Preface

Asymptotic geometry is the study of metric spaces from a large scale point of view, where the local geometry does not come into play. An important class of spaces to be studied are the hyperbolic spaces (in the sense of Gromov), for which it turns out that the asymptotic geometry is almost completely encoded in the boundary at infinity.

The basic example of these spaces is the classical hyperbolic space H^n. A main feature of this classical space is the deep relation between the geometry of H^n and the Möbius geometry of its boundary $\partial_\infty \mathrm{H}^n$. For example the isometries of H^n correspond to Möbius transformations of $\partial_\infty \mathrm{H}^n$. The classical space itself has different realizations, but there are natural isomorphisms between these models, which induce Möbius transformations between the boundaries at infinity.

Mikhael Gromov realized that the essential asymptotic properties of H^n can be encoded in a simple condition for quadruples of points. A metric space X is called (Gromov) hyperbolic, if there exists some $\delta \geq 0$ such that every quadruple $Q = \{x, y, z, w\} \subset X$ satisfies the following inequality only involving the six distances between the four points:

$$|xz| + |yu| \leq \max\{|xy| + |zu|, |xu| + |yz|\} + 2\delta.$$

It is a remarkable fact that this inequality suffices to build up a theory of general hyperbolic spaces, which is very similar to the classical theory of the classical hyperbolic space but which allows much more flexibility and can be applied to many situations.

We develop the basics of this theory of general hyperbolic spaces in the first eight chapters of the book. In our account we stress the analogy between a Gromov hyperbolic space X and the classical hyperbolic space H^n. We describe the boundary at infinity $\partial_\infty X$ in different realizations as a bounded and as an unbounded metric space in analogy to the unit disc and the upper half-space model of H^n. We introduce a *quasi-Möbius structure* on $\partial_\infty X$ and discuss in detail the relation between the morphisms of X and the quasi-Möbius transformations of the boundary.

In the second part of the book we focus on several aspects of the asymptotic geometry of arbitrary metric spaces. It turns out that the simple philosophy to study "a boundary at infinity" does not work in this general situation.

Instead we introduce various dimension type asymptotic invariants and give several interesting applications in particular for embedding and non-embedding results.

In this book we only discuss a few elements of asymptotic geometry and our viewpoint is in no way exhaustive. For example, our book has little intersection with the recent book of John Roe [Ro] about the same subject and can be considered as a complement. Almost all of the results in this book are in the literature, but the presentation and some of the proofs are new.

This book grew out of lectures which we gave at the Steklov Institute in St. Petersburg and the University of Zürich. We want to thank the audience of these lectures, in particular Kathrin Haltiner, Alina Rull and Deborah Ruoss, for their questions and suggestions.

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