Preface

It would not be an exaggeration to say that since the 1940s Feynman's path integrals substantially changed quantum physics. This concept led to the implementation of many path integration methods of different levels of mathematical sophistication, see [6], [84] and the numerous references therein. Spectacular progress in rigorous quantum field theory (at least for low-dimensional space-time models) was achieved through the Euclidean strategy in which the Minkowski space was converted into a Euclidean space by passing to imaginary values of time. The corresponding quantum field was constructed and studied in this Euclidean domain and then transferred to real time by a certain procedure. Due to this development, Feynman–Wiener path integrals and hence the theory of Markov processes as well as methods of classical statistical mechanics were widely applied. The state of the art in this domain up to the time of their respective publication was presented in the monographs by B. Simon [273] and J. Glimm and A. Jaffe [135]. The introduction to the former book gives a profound survey of ideas and historical facts behind the Euclidean strategy.

Quantum statistical mechanics is close, both conceptually and technically, to quantum field theory. Its rigorous version has been developed on the basis of the theory of operator algebras, whose fundamentals can be found in the monographs by O. Bratteli and D. W. Robinson [76], [77], G. G. Emch [114], and by M. Takesaki [300], [301], [302]. However, for a big class of important quantum models, especially those described by unbounded operators, these methods encountered considerable difficulties; see the discussion on page 241 of [77] and also in [160], [161].

The present book is dedicated to the rigorous statistical mechanics of infinite systems of interacting quantum anharmonic oscillators. It can be considered as a natural continuation of B. Simon's book "The Statistical Mechanics of Lattice Gases: I", where both classical and quantum models of this kind are on the list of *'models not to be discussed',* see pp. 19–26 in [277]. In addition, our book is connected with quantum field theory by the fact that the free quantum field can be interpreted as an infinite system of interacting quantum harmonic oscillators.

There is, however, one more important reason to develop the theory presented in this book. Since the 1960s, systems of quantum oscillators have been widely used in models of quantum solid state physics where they describe vibrations of light particles localized near sites of crystal lattices and their interaction with other particles and fields. In this context, we mention the book by A. A. Maradudin *et al.* [212], see also [154], [155], and the series of articles by A. Verbeure and his collaborators [310], [316]. The theory of quantum harmonic oscillators is relatively simple and therefore is quite well elaborated. The case of anharmonic oscillators is more complex. However, systems of anharmonic oscillators possess much richer properties and hence have much wider applications, which strongly stimulates the development of their theory, including its mathematically rigorous versions. Clearly, the properties of such systems depend on the geometry of interactions and hence on the configuration of the equilibrium positions of the oscillators. In the model studied in this book, they constitute a countable set $\mathbb{L} \subset \mathbb{R}^d$, equipped with the Euclidean distance and obeying a certain regularity condition. A particular case is the model where $\mathbb{L}$ is a crystal lattice. More general cases of $\mathbb{L}$ correspond to quantum particles irregularly distributed in $\mathbb{R}^d$. Most of our results apply to this general case, however, a number of them are valid for crystal lattices only.

In classical (i.e., non-quantum) statistical mechanics, a complete description of the equilibrium thermodynamic properties of an infinite-particle system can be given by constructing its Gibbs states. For some quantum models with bounded Hamiltonians, equilibrium thermodynamic states are defined as functionals on algebras of observables satisfying the Kubo–Martin–Schwinger (KMS) condition, see [145] and [77], which is an equilibrium condition reflecting a consistency between the dynamics and thermodynamics of the model. However, for an infinite system of interacting quantum anharmonic oscillators, the KMS condition cannot be formulated and hence the KMS states cannot even be defined. In this situation, a natural alternative is given by a version of the Euclidean strategy based on path integral techniques, which was successful in low dimensional quantum field theory. Because of their intuitive appeal, methods employing integration in function spaces on the 'physical' level of strictness enjoy great popularity among theoretical physicists working in quantum physics. This is assured by numerous monographs and textbooks in this field having appeared or having been reprinted recently, see e.g., [165], [178], [213], [220], [322]. Thus, the second goal of this book is to provide a firm mathematical background of path integral methods used in quantum statistical mechanics, based on the latest achievements in stochastic and functional analysis. We also believe that the mathematical problems arising here will stimulate development of the corresponding fields of mathematics.

In accordance with these goals, we address the book to both communities – physicists and mathematicians. Theoretical physicists, especially those who are concerned with the rigorous mathematical background of their results, can find here a concise collection of facts, concepts, and tools relevant for the application of path integrals and other methods based on measure and integration theory to problems of quantum physics. They can also find the latest results in the mathematical theory of quantum anharmonic crystals, which can be used as a basis for the study of equilibrium and non-equilibrium statistical mechanical properties of models employing quantum anharmonic oscillators. Mathematicians are given an opportunity to learn what kind of problems arise in quantum statistical mechanics and how to attack them. We believe that our methods are also applicable to other problems involving infinitely many variables, for example, in biology and economics.

In view of its interdisciplinary nature, this book consists of 'mathematical' and 'physical' parts, preceded by an introduction, where we outline the ideas on which our approach rests, formulate its main aspects, and briefly describe physical consequences of the theory developed on the basis of this approach.

The first part, comprising three chapters, starts with a description of the model considered throughout the book. For this model, we define local Gibbs states as functionals on the corresponding algebras of local observables and give the mathematical background of the theory of such states, which includes elements of the theory of linear operators in Hilbert spaces. Then we present a detailed description of the properties of Schrödinger operators of single quantum oscillators, both harmonic and anharmonic. Afterwards, we prepare the description of the local Gibbs states of our model in terms of stochastic processes and associated path measures. Here we present a number of facts from the theory of probability measures on topological spaces coming from various sources. Most of the statements are proven here, addressing those readers who would like to get into the details without using additional sources. Thereby, we develop a description of local Gibbs states in terms of path space measures, which in this book are called local Euclidean Gibbs measures. They have the same structure as the local Gibbs measures of the corresponding classical models of unbounded spins. Here, however, the 'spins' are not only unbounded, but also belong to an infinite-dimensional Banach space. In the next chapter, we develop tools for studying local Euclidean Gibbs measures, based on their approximation by the Gibbs measures of classical models with unbounded finite-dimensional spins. With the help of this approximation, we derive a number of correlation inequalities, which are then crucially used throughout the book. In Chapter 3, which is the main point of the first part and perhaps of the whole book, we introduce and study the (global) Euclidean Gibbs measures of our model. These measures contain all information about equilibrium thermodynamic properties of the model and play the same role as the KMS states do in the algebraic formulation of quantum statistical mechanics.

The second part of the book is dedicated to a description of some physical properties of our model which is based on the Euclidean Gibbs measures constructed in the first part. Here we present a complete theory of phase transitions and quantum effects. This theory is mainly based on various correlation inequalities, on regularity properties of the paths of the underlying stochastic processes, and on the spectral properties of the corresponding Schrödinger operators. It explains a large number of relevant experimental data, confirming our approach. In this context, one has to mention powerful methods of studying Gibbs states and phase transitions in classical lattice systems based on cluster, polymer, and other expansions and estimates. Some of them, like cluster expansions, have also found applications to quantum anharmonic crystals, see the works by R. A. Minlos, e.g. [217], and the bibliographic notes below. We expect that the general framework developed in this book will lead to a more effective use of these methods in the future. At the same time, a number of such methods, for instance, the Pirogov–Sinai theory of phase transitions [245], [321], are applicable to classical models only. In our approach, quantum anharmonic crystals are described as systems of classical albeit infinite-dimensional 'spins'. Thus, we hope that by means of our techniques the development of a version of the Pirogov–Sinai theory, applicable to our and similar models, will be possible. A relevant problem which we also leave for the future is the interpretation of our results in terms of states on von Neumann algebras in the spirit of works by the groups of J. Fröhlich [66], R. Gielerak [131], [132], and A. Verbeure [79], [313].

Although we did our best to make the book self-contained, the reader is supposed to have certain preliminary knowledge at a graduate level, both in mathematics and physics. As a book of great impact on this area we strongly recommend B. Simon's monograph [274], and also the monographs [76], [77], [114], [145], [300], [301], [302] as sources on the algebraic methods of quantum statistical mechanics. Since the construction of our Euclidean Gibbs measures is carried out in the framework of the Dobrushin–Lanford–Ruelle approach, we recommend learning its fundamentals from the books [129], [249], [281].

The line of research described in this book has its roots in original work by the late Raphael Høegh-Krohn, who discovered a fundamental duality in relativistic quantum statistical mechanics by representing the basic correlation functions in terms of a certain stochastic process, see [1]. For this, it seems more than appropriate to call this stochastic process the Høegh-Krohn process, as we do in this book. This should be understood as an expression of our admiration for a great mathematician, who departed much too early. Our work on the book was completed on the eve of the 20-th anniversary of Raphael's death to confirm that his spirit is still among us.

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