Preface

Elliptic mixed, transmission, or crack problems belong to the analysis on manifolds with singularities, more precisely, to the calculus of boundary value problems, where the data or the coefficients have singularities. A classical example is the Zaremba problem for the Laplace equation with mixed Dirichlet and Neumann conditions. Mixed problems in general are characterised by boundary conditions that have a jump along an interface on the boundary. At the same time the configuration may have singularities, e.g., conical points or edges, and it is an interesting task of the mathematical analysis to establish the properties of solvability.

Problems of that kind are natural in many models of physics and the applied sciences. Numerous authors have contributed to this field and developed approaches of different generality. In recent years it became more and more clear that boundary value problems with singularities can be understood from the viewpoint of the analysis on manifolds with singularities and that a transparent description of solvability can be achieved by applying suitable extensions of the pseudo-differential calculus, combined with operator algebra aspects, elements of the index theory, and other areas of pure mathematics. Singularities are employed as a source of symbolic information in the form of higher generations of (operator-valued) amplitude functions, which are involved in the construction of parametrices and the characterisation of regularity and asymptotics of solutions.

The present exposition is aimed at developing these ideas in a way that mixed, transmission and crack problems appear as a natural generalisation of standard boundary value problems, here as edge and corner problems. Our approach is completely general; examples and applications in mixed and crack problems will mainly concern second order elliptic differential operators with specific boundary conditions, e.g., of Zaremba type. First we treat the case of smooth interfaces in the framework of the edge calculus and then admit interfaces with conical singularities; this requires the tools of the corner calculus of boundary value problems.

The task to express parametrices on the level of symbols generates these calculi in large generality, because the amplitude functions are families of boundary value problems combined with different degeneracies in the parameters. In the description of the structures it may be even more convenient to refer to the general properties.

In Chapter 1 we give an introduction into mixed and transmission problems for differential operators and their symbolic structure.

Chapter 2 contains general tools of the pseudo-differential calculus. This will partly be done in a 'non-orthodox' form, although the starting point are operators with standard interior and boundary symbols. However, the applications require a variety of new structures, such as symbols with twisted homogeneity, exit properties at infinity, meromorphy in the covariables, etc.

In Chapter 3 we outline pseudo-differential boundary value problems with the transmission property at the boundary, including the calculus on a manifold with conical exit to infinity.

In Chapter 4 we treat mixed elliptic problems in standard Sobolev spaces and with extra interface conditions. Here we employ specific reductions of orders and a kind of plus/minus calculus of boundary value problems.

Chapter 5 is devoted to mixed problems in weighted edge spaces, also with extra conditions of trace and potential type at the interface.

In Chapters 6 and 7 we develop the calculus of boundary value problems on manifolds with conical singularities and smooth edges, i.e., we establish the corresponding cone and edge algebras.

In Chapter 8 we consider mixed and crack problems with conical singularities at the interface. We give a homotopy argument that allows us to reduce crack problems to mixed problems.

In Chapter 9 we study operators on infinite cylinders with meromorphic amplitude functions for different cases of cross sections, especially, with boundary, and also with conical singularities. Moreover, we formulate relative index results.

Chapter 10 contains a general discussion on motivations of the approach and on expected future developments, with open problems and new challenges.

Compared with other monographs in the field of the (pseudo-differential) analysis of singular problems, e.g., [90], the present book gives for the first time a general approach to mixed elliptic problems with singularities of the interface, together with explicit tools for computing the number of extra interface conditions. The new results are mainly based on the authors' joint works [71]–[78] and [34], the latter in cooperation with Dines.

This book is addressed to mathematicians and physicists interested in models with singularities, associated boundary value problems, and their solvability strategies based on pseudo-differential operators. The material is also useful for students in higher semesters and young researchers as well as for experienced specialists who are active in the fields of the analysis on manifolds with geometric singularities, applications of index theory and spectral theory, operator algebras with symbolic structures, quantisation, and asymptotic analysis.

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