1 Introduction

1.1 Historical overview

Even though Einstein introduced his equations in 1915, cf. [35] and [36], it was not until 1952 that it was firmly established that they allow a formulation as an initial value problem. The seminal paper was written by Yvonne Choquet-Bruhat, cf. [39], and it contains a proof of local existence of solutions. Due to the diffeomorphism invariance of the equations, the step from local existence to the existence of a maximal development is non-trivial. The reason is that even if there is a maximal development in the sense that it cannot be extended, there is no reason to expect this development to be unique. In the end, one does in fact need to restrict one's attention to a special class of developments in order to get an element which is maximal and unique in the given class. Partly as a consequence of this, it was not until 1969 that Choquet-Bruhat and Robert Geroch, cf. [10], demonstrated that, given initial data, there is a unique maximal globally hyperbolic development (MGHD). The existence of a MGHD does not say anything about the "global" properties of solutions, but it is nevertheless a fundamental theoretical starting point for any analysis of solutions to Einstein's equations. To take but one example, the question of predictability in general relativity, as embodied in the strong cosmic censorship conjecture, is phrased in terms of the MGHD.

The initial value point of view illustrates the strong connection between Einstein's general theory of relativity and the theory of hyperbolic partial differential equations (PDE's). Historically, this is a connection which has not received as much attention as one might have expected. In the beginning, most people working in the field focused on writing down different explicit solutions. Obviously, this was a natural starting point, and some of the solutions written down in this period will with all probability always be of fundamental importance. However, it is possible to work with such spacetimes without even being aware of the hyperbolic PDE character of the equations.

The question of singularities in general relativity has always been important. At an early stage, it was suggested that some of the singularities present in the model solutions of the universe and of isolated systems would not be present in less symmetric solutions. Remarkably, the existence of singularities in a more general situation was demonstrated by methods which avoid the PDE aspect of the problem altogether. The singularity theorems of Hawking and Penrose, cf. [66], [46], [47] for some of the original papers and [48], [87], [65] for textbook presentations, demonstrate that singularities are generic, assuming one is prepared to equate the existence of a singularity with the existence of an incomplete causal geodesic. These results had an important impact on the field, not least because they illustrated the importance of the geometry, and partly as a consequence of them, Lorentz geometry has become an important subject in its own right. Surprisingly enough, not much information concerning Einstein's equations is required in order to obtain these theorems; some general energy conditions concerning the matter present suffice. On the other hand, one does not obtain that much information. It would of course be of interest to know if the gravitational fields become arbitrarily strong as one approaches a singularity, i.e., in the incomplete directions of causal geodesics. To address this question, it would seem that it is necessary to take the PDE aspect into account.

In the formulation of Einstein's equations as an initial value problem, the initial data cannot be chosen freely; they have to satisfy certain constraint equations. The various methods introduced for solving these equations in the end lead to non-linear elliptic PDE's on manifolds, so that even before coming to the evolutionary aspect of the equations, one is faced with a difficult analysis problem. Given initial data, the problem of local existence has been solved by the work of Yvonne Choquet-Bruhat as mentioned above. However, in the end one is interested in the global structure. In terms of analysis, this often makes it necessary to prove global existence of solutions to a non-linear hyperbolic PDE, which is usually very difficult. In view of these complications, it is not surprising that it has taken such a long time for the hyperbolic PDE aspect of the equations to become the focus of attention.

1.2 Some global results, recent developments

To our knowledge, the first global results in the absence of symmetries are due to Helmut Friedrich, cf. [40]. In this paper, he demonstrates, among other things, the stability of de Sitter space. Furthermore, he proves that initial data close to those induced on a hyperboloid in Minkowski space yield developments that are future null geodesically complete. In addition to these results, he derives detailed information concerning the asymptotics. However, it is interesting to note that the methods he uses are very geometric in nature. The proof is based on a very intelligent and geometric choice of equations which makes it possible to reduce a global problem, from the geometric point of view, to a problem of local stability in the PDE setting.

The proof of the global non-linear stability of Minkowski space, see [20], by Demetrios Christodoulou and Sergiu Klainerman is based on very different methods. In [20], the authors solve a problem which is global in nature also from the non-linear hyperbolic PDE point of view, and for this reason more work on the analysis part of the problem is required. Beyond being an important result in its own right, the work of Christodoulou and Klainerman seems to have had the effect of bringing Einstein's equations to the attention of the non-linear hyperbolic PDE community, as illustrated by e.g. [57] and [58]. Furthermore, the interest in the question of local existence has been revived.

In the case of non-linear hyperbolic PDE's, it is of interest to prove well posedness for as low a regularity of the data as possible. One reason is that a local existence result is usually associated with a continuation criterion; e.g. if the maximal existence time to the future is T_+, then there is a statement of the form: either $T_+ = \infty$ or a suitable norm of the solution is unbounded as $t \to T_+-$. Proving local existence in a lower degree of regularity usually results in a continuation criterion involving a weaker norm. In the optimal situation, it is possible to prove local existence in such a weak regularity that the norm appearing in the continuation criterion is controlled by a quantity which is preserved by the evolution. In such a situation, one is allowed to conclude that solutions do not blow up in a finite time simply due to the local existence result. A striking illustration of this perspective is given by the work of Klainerman and Matei Machedon, cf. [51], which proves that solutions to the Yang–Mills equations in $3 + 1$ dimensional Minkowski space do not blow up in finite time due to such an argument. However, it should be noted that the latter conclusion had already been obtained by the work of Douglas Eardley and Vincent Moncrief in [33] and [34]. In the case of Einstein's equations, there is no hope for a similar result, but the question of local existence for as rough data as possible is nevertheless of interest. Since the result of Choquet-Bruhat, [39], there have been many papers concerned with the question of local existence. One classical reference is [38]. More recently, Klainerman and Igor Rodnianski have undertaken an ambitious programme in order to obtain the optimal regularity result, cf. [52] and the references cited therein.

The results [40] and [20] are the first examples of results that are global in nature, concern a situation without symmetry and have been obtained by PDE methods. Some examples of more recent results in this category are [3], [57] and [77]. The number of results that are global in nature, concern a situation with symmetry and have been obtained by PDE methods is, due to the work carried out the last twenty years or so, quite large.

1.3 Purpose

The above sections were intended to illustrate that in recent years, the perspective on Einstein's equations taken by researchers has tended toward the initial value point of view. Unfortunately, it seems that the main references available relevant to this perspective are of one of the following types: textbooks on geometric aspects, i.e., books on Lorentz geometry, textbooks on non-linear hyperbolic PDE's, research papers and overview papers. The reason this is unfortunate is that the books on Lorentz geometry typically ignore the hyperbolic PDE aspects (or, as in the case of [48] and [87], give an outline rather than a detailed exposition) whereas the books on non-linear hyperbolic PDE's typically ignore the geometry. That geometry is important in the context of non-linear hyperbolic PDE's even outside the field of general relativity can be illustrated in the following way. Consider a non-linear wave equation of the form

$$g^{\mu\nu}(u, \partial u)\partial_\mu \partial_\nu u = f(u, \partial u)$$

on $\mathbb{R}^{n+1}$, where ∂u is used to represent the first derivatives and we use Einstein's summation convention, cf. Section A.1. Then $g^{\mu\nu}(u, \partial u)$ are the components of the inverse of a Lorentz metric, and such fundamental aspects of the equations as uniqueness are most naturally expressed in terms of the causal structure of this metric. The research papers are of course worth reading, but at some stage a coherent presentation is preferable. The overview papers fill an important function in that they provide the intuition behind the results, but in the end one would like to have complete proofs.

The purpose of these notes is, among other things, to give a complete proof of the existence of a maximal globally hyperbolic development. For each matter model one couples to Einstein's equations, it is necessary to give a proof for the resulting combination of equations. Since it is not the ambition of these notes to give an exhaustive list of matter models which can be coupled to Einstein's equations, we shall restrict our attention to the non-linear scalar field case. This is a matter model which allows vacuum with a cosmological constant as a special case. The reason we have chosen it is that it has attracted attention in recent years in the cosmological setting as a model which produces different types of accelerated expansion, something recent observational data indicate that our universe is undergoing. Beyond this main goal, there are, however, several other results that are of fundamental importance, but seem to fall outside of the scope of most textbooks. For instance, uniqueness of solutions to linear wave equations expressed in terms of the causal structure of the Lorentz metric involved is something standard textbooks on PDE's typically avoid. Cauchy stability in the context of general relativity is another result which is of fundamental importance, but detailed proofs are rarely written down in an accessible way. Finally, there are recent results concerning the structure of globally hyperbolic spacetimes, cf. [4], [5], [6], which deserve to be presented in the present context.

It is not the ambition of these notes to give an up-to-date overview of the methods developed to prove local existence in low regularity. In fact, the methods presented here for proving local existence are quite old. The main purpose is rather to present some elementary theory for hyperbolic PDE's and some elementary Lorentz geometry in a unified way, the hope being that this will be of use to those working in the field. The reason for doing so at this point in time is that the importance of the combination has become apparent in the last fifteen years or so.

As we mentioned above, researchers working on problems related to Einstein's equations come from many different backgrounds. In particular, there are those with a geometric background but not such a strong background in PDE's, and there are those who come from the non-linear hyperbolic PDE community. For this reason, these notes have been written with the ambition of not presupposing more of the reader than a knowledge of measure and integration theory, some very elementary functional analysis, basic Lorentz geometry and elementary differential geometry. To be more specific, it is, from a logical point of view, possible to read these notes given a knowledge of analysis corresponding to *Principles of Mathematical Analysis* and *Real and Complex Analysis* by Walter Rudin, cf. [78] and [79]. In fact, only the first eight chapters of the latter book are needed. In particular, no previous familiarity with PDE's is required. Concerning Lorentz geometry, we assume the reader is familiar with the material contained in Barrett O'Neill's book *Semi-Riemannian Geometry*, [65], though no mention of manifolds will be made until Chapter 10. Concerning differential geometry, we presuppose the basics, including familiarity with Stokes' theorem, as contained in, e.g., [25], and, in the last part of these notes, a familiarity with Lie groups corresponding to the material given in, e.g., Lee's book [55]. Assuming less than the material contained in these references seems unreasonable, since they are all very well written and much more fundamental than the results presented in these notes. However, assuming more also seems unwarranted, given the goal already mentioned. Let us mention that, recently, two related books were published; see [19] and [9]. Finally, let us recommend the book [69], treating similar material from a different perspective, as a good companion to these notes.

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