Preface

Characterization problems in mathematical statistics are statements in which the description of possible distributions of random variables follows from properties of some functions in these variables. One of the famous examples of a characterization problem is the classical Kac–Bernstein theorem ([65], [13]). This theorem characterizes a Gaussian distribution by the independence of the sum $\xi_1 + \xi_2$ and of the difference $\xi_1 - \xi_2$ of independent random variables ξ_j. Taking into account that the characteristic function of the random variable ξ_j with distribution μ_j is the expectation $f_j(y) = \hat{\mu}_j(y) = \mathbf{E}[e^{i\xi_j y}]$, it is easily verified that the Kac–Bernstein theorem is equivalent to the statement that, in the class of normalized continuous positive definite functions, all solutions to the Kac–Bernstein functional equation

$$f_1(u+v)f_2(u-v) = f_1(u)f_1(v)f_2(u)f_2(-v), \quad u, v \in \mathbb{R},$$

are of the form $f_j(y) = \exp\{-\sigma y^2 + ib_j y\}$, where $\sigma \geq 0$, and $b_j \in \mathbb{R}$.

The Kac–Bernstein theorem was the first among characterization theorems where independent linear forms of independent random variables ξ_j under different restrictions on ξ_j were studied. These studies were completed with the following Skitovich–Darmois theorem ([98], [23]): Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables, and α_j, β_j be nonzero real numbers. If the linear forms $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ are independent, then all random variables ξ_j are Gaussian. Just as in the case of the Kac–Bernstein theorem, the Skitovich–Darmois theorem is equivalent to the statement that, in the class of normalized continuous positive definite functions, all solutions to the Skitovich–Darmois functional equation

$$\prod_{j=1}^{n} f_j(\alpha_j u + \beta_j v) = \prod_{j=1}^{n} f_j(\alpha_j u) \prod_{j=1}^{n} f_j(\beta_j v), \quad u, v \in \mathbb{R},$$

are of the form

$$f_j(y) = \exp\{-\sigma_j y^2 + ib_j y\}, \tag{1}$$

where $\sigma_j \geq 0$ and $b_j \in \mathbb{R}$.

The Skitovich–Darmois theorem was generalized by Ghurye and Olkin ([54]) to the multivariable case when, instead of random variables, random vectors ξ_j in the space $\mathbb{R}^m$ are considered, and coefficients of the linear forms L_1 and L_2 are nonsingular matrices. In this case the independence of L_1 and L_2 also implies that all independent random vectors ξ_j are Gaussian. The proof of the Ghurye–Olkin theorem is also reduced to solving the corresponding functional equation. It should be noted that nonsingular matrices are topological automorphisms of the group $\mathbb{R}^m$.

Next Heyde proved the following result ([61]): Let ξ_j, $j = 1, 2, \dots, n$, $n \geq 2$, be independent random variables, and let α_j, β_j be nonzero real numbers such that

$\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1} \neq 0$ for all $i \neq j$. If the conditional distribution of $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$ given $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ is symmetric, then all ξ_j are Gaussian. This statement is closely related to the Skitovich–Darmois theorem. The Heyde theorem is also equivalent to the assertion that, in the class of normalized continuous positive definite functions, all solutions to the Heyde functional equation

$$\prod_{j=1}^{n} f_j(\alpha_j u + \beta_j v) = \prod_{j=1}^{n} f_j(\alpha_j u - \beta_j v), \quad u, v \in \mathbb{R},$$

are of the form (1).

In the last 30 years much attention has been devoted to generalizing of the classical characterization theorems into various algebraic structures such as locally compact Abelian groups, Lie groups, quantum groups, symmetric spaces (see e.g., [2], [29], [31]–[44], [46], [47], [49], [52], [55], [56], [64], [78]–[82], [84]–[87], [91], [93], [94]).

These investigations were motivated, first of all, by the desire to find the natural limits for possible extensions of the classical results. The present book is devoted to generalization of the Kac–Bernstein, Skitovich–Darmois, and Heyde characterization theorems to the case where independent random variables take values in a second countable locally compact Abelian group X, and coefficients of linear forms are topological automorphisms of X. It turns out that the possibility to prove a characterization theorem for X not only depends on the structure of X but also determines its structure. For example, assume that independent random variables ξ_1 and ξ_2 take values in X and their characteristic functions do not vanish, then the independence of $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ implies that ξ_j are Gaussian if and only if X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Note that in the case of groups as well as in the classical case, the proof of this theorem can be reduced to solving of some functional equation in the class of normalized continuous positive definite functions on the character group of X.

We describe and comment on the main contents of the book.

Chapter I contains mainly well-known facts from abstract harmonic analysis and the theory of infinite Abelian groups. In Section 1 we give the basic definitions and consider some examples of locally compact Abelian groups. In particular, we describe all subgroups of the additive group of the rational numbers $\mathbb{Q}$, the groups of p-adic integers Δ_p, and $\boldsymbol{a}$-adic solenoids $\Sigma_{\boldsymbol{a}}$. We formulate structure theorems for various classes of locally compact Abelian groups and describe the topological automorphism groups of the groups $\mathbb{R}^n$, $\mathbb{T}^n$, Δ_p, $\Sigma_{\boldsymbol{a}}$. The basic results of Ulm theory for countable p-primary Abelian groups are also presented. In Section 2 we discuss some aspects of probability distributions on locally compact Abelian groups (the Bochner theorem, properties of characteristic functions, the Lévy–Khinchin formula, idempotent distributions).

Chapter II is devoted to Gaussian distributions on a locally compact Abelian group X. In Section 3 we define Gaussian distributions and study their properties. In Section 4 we describe locally compact Abelian groups where every Gaussian distribution has only Gaussian factors (the group analogue of the Cramér theorem of decomposition of a Gaussian distribution). In Section 5 we study properties of continuous polynomials on locally compact Abelian groups. We describe locally compact Abelian groups where any distribution μ with a characteristic function of the form $\hat{\mu}(y) = e^{P(y)}$, where $P(y)$ is a continuous polynomial, is Gaussian (the group analogue of the Marcinkiewicz theorem). The group analogues of the Cramér and Marcinkiewicz theorems are basic tools for proving characterization theorems in Chapters III–VI. In Section 6 we consider Gaussian distributions in the sense of Urbanik, i.e., such distributions on X which any character transforms into Gaussian distributions on the circle group $\mathbb{T}$. We describe locally compact Abelian groups for which the class of Gaussian distributions coincides with the class of Gaussian distributions in the sense of Urbanik.

In Chapter III we study distributions of independent random variables ξ_1 and ξ_2 taking values in a locally compact Abelian group X such that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent. In Section 7 we describe all groups X where such distributions are invariant with respect to a compact subgroup K of X and that, under the natural homomorphism $X \mapsto X/K$, induce Gaussian distributions on the factor group X/K. This is the widest subclass of locally compact Abelian groups on which the Kac–Bernstein type theorem can be proved. It consists of all groups X having the connected component of zero without elements of order 2.

If the connected component of zero of a group X contains elements of order 2, then for such groups the following natural problem arises: to describe all possible distributions of independent random variables ξ_j taking values in X and such that the sum $\xi_1 + \xi_2$ and the difference $\xi_1 - \xi_2$ are independent. We present a solution to this problem in Section 8 for the group $\mathbb{R} \times \mathbb{T}$ and the $\boldsymbol{a}$-adic solenoids $\Sigma_{\boldsymbol{a}}$. In Section 9 we study distributions of independent identically distributed random variables ξ_1 and ξ_2 taking values in a locally compact Abelian group X such that $\xi_1 + \xi_2$ and $\xi_1 - \xi_2$ are independent (Gaussian distributions in the sense of Bernstein).

Chapters IV and V are devoted to some group analogues of the Skitovich–Darmois theorem. Let ξ_j, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables taking values in a locally compact Abelian group X, and α_j, β_j be topological automorphisms of X. Put $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$.

In Chapter IV we assume that the characteristic functions of independent random variables ξ_j do not vanish. We prove in Section 10 that independence of the linear forms L_1 and L_2 implies that all ξ_j are Gaussian if and only if X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$. Under the condition that the characteristic functions of ξ_j do not vanish, this is the widest subclass of locally compact Abelian groups on which the Skitovich–Darmois theorem can be extended.

Assume that a group X contains a subgroup topologically isomorphic to the circle group $\mathbb{T}$. Then the following natural problem arises: to describe all possible distributions of independent random variables ξ_j taking values in X and having the property that the linear forms L_1 and L_2 are independent. The remainder of Chapter IV is devoted to solution of this problem. It turns out that if the characteristic functions of independent random variables ξ_j with distributions μ_j do not vanish and L_1 and L_2 are independent, then the distributions μ_j can be replaced by their shifts μ'_j in such a way that all μ'_j are supported in the connected component of zero of the group X. Hence the problem can be reduced to the case when X is connected. An important characteristic of a connected locally compact Abelian group X is its dimension dim X. If dim $X = 1$, then X is topologically isomorphic to one of the following groups: the real line $\mathbb{R}$, an $\boldsymbol{a}$-adic solenoid $\Sigma_{\boldsymbol{a}}$ or the circle group $\mathbb{T}$. If dim $X = 2$, then either X contains no subgroup topologically isomorphic to the circle group $\mathbb{T}$ or X is topologically isomorphic to one of the following groups: $\mathbb{T}^2$, $\mathbb{R} \times \mathbb{T}$ or $\Sigma_{\boldsymbol{a}} \times \mathbb{T}$. Assume that the number of random variables ξ_j is equal to 2. In Section 11 we describe for the two-dimensional torus $\mathbb{T}^2$ all possible distributions of the random variables ξ_j in the case when linear forms L_1 and L_2 are independent. Generally speaking, these distributions are not Gaussian, and we describe, in particular, all topological automorphisms α_j, β_j of the two-dimensional torus $\mathbb{T}^2$ for which the corresponding distributions are Gaussian. In Section 12 we solve the same problem for the groups $\mathbb{R} \times \mathbb{T}$ and $\Sigma_{\boldsymbol{a}} \times \mathbb{T}$.

In Chapter V we omit the assumption that the characteristic functions of independent random variables ξ_j do not vanish and suppose that ξ_j take values in different classes of locally compact Abelian groups (finite, discrete, discrete torsion, compact totally disconnected, etc.) Since Gaussian distributions on a totally disconnected group are degenerate, idempotent distributions play an important role on such groups. It turns out that, in contrast to the classical situation, there are essential distinctions between the cases when we deal with linear forms of two random variables, of three random variables, or of $n \geq 4$ random variables. In the group situation some new effects appear which do not hold on the real line. To show this consider the following example. Let $\mathbb{Z}(5)$ be the group of residues modulo 5, and ξ_j, $j = 1, 2, \ldots, n, n \geq 2$, be independent random variables with values in $\mathbb{Z}(5)$. If $n = 2$ and the linear forms L_1 and L_2 are independent, then all random variables ξ_j have idempotent distributions. If $n = 3$ and the linear forms L_1 and L_2 are independent, then we can only assert that at least one of the random variables ξ_j has an idempotent distribution. On the other hand for every $n \geq 4$ there exist independent random variables ξ_j, $j = 1, 2, \ldots, n$, taking values in $\mathbb{Z}(5)$ and automorphisms α_j, β_j of $\mathbb{Z}(5)$ such that the linear forms L_1 and L_2 are independent, but all ξ_j have non-idempotent distributions.

In Section 13 we consider two independent random variables ξ_1 and ξ_2. We prove that if X is a discrete group and the linear forms L_1 and L_2 are independent, then ξ_1 and ξ_2 have idempotent distributions. We also describe compact totally disconnected groups for which this property holds true. We prove that the Skitovich–Darmois theorem fails for compact connected groups. In Section 14 we study the case when the number n of random variables ξ_j, $j = 1, 2, \ldots, n$, is greater than 2. First we assume that the ξ_j take values in a finite group. Then we study the case when X is either a compact totally disconnected group or a discrete torsion group. We describe in both cases the groups X for which independence of the linear forms L_1 and L_2 implies that either all random variables ξ_j have idempotent distributions or at least two of the random variables ξ_j have idempotent distributions, or at least one of the random variables ξ_j has an idempotent distribution. Further we consider an arbitrary number $n \geq 4$ of random variables ξ_j, and X is assumed to be either a discrete torsion group or a compact group. We note that our proof of the Skitovich–Darmois theorem for discrete Abelian groups in the case when $n = 2$, in contrast to the proof of the Skitovich–Darmois theorem for discrete Abelian groups in the case when $n > 2$, does not use the Ulm theory.

In Section 15 we consider independent random variables ξ_1 and ξ_2 taking values in an **a**-adic solenoid $\Sigma_{\boldsymbol{a}}$. We describe all possible distributions of random variables ξ_j assuming that the linear forms L_1 and L_2 are independent. The result depends on both an **a**-adic solenoid and topological automorphisms α_j, β_j.

In Chapter VI we study group analogues of the Heyde theorem. Let ξ_j, $j = 1, 2, \ldots, n$, $n \geq 2$, be independent random variables taking values in a locally compact Abelian group X. Let α_j, β_j be topological automorphisms of X satisfying the condition

(i) $\beta_i\alpha_i^{-1} \pm \beta_j\alpha_j^{-1}$ are topological automorphisms of X for all $i \neq j$.

Let $L_1 = \alpha_1\xi_1 + \cdots + \alpha_n\xi_n$ and $L_2 = \beta_1\xi_1 + \cdots + \beta_n\xi_n$. Assume first that the characteristic functions of the independent random variables ξ_j do not vanish. In Section 16 we prove that symmetry of the conditional distribution of the linear form L_2 given L_1 implies that all ξ_j are Gaussian if and only if X contains no elements of order 2. This result can not be improved. If a group X contains elements of order 2, then the following natural problem arises: to describe all possible distributions of independent random variables ξ_j taking values in X and having the property that the conditional distribution of the linear form L_2 given L_1 is symmetric. We assume that on the group X there exist topological automorphisms α_j, β_j satisfying condition (i). A simple example of such a group is the two-dimensional torus $X = \mathbb{T}^2$, and even in this case the problem is very interesting. It turns out that the distributions which are characterized by the symmetry of the conditional distribution of the linear form L_2 given L_1 are convolutions of Gaussian distributions concentrated on a dense one-parameter subgroup of $\mathbb{T}^2$ and distributions supported in the subgroup of $\mathbb{T}^2$ generated by elements of order 2.

In Section 17 we drop the assumption that the characteristic functions of independent random variables ξ_j do not vanish. We first study the case when ξ_j take values in a finite group and then ξ_j take values in a discrete group X. In the case when X is discrete and the number of random variables ξ_j is 2, we prove in particular that the symmetry of the conditional distribution of the linear form L_2 given L_1 implies that the random variables ξ_1 and ξ_2 have idempotent distributions if and only if the group X contains no elements of order 2.

In the appendix we study the Kac–Bernstein and Skitovich–Darmois functional equations on locally compact Abelian groups in the classes of continuous and measurable functions.

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