1 Introduction

These notes are taken from the final part of a class on rectifiability given at the University of Zürich during the summer semester 2004. The main aim is to provide a self-contained reference for the proof of the following remarkable theorem,

Theorem 1.1. *Let μ be a locally finite measure on $\mathbb{R}^n$ and α a nonnegative real number. Assume that the following limit exists, is finite and nonzero for μ-a.e. x:*

$$\lim_{r\downarrow 0} \frac{\mu(B_r(x))}{r^\alpha}\,.$$

Then either $\mu = 0$, or α is a natural number $k \leq n$. In the latter case, a measure μ satisfies the requirement above if and only if there exists a Borel measurable function f and a countable collection $\{\Gamma_i\}$ of Lipschitz k-dimensional submanifolds of $\mathbb{R}^n$ such that

$$\mu(A) \;=\; \sum_i \int_{\Gamma_i\cap A} f(x)\, d\mathrm{Vol}^k(x) \quad \textit{for any Borel set } A.$$

Here Vol^k denotes the natural k-dimensional volume measure that a Lipschitz submanifold inherits as a subset of $\mathbb{R}^n$.

The first part of Theorem 1.1, (i.e. if μ is nontrivial then α must be integer) was proved by Marstrand in [**17**]. The second part is trivial when $k = 0$ and $k = n$. The first nontrivial case, $k = 1$ and $n = 2$, was proved by Besicovitch in his pioneering work [**2**], though in a different framework (Besicovitch's statement dealt with sets instead of measures). Besicovitch's theorem was recast in the framework above in [**24**], and in [**23**] it was extended to the case $k = 1$ and generic n. The higher dimensional version remained a long standing problem. Marstrand in [**16**] made a major contribution to its solution. His ideas were sufficient to prove a weaker theorem for 2-dimensional sets in $\mathbb{R}^3$, which was later generalized by Mattila in [**18**] to arbitrary dimensions and codimensions.

The problem was finally solved by Preiss in [**25**]. His proof starts from Marstrand's work but he introduces many new and interesting ideas. Although the excellent book of Mattila [**21**] gives a summary of this proof, many details and some important ideas were not documented. As far as I know, the only reference for the proof of the second part of Theorem 1.1 is Preiss' paper itself.

As a measure of the complexity of the subject, we remark that natural generalizations of Marstrand, Mattila, and Preiss' theorems proved to be quite hard; see for instance [**12**] and [**13**].

Actually, in [**25**] Preiss proved the following stronger quantitative version of the second part of Theorem 1.1:

Theorem 1.2. *For any pair of nonnegative integers* $k \leq n$ *there exists a constant* $c(k,n) > 1$ *such that the following holds. If* μ *is a locally finite measure on* $\mathbb{R}^n$ *and*

$$0 < \limsup_{r\downarrow 0} \frac{\mu(B_r(x))}{r^k} < c(k,n) \liminf_{r\downarrow 0} \frac{\mu(B_r(x))}{r^k} < \infty \quad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n,$$

then the same conclusion as for Theorem 1.1 holds.

The proof of this statement is longer and more difficult. On the other hand, most of the deep ideas contained in [**25**] are already needed to prove Theorem 1.1. Therefore, I decided to focus on Theorem 1.1.

Despite the depth of Theorem 1.1, no substantial knowledge of geometric measure theory is needed to read these notes. Indeed, the only prerequisites are:

- Some elementary measure theory;
- Some classical covering theorems and the Besicovitch Differentiation Theorem;
- Rademacher's Theorem on the almost everywhere differentiability of Lipschitz maps;
- The definition of Hausdorff measures and a few of their elementary properties.

All the fundamental definitions, propositions, and theorems are given in Chapter 2, together with references on where to find them.

The reader will note that I do not assume any knowledge of rectifiable sets. I define them in Chapter 4, where I prove some of their basic properties. The material of Chapter 4 can be found in other books and Mattila's book is a particularly good reference for Chapter 3 and Chapter 5. However, there are two good reasons for including Chapters 3, 4, and 5 in these notes:

(a) To make these notes accessible to people who are not experts in the field;
(b) To show the precursors of some ideas of Preiss' proof, in the hope that it makes them easier to understand.

These two reasons have also been the main guidelines in presenting the proofs of the various propositions and theorems. Therefore, some of the proofs are neither the shortest nor the most elegant available in the literature. For instance, as far as I know, the shortest and most elegant proof of Marstrand's Theorem (see Theorem 3.1) uses a beautiful result of Kirchheim and Preiss (see Theorem 3.11 in [**10**]). However, I have chosen to give Marstrand's original proof because the "moments" introduced by Preiss (which play a major role in his proof; see Chapters 7, 8, and 9) are reminiscent of the "barycenter" introduced by Marstrand (see (3.17)).

Similarly, I have not hesitated to sacrifice generality, whenever this seemed to make the statements, the notation, or the ideas more transparent. Therefore, many other remarkable facts proved by Preiss in [**25**] are not mentioned in these notes.

As already mentioned above, Chapter 2 is mostly a list of prerequisites on measure theory. In Chapter 3, we prove the classical result of Marstrand that if $\alpha \in \mathbb{R}$ and $\mu \neq 0$ satisfy the assumption of Theorem 1.1, then α is an integer. In this chapter we also introduce the notion of tangent measure.

In Chapter 4 we define rectifiable sets and rectifiable measures and we prove the Area Formula and a classical rectifiability criterion. As an application of these tools we give a first characterization of rectifiable measures in terms of their tangent measures.

In Chapter 5 we prove a deeper rectifiability criterion, due to Marstrand for 2-dimensional sets in $\mathbb{R}^3$ and extended by Mattila to general dimension and codimension. This rectifiability criterion plays a crucial role in the proof of Theorem 1.1.

In Chapter 6 we give an overview of Preiss' proof of Theorem 1.1. In this chapter we motivate some of its difficulties and we split the proof into three main steps, each of which is taken in one of the subsequent three chapters. Chapter 10 is a collection of open problems connected to the various topics of the notes, which I collected together with Bernd Kirchheim.

In Appendix A we prove the Kirchheim–Preiss Theorem on the analyticity of the support of uniformly distributed euclidean measures, whereas Appendix B contains some useful elementary computations on Gaussian integrals.

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