Introduction

Boundary value problems on configurations with (geometric) singularities are an important and beautiful area of mathematical research, motivated by models of the applied sciences and engineering, for instance, of mechanics, elasticity, crack theory, and mathematical physics. Operators on manifolds with singularities are also of interest in pure mathematics, such as geometry, topology, index theory, spectral theory, and operator algebras. It is known that the parametrices of elliptic boundary value problems for differential operators in smooth domains are pseudo-differential operators (more precisely, pseudo-differential boundary value problems with the transmission property). The classical pseudo-differential techniques contain many elements of the general strategy to studying boundary value problems on singular configurations (stratified spaces), in particular, with conical, edge, corner or higher polyhedral singularities.

A category of interesting models are the so-called mixed problems in which the boundary conditions to an elliptic equation have a jump along a submanifold of the boundary of codimension 1. The reduction of such problems to the boundary gives rise to pseudo-differential equations without the transmission property at the interface on the boundary. Examples are operators with symbols associated with fractional powers of the symbol of (minus) the Laplacian. For instance, the square root belongs to the Zaremba problem (i.e., with mixed Dirichlet and Neumann conditions). Boundary value problems (with or without the transmission property) have much in common with edge problems, with the boundary as the edge and the inner normal as the model cone. Also mixed or crack problems can be interpreted as edge problems; in this case the interface with the jump of the mixed conditions or the boundary of the crack play the role of an edge, and the normal planes contain the corresponding model cones.

The present book is aimed at studying problems of that type, based on the general calculus of operators on a manifold with edges. First this concerns the case of smooth edges (i.e., interfaces or crack boundaries). We also treat mixed and crack problems when the interfaces or crack boundaries have conical singularities, using a corresponding corner calculus of boundary value problems.

In Chapter 1 we sketch a number of mixed and transmission problems and illustrate their symbolic structure. The original problems essentially refer to differential operators with differential conditions. However, reductions to the boundary give rise to pseudo-differential operators and associated boundary or transmission problems. Let us briefly recall a few ideas from the pseudo-differential calculus.

If X is a C^∞ manifold and A a differential operator of order μ on X, we have the homogeneous principal symbol $\sigma_\psi(A)$ of A of order μ which is a function in $C^\infty(T^*X \setminus 0)$, with T^*X being the cotangent bundle of X and 0 the zero section of T^*X. Ellipticity of A is defined by the condition $\sigma_\psi(A) \neq 0$ on $T^*X \setminus 0$. The solvability of the equation $Au = f$ and the regularity of solutions u are essentially characterised by the properties of a parametrix of A. It is therefore an interesting task to construct parametrices of elliptic differential operators. Those belong to the space of classical pseudo-differential operators on X. In other words, the problem of describing parametrices leads to pseudo-differential operators. They enlarge the class of differential operators. Classical pseudo-differential operators A of order $\nu \in \mathbb{R}$ form a space $L^{\nu}_{\mathrm{cl}}(X)$ with principal symbols $\sigma_{\psi}(A)$. The notion of ellipticity is analogous to that of differential operators. Parametrices of elliptic elements belong to $L^{-\nu}_{\mathrm{cl}}(X)$. Another important property is that the pseudo-differential operators form an algebra. In particular, compositions are defined, and the principal symbols satisfy the relation $\sigma_{\psi}(AB) = \sigma_{\psi}(A)\sigma_{\psi}(B)$ (for convenience, we talk about an algebra, even if the algebraic operations are only possible under some natural extra assumptions; for instance, the sum of two classical operators is classical when the difference of their orders is an integer; in matrix-valued scenarios we assume that rows and columns of the involved operators fit together, etc.). All these aspects have been well known for a long time and belong to the basics of the pseudo-differential calculus. There is a large variety of generalisations, e.g., operators with other kinds of symbols than classical ones.

In Chapter 2 we recall some technicalities from the standard calculus. We also outline a number of specific variants, especially, with operator-valued symbols and twisted homogeneity, the calculus on manifolds with conical exit to infinity, and operators based on the Mellin transform. Our approach of mixed, transmission or singular crack problems is mainly oriented towards the machinery of operators on manifolds with geometric singularities as is developed in [180] and [183]; see also the monographs [182] or [43]. An element of the philosophy here is that singularities of the underlying configuration or discontinuities of coefficients in the involved operators are recognised and employed as a source of extra symbolic information, apart from the ‘interior’ symbol σ_{ψ}. In that sense the problems are described (modulo lower order terms) by a principal symbolic hierarchy with different components that take part in the definition of ellipticity and the construction of parametrices. A non-trivial example is the pseudo-differential calculus of boundary value problems with the transmission property at the boundary, cf. [15].

In Chapter 3 we outline some elements of that calculus which is a necessary tool for establishing the more singular ‘higher’ calculi connected with mixed and crack problems. The operators representing boundary value problems consist of 2×2 block matrices $\mathcal{A} = \begin{pmatrix} A+G & K \\ T & Q \end{pmatrix}$, where A is a pseudo-differential operator on a C^{∞} manifold X with boundary, with the transmission property at ∂X, and G is a so-called Green operator. T is a trace operator, responsible for boundary conditions; it maps distributions on $\operatorname{int} X$ to distributions on ∂X. Moreover, K is a potential operator, mapping in the opposite direction, and Q is a classical pseudo-differential operator on ∂X. The principal symbolic hierarchy in the case of boundary value problems consists of two components, namely,

$$\sigma(\mathcal{A}) = (\sigma_{\psi}(\mathcal{A}), \sigma_{\partial}(\mathcal{A})),$$

with the interior symbol $\sigma_{\psi}(\mathcal{A}) := \sigma_{\psi}(A)$ and the boundary symbol $\sigma_{\partial}(\mathcal{A})$. First simple examples may be found in Section 1.1.1; we see, in particular, how an operator A in $\mathbb{R}^n$ acquires a boundary symbol from any given (smooth) boundary, prescribed as an additional information, and how extra trace operators become natural in the ellipticity. Further intuitive ideas in connection with symbols which are contributed by hypersurfaces of a configuration are given in Section 10.1.1.

Chapter 4 is devoted to mixed elliptic boundary value problems, realised as continuous operators in standard Sobolev spaces. An example is the Zaremba problem for a second order elliptic differential operator A in a smooth bounded domain G. The boundary $Y := \partial G$ is subdivided into smooth submanifolds $Y_\pm$ with a common boundary $Z = Y_- \cap Y_+$. The Zaremba problem is represented by a column matrix $\mathcal{A} := {}^{\mathrm{t}}(A\ T_-\ T_+)$, where $T_\mp$ are the trace operators describing Dirichlet and Neumann boundary conditions on the minus and the plus side, respectively. The interpretation of $\mathcal{A}$ as an operator on standard Sobolev spaces $H^s(G)$ for $s > \frac{3}{2}$ rules out specific singularities of solutions that may be expected when we prescribe independent Dirichlet and Neumann data $g_\mp$ on $Y_\mp$. Therefore, the image im $\mathcal{A}$ has infinite codimension in the space $H^{s-2}(G) \oplus H^{s-\frac{1}{2}}(\mathrm{int}\, Y_-) \oplus H^{s-\frac{3}{2}}(\mathrm{int}\, Y_+)$; thus the solvability in this case is only guaranteed when we impose suitable restrictions on the data on the right-hand sides. Nevertheless, mixed problems in this formulation already give us important partial answers for the general case with arbitrary $g_\mp$. Depending on s we complete $\mathcal{A}$ by a potential operator $\mathcal{K}$ acting on distributions on the interface such that the row matrix $\mathfrak{A} := (\mathcal{A}\ \ \mathcal{K})$ is a Fredholm operator between the respective Sobolev spaces.

The solutions of mixed elliptic problems with arbitrary conditions $g_\mp$ may have singularities along the interface Z, inherited by the jumping boundary conditions. This means that the standard Sobolev spaces in G should be replaced by other spaces that admit such singularities.

In Chapter 5 we show that weighted edge spaces are an adequate choice, where we interpret $X := \overline{G}$ as a manifold with edge Z and boundary Y. Mixed problems are identified as elements of the edge algebra of boundary value problems on such a manifold. The tools around the edge algebra are developed in Chapter 7; basic notions are already introduced in Chapter 2. By 'edge algebra' we understand a calculus of pseudo-differential boundary value problems on a manifold with edge Z and boundary. Compared with the block matrices $\mathcal{A} = \begin{pmatrix} A+G & K \\ T & Q \end{pmatrix}$ of Chapter 3 we now have larger block matrices $\mathfrak{A} = \begin{pmatrix} \mathcal{A}+\mathcal{M}+\mathcal{G} & \mathcal{K} \\ \mathcal{T} & \mathcal{Q} \end{pmatrix}$ with $\mathcal{A}$ in the upper left corner, together with a so-called Green operator $\mathcal{G}$ with respect to Z and a smoothing Mellin operator $\mathcal{M}$, also referring to Z. The meaning of $\mathcal{T}$, $\mathcal{K}$ and $\mathcal{Q}$ with respect to the edge is similar to that of the corresponding operators T, K and Q in $\mathcal{A}$ with respect to the boundary. $\mathcal{T}$ is a trace operator, $\mathcal{K}$ a potential operator with respect to Z, and $\mathcal{Q}$ is a classical pseudo-differential operator on Z. Modulo lower order terms an operator $\mathfrak{A}$ in the edge algebra is described by a principal symbolic hierarchy

$$\sigma(\mathfrak{A}) = (\sigma_\psi(\mathfrak{A}), \sigma_\partial(\mathfrak{A}), \sigma_\wedge(\mathfrak{A})), \tag{0.0.1}$$

consisting of the interior symbol $\sigma_\psi(\mathfrak{A}) = \sigma_\psi(A)$, the boundary symbol $\sigma_\partial(\mathfrak{A}) = \sigma_\partial(\mathcal{A})$ and the edge symbol $\sigma_\wedge(\mathfrak{A})$.

In the case of mixed problems the operators $\mathfrak{A}$ act between spaces of the kind

$$\mathcal{W}^{s,\gamma}(\mathbb{X}) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y_-) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y_+) \oplus H^{s-1}(Z)$$

(in general, the components may also be vector-valued; here, for convenience, of unified smoothness on Y_- and Y_+ by means of suitable reductions of orders). $\mathbb{X}$ is the stretched manifold associated with X and the chosen edge Z of codimension 1 on the boundary Y, and $\mathcal{W}^{s,\gamma}(\mathbb{X})$ are weighted edge spaces on $\mathbb{X}$ of smoothness $s \in \mathbb{R}$ and weight $\gamma \in \mathbb{R}$; the spaces $\mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(Y_\pm)$ are of a similar meaning; the edge Z in this case is the boundary of $Y_\pm$. The shifts of s and γ by $\frac{1}{2}$ make sense in connection with restriction operators to $Y_\pm$ contained in the trace operators in $\mathfrak{A}$ on the $\pm$ sides of the boundary. In contrast to Chapter 4 the operators $\mathfrak{A}$ may contain trace and potential operators with respect to Z at the same time, when the weight γ is negative; such weights correspond to singularities of solutions at Z. In the theory of elliptic (with respect to (0.0.1)) operators $\mathfrak{A}$ the additional trace and potential conditions occur for similar reasons as standard boundary and potential conditions in the context of boundary value problems. However, because of the influence of the choice of weights we may have potentials already in problems with differential operators. In differential boundary value problems in standard Sobolev spaces the potentials do not occur, however, they are generated in parametrices. Apart from the observation that mixed problems are particular edge problems, in Chapter 5 we compute the number of trace and potential operators for concrete examples, e.g., the Zaremba problem, or problems with jumping oblique derivatives. The calculations are based on the results of Chapter 4, together with reformulations of standard Sobolev spaces in terms of weighted edge spaces for $\gamma = s$ and combined with relative index results for the case $\gamma < s$.

Chapter 6 contains a survey on the calculus of (pseudo-differential) boundary value problems on a manifold with conical singularities, with the transmission property at the boundary. We establish several variants of the cone algebra of boundary value problems, including the case of an infinite cone which is necessary for the edge symbolic calculus. The operators have a principal symbolic hierarchy of the form $\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_c(\mathcal{A}))$ consisting of the interior symbol $\sigma_\psi(\mathcal{A}) = \sigma_\psi(A)$, the boundary symbol $\sigma_\partial(\mathcal{A})$ and the so-called conormal symbol $\sigma_c(\mathcal{A})$. The cone algebras are closed under parametrix construction of elliptic elements and under inversion (in the case of invertibility of operators in weighted Sobolev spaces). As an example we treat the Laplacian in a spindle (represented in stretched form by $I^\wedge := \mathbb{R}_+ \times I$ for an interval I, with two conical points corresponding to $\{0\}$ and $\{\infty\}$) with Dirichlet, Neumann, or Zaremba conditions, and show a number of results on unique solvability in $\mathcal{H}^{s,\gamma}(I^\wedge)$-spaces, with weights γ depending on the length of I (concerning notation, see Section 2.4.2).

In the case of an infinite (open stretched) cone $X^\wedge = \mathbb{R}_+ \times X$, where the base X is a compact C^∞ manifold with boundary ∂X, the cone operators $\mathcal{A}$ act between spaces of the kind

$$\mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge)$$

(the components may also be vector-valued). The $\mathcal{K}^{s,\gamma}$-spaces are based on the Mellin transform in the axial variable r near $r = 0$, while for $r \to \infty$ they are modelled on the standard Sobolev spaces up to infinity. The principal symbolic hierarchy in this case also contains a component that controls the behaviour of $\mathcal{A}$ up to infinity, called the exit symbol $(\sigma_{\mathrm{E}}(\mathcal{A}), \sigma_{\mathrm{E}'}(\mathcal{A}))$. The principal symbolic hierarchy for the infinite cone $X^\wedge$ is a tuple

$$\sigma(\mathcal{A}) = (\sigma_\psi(\mathcal{A}), \sigma_\partial(\mathcal{A}), \sigma_{\mathrm{c}}(\mathcal{A}), \sigma_{\mathrm{E}}(\mathcal{A}), \sigma_{\mathrm{E}'}(\mathcal{A})).$$

In Chapter 7 we formulate the (pseudo-differential) calculus of boundary value problems on a manifold W with boundary V and edge Y, with the transmission property at the boundary ($\mathbb{W}$ and $\mathbb{V}$ will denote the stretched manifolds associated with W and V, respectively, and n the dimension of the base X of the model cones, locally near Y). As a typical element of that calculus we establish the 3×3 block matrix structure of operators with the principal symbolic hierarchy (0.0.1). We study the edge algebra in weighted edge spaces $\mathcal{W}^{s,\gamma}(\mathbb{W}) \oplus \mathcal{W}^{s-\frac{1}{2},\gamma-\frac{1}{2}}(\mathbb{V}) \oplus H^{s-\frac{n}{2}}(Y)$ and consider ellipticity with respect to the components of the symbols. The edge algebra is closed under parametrix construction of elliptic elements and under inverses (in the case of invertibility of operators in weighted edge spaces). Locally near the singularity the weighted edge spaces are modelled on abstract edge spaces $\mathcal{W}^s(\mathbb{R}^q, H)$ for Hilbert spaces H which are equipped with the action of a strongly continuous group of isomorphisms $\{\kappa_\lambda\}_{\lambda\in\mathbb{R}_+}$, $\kappa_\lambda\colon H \to H$, $\lambda \in \mathbb{R}_+$, here for the case $H = \mathcal{K}^{s,\gamma}(X^\wedge)$. We show that there are several scales of such spaces in which the concept of edge operators with twisted homogeneity applies. Although in this exposition we mainly prefer spaces modelled on $\mathcal{W}^s(\mathbb{R}^q, \mathcal{K}^{s,\gamma}(X^\wedge))$ which are defined in terms of group actions on the parameter spaces $\mathcal{K}^{s,\gamma}(X^\wedge)$ independent of s, γ, we also discuss alternative possibilities with parameter spaces $\langle r\rangle^{s-\gamma}\mathcal{K}^{s,\gamma}(X^\wedge)$ and (s,γ)-dependent group actions. In a final section we show that the edge calculus also covers mixed problems in wedges of arbitrary opening angle for the Laplacian and with Dirichlet, Neumann, or Zaremba conditions, where we explicitly compute the (difference of the) number of additional trace and potential conditions on the edge, depending on the chosen weight.

Chapter 8 is devoted to mixed elliptic and crack problems when the interface (the boundary of the crack) has conical singularities. We essentially content ourselves with a number of concrete cases for the Laplacian, especially, problems of Zaremba type; but after the material of Chapters 6 and 7 it becomes clear that the approach is completely general.

In Chapter 9 operators with meromorphic symbols on infinite cylinders are studied. These operators are generated in the Mellin symbolic structure on manifolds with conical singularities, edges or corners. Applying the work of Gohberg and Sigal [60] we specify the structures for meromorphic pseudo-differential families when the cross section is closed and compact, and, moreover, for boundary value problems for a compact smooth cross section with boundary, and for families in the cone algebra when the cross section is a manifold with conical singularities.

Chapter 10 contains a general discussion of the approach with operator algebras and symbolic structures on manifolds with singularities. We describe motivations and intentions that have played a role in the development of the past years, and we show that the theory as a whole is at a new beginning, with new challenges and open problems.

The analysis of operators on configurations with singularities has a long history. It goes back to the 19th century, starting from observations in models of physics, such as potential theory, heat distribution, wave propagation, etc. For instance, the solutions of elliptic boundary value problems in domains with conical singularities at the boundary may have singularities which can be expressed in terms of asymptotic expansions in the distance variable r to the conical point. The data in such expansions (powers of r, logarithmic terms, and coefficients) are determined by the poles of the inverse of the conormal symbol of the given operator. There is a huge literature on this subject matter, and we cannot give a complete appreciation of all merits and achievements of the past. The work of Kondratyev [100] on elliptic boundary value problems for the case with conical singularities at the boundary is quoted here as a substitute for papers of many other authors. The subsequent development has extended and deepened the ideas for large classes of concrete boundary value problems, also with edges and for many other cases of non-compact domains; more references will be given below.

The present book is based on pseudo-differential strategies. Their design is directly deduced from the task to express parametrices of elliptic problems on manifolds with singularities, for instance, of elliptic mixed, transmission and crack problems. As is known from 'standard' elliptic boundary value problems, the parametrices, together with their continuity in Sobolev spaces, give rise to the regularity of solutions and to other qualitative results. This concerns, in particular, the nature of Green functions. Applied to $Au = f$ these produce solutions u with vanishing boundary data.

Apart from a fundamental solution (or a parametrix) of the given elliptic operator A the Green function of a boundary value problem can be characterised as a pseudo-differential operator on the boundary. Its symbol is operator-valued and acts on spaces in normal direction to the boundary. Such so-called Green symbols are very close to the symbols of trace and potential operators referring to the boundary. Knowing the latter ones we can predict the structure of the Green functions, and vice versa.

In other words, if (for some practical reason) we are interested in solutions of a boundary value problem with vanishing boundary data, we should be aware of the case with arbitrary data, since the Green functions contain the same 'complexity' of information from the boundary anyway.

This may be a key for understanding the relevance of additional trace and potential data on interfaces also in the case of mixed, transmission, or crack problems. Formally they are generated from the notion of ellipticity of the (in this case edge) symbol itself as an analogue of Shapiro–Lopatinskij elliptic conditions, because ellipticity should mean that all components of the principal symbolic hierarchy are bijective. The analytic structure of trace and potential operators of the calculus in the edge case generates analogues of the Green functions, here referring to the interfaces of the singular configuration (i.e., discontinuities of boundary data in mixed problems or crack boundaries).

The program to investigate parametrices and their symbolic structure is interesting also on configurations with 'higher' singularities. They occur, for instance, when the above-mentioned interfaces themselves have singularities, e.g., conical ones as in Chapter 8. A transparent management of corresponding operator calculi requires a well organised building of symbolic and operator levels. Let us continue illustrating several features of this approach by the aspect of Green, trace, and potential symbols. Their role does not only consist of encoding extra data on the lower-dimensional strata of a configuration with singularities. They contain the information of the corresponding symbolic and operator calculi on the lower-dimensional strata which are themselves manifolds with singularities. For instance, the set of all elliptic boundary value problems for a fixed elliptic operator on a compact C^∞ manifold X with boundary Y is classified by the space of all elliptic pseudo-differential operators on Y (via reduction to the boundary). More generally, if the boundary Y has, say, conical singularities, elliptic boundary value problems for a fixed elliptic (Fuchs type) operator on X are classified by the elliptic elements of the cone algebra on Y, etc. In general, the calculus contains reductions to all lower-dimensional strata, and there are Green, trace, potential, etc., operators for all of them. The technical tool to express these elements of the calculi are operator-valued symbols with 'twisted homogeneity'. This concept was first systematically developed in connection with edge algebras, cf. [163], [180], and then applied to several higher algebras, e.g., on manifolds with corners, cf. [183], [192]. At the same time the Green, trace, and potential symbols are a convenient tool to describe phenomena with asymptotics of solutions, especially, edge asymptotics and their singular functions. This is a voluminous program, worth to be presented in a separate exposition in more detail. The technique with asymptotics in that spirit may be found, e.g., in [182], [186], [187], [185], [188], or [90]. We comment more on that in Chapter 10.

The story with additional elliptic trace and potential conditions on lower-dimensional strata has also other remarkable sides. For instance, it may be an ambitious task to determine the necessary number of such conditions (depending on the chosen weights γ in the distribution spaces). Therefore, we study explicit examples in connection with mixed and crack problems (see Chapters 4, 5 and 8), but also other concrete elliptic cone and edge operators; see, for instance, Sections 6.3.3, 6.3.4, 6.4.1, or 7.3.3, 7.3.4.

If W is a manifold with boundary and (say, a compact smooth) edge Y, the consideration starts from the homogeneous principal edge symbol

$$\sigma_\wedge(\mathcal{A})(y,\eta) : \mathcal{K}^{s,\gamma}(X^\wedge) \oplus \mathcal{K}^{s-\frac{1}{2},\gamma-\frac{1}{2}}((\partial X)^\wedge) \to \mathcal{K}^{s-\mu,\gamma-\mu}(X^\wedge) \oplus \mathcal{K}^{s-\mu-\frac{1}{2},\gamma-\mu-\frac{1}{2}}((\partial X)^\wedge), \tag{0.0.2}$$

$(y,\eta) \in T^*Y \setminus 0$, which is for elliptic $\mathcal{A}$ on $W \setminus Y$ (in the sense of Chapter 3) and under a suitable choice of the weight γ a family of Fredholm operators (for simplicity, at the moment we consider spaces that only contain one component over $X^\wedge$ and $(\partial X)^\wedge$, respectively; in general, we have spaces of distributional sections in several vector bundles).

The additional trace symbols, potential symbols, etc., $\sigma_\wedge(\mathcal{T})(y,\eta)$, $\sigma_\wedge(\mathcal{K})(y,\eta)$ and $\sigma_\wedge(\mathcal{Q})(y,\eta)$ play the role to fill up (0.0.2) to a block matrix family

$$\begin{pmatrix} \sigma_\wedge(\mathcal{A}) & \sigma_\wedge(\mathcal{K}) \\ \sigma_\wedge(\mathcal{T}) & \sigma_\wedge(\mathcal{Q}) \end{pmatrix}(y,\eta)\colon \boldsymbol{\mathcal{K}}^{s,\gamma}(X^\wedge) \oplus \mathbb{C}^{j_-} \to \boldsymbol{\mathcal{K}}^{s-\mu,\gamma-\mu}(X^\wedge) \oplus \mathbb{C}^{j_+}$$

for suitable $j_\pm \in \mathbb{N}$ (where $\boldsymbol{\mathcal{K}}^{s,\gamma}(X^\wedge)$, $\boldsymbol{\mathcal{K}}^{s-\mu,\gamma-\mu}(X^\wedge)$ are abbreviations for the spaces in (0.0.2)).

Because of the (twisted) homogeneity of $\sigma_\wedge(\mathcal{A})(y,\eta)$ in $\eta \neq 0$ it suffices to do that for $(y,\eta) \in S^*Y$, the unit cosphere bundle in $T^*Y \setminus 0$. In order to find $\sigma_\wedge(\mathcal{T}), \sigma_\wedge(\mathcal{K})$, $\sigma_\wedge(\mathcal{Q})$ we have to know at least the index $\operatorname{ind} \sigma_\wedge(\mathcal{A})(y,\eta)$ which is equal to $j_+ - j_-$. In other words, in order to characterise the extra conditions, we suddenly have to solve an index problem on the infinite cone $X^\wedge$ with boundary (and, of course, much more, to know the dimensions $j_\pm$ themselves, together with kernels and cokernels).

Let us stress at this moment that, in general, the above-mentioned spaces $\mathbb{C}^{j_\pm}$ have to be interpreted as the fibres of (smooth complex) vector bundles $J_\pm$ over the edge Y (they may be non-trivial, unless we do not impose corresponding assumptions on $\mathcal{A}$). In analogous form, this is also the case in boundary value problems for an elliptic operator A on a compact C^∞ manifold X with boundary, where the role of trace, potential, etc., boundary symbols $\sigma_\partial(T)(x',\xi'), \sigma_\partial(K)(x',\xi')$ and $\sigma_\partial(Q)(x',\xi')$ is to fill up the Fredholm family $\sigma_\partial(A)(x',\xi')\colon H^s(\mathbb{R}_+) \to H^{s-\mu}(\mathbb{R}_+)$ to a block matrix family of isomorphisms $\sigma_\partial(\mathcal{A})(x',\xi') = \begin{pmatrix} \sigma_\partial(A) & \sigma_\partial(K) \\ \sigma_\partial(T) & \sigma_\partial(Q) \end{pmatrix}(x',\xi')\colon H^s(\mathbb{R}_+) \oplus \mathbb{C}^{l_-} \to H^{s-\mu}(\mathbb{R}_+) \oplus \mathbb{C}^{l_+}$, $(x',\xi') \in T^*(\partial X) \setminus 0$. Clearly, the formalism is possible in analogous form for operators A, acting between distributional sections of vector bundles. If $\mathbb{C}^{l_\pm}$ are the fibres of vector bundles $L_\pm$ on ∂X, then the Fredholm family $\sigma_\partial(A)(x',\xi')$, parametrised by the compact space $S^*(\partial X)$, has an index element

$$\operatorname{ind}_{S^*(\partial X)} \sigma_\partial(A) = [L_+] - [L_-] \in \pi_{\partial X}^* K(\partial X), \tag{0.0.3}$$

where $K(\cdot)$ denotes the K-group over the space in parentheses and $\pi_{\partial X}^*\colon K(\partial X) \to K(S^*(\partial X))$ is the pull back under the canonical projection $\pi_{\partial X}\colon S^*(\partial X) \to \partial X$. It is well known that there are elliptic operators A such that

$$\operatorname{ind}_{S^*(\partial X)} \sigma_\partial(A) \in K(S^*(\partial X)) \tag{0.0.4}$$

is not the pull back of an element of $K(\partial X)$. This is, for instance, the case when A is a Dirac operator in even dimensions.

Then the condition (0.0.3) represents a topological obstruction for the existence of additional trace and potential operators which complete A to a 2×2 block matrix Fredholm operator $\mathcal{A}$ between standard Sobolev spaces, with A in the upper left corner. More precisely, (0.0.3) means vanishing of the obstruction, and then $\mathcal{A}$ exists as desired, cf. Atiyah and Bott [8] and Boutet de Monvel [15]. We do not study here the general case, i.e., when (0.0.3) is violated, and refer to the article [190], see also [194], with global projection conditions of trace and potential type rather than usual ones.

What concerns the existence of additional trace, potential, etc., conditions for $\mathcal{A}$ in the case of a manifold with edge there is a similar (necessary and sufficient) condition, namely,

$$\operatorname{ind}_{S^*Y} \sigma_\wedge(\mathcal{A}) \in \pi_Y^* K(Y) \tag{0.0.5}$$

with $\pi_Y : S^*Y \to Y$ being the canonical projection. This situation for the analogous case of a manifold without boundary and edges has been investigated in [182] and in Schulze and Seiler [198]; in the latter article this has been performed also in the sense of global projection conditions with respect to the edge Y.

Throughout this book we assume that (0.0.5) is satisfied. Let us note for completeness that the phenomenon of edge operators where (0.0.5) is violated has been studied under different aspects in Nazaikinskij, Savin, Schulze and Sternin [131], [132], (see also Nazaikinskij, Schulze, Sternin, and Shatalov [138] or Schulze, Sternin, and Shatalov [199] for the case of boundary value problems in the case of differential operators). Operators on manifolds with singularities from the viewpoint of ellipticity and index lead to a large variety of beautiful and new questions of K-theoretic nature that are far from being solved in general. For instance, in connection with such problems on stratified spaces we have to expect hierarchies of topological obstructions for the extra conditions, cf. Section 10.5.3.

The complexity of several phenomena makes it desirable to establish explicit examples of reasonable generality. That is why in Chapter 7 of this book we report the construction of extra conditions by reformulating elliptic operators on a smooth manifold. The results belong to a cycle of papers [36], [35], [113] of Schulze in cooperation with Dines and Liu, in which such questions are studied in different situations. The technique is also used here in Chapters 5 and 8 to explicitly computing the above-mentioned index of the Fredholm family $\sigma_\wedge(\mathcal{A})$. It is a typical effect that $\operatorname{ind} \sigma_\wedge(\mathcal{A})$ depends on the chosen weights γ; the difference of indices for different weights can be characterised by relative index formulas. This is also illustrated here in Chapter 5 for the case of mixed problems.

In the present book we mainly focus on the analytic part of the elliptic theory on manifolds with singularities, here in the context of mixed, transmission and crack problems; those can be regarded as examples of the edge and corner calculus of boundary value problems. The symbolic structures show that concrete cases require in a very early phase of the consideration the cone, edge and corner calculi of high generality, because the operator-valued symbols are families operating on configurations that have again singularities. As amplitude functions they have to be composed and inverted within the calculus, and this generates large operator algebras. Therefore, a part of this exposition is devoted to the general aspects of operators on manifolds with boundary and conical points and edges. In particular, (pseudo-differential) boundary value problems are a necessary element of the theory both in local form on a C^∞ manifold with boundary, and from the point of view of the exit calculus, i.e., when the underlying manifold is an infinite cone with conical exit to infinity.

As noted before, all these structures are motivated by the various ingredients of parametrices of elliptic operators, realised here as edge boundary value problems. A short characterisation of our calculus is that it describes in advance the structure of parametrices. This is connected with the functional analytic aspects around weighted cone and edge spaces which determine in advance different facets of the regularity of solutions. Thus the 'technical' parts of this exposition directly belong to the concrete problems. Moreover, we believe that the tools can be employed to other areas of the analysis on manifolds with singularities, much more than it is already the case in the papers of Krainer [104], [105], [106] on parabolic operators, of Gil, Krainer and Mendoza [55], [54] on resolvents of elliptic operators on manifolds with singularities, of Liu and Witt [114], [115] on explicit information on asymptotic types, or Dreher and Witt [40], [42], [41] on modifications and applications of edge spaces in hyperbolic equations.

The ellipticity of interesting operator classes is often a starting point to develop the corresponding index theories and to derive index formulas, see Nazaikinskij, Savin, Schulze and Sternin [133], [134], [135], and Fedosov, Schulze, and Tarkhanov [48], [50], [51] and more references below. Other interesting results on the Mellin-edge calculus with meromorphic symbols and asymptotic data are contained in papers of Witt [227], [228].

Relations to models motivated by problems of elasticity and material sciences may be found in Kapanadze and Schulze [96], [92], [93]. Contributions to the functional analytic machinery of the cone and edge calculus are given in Hirschmann [79] on edge spaces, Gil, Schulze and Seiler [56], [57] on edge quantisations, Seiler [205] on a Calderón–Vaillancourt theorem in abstract edge spaces, Seiler [206], Schulze and Volpato [201] on kernel characterisations of Green operators, Kapanadze, Schulze, and Seiler [95] on edge operators with singular trace conditions, see also Liu and Witt [115], moreover, Coriasco and Schulze [28] on the edge calculus with model cones of different dimensions, Airapetyan and Witt [4], or Tarkhanov [217] on a motivated choice of edge spaces. Let us also mention the development towards calculi with higher singularities, especially, [183], [192], [75], and the joint papers of Schulze with Maniccia [118], Krainer [109], [108], and Calvo and Martin [20], [18].

Concerning the literature on mixed, crack, or other singular boundary value problems, there is a huge number of investigations devoted to specific topics or different approaches, and we cannot give a complete review here.

In the past decades the analysis around manifolds with singularities, especially extensions of the index theory, attracted many mathematicians. There are now different schools and confessions, partly with a common terminology, such as corner manifolds or manifolds with boundary, although for quite different species of operators, see also the remarks in Section 10.3.4. If we now give a list of more references that have from different point of view connections with this exposition, we are aware of such ambiguities. Let us mention, in particular, Agmon [2], Agranovich and Vishik [3], Kohn and Nirenberg [98], Vishik and Eskin [221], Atiyah and Singer [11], Atiyah [7], Kasparov [97], Eskin [44], Vishik and Grushin [224], Sternin [214], [215], Kondratyev [100], Plamenevskij [150], [152], [151], Rabinovich [153], Gramsch [61], Gohberg and Sigal [60], Fedosov [46], Seeley [202], [203], Grushin [70], Boutet de Monvel [14],

[15], Atiyah, Patodi and Singer [9], Maz'ja and Paneah [121], Shubin [208], Parenti [145], Cordes [25], [26], Fichera [52], Teleman [218], Cheeger [21], Melrose [124], [125], Melrose and Mendoza [127], Rempel and Schulze [159], [162], [163], [155], Luke [117], Grubb [69], [68], Kondratyev and Oleynik [101], Grisvard [65], Maz'ja and Rossmann [122], Dauge [30], Chkadua and Duduchava [22], [23], Costabel and Dauge [29], Shaw [207], Brüning and Seeley [16], Mazzeo [123], Piazza [147], Roe [165], Melrose [126], Rozenblum [166], Egorov and Schulze [43], Booss-Bavnbek and Wojciechowski [13], Mantlik [119], [120], Fedosov and Schulze [47], Nistor [142], [143], Melrose and Nistor [128], Melrose and Piazza [129], Witt [228], [227], Fedosov, Schulze, and Tarkhanov [50], [49], [51], Nazaikinskij and Sternin [140], [141], Nazaikinskij, Savin and Sternin [135], [136], [137], Grieser and Lesch [64], Krainer [106], [103], Seiler [205], [206], [204], Gil, Krainer, and Mendoza [55], [54], Ammann, Lauter, and Nistor [5].

Let us finally make a few remarks on how the present exposition is organised. Our approach to mixed elliptic boundary value problems in Chapters 1, 4 and 5 employs elements of the pseudo-differential calculus and of the formalism of boundary value problems with the transmission property at the boundary. These theories, developed in Chapters 2 and 3 before, are independently readable as introductions. In Chapter 5 on mixed problems in weighted edge spaces we make use of boundary value problems on manifolds with conical singularities and edges; they belong to the necessary tools for the construction of parametrices. In order to come to mixed problems in this exposition as early as possible we develop the details on cone and edge algebras afterwards in Chapters 6 and 7. They can be consulted if something remains to be completed in the considerations before. This material on cone and edge boundary value problems is also an introduction; at the same time it prepares the applications in Chapter 8, namely, mixed problems with singular interfaces and crack problems with singular crack boundaries. Such problems belong to the corner theory of singularity order 2 and with boundary. As such they are of some complexity. In order to illustrate the main ideas which are nevertheless simple, in Chapter 10 we try to give a feeling for the iterative way of treating problems with higher singularities and for the several sources and mathematical ingredients in connection with the building of higher singular algebras.