Preface

The goal of these lecture notes is to introduce you to the central concepts surrounding wavelets and their applications as quickly as possible. They are suitable for beginning graduate students and above. We focus here on *ideas* and then indicate where the details can be found. Thus these notes do not attempt to replace a comprehensive textbook for a course. We hope that these notes will help you to begin your adventures with wavelets.

A main purpose of language is to give us the ability to encode, transmit, and extract information efficiently. In situations when words fail us, other methods, such as the musical score, become a type of language. Mathematics is a language (or collection of languages), since it gives us an ability to 'talk' about functions, operators, and other objects. When people encounter things that are 'indescribable' in their current language(s), they try to develop a new language to capture it. For this reason, harmonic analysts developed powerful time/frequency tools, electrical engineers developed subband coding, and quantum physicists developed tools to understand coherent states. In the late 1980s they started to realize that the languages they were creating had much in common, and the core of these languages could be combined into a single language. Hence the language that is now called wavelet theory was born.

Among the most spectacular and well-known early applications of wavelets are the wavelet-based FBI standard for storing, searching and retrieving fingerprints, and the wavelet-based JPEG-2000 standard for image compression and transmission used widely on the internet. In the initial excitement, there were highly exaggerated claims made about the power of wavelets. Wavelets are not a miracle solution, however. They are just a library of bases that is appropriate for a large number of situations where the traditional tools, Fourier analysis for example, do not work very well. Wavelets have now matured to become part of the standard curriculum in electrical engineering, statistics, physics, mathematical analysis and applied mathematics. They have become part of the toolbox used for statistics, signal and image processing, medical imaging, geophysics, speech recognition, video coding, internet communications, economics, etc. There are still many problems that cannot be described well with our current language. People keep designing new tools, and the wavelet language still evolves.

There is a lot of on-going research on very intensive computational problems, like the N-body problem, turbulence, weather prediction, etc., where people are attempting to incorporate wavelet components into their models, with varied success. The idea is to exploit the multiresolution structure of wavelets, their compression and denoising properties, the flexibility in their construction, their ability to represent certain operators efficiently, and the associated fast algorithms to compute derivatives, products, etc. If one could capitalize on some or all of these areas, it could potentially make a tremendous difference in such large computational problems. To attempt such large projects multidisciplinary teams are needed: experimentalists, numerical analysts, scientific computing specialists, physicists, engineers, mathematicians, and statisticians, all willing to cooperate and communicate with one another.

Organization of the book

The book is organized as follows:

In the Preliminary Chapter we collect definitions that we will use throughout the book, but which are not really about wavelets. In particular we lay down the concepts of bases, orthogonal and biorthogonal bases, Riesz bases, and frames.

In Chapter 2 we review time-frequency analysis. We start with Fourier series and the Fourier transform, which provide an analysis perfectly localized in frequency but not in space. We discuss the windowed Fourier transform, Gabor bases and the local trigonometric bases that introduce some localization in space and are appropriate for many situations. Finally we gain localization at all scales with the wavelet transform. We emphasize the notion of time-frequency localization in the phase-plane for each decomposition. We also point out the existence of unavoidable obstructions for perfect localization in both time and frequency epitomized by the celebrated Heisenberg uncertainty principle.

In Chapter 3 we introduce the notion of a multiresolution analysis (MRA). We describe carefully the Haar wavelet and MRA. This example contains most of the important ideas behind multiresolution analysis and demonstrates the fast wavelet transform. We discuss the connection to filter banks, as well as how to design wavelets with certain attributes. Daubechies' compactly supported wavelets are still the most widely used wavelets; we describe them briefly and give pointers to the literature. We try to emphasize the competing properties of compact support, smoothness, vanishing moments and symmetry, and the costs of choosing one property over another.

In Chapter 4 we discuss variants of the classical orthogonal wavelets and MRA. These variations give flexibility in the design of wavelets at the expense of orthogonality (biorthogonal wavelets), or by increasing the number of wavelets used (multiwavelets). We describe carefully the MRA associated to biorthogonal and multiwavelets as well as the corresponding filter banks, and note that fast algorithms are still available. We also explain how to construct wavelets in two dimensions by tensor products; such tensor product wavelets are the most widely used wavelets for image processing. For many applications (images, differential equations on domains) one needs wavelets confined to an interval or a domain in space. We describe very briefly the options that have been explored and give pointers to the literature. We also discuss wavelet packets, which are a library of bases that includes the wavelets, but also many intermediate bases that could be better adjusted to a given function. Next we mention, without going into great detail, a host of friends, relatives and mutations of wavelets that have been constructed to tackle more specialized problems. These objectlets obey some sort of multiresolution structure on graphs, domains, weighted spaces, clouds of data, or other settings where there is no translation/scale invariance. Finally we discuss the prolate spheroidal wave functions, which are based on a different notion of time-frequency localization.

In Chapter 5 we describe a few applications, without attempting to be systematic or comprehensive. The choice of sample applications is dictated by the authors' experiences in this area. We touch on the basics of signal compression and denoising. We describe in some detail how to calculate derivatives using biorthogonal wavelets, and how one can construct wavelets with fancier differential properties. We describe how wavelets can characterize a variety of function spaces that trigonometric bases cannot, and how well adapted wavelets are to identifying very fine local regularity properties of functions. Finally we very briefly describe how one can attempt to use wavelets in the numerical study of differential equations.

Prerequisites

We assume the reader knows calculus and linear algebra. In particular the reader should be proficient with vector spaces, linearly independent vectors, pointwise convergence of a function, continuity, differentiation, and integration. We also expect the reader to have been exposed to some real and complex analysis, such as point set topology, epsilon-delta arguments, uniform convergence, complex numbers and calculus of complex valued functions. We do not assume the reader has taken formal courses in measure theory or functional analysis, but we expect some familiarity with the concepts of orthonormal basis, orthogonal projections, and orthogonal complements on Hilbert spaces. We have collected some of these basic concepts and definition in the Preliminary Chapter 1, mainly for reference. A reader without these prerequisites may still be able to pick up the main ideas, but will not be able to attain a thorough understanding of the theory presented in this book.

Origins and acknowledgments

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