of the Levi-Civita connection ∇. Differentiating this once more yields $\frac{\nabla^3}{dt^3}J(t) = -\frac{\nabla R}{dt}(J(t), \dot c(t))\dot c(t) - R(\frac{\nabla}{dt}J(t), \dot c(t))\dot c(t)$. For $J = J_j$ and $t = 0$ we have $J_j(0) = 0$, $\frac{\nabla J_j}{dt}(0) = e_j$, $\frac{\nabla^2 J_j}{dt^2}(0) = -R(0, \dot c(0))\dot c(0) = 0$, and $\frac{\nabla^3 J_j}{dt^3}(0) = -R(e_j, X)X$ where $X = \dot c(0)$. Identifying $J_j(t)$ with its parallel translate to $T_x\Omega$ along c the Taylor expansion of J_j up to order 3 reads as

$$J_j(t) = te_j - \frac{t^3}{6}R(e_j, X)X + \mathrm{O}(t^4).$$

This implies

$$\begin{aligned}\frac{1}{t}J_1(t) \wedge \cdots \wedge \frac{1}{t}J_n(t) &= e_1 \wedge \cdots \wedge e_n\\ &\qquad - \frac{t^2}{6}\sum_{j=1}^n e_1 \wedge \cdots \wedge R(e_j, X)X \wedge \cdots \wedge e_n + \mathrm{O}(t^3)\\ &= e_1 \wedge \cdots \wedge e_n\\ &\qquad - \frac{t^2}{6}\sum_{j=1}^n \varepsilon_j \langle R(e_j, X)X, e_j\rangle e_1 \wedge \cdots \wedge e_n + \mathrm{O}(t^3)\\ &= \Big(1 - \frac{t^2}{6}\,\mathrm{ric}(X, X) + \mathrm{O}(t^3)\Big)e_1 \wedge \cdots \wedge e_n.\end{aligned}$$

Thus

$$\mu_x(\exp_x(tX)) = 1 - \frac{t^2}{6}\operatorname{ric}(X,X) + O(t^3)$$

and therefore

$$\mu_x(x) = 1, \quad d\mu_x(X) = 0, \quad \operatorname{Hess}(\mu_x)(X,X) = -\frac{1}{3}\operatorname{ric}(X,X).$$

Taking a trace yields the result for the d'Alembertian. □

Lemma 1.3.17 and (1.16) with $f = \mu_x$ and $F(t) = t^{-1/2}$ yield:

Corollary 1.3.18. *Under the assumptions of Lemma 1.3.17 one has*

$$(\Box\mu_x^{-1/2})(x) = -\frac{1}{6}\operatorname{scal}(x). \qquad \Box$$

Let Ω be a domain in a Lorentzian manifold M, geodesically starshaped with respect to $x \in \Omega$. Set

$$\Gamma_x := \gamma \circ \exp_x^{-1} : \Omega \to \mathbb{R} \tag{1.17}$$

where γ is defined as in (1.6) with $V = T_x\Omega$.

Lemma 1.3.19. *Let M be a time-oriented Lorentzian manifold. Let the domain $\Omega \subset M$ be geodesically starshaped with respect to $x \in \Omega$. Then the following holds on Ω:*

(1) $\langle \operatorname{grad}\Gamma_x, \operatorname{grad}\Gamma_x\rangle = -4\Gamma_x$.

(2) *On $I_+^\Omega(x)$ (or on $I_-^\Omega(x)$) the gradient* $\operatorname{grad}\Gamma_x$ *is a past directed (or future directed respectively) timelike vector field.*

(3) $\Box\Gamma_x - 2n = -\langle \operatorname{grad}\Gamma_x, \operatorname{grad}(\log(\mu_x))\rangle$.

Proof. (1) Let $y \in \Omega$ and $Z \in T_y\Omega$. The differential of γ at a point p is given by $d_p\gamma = -2\langle p, \cdot\rangle$. Hence

$$\begin{aligned} d_y\Gamma_x(Z) &= d_{\exp_x^{-1}(y)}\gamma \circ d_y \exp_x^{-1}(Z) \\ &= -2\langle \exp_x^{-1}(y), d_y\exp_x^{-1}(Z)\rangle. \end{aligned}$$

Applying the Gauss Lemma [O'Neill1983, p. 127], we obtain

$$d_y\Gamma_x(Z) = -2\langle d_{\exp_x^{-1}(y)}\exp_x(\exp_x^{-1}(y)), Z\rangle.$$

Thus

$$\operatorname{grad}_y\Gamma_x = -2 d_{\exp_x^{-1}(y)}\exp_x(\exp_x^{-1}(y)). \tag{1.18}$$

It follows again from the Gauss Lemma that

$$\begin{aligned} \langle \operatorname{grad}_y\Gamma_x, \operatorname{grad}_y\Gamma_x\rangle &= 4\langle d_{\exp_x^{-1}(y)}\exp_x(\exp_x^{-1}(y)), d_{\exp_x^{-1}(y)}\exp_x(\exp_x^{-1}(y))\rangle \\ &= 4\langle \exp_x^{-1}(y), \exp_x^{-1}(y)\rangle \\ &= -4\Gamma_x(y). \end{aligned}$$

(2) On $I_+^\Omega(x)$ the function Γ_x is positive, hence $\langle \operatorname{grad}\Gamma_x, \operatorname{grad}\Gamma_x\rangle = -4\Gamma_x < 0$. Thus $\operatorname{grad}\Gamma_x$ is timelike. For a future directed timelike tangent vector $Z \in T_x\Omega$ the curve $c(t) := \exp_x(tZ)$ is future directed timelike and Γ_x increases along c. Hence $0 \le \frac{d}{dt}(\Gamma_x \circ c) = \langle \operatorname{grad}\Gamma_x, \dot c\rangle$. Thus $\operatorname{grad}\Gamma_x$ is past directed along c. Since every point in $I_+^\Omega(x)$ can be written in the form $\exp_x(Z)$ for a future directed timelike tangent vector Z this proves the assertion for $I_+^\Omega(x)$. The argument for $I_-^\Omega(x)$ is analogous.

(3) Using (1.13) with $f = \mu_x^{-1}$ and $X = \operatorname{grad}\Gamma_x$ we get

$$\operatorname{div}(\mu_x^{-1} \operatorname{grad}\Gamma_x) = \mu_x^{-1} \operatorname{div}\operatorname{grad}\Gamma_x + \langle \operatorname{grad}(\mu_x^{-1}), \operatorname{grad}\Gamma_x\rangle$$

and therefore

$$\begin{aligned}\Box\Gamma_x &= \langle \operatorname{grad}(\log(\mu_x^{-1})), \operatorname{grad}\Gamma_x\rangle - \mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad}\Gamma_x)\\ &= -\langle \operatorname{grad}(\log(\mu_x)), \operatorname{grad}\Gamma_x\rangle - \mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad}\Gamma_x).\end{aligned}$$

It remains to show $\mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad}\Gamma_x) = -2n$. We check this in normal coordinates $\xi^1, \dots, \xi^n$ about x. By (1.18) we have $\operatorname{grad}\Gamma_x = -2\sum_j \xi^j \frac{\partial}{\partial \xi^j}$ so that (1.15) implies

$$\mu_x \operatorname{div}(\mu_x^{-1} \operatorname{grad}\Gamma_x) = -2\sum_j \frac{\partial \xi^j}{\partial \xi^j} = -2n. \qquad \Box$$

Remark 1.3.20. If Ω is convex and τ is the time-separation function of Ω, then one can check that

$$\tau(p,q) = \begin{cases} \sqrt{\Gamma(p,q)}, & \text{if } p < q\\ 0, & \text{otherwise.}\end{cases}$$

1.4 Riesz distributions on a domain

Riesz distributions have been defined on all spaces isometric to Minkowski space. They are therefore defined on the tangent spaces at all points of a Lorentzian manifold. We now show how to construct Riesz distributions defined in small open subsets of the Lorentzian manifold itself. The passage from the tangent space to the manifold will be provided by the Riemannian exponential map.

Let Ω be a domain in a time-oriented n-dimensional Lorentzian manifold, $n \ge 2$. Suppose Ω is geodesically starshaped with respect to some point $x \in \Omega$. In particular, the Riemannian exponential function $\exp_x$ is a diffeomorphism from $\Omega' := \exp^{-1}(\Omega) \subset T_x\Omega$ to Ω. Let $\mu_x : \Omega \to \mathbb{R}$ be defined as in (1.9). Put

$$R_\pm^\Omega(\alpha, x) := \mu_x \exp_x^* R_\pm(\alpha),$$

that is, for every testfunction $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$,

$$R_\pm^\Omega(\alpha, x)[\varphi] := R_\pm(\alpha)[(\mu_x\varphi) \circ \exp_x].$$

Note that $\mathrm{supp}((\mu_x\varphi)\circ\exp_x)$ is contained in Ω'. Extending the function $(\mu_x\varphi)\circ\exp_x$ by zero we can regard it as a testfunction on $T_x\Omega$ and thus apply $R_\pm(\alpha)$ to it.

Definition 1.4.1. We call $R_+^\Omega(\alpha,x)$ the *advanced Riesz distribution* and $R_-^\Omega(\alpha,x)$ the *retarded Riesz distribution* on Ω at x for $\alpha\in\mathbb{C}$.

The relevant properties of the Riesz distributions are collected in the following proposition.

Proposition 1.4.2. *The following holds for all $\alpha\in\mathbb{C}$ and all $x\in\Omega$:*

(1) *If $\Re\mathrm{e}(\alpha)>n$, then $R_\pm^\Omega(\alpha,x)$ is the continuous function*

$$R_\pm^\Omega(\alpha,x)=\begin{cases}C(\alpha,n)\,\Gamma_x^{\frac{\alpha-n}{2}} & \text{on } J_\pm^\Omega(x),\\ 0 & \text{elsewhere.}\end{cases}$$

(2) *For every fixed testfunction φ the map $\alpha\mapsto R_\pm^\Omega(\alpha,x)[\varphi]$ is holomorphic on $\mathbb{C}$.*

(3) $\Gamma_x\cdot R_\pm^\Omega(\alpha,x)=\alpha(\alpha-n+2)\,R_\pm^\Omega(\alpha+2,x)$.

(4) $\mathrm{grad}\,(\Gamma_x)\cdot R_\pm^\Omega(\alpha,x)=2\alpha\,\mathrm{grad}\,R_\pm^\Omega(\alpha+2,x)$.

(5) *If $\alpha\neq 0$, then $\Box R_\pm^\Omega(\alpha+2,x)=\big(\frac{\Box\Gamma_x-2n}{2\alpha}+1\big)R_\pm^\Omega(\alpha,x)$.*

(6) $R_\pm^\Omega(0,x)=\delta_x$.

(7) *For every $\alpha\in\mathbb{C}\setminus(\{0,-2,-4,\dots\}\cup\{n-2,n-4,\dots\})$ we have*

$$\mathrm{supp}\big(R_\pm^\Omega(\alpha,x)\big)=J_\pm^\Omega(x)\quad\textit{and}\quad\mathrm{sing\,supp}\big(R_\pm^\Omega(\alpha,x)\big)\subset C_\pm^\Omega(x).$$

(8) *For every $\alpha\in\{0,-2,-4,\dots\}\cup\{n-2,n-4,\dots\}$ we have*

$$\mathrm{supp}\big(R_\pm^\Omega(\alpha,x)\big)=\mathrm{sing\,supp}\big(R_\pm^\Omega(\alpha,x)\big)\subset C_\pm^\Omega(x).$$

(9) *For $n\geq 3$ and $\alpha=n-2,n-4,\dots,1$ or 2 respectively we have*

$$\mathrm{supp}\big(R_\pm^\Omega(\alpha,x)\big)=\mathrm{sing\,supp}\big(R_\pm^\Omega(\alpha,x)\big)=C_\pm^\Omega(x).$$

(10) *For $\Re\mathrm{e}(\alpha)>0$ we have $\mathrm{ord}(R_\pm^\Omega(\alpha,x))\leq n+1$. Moreover, there exists a neighborhood U of x and a constant $C>0$ such that*

$$|R_\pm^\Omega(\alpha,x')[\varphi]|\leq C\cdot\|\varphi\|_{C^{n+1}(\Omega)}$$

for all $\varphi\in\mathcal{D}(\Omega,\mathbb{C})$ and all $x'\in U$.

(11) *If $U\subset\Omega$ is an open neighborhood of x such that Ω is geodesically starshaped with respect to all $x'\in U$ and if $V\in\mathcal{D}(U\times\Omega,\mathbb{C})$, then the function $U\to\mathbb{C}$, $x'\mapsto R_\pm^\Omega(\alpha,x')[y\mapsto V(x',y)]$, is smooth.*

(12) *If $U\subset\Omega$ is an open neighborhood of x such that Ω is geodesically starshaped with respect to all $x'\in U$, if $\Re\mathrm{e}(\alpha)>0$, and if $V\in\mathcal{D}^{n+1+k}(U\times\Omega,\mathbb{C})$, then the function $U\to\mathbb{C}$, $x'\mapsto R_\pm^\Omega(\alpha,x')[y\mapsto V(x',y)]$, is C^k.*

(13) *For every* $\varphi \in \mathcal{D}^k(\Omega, \mathbb{C})$ *the map* $\alpha \mapsto R_\pm^\Omega(\alpha, x)[\varphi]$ *is a holomorphic function on* $\{\alpha \in \mathbb{C} \mid \mathfrak{Re}(\alpha) > n - 2[\frac{k}{2}]\}$.

(14) *If* $\alpha \in \mathbb{R}$, *then* $R_\pm^\Omega(\alpha, x)$ *is real, i.e.,* $R_\pm^\Omega(\alpha, x)[\varphi] \in \mathbb{R}$ *for all* $\varphi \in \mathcal{D}(\Omega, \mathbb{R})$.

Proof. It suffices to prove the statements for the advanced Riesz distributions.

(1) Let $\mathfrak{Re}(\alpha) > n$ and $\varphi \in \mathcal{D}(\Omega, \mathbb{C})$. Then

$$\begin{aligned} R_+^\Omega(\alpha, x)[\varphi] &= R_+^\Omega(\alpha, x)[(\mu_x \circ \exp_x) \cdot (\varphi \circ \exp_x)] \\ &= C(\alpha, n) \int_{J_+(0)} \gamma^{\frac{\alpha - n}{2}} \cdot (\varphi \circ \exp_x) \cdot \mu_x \, dz \\ &= C(\alpha, n) \int_{J_+^\Omega(x)} \Gamma_x^{\frac{\alpha - n}{2}} \cdot \varphi \, \mathrm{dV}. \end{aligned}$$

(2) This follows directly from the definition of $R_+^\Omega(\alpha, x)$ and from Lemma 1.2.2 (4).

(3) By (1) this obviously holds for $\mathfrak{Re}(\alpha) > n$ since $C(\alpha, n) = \alpha(\alpha - n + 2) \cdot C(\alpha + 2, n)$. By analyticity of $\alpha \mapsto R_+^\Omega(\alpha, x)$ it must hold for all α.

(4) Consider α with $\mathfrak{Re}(\alpha) > n$. By (1) the function $R_+^\Omega(\alpha + 2, x)$ is then C^1. On $J_+^\Omega(x)$ we compute

$$\begin{aligned} 2\alpha \, \mathrm{grad} \, R_+^\Omega(\alpha + 2, x) &= 2\alpha C(\alpha + 2, n) \, \mathrm{grad} \left(\Gamma_x^{\frac{\alpha + 2 - n}{2}} \right) \\ &= \underbrace{2\alpha C(\alpha + 2, n) \frac{\alpha + 2 - n}{2}}_{C(\alpha, n)} \Gamma_x^{\frac{\alpha - n}{2}} \, \mathrm{grad} \, \Gamma_x \\ &= R_+^\Omega(\alpha, x) \, \mathrm{grad} \, \Gamma_x. \end{aligned}$$

For arbitrary $\alpha \in \mathbb{C}$ assertion (4) follows from analyticity of $\alpha \mapsto R_+^\Omega(\alpha, x)$.

(5) Let $\alpha \in \mathbb{C}$ with $\mathfrak{Re}(\alpha) > n + 2$. Since $R_+^\Omega(\alpha + 2, x)$ is then C^2, we can compute $\Box R_+^\Omega(\alpha + 2, x)$ classically. This will show that (5) holds for all α with $\mathfrak{Re}(\alpha) > n + 2$. Analyticity then implies (5) for all α.

$$\begin{aligned} &\Box R_+^\Omega(\alpha + 2, x) \\ &\quad = -\,\mathrm{div} \left(\mathrm{grad} \, R_+^\Omega(\alpha + 2, x) \right) \\ &\quad \overset{(4)}{=} -\frac{1}{2\alpha} \, \mathrm{div} \left(R_+^\Omega(\alpha, x) \cdot \mathrm{grad}(\Gamma_x) \right) \\ &\quad \overset{(1.13)}{=} \frac{1}{2\alpha} \Box \Gamma_x \cdot R_+^\Omega(\alpha, x) - \frac{1}{2\alpha} \langle \mathrm{grad} \, \Gamma_x, \mathrm{grad} \, R_+^\Omega(\alpha, x) \rangle \\ &\quad \overset{(4)}{=} \frac{1}{2\alpha} \Box \Gamma_x \cdot R_+^\Omega(\alpha, x) - \frac{1}{2\alpha \cdot 2(\alpha - 2)} \langle \mathrm{grad} \, \Gamma_x, \mathrm{grad} \, \Gamma_x \cdot R_+^\Omega(\alpha - 2, x) \rangle \\ &\quad \overset{\text{Lemma } 1.3.19(1)}{=} \frac{1}{2\alpha} \Box \Gamma_x \cdot R_+^\Omega(\alpha, x) + \frac{1}{\alpha(\alpha - 2)} \Gamma_x \cdot R_+^\Omega(\alpha - 2, x) \end{aligned}$$

$$\overset{(3)}{=} \frac{1}{2\alpha} \Box\Gamma_x \cdot R_+^\Omega(\alpha, x) + \frac{(\alpha-2)(\alpha-n)}{\alpha(\alpha-2)} R_+^\Omega(\alpha, x)$$
$$= \left(\frac{\Box\Gamma_x - 2n}{2\alpha} + 1\right) R_+^\Omega(\alpha, x).$$

(6) Let φ be a testfunction on Ω. Then by Proposition 1.2.4 (7)

$$\begin{aligned} R_+^\Omega(0,x)[\varphi] &= R_+(0)[(\mu_x\varphi)\circ \exp_x] \\ &= \delta_0[(\mu_x\varphi)\circ\exp_x] \\ &= ((\mu_x\varphi)\circ\exp_x)(0) \\ &= \mu_x(x)\varphi(x) \\ &= \varphi(x) \\ &= \delta_x[\varphi]. \end{aligned}$$

(11) Let $A(x,x')\colon T_x\Omega \to T_{x'}\Omega$ be a time-orientation preserving linear isometry. Then

$$R_+^\Omega(\alpha, x')[V(x',\cdot)] = R_+(\alpha)[(\mu_{x'}\cdot V(x',\cdot))\circ\exp_{x'}\circ A(x,x')]$$

where $R_+(\alpha)$ is, as before, the Riesz distribution on $T_x\Omega$. Hence if we choose $A(x,x')$ to depend smoothly on x', then $(\mu_{x'}\cdot V(x',y))\circ\exp_{x'}\circ A(x,x')$ is smooth in x' and y and the assertion follows from Lemma 1.1.6.

(10) Since $\mathrm{ord}(R_\pm(\alpha)) \le n+1$ by Proposition 1.2.4 (8) we have $\mathrm{ord}(R_\pm^\Omega(\alpha,x)) \le n+1$ as well. From the definition $R_\pm^\Omega(\alpha,x) = \mu_x \exp_x^* R_\pm(\alpha)$ it is clear that the constant C may be chosen locally uniformly in x.

(12) By (10) we can apply $R_\pm^\Omega(\alpha,x')$ to $V(x',\cdot)$. Now the same argument as for (11) shows that the assertion follows from Lemma 1.1.6.

The remaining assertions follow directly from the corresponding properties of the Riesz distributions on Minkowski space. For example (13) is a consequence of Corollary 1.2.5. □

Advanced and retarded Riesz distributions are related as follows.

Lemma 1.4.3. *Let Ω be a convex time-oriented Lorentzian manifold. Let $\alpha \in \mathbb{C}$. Then for all $u \in \mathcal{D}(\Omega\times\Omega, \mathbb{C})$ we have*

$$\int_\Omega R_+^\Omega(\alpha,x)\,[y\mapsto u(x,y)]\;\mathrm{dV}(x) = \int_\Omega R_-^\Omega(\alpha,y)\,[x\mapsto u(x,y)]\;\mathrm{dV}(y).$$

Proof. The convexity condition for Ω ensures that the Riesz distributions $R_\pm^\Omega(\alpha,x)$ are defined for all $x\in\Omega$. By Proposition 1.4.2 (11) the integrands are smooth. Since u has compact support contained in $\Omega\times\Omega$ the integrand $R_+^\Omega(\alpha,x)\,[y\mapsto u(x,y)]$ (as a function in x) has compact support contained in Ω. A similar statement holds for the integrand of the right-hand side. Hence the integrals exist. By Proposition 1.4.2 (13) they are holomorphic in α. Thus it suffices to show the equation for α with $\mathfrak{Re}(\alpha) > n$.

For such an $\alpha \in \mathbb{C}$ the Riesz distributions $R_+(\alpha, x)$ and $R_-(\alpha, y)$ are continuous functions. From the explicit formula (1) in Proposition 1.4.2 we see

$$R_+(\alpha, x)(y) = R_-(\alpha, y)(x)$$

for all $x, y \in \Omega$. By Fubini's theorem we get

$$\begin{aligned}\int_\Omega R_+^\Omega(\alpha, x)[y \mapsto u(x,y)]\,\mathrm{dV}(x) &= \int_\Omega \left(\int_\Omega R_+^\Omega(\alpha, x)(y)\, u(x,y)\, \mathrm{dV}(y)\right) \mathrm{dV}(x) \\ &= \int_\Omega \left(\int_\Omega R_-^\Omega(\alpha, y)(x)\, u(x,y)\, \mathrm{dV}(x)\right) \mathrm{dV}(y) \\ &= \int_\Omega R_-^\Omega(\alpha, y)[x \mapsto u(x,y)]\,\mathrm{dV}(y). \qquad \square\end{aligned}$$

As a technical tool we will also need a version of Lemma 1.4.3 for certain nonsmooth sections.

Lemma 1.4.4. *Let Ω be a causal domain in a time-oriented Lorentzian manifold of dimension n. Let $\Re\mathrm{e}(\alpha) > 0$ and let $k \geq n+1$. Let K_1, K_2 be compact subsets of $\overline{\Omega}$ and let $u \in C^k(\overline{\Omega} \times \overline{\Omega})$ so that $\mathrm{supp}(u) \subset J_+^\Omega(K_1) \times J_-^\Omega(K_2)$. Then*

$$\int_\Omega R_+^\Omega(\alpha, x)\,[y \mapsto u(x,y)]\;\mathrm{dV}(x) = \int_\Omega R_-^\Omega(\alpha, y)\,[x \mapsto u(x,y)]\;\mathrm{dV}(y).$$

Proof. For fixed x, the support of the function $y \mapsto u(x,y)$ is contained in $J_-^\Omega(K_2)$. Since Ω is causal, it follows from Lemma A.5.3 that the subset $J_-^\Omega(K_2) \cap J_+^\Omega(x)$ is relatively compact in $\overline{\Omega}$. Therefore the intersection of the supports of $y \mapsto u(x,y)$ and $R_+^\Omega(\alpha, x)$ is compact and contained in $\overline{\Omega}$. By Proposition 1.4.2 (10) one can then apply $R_+^\Omega(\alpha, x)$ to the C^k-function $y \mapsto u(x,y)$. Furthermore, the support of the continuous function $x \mapsto R_+^\Omega(\alpha, x)\,[y \mapsto u(x,y)]$ is contained in $J_+^\Omega(K_1) \cap J_-^\Omega(\mathrm{supp}(y \mapsto u(x,y))) \subset J_+^\Omega(K_1) \cap J_-^\Omega(J_-^\Omega(K_2)) = J_+^\Omega(K_1) \cap J_+^\Omega(K_2)$, which is relatively compact in $\overline{\Omega}$, again by Lemma A.5.3. Hence the function $x \mapsto R_+^\Omega(\alpha, x)\,[y \mapsto u(x,y)]$ has compact support in $\overline{\Omega}$, so that the left-hand side makes sense. Analogously the right-hand side is well defined. Our considerations also show that the integrals depend only on the values of u on $\left(J_+^\Omega(K_1) \cap J_-^\Omega(K_2)\right) \times \left(J_+^\Omega(K_1) \cap J_-^\Omega(K_2)\right)$ which is a relatively compact set. Applying a cut-off function argument we may assume without loss of generality that u has compact support. Proposition 1.4.2 (13) says that the integrals depend holomorphically on α on the domain $\{\Re\mathrm{e}(\alpha) > 0\}$. Therefore it suffices to show the equality for α with sufficiently large real part, which can be done exactly as in the proof of Lemma 1.4.3. $\square$

1.5 Normally hyperbolic operators

Let M be a Lorentzian manifold and let $E \to M$ be a real or complex vector bundle. For a summary on basics concerning linear differential operators see Appendix A.4.

A linear differential operator $P\colon C^\infty(M,E) \to C^\infty(M,E)$ of second order will be called *normally hyperbolic* if its principal symbol is given by the metric,

$$\sigma_P(\xi) = -\langle \xi, \xi \rangle \cdot \mathrm{id}_{E_x}$$

for all $x \in M$ and all $\xi \in T_x^*M$. In other words, if we choose local coordinates $x^1, \dots, x^n$ on M and a local trivialization of E, then

$$P = -\sum_{i,j=1}^{n} g^{ij}(x) \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{j=1}^{n} A_j(x) \frac{\partial}{\partial x^j} + B_1(x)$$

where A_j and B_1 are matrix-valued coefficients depending smoothly on x and $(g^{ij})_{ij}$ is the inverse matrix of $(g_{ij})_{ij}$ with $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$.

Example 1.5.1. Let E be the trivial line bundle so that sections in E are just functions. The d'Alembert operator $P = \Box$ is normally hyperbolic because

$$\sigma_{\mathrm{grad}}(\xi) f = f \xi^\sharp, \quad \sigma_{\mathrm{div}}(\xi) X = \xi(X)$$

and so

$$\sigma_\Box(\xi) f = -\sigma_{\mathrm{div}}(\xi) \circ \sigma_{\mathrm{grad}}(\xi) f = -\xi(f\xi^\sharp) = -\langle \xi, \xi \rangle f.$$

Recall that $\xi \mapsto \xi^\sharp$ denotes the isomorphism $T_x^*M \to T_xM$ induced by the Lorentzian metric, compare (1.11).

Example 1.5.2. Let E be a vector bundle and let ∇ be a connection on E. This connection together with the Levi-Civita connection on T^*M induces a connection on $T^*M \otimes E$, again denoted ∇. We define the *connection-d'Alembert operator* $\Box^\nabla$ to be minus the composition of the following three maps

$$C^\infty(M,E) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes E) \xrightarrow{\nabla} C^\infty(M, T^*M \otimes T^*M \otimes E) \xrightarrow{\mathrm{tr} \otimes \mathrm{id}_E} C^\infty(M,E)$$

where $\mathrm{tr}\colon T^*M \otimes T^*M \to \mathbb{R}$ denotes the metric trace, $\mathrm{tr}(\xi \otimes \eta) = \langle \xi, \eta \rangle$. We compute the principal symbol,

$$\sigma_{\Box^\nabla}(\xi)\varphi = -(\mathrm{tr} \otimes \mathrm{id}_E) \circ \sigma_\nabla(\xi) \circ \sigma_\nabla(\xi)(\varphi) = -(\mathrm{tr} \otimes \mathrm{id}_E)(\xi \otimes \xi \otimes \varphi) = -\langle \xi, \xi \rangle \varphi.$$

Hence $\Box^\nabla$ is normally hyperbolic.

Example 1.5.3. Let $E = \Lambda^k T^*M$ be the bundle of k-forms. Exterior differentiation $d\colon C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ increases the degree by one while the codifferential $\delta\colon C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k-1} T^*M)$ decreases the degree by one, see [Besse1987, p. 34] for details. While d is independent of the metric, the codifferential δ does depend on the Lorentzian metric. The operator $P = d\delta + \delta d$ is normally hyperbolic.

Example 1.5.4. If M carries a Lorentzian metric and a spin structure, then one can define the spinor bundle ΣM and the Dirac operator

$$D : C^\infty(M, \Sigma M) \to C^\infty(M, \Sigma M),$$

see [Bär–Gauduchon–Moroianu2005] or [Baum1981] for the definitions. The principal symbol of D is given by Clifford multiplication,

$$\sigma_D(\xi)\psi = \xi^\sharp \cdot \psi.$$

Hence

$$\sigma_{D^2}(\xi)\psi = \sigma_D(\xi)\sigma_D(\xi)\psi = \xi^\sharp \cdot \xi^\sharp \cdot \psi = -\langle \xi, \xi \rangle\, \psi.$$

Thus $P = D^2$ is normally hyperbolic.

The following lemma is well-known, see e.g. [Baum–Kath1996, Prop. 3.1]. It says that each normally hyperbolic operator is a connection-d'Alembert operator up to a term of order zero.

Lemma 1.5.5. *Let $P : C^\infty(M, E) \to C^\infty(M, E)$ be a normally hyperbolic operator on a Lorentzian manifold M. Then there exists a unique connection ∇ on E and a unique endomorphism field $B \in C^\infty(M, \mathrm{Hom}(E, E))$ such that*

$$P = \Box^\nabla + B.$$

Proof. First we prove uniqueness of such a connection. Let ∇' be an arbitrary connection on E. For any section $s \in C^\infty(M, E)$ and any function $f \in C^\infty(M)$ we get

$$\Box^{\nabla'}(f \cdot s) = f \cdot (\Box^{\nabla'} s) - 2\nabla'_{\mathrm{grad} f} s + (\Box f) \cdot s. \tag{1.19}$$

Now suppose that ∇ satisfies the condition in Lemma 1.5.5. Then $B = P - \Box^\nabla$ is an endomorphism field and we obtain

$$f \cdot \big(P(s) - \Box^\nabla s\big) = P(f \cdot s) - \Box^\nabla (f \cdot s).$$

By (1.19) this yields

$$\nabla_{\mathrm{grad} f} s = \tfrac{1}{2} \{ f \cdot P(s) - P(f \cdot s) + (\Box f) \cdot s \}. \tag{1.20}$$

At a given point $x \in M$ every tangent vector $X \in T_x M$ can be written in the form $X = \mathrm{grad}_x f$ for some suitably chosen function f. Thus (1.20) shows that ∇ is determined by P and $\Box$ (which is determined by the Lorentzian metric).

To show existence one could use (1.20) to define a connection ∇ as in the statement. We follow an alternative path. Let ∇' be some connection on E. Since P and $\Box^{\nabla'}$ are both normally hyperbolic operators acting on sections in E, the difference $P - \Box^{\nabla'}$ is a differential operator of first order and can therefore be written in the form

$$P - \Box^{\nabla'} = A' \circ \nabla' + B',$$

for some $A' \in C^\infty(M, \mathrm{Hom}(T^*M \otimes E, E))$ and $B' \in C^\infty(M, \mathrm{Hom}(E, E))$. Set for every vector field X on M and section s in E

$$\nabla_X s := \nabla'_X s - \frac{1}{2} A'(X^\flat \otimes s).$$

This defines a new connection ∇ on E. Let $e_1, \dots, e_n$ be a local Lorentz orthonormal basis of TM. Write as before $\varepsilon_j = \langle e_j, e_j \rangle = \pm 1$. We may assume that at a given point $p \in M$ we have $\nabla_{e_j} e_j(p) = 0$. Then we compute at p

$$\begin{aligned}
\Box^{\nabla'} s + A' \circ \nabla' s &= \sum_{j=1}^n \varepsilon_j \Big\{ -\nabla'_{e_j} \nabla'_{e_j} s + A'(e_j^\flat \otimes \nabla'_{e_j} s) \Big\} \\
&= \sum_{j=1}^n \varepsilon_j \Big\{ -(\nabla_{e_j} + \frac{1}{2} A'(e_j^\flat \otimes \cdot))(\nabla_{e_j} s + \frac{1}{2} A'(e_j^\flat \otimes s)) \\
&\qquad + A'(e_j^\flat \otimes \nabla_{e_j} s) + \frac{1}{2} A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) \Big\} \\
&= \sum_{j=1}^n \varepsilon_j \Big\{ -\nabla_{e_j} \nabla_{e_j} s - \frac{1}{2} \nabla_{e_j} (A'(e_j^\flat \otimes s)) \\
&\qquad + \frac{1}{2} A'(e_j^\flat \otimes \nabla_{e_j} s) + \frac{1}{4} A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) \Big\} \\
&= \Box^\nabla s + \frac{1}{4} \sum_{j=1}^n \varepsilon_j \Big\{ A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) - 2(\nabla_{e_j} A')(e_j^\flat \otimes s) \Big\},
\end{aligned}$$

where ∇ in $\nabla_{e_j} A'$ stands for the induced connection on $\mathrm{Hom}(T^*M \otimes E, E)$. We observe that $Q(s) := \Box^{\nabla'} s + A' \circ \nabla' s - \Box^\nabla s = \frac{1}{4} \sum_{j=1}^n \varepsilon_j \big\{ A'(e_j^\flat \otimes A'(e_j^\flat \otimes s)) - 2(\nabla_{e_j} A')(e_j^\flat \otimes s) \big\}$ is of order zero. Hence

$$P = \Box^{\nabla'} + A' \circ \nabla' + B' = \Box^\nabla s + Q(s) + B'(s)$$

is the desired expression with $B = Q + B'$. $\Box$

The connection in Lemma 1.5.5 will be called the *P-compatible* connection. We shall henceforth always work with the P-compatible connection. We restate (1.20) as a lemma.

Lemma 1.5.6. *Let $P = \Box^\nabla + B$ be normally hyperbolic. For $f \in C^\infty(M)$ and $s \in C^\infty(M, E)$ one gets*

$$P(f \cdot s) = f \cdot P(s) - 2\, \nabla_{\mathrm{grad}\, f}\, s + \Box f \cdot s. \qquad \Box$$