Introduction

The aim of this book is to give an introduction to the duality of quantum groups and to quantum groups in the setting of C^*-algebras and von Neumann algebras.

Roughly, a Hopf algebra or quantum group is the natural generalization of a group within the setting of non-commutative geometry: following the general principle of non-commutative geometry, the underlying space of the group is replaced by an algebra, and the group operations are replaced by additional structure maps on this algebra.

In the setting of C^*-algebras and von Neumann algebras, the term "quantum group" refers to generalizations of locally compact groups. In other fields of mathematics, the term "quantum group" is usually applied to a wide range of mathematical objects, which are studied by quite different methods. Therefore it seems appropriate to give an overview before we outline the approach adopted in this book.

Hopf algebras and quantum groups in the algebraic setting

Initially, Hopf algebras were studied in a purely algebraic setting. The first examples appeared in the following situations:

Algebraic topology. In the study of the cohomology ring $H^*(G)$ of a compact Lie group G, Hopf investigated the map $\Delta\colon H^*(G) \to H^*(G) \otimes H^*(G)$ induced by the multiplication map $G \times G \to G$, and used algebraic properties of this map to determine the structure of $H^*(G)$. More generally, if X is "a group up to homotopy", more precisely, a Hopf space, then $H^*(X)$ is a Hopf algebra, and this algebraic structure can be used to show that X has the same cohomology like a product of spheres [66].

Affine algebraic groups. A basic principle in algebraic geometry is to describe an affine space X in terms of the algebra of regular functions $\mathcal{O}(X)$. Now, almost by definition, algebraic group structures on X correspond bijectively with Hopf algebra structures on $\mathcal{O}(X)$. Applications of the Hopf algebra point of view are given, for example, in [1].

Representation theory of groups. Further natural examples of Hopf algebras are the group algebra $\Bbbk G$ of a finite group G and the universal enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$.

These examples fall into two classes: the first two examples of Hopf algebras are commutative, and the last examples satisfy a cocommutativity condition that is, in some sense, dual to commutativity. In particular, both classes are closely related to classical groups.

The theory of Hopf algebras received strong new impulses when new classes of examples were constructed that were neither commutative nor cocommutative:

Deformations. One of the most influential developments in the theory of Hopf algebras was the introduction of q-deformations of universal enveloping algebras associated to certain Lie algebras. First examples were constructed by Faddeev and the Leningrad school in connection with work on the quantum inverse scattering method; later, Drinfeld and Jimbo produced a q-deformed Hopf algebra for every semisimple complex Lie algebra [37], where the deformation is related to a certain Poisson structure on the initial Lie algebra. These Drinfeld–Jimbo Hopf algebras and their representation theory are very well understood, see, for example, [23], [24], [68], [79], [80], [84], [103], [140].

Knot invariants and the Yang–Baxter equation. There exists an intriguing connection between physics, low-dimensional topology, and the corepresentation theory of certain Hopf algebras. The starting point is that the category of corepresentations of a Hopf algebra carries a natural tensor product, very much like the category of representations of a group. This tensor product is symmetric only if the Hopf algebra is cocommutative. But for certain Hopf algebras called braided or triangular, there exists a braiding, which is an isomorphism $c_{V,W} \colon V \otimes W \to W \otimes V$, natural in the corepresentations V and W. The coherence constraints on such a braiding can be related to planar braid diagrams and to the quantum Yang–Baxter equation known from physics. In particular, one can construct knot invariants and solutions of the Yang–Baxter equation out of braided Hopf algebras. Conversely, solutions of the quantum Yang–Baxter equation give rise to bialgebras, and, in special cases, to Hopf algebras. A very nice account of these topics can be found in [79].

Unlike the first commutative and cocommutative examples of Hopf algebras, the new examples listed above are no longer directly related to classical groups; therefore they are usually called quantum groups.

Algebraic quantum groups and their duality. An algebraic framework for the study of quantum groups and their duality was developed by Van Daele [174], [177]. In his theory, a quantum group is a non-unital Hopf algebra equipped with an integral, which is an analogue of the Haar measure of a locally compact group, and to every such quantum group, one can associate a dual quantum group.

Quantum groups in the setting of C^*-algebras and von Neumann algebras

A major motivation for the introduction of quantum groups in the setting of C^*-algebras and von Neumann algebras was the generalization of Pontrjagin duality to non-abelian locally compact groups:

Kac algebras and generalized Pontrjagin duality. For every locally compact abelian group G, the set of characters $\widehat{G}$ is a locally compact abelian group again, and the Pontrjagin–van Kampen theorem says that $\widehat{\widehat{G}} \cong G$ (see Section 1.1). For a non-abelian locally compact group, a generalized dual can no longer be defined in the form of a group, and one has to look for a larger category (of "quantum groups") that includes both locally compact groups and their generalized duals. This problem was solved by Vainerman and Kac [167], [168], and by Enock and Schwartz [47]: they defined the notion of a Kac algebra, which is a von Neumann algebra equipped with similar structure maps like a Hopf algebra, and constructed for every Kac algebra A a dual Kac algebra $\widehat{A}$, such that $\widehat{\widehat{A}} \cong A$. An important rôle in their theory is played by the analogue of the Haar measure of a locally compact group, which is part of the structure of a Kac algebra. A C^*-algebraic counterpart of the theory was developed by Vallin and Enock [49], [170].

The concept of a Kac algebra, however, turned out to be too restrictive to include all interesting examples of quantum groups in the setting of C^*-algebras:

Compact quantum groups. Woronowicz developed a general theory of compact quantum groups in the setting of C^*-algebras [193], [202], which contains examples that do not satisfy all axioms of a Kac algebra. This theory is very appealing: the definition of a compact quantum group is concise, the existence of a Haar measure on every compact quantum group can be deduced from the axioms, and the corepresentation theory of every such quantum group is very similar to the representation theory of a compact group.

A new perspective on quantum groups in the setting of C^*-algebras and von Neumann algebras was introduced by Baaj and Skandalis:

Multiplicative unitaries. Examples of multiplicative unitaries were used for a long time in the theory of quantum groups, till Baaj and Skandalis put them center-stage, formulated an abstract definition, and gave a comprehensive treatment [7]. Roughly, a multiplicative unitary simultaneously encodes a quantum group and the dual of that quantum group; conversely, to every "reasonable" quantum group, one can associate a multiplicative unitary.

Finally, comprehensive theories of locally compact quantum groups were developed, which cover all known examples:

Locally compact quantum groups / weighted Hopf algebras. The theories developed by Vaes and Kustermans [91], [93], and Masuda, Nakagami, and Woronowicz [110], seem to give a definite answer to the question "What is a locally compact quantum group in the setting of C^*-algebras/von Neumann algebras?".

Organization of the book

The aim of this book, as stated above, is to give an introduction to the duality of quantum groups, and to quantum groups in the setting of C^*-algebras and von Neumann algebras.

One possible approach would be to start immediately with a study of locally compact quantum groups, which form the most general framework. For someone who is familiar with Hopf algebras and with the high-level analytic techniques used in the theory of locally compact quantum groups, this is probably the best choice. In this book, however, we shall adopt another approach, which may be better suited for graduate students and researchers from other fields.

Contents of the book. Part I of this book provides an introduction to quantum groups in a purely algebraic setting. After a review of Hopf algebras and their duality (Chapter 1), we discuss the duality of algebraic quantum groups developed by Van Daele (Chapter 2). This theory provides a very nice model for the generalizations of Pontrjagin duality that will be considered in Part II and yields many fundamental formulas. Finally, we investigate algebraic compact quantum groups (Chapter 3). This class of quantum groups can be studied not only in an algebraic, but also in a C^*-algebraic setting, and will serve us as a bridge for the passage to the setting of C^*-algebras.

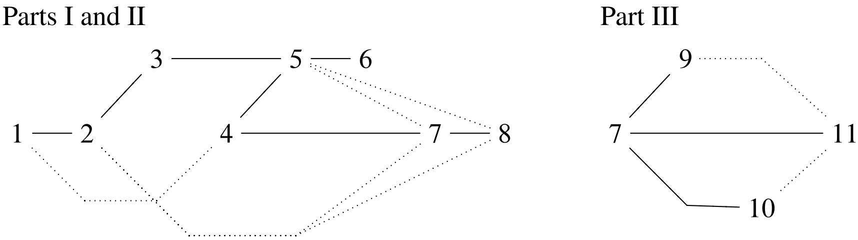
In Part II, we turn to quantum groups in the setting of C^*-algebras and von Neumann algebras. First, we discuss the problems that arise in the definition of a Hopf C^*-algebra or Hopf–von Neumann algebra, consider examples related to locally compact groups, and list the existing approaches (Chapter 4). Then, we present Woronowicz's theory of C^*-algebraic compact quantum groups (Chapter 5), which is particularly accessible and close to the algebraic setting, and consider important examples (Chapters 6). Closely related to quantum groups in the setting of C^*-algebras and von Neumann algebras are multiplicative unitaries, which are studied subsequently (Chapter 7). Part II ends with an overview of the theory of locally compact quantum groups and some examples (Chapter 8). We focus on motivation, which can often be found in the theory of algebraic quantum groups, and explain the central analytic tools, but do not give any proof.

Part III of this book is devoted to selected topics. First, we discuss coactions of quantum groups on C^*-algebras, reduced crossed products, and a generalization of the Takesaki–Takai duality theorem for group actions (Chapter 9). The crossed product construction and the duality theorem make essential use of multiplicative unitaries, and are due to Baaj and Skandalis. Next, we give an introduction to measurable quantum groupoids, or, more precisely, to pseudo-multiplicative unitaries on Hilbert spaces (Chapter 10). In particular, we present the relative tensor product of Hilbert spaces, which is also known as Connes' fusion. Finally, we summarize some results of the author's thesis on pseudo-multiplicative unitaries on C^*-modules and quantum groupoids in the setting of C^*-algebras (Chapter 11).

Frequently used notation and important terms used in this book are listed in separate indices, and some background is compiled in a short appendix.

Prerequisites. This book should be accessible to graduate students and researchers from other fields of mathematics. For Part I, no special background is needed. Part II assumes some familiarity with Hilbert spaces, C^*-algebras, and von Neumann algebras, as summarized in the appendix; at some points, we use the language of C^*-modules, which is also summarized in the appendix. Part III contains advanced topics and is addressed to readers with some background in the field of C^*-algebras or von Neumann algebras.

Logical dependence of the chapters. The logical dependence of the individual chapters of this book is sketched in the following diagram:



The dotted lines indicate that a chapter provides examples or motivation for the developments in a subsequent chapter, without that an understanding of the first chapter is needed for an understanding of the second one.

Preliminaries and notation

Let us fix some notation and terminology.

As usual, the letters $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ denote the sets of natural, integer, real, and complex numbers, respectively. We put $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. The letter $\Bbbk$ will stand for an arbitrary field.

We adopt the following convention. A *sesquilinear form* on a complex vector space H is a map $\langle \cdot | \cdot \rangle \colon H \times H \to H$ that is conjugate-linear in the first variable and linear in the second variable. In particular, we apply this convention to inner products on Hilbert spaces.

The domain of definition of a map ϕ is denoted by $\mathrm{Dom}(\phi)$, and the image is denoted by $\mathrm{Im}(\phi)$. The identity map on a set X is denoted by id_X or shortly by id.

We denote the set of all bounded linear operators from a Hilbert space H_1 to a Hilbert space H_2 by $\mathcal{L}(H_1, H_2)$, and the subset of all compact linear operators by

$\mathcal{K}(H_1, H_2)$. Furthermore, we use the ket-bra notation: for every element ξ of a Hilbert space H, we define maps

$$|\xi\rangle\colon \mathbb{C} \to H,\, \lambda \mapsto \lambda\xi, \quad \text{and} \quad \langle\xi|\colon H \to \mathbb{C},\, \zeta \mapsto \langle\xi|\zeta\rangle.$$

Considering $\mathbb{C}$ as a Hilbert space, we have $|\xi\rangle \in \mathcal{K}(\mathbb{C}, H)$, $\langle\xi| \in \mathcal{K}(H, \mathbb{C})$, and $|\xi\rangle^* = \langle\xi|$.

Given a subset X of a vector space V, we denote by span $X \subseteq V$ the linear span of X. If V is a topological vector space, we denote by $\overline{\operatorname{span}}\, X$ and $[X]$ the closed linear span of X. We say that $X \subseteq V$ is *linearly dense* in $Y \subseteq V$ if $\overline{\operatorname{span}}\, X = Y$.

In Part II and III, we denote the algebraic tensor product by the symbol "$\odot$" to distinguish it from the minimal tensor product of C^*-algebras.