Introduction

A one-dimensional quantum harmonic oscillator is a quantized version of the model of a particle performing sinusoidal oscillations in a parabolic potential field, corresponding to Hooke's law. A ν-dimensional harmonic oscillator, $\nu \in \mathbb{N}$, performs independent simultaneous oscillations in all ν dimensions; its momentum and displacement are ν-dimensional vectors. For anharmonic oscillators, the corresponding potential fields are usually super-quadratic and may have multiple minima. The latter peculiarity entails essential changes in the particle dynamics as compared with the case of convex potentials. A typical example from physics is a hydrogen bound $\mathrm{O}-\mathrm{H}-\mathrm{O}$, consisting of two negative oxygen ions and a positive hydrogen ion (proton), which here stands for a quantum particle. Such a bound is the key structure element of many inorganic and organic compounds. The potential field created by the oxygen ions has two minima (wells), close to each of the ions. Then such a bound with the proton localized in one of the wells is an electric dipole. These dipoles interact with each other, which can force the protons to stay in one of the corresponding two wells. At the same time, the protons oscillate between the wells, even in low-energy states. This motion through a potential barrier, forbidden for classical particles, is called *quantum mechanical tunneling*, see Subsection 1.1.3 below. It produces a strong delocalizing effect, especially at low temperature.

Along with modeling localized quantum particles, quantum anharmonic oscillators are also involved in models describing the interaction of vibrating quantum particles with a radiation (photon) field or strong electron-electron correlations caused by the interaction of electrons with vibrating light ions. Infinite systems of interacting quantum particles of this kind possess interesting physical properties connected with ordering (phase transitions) and quantum effects. Most of them are related to solids, such as ionic crystals containing localized light particles oscillating in the field created by heavy ionic complexes, like the hydrogen bounds mentioned above, or quantum crystals consisting entirely of such particles, e.g., crystalline helium. In the Born–Oppenheimer (called also adiabatic) approximation, the motion of heavy ions is neglected and the oscillators are attached to the sites of a regular crystal lattice – one oscillator per site. Other important physical objects of this kind are systems of localized light particles irregularly distributed (admixed) in a certain medium. In the corresponding model, the sites the oscillators are attached to constitute an irregular set and the localization potentials may vary from site to site. This can also include the case where $\mathbb{L}$ is a lattice but V_ℓ, as well as $J_{\ell\ell'}$, are random. Often, as in the case of hydrogen bounds, the described particles carry electric charges and their displacements from equilibrium positions produce dipole moments. Then the interaction between the particles is of dipole–dipole type and thereby has slow spatial decay. In what follows, infinite systems of interacting quantum anharmonic oscillators with possibly irregular spatial distribution of their equilibrium positions and with long-range interactions can be used

in modeling a wide variety of physical objects. A rigorous mathematical description of such systems is still a challenging task, and one of the aims of the present book is to provide a framework for such a construct.

In the sequel, by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_0$ we denote the sets of complex, real, integer, positive integer, and nonnegative integer numbers, respectively. The main object of our study is a system of interacting quantum anharmonic oscillators attached to the elements of a countable set $\mathbb{L} \subset \mathbb{R}^d$ equipped with the Euclidean distance $|\cdot|$ inherited from $\mathbb{R}^d$. We suppose that

$$\sup_{\ell \in \mathbb{L}} \sum_{\ell' \in \mathbb{L}} \frac{1}{(1 + |\ell - \ell'|)^{d+\epsilon}} < \infty, \tag{1}$$

for every $\epsilon > 0$, which in particular means that $\mathbb{L}$ has no accumulation points. The condition (1) implies that subsets of $\mathbb{R}^d$ of small volume cannot contain a large number of elements of $\mathbb{L}$. In general, this will be the only condition imposed on the set $\mathbb{L}$. However, some of our results have been obtained in the case where $\mathbb{L}$ is a crystal lattice, which is clearly indicated in the text. For simplicity, in such cases we always assume that $\mathbb{L} = \mathbb{Z}^d$. With a slight abuse of terminology we call our model a *quantum anharmonic crystal,* even if $\mathbb{L}$ is not a lattice. The heuristic Hamiltonian of our model is

$$H = -\frac{1}{2} \sum_{\ell, \ell'} J_{\ell\ell'} \cdot (q_\ell, q_{\ell'}) + \sum_{\ell} H_\ell, \tag{2}$$

where the interaction term is harmonic – the simplest possible choice, which, however, has a physical motivation (it is of dipole–dipole type). The indices in the sums run through the set $\mathbb{L}$, the displacement q_ℓ of the oscillator attached to a given $\ell \in \mathbb{L}$ is a ν-dimensional vector, whose components $q_\ell^{(j)}$, $j = 1, \dots, \nu$, are position operators. By $(\cdot, \cdot)$ and $|\cdot|$ we denote the scalar product and norm in $\mathbb{R}^\nu$. The one-site Hamiltonian

$$H_\ell = H_\ell^{\mathrm{har}} + V_\ell(q_\ell) \stackrel{\mathrm{def}}{=} \frac{1}{2m} |p_\ell|^2 + \frac{a}{2} |q_\ell|^2 + V_\ell(q_\ell), \quad a > 0, \tag{3}$$

describes an isolated quantum anharmonic oscillator of mass m and momentum $p_\ell = (p_\ell^{(1)}, \dots, p_\ell^{(\nu)})$. It is also called the *Schrödinger operator* of the oscillator. H_ℓ^{har} is the Schrödinger operator of a ν-dimensional harmonic oscillator of rigidity a. The components of p_ℓ and q_ℓ, which are operators in $L^2(\mathbb{R}^\nu)$, obey the canonical commutation relation

$$p_\ell^{(j)} q_{\ell'}^{(j')} - q_{\ell'}^{(j')} p_\ell^{(j)} = -\mathrm{i} \delta_{\ell\ell'} \delta_{jj'}, \quad j, j' = 1, \dots, \nu, \quad \mathrm{i} = \sqrt{-1}.$$

In our presentation of this relation, Planck's constant $\hbar$ is included into the mass parameter

$$m = m_{\mathrm{ph}} / \hbar^2, \tag{4}$$

where m_{ph} is the physical mass of the particle. The anharmonic potentials $V_\ell \colon \mathbb{R}^\nu \to \mathbb{R}$, which may vary from site to site, are continuous functions obeying

$$b_V |x|^{2r} - c_V \le V_\ell(x) \le V(x),$$

with constants $b_V > 0$, $c_V \geq 0$, $r \in \mathbb{N} \setminus \{1\}$, and a continuous function $V : \mathbb{R}^\nu \to \mathbb{R}$. These bounds are responsible for the system stability. As for the interaction intensities, the only general restriction is

$$\hat{J}_0 \stackrel{\text{def}}{=} \sup_\ell \sum_{\ell'} |J_{\ell\ell'}| < \infty,$$

which is a stability condition as well. By imposing this condition we shall avoid problems with infinite forces acting on a given oscillator. In general, we do not assume that the model has special properties like translation invariance or that the interaction has finite range. Therefore, our model can describe also systems with long-range interactions and with spatial irregularities like impurities or the ones with random components.

The Hamiltonian (2) has no direct mathematical meaning and usually is 'represented' by local Hamiltonians H_Λ corresponding to finite $\Lambda \subset \mathbb{L}$. Here and in the sequel, the adjective 'local' characterizes a property, related to a finite $\Lambda \subset \mathbb{L}$, whereas 'global' will always refer to the whole 'lattice' $\mathbb{L}$. Cases of infinite $\Lambda \subsetneq \mathbb{L}$ are indicated explicitly. Each H_Λ describes the subsystem of oscillators attached to the lattice points $\ell \in \Lambda$, and hence is obtained from (2) by restricting the corresponding sums to Λ. It is a self-adjoint lower bounded operator in the physical Hilbert space $L^2(\mathbb{R}^{\nu|\Lambda|})$, the elements of which are called *wave functions*. The operator H_Λ has discrete spectrum and is such that

$$\text{trace}[\exp(-\tau H_\Lambda)] < \infty, \quad \text{for all } \tau > 0. \tag{5}$$

The quantum-mechanical states of the subsystem in Λ are defined by the wave functions $\psi \in L^2(\mathbb{R}^{\nu|\Lambda|})$ of unit norm in the following sense. Let $\mathfrak{C}_\Lambda$ be the algebra of all bounded linear operators in $L^2(\mathbb{R}^{\nu|\Lambda|})$. Its elements are called local observables. For the mentioned ψ, the state ω_ψ is defined on $\mathfrak{C}_\Lambda$ as the linear functional

$$\mathfrak{C}_\Lambda \ni A \mapsto \omega_\psi(A) = (\psi, A\psi)_{L^2(\mathbb{R}^{\nu|\Lambda|})},$$

where $(\cdot,\cdot)_{L^2(\mathbb{R}^{\nu|\Lambda|})}$ is the scalar product in $L^2(\mathbb{R}^{\nu|\Lambda|})$. Such a state can be extended to unbounded operators, which contain ψ in their domains. The state ω_ψ is *pure* (also called *extreme*), which means that it cannot be expressed as a nontrivial convex combination of other states. If ψ is the eigenfunction of H_Λ corresponding to the eigenvalue E, then the energy of the subsystem in the state ω_ψ is $\omega_\psi(H_\Lambda) = E$. By (5) and the Hilbert–Schmidt theorem it follows that there exists an orthonormal basis $\{\psi_n\}_{n\in\mathbb{N}_0}$ of $L^2(\mathbb{R}^{\nu|\Lambda|})$, consisting of eigenvectors of H_Λ. Let $\{E_n\}_{n\in\mathbb{N}_0}$ be the set of the corresponding eigenvalues of H_Λ. According to the fundamental law of statistical mechanics, the equilibrium state $\varrho_{\beta,\Lambda}$ at a given value of the parameter $\beta = 1/k_B T$, called *inverse temperature*, is a mixture of the pure states ω_{ψ_n} with coefficients proportional to $\exp(-\beta E_n)$. Here $k_B > 0$ and $T > 0$ are Boltzmann's constant and absolute temperature, respectively. By these arguments we are immediately led to the formula

$$\varrho_{\beta,\Lambda}(A) = \frac{\text{trace}(Ae^{-\beta H_\Lambda})}{\text{trace}(e^{-\beta H_\Lambda})}. \tag{6}$$

This is the Gibbs state corresponding to the *canonical ensemble*. In the *grand canonical ensemble*, one includes also states with different numbers of particles. As in our case this number is constant and equal to $|\Lambda|$, we shall consider canonical ensembles only. Along with the thermodynamics of the considered system of oscillators, the Hamiltonian H_Λ determines its dynamics as well. There exist two equivalent approaches to the description of the dynamics of a quantum system. In the Schrödinger approach, the states ω_ψ evolve according to the Schrödinger equation, whereas the observables remain constant in time. In the Heisenberg picture, the states are constant but the observables evolve according to the following rule[1]

$$\mathfrak{C}_\Lambda \ni A \mapsto \alpha_t^\Lambda(A) \stackrel{\text{def}}{=} e^{\mathrm{i}tH_\Lambda} A e^{-\mathrm{i}tH_\Lambda}, \quad t \in \mathbb{R}. \tag{7}$$

As H_Λ is self-adjoint, the operators $e^{\mathrm{i}tH_\Lambda}$ are unitary; hence, the mappings α_t^Λ, $t \in \mathbb{R}$, constitute a one-parameter group of automorphisms of $\mathfrak{C}_\Lambda$. In our context, it is more appropriate to adopt the Heisenberg approach, at least because in both cases (6) and (7), one deals with mappings defined on one and the same set $\mathfrak{C}_\Lambda$. Note that the fact, that they are defined by the same operator H_Λ, is crucial. One might observe, however, that the picture just drawn has some deficiency since therein the subsystem in Λ is described separately from the rest of the system, the influence of which is thereby ignored. In the path integral approach developed below, this problem is settled by considering conditional Gibbs measures, in which the interaction of the subsystem in Λ with the remaining part of the system is taken into account.

We have come to the point where we can start to build up our Euclidean approach. In its first stage, we realize the state $\varrho_{\beta,\Lambda}$ with the help of a path measure. Here multiplication operators play a significant role. For a bounded Borel function, $F\colon \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$, the corresponding multiplication operator F acts according to

$$(F\psi)(x) = F(x)\psi(x), \quad \psi \in L^2(\mathbb{R}^{\nu|\Lambda|}).$$

In this case, we can write

$$\varrho_{\beta,\Lambda}(F) = \frac{\int_{\mathbb{R}^{\nu|\Lambda|}} F(x) K_\beta(x,x)\mathrm{d}x}{\int_{\mathbb{R}^{\nu|\Lambda|}} K_\beta(x,x)\mathrm{d}x}, \tag{8}$$

where $K_\tau(x,y)$ is the integral kernel of $\exp(-\tau H_\Lambda)$. This defines the restriction of the state (6) to the abelian subalgebra consisting of all multiplication operators by bounded Borel functions. Of course, such a result is not sufficient. To extend this kind of representation to the remaining elements of $\mathfrak{C}_\Lambda$ we proceed as follows. First we prove that the linear span of the products

$$\alpha_{t_1}^\Lambda(F_1)\ldots\alpha_{t_n}^\Lambda(F_n)$$

with all possible choices of $n \in \mathbb{N}$, $t_1,\ldots,t_n \in \mathbb{R}$, $F_1,\ldots,F_n \in C_\mathrm{b}(\mathbb{R}^{\nu|\Lambda|})$, is dense in $\mathfrak{C}_\Lambda$ in a certain (σ-weak) topology, in which the state (6) is continuous. Here $C_\mathrm{b}(\mathbb{R}^{\nu|\Lambda|})$

[1] For convenience, we set $t = \text{time}/\hbar$, where $\hbar$ is Planck's constant.

is the set of all bounded continuous functions $F\colon \mathbb{R}^{\nu|\Lambda|} \to \mathbb{C}$. Thereby, this state is fully determined by its values on such products, that is, by the Green functions

$$G^{\Lambda}_{F_1,\dots,F_n}(t_1,\dots,t_n) \stackrel{\text{def}}{=} \varrho_{\beta,\Lambda}[\alpha^{\Lambda}_{t_1}(F_1)\dots\alpha^{\Lambda}_{t_n}(F_n)], \quad F_1,\dots,F_n \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|}). \tag{9}$$

Let us formally set here $t_k = \mathrm{i}\tau_k$, $\tau_k \in \mathbb{R}$, $k = 1,\dots,n$, and consider

$$\begin{aligned}\Xi(\tau_1,\dots,\tau_n) \stackrel{\text{def}}{=} \mathrm{trace}\big\{&e^{-\tau_1 H_\Lambda} F_1 e^{-(\tau_2-\tau_1)H_\Lambda} \\ &\times\dots\times F_{n-1} e^{-(\tau_n-\tau_{n-1})H_\Lambda} F_n e^{-(\beta-\tau_n)H_\Lambda}\big\},\end{aligned} \tag{10}$$

which can be written in the form (8) provided $0 \le \tau_1 \le \dots \le \tau_n \le \beta$. Now the problem of relating (10) to (9) can be settled by means of an analytic continuation from the real to imaginary values of time. This is done by proving that each Green function is the restriction of a function $G^{\Lambda}_{F_1,\dots,F_n}$, which is analytic in the following complex tubular domain[2]

$$\mathcal{D}^n_\beta = \{(z_1,\dots,z_n) \in \mathbb{C}^n \mid 0 < \Im(z_1) < \dots < \Im(z_n) < \beta\}, \tag{11}$$

and continuous on its closure $\bar{\mathcal{D}}^n_\beta \subset \mathbb{C}^n$. Thereby, one shows that for any $n \in \mathbb{N}$, the 'imaginary time' subset

$$\{(z_1,\dots,z_n) \in \mathcal{D}^n_\beta \mid \Re(z_1) = \dots = \Re(z_n) = 0\}$$

is a set of uniqueness for functions analytic in $\mathcal{D}^n_\beta$. This means that if two such functions take equal values on this set, then they are equal everywhere and thus equal as functions. Therefore, the Green functions (9), and hence the state (6), are determined by the so-called Matsubara functions

$$\begin{aligned}\Gamma^{\Lambda}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n) &\stackrel{\text{def}}{=} G^{\Lambda}_{F_1,\dots,F_n}(\mathrm{i}\tau_1,\dots,\mathrm{i}\tau_n) \\ &= \Xi(\tau_1,\dots,\tau_n)/\mathrm{trace}[e^{-\beta H_\Lambda}] \\ &= \mathrm{trace}[F_1 e^{-(\tau_2-\tau_1)H_\Lambda} F_2 e^{-(\tau_3-\tau_2)H_\Lambda}\dots F_n e^{-(\tau_{n+1}-\tau_n)H_\Lambda}]/\mathrm{trace}[e^{-\beta H_\Lambda}],\end{aligned} \tag{12}$$

taken at ordered arguments $0 \le \tau_1 \le \dots \le \tau_n \le \tau_1 + \beta \stackrel{\text{def}}{=} \tau_{n+1}$, with all possible choices of $n \in \mathbb{N}$ and $F_1,\dots,F_n \in C_{\mathrm{b}}(\mathbb{R}^{\nu|\Lambda|})$. Their extensions to $[0,\beta]^n$ are defined as

$$\Gamma^{\Lambda}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n) = \Gamma^{\Lambda}_{F_{\sigma(1)},\dots,F_{\sigma(n)}}(\tau_{\sigma(1)},\dots,\tau_{\sigma(n)}),$$

where σ is the permutation of $\{1,2,\dots,n\}$ such that $\tau_{\sigma(1)} \le \tau_{\sigma(2)} \le \dots \le \tau_{\sigma(n)}$. This multiple-time analyticity can be thought of as a consequence of the above mentioned fact that the dynamics and thermodynamics of the subsystem are determined by the same local Hamiltonian and hence are in equilibrium. As follows from the representation (12), the Matsubara function $\Gamma^{\Lambda}_{F_1,\dots,F_n}$ can be written in the form

$$\Gamma^{\Lambda}_{F_1,\dots,F_n}(\tau_1,\dots,\tau_n) = \int_{\Omega_{\beta,\Lambda}} F_1(x_\Lambda(\tau_1))\dots F_n(x_\Lambda(\tau_n))\nu_{\beta,\Lambda}(\mathrm{d}x_{\beta,\Lambda}), \tag{13}$$

[2]For a $z = x + \mathrm{i}y \in \mathbb{C}$, $x, y \in \mathbb{R}$, we write $x = \Re(z)$, $y = \Im(z)$.

where $\nu_{\beta,\Lambda}$ is a certain probability measure on the local path space $\Omega_{\beta,\Lambda}$. It is canonically associated with a β-periodic Markov process[3], for which the transition probabilities are defined by the kernels $K_\tau(x, y)$ mentioned above. This is the main point of the first stage of our approach. As was mentioned above, this approach is called Euclidean in view of the passage from the real to imaginary values of time. Correspondingly, the measure $\nu_{\beta,\Lambda}$ is called a local Euclidean Gibbs measure. By standard arguments, it is uniquely determined by the integrals (13); hence, since the Matsubara functions $\Gamma^{\Lambda}_{F_1,\dots,F_n}$ uniquely determine the state $\varrho_{\beta,\Lambda}$, the representation (13) establishes a one-to-one correspondence between the local Gibbs states and local Euclidean Gibbs measures.

Now suppose that we are given an algebra of observables $\mathfrak{C}$ and a one-parameter group of time automorphisms $\alpha_t : \mathfrak{C} \to \mathfrak{C}$, $t \in \mathbb{R}$, which determines the dynamics of the underlying system. How can one find a σ-weakly continuous state ω on $\mathfrak{C}$ such that the continuation of the Green functions of this state to imaginary values of time is possible and thereby a kind of equilibrium can be established? Here we note that the family of such states need not be a singleton. The answer to the above question is related to the Kubo–Martin–Schwinger (KMS) property of ω. For $A, B \in \mathfrak{C}$, let us set

$$F^{\omega}_{A,B}(t) = \omega(B\alpha_t(A)), \qquad G^{\omega}_{A,B}(t) = \omega(\alpha_t(A)B), \quad t \in \mathbb{R}. \tag{14}$$

Then ω is called a β-KMS state if there exists a function F, analytic in $\mathcal{D}^1_\beta$ and continuous on its closure $\bar{\mathcal{D}}^1_\beta$, such that

$$F^{\omega}_{A,B}(t) = F(t), \quad G^{\omega}_{A,B}(t) = F(t + \mathrm{i}\beta), \quad \text{for all } t \in \mathbb{R}.$$

In [144], see also page 202 in [145], it was suggested to use the KMS property of ω as the defining property of an equilibrium state at a given value of β. It turns out that if ω is a KMS state, then each Green function

$$G^{\omega}_{A_1,\dots,A_n}(t_1, \dots, t_n) = \omega(\alpha_{t_1}(A_1) \dots \alpha_{t_n}(A_n)),$$

for any $n \in \mathbb{N}$ and $A_1, \dots, A_n \in \mathfrak{C}$, has a multiple-time analyticity property, the same as the Green functions (9). This fact was proven in [176]. Thus, a σ-weakly continuous KMS state is uniquely determined by the Matsubara functions corresponding to the operators from a maximal abelian subalgebra of $\mathfrak{C}$. Let us now analyze the possibility of using the idea just outlined in constructing global equilibrium states. As we have seen, the crucial elements of this construction are the algebra of observables and the group of time automorphisms. A candidate for such an algebra could be the norm-completion of the algebra of local observables

$$\mathfrak{C}^{\mathrm{loc}} = \bigcup_{\Lambda} \mathfrak{C}_\Lambda, \tag{15}$$

where the union is taken over all finite Λ. It is a C^*-algebra, but need not be a von Neumann algebra. The group of time automorphisms could be obtained in the

[3]The periodic Markov property was introduced and studied in [177].

infinite-volume limit $\Lambda \nearrow \mathbb{L}$ from the automorphisms (7). For some systems with bounded local Hamiltonians H_Λ, e.g., quantum spin models or the ideal Fermi gas, this 'algebraic' way of constructing equilibrium states can be realized, see [77]. However, in the case of the model (2), (3), it does not work since the construction of corresponding infinite-volume time automorphisms is beyond the technical possibilities existing at this time. As a consequence, the global KMS condition for this model cannot be formulated and hence the KMS states cannot even be defined[4].

What we have also learned from the above consideration is that the Matsubara functions can determine equilibrium states. For our model, this can be done for the local states by (12) and (13), where these functions are obtained as integrals with respect to local Euclidean Gibbs measures. On the global level, general abstract techniques of constructing equilibrium states from given (complete) sets of Matsubara functions were elaborated in [66], [131], [132], [133]. As follows from these works, the number of equilibrium states existing for the same values of the model parameters and temperature is in correspondence with the number of sets of Matsubara functions, which can be constructed for these values. Therefore, all the information about the thermodynamics of the considered model is contained in these functions. Our approach gives a way how to obtain them. Here we exploit the fact that the local states are represented by probability measures and hence can be interpreted as local Gibbs measures of classical lattice systems of unbounded spins. For such systems, a complete description of the equilibrium thermodynamic properties is achieved by constructing their Gibbs states as probability measures on appropriate configuration spaces. Here the use of the distributions of configurations in a finite $\Lambda \subset \mathbb{L}$ conditioned by configurations outside Λ is standard. The corresponding techniques constitute the Dobrushin–Lanford–Ruelle (DLR) theory, which now is well-elaborated and widely used. By virtue of the Feynman–Kac formula employed in its construction, each of the local Euclidean Gibbs measures $\nu_{\beta,\Lambda}$ has the same structure as the local Gibbs measure of a classical lattice model. The only, but essential, difference is that here even the single-site spaces are infinite-dimensional (spaces of continuous paths). Therefore, the reference measure employed in the construction of $\nu_{\beta,\Lambda}$ cannot be Lebesgue measure, which does not exist for such spaces. Instead, we use a Gaussian measure, which serves as a local Euclidean Gibbs measure of a single harmonic oscillator. In spite of the mentioned difficulty, the local Euclidean Gibbs measures, as well as the corresponding local conditional Gibbs measures, possess properties which allow for employing most of the DLR techniques adapted, however, to infinite-dimensional single-site spaces. This is realized in the first part of the book.

As was mentioned above, the model (2) has various physical applications and the corresponding physical objects are well studied, both experimentally and theoretically, e.g., by means of numerical methods and computer simulations. At the same time, the rigorous mathematical description of its equilibrium thermodynamic properties based on a widely recognized method has not been given yet. In the first part of the present book we develop a version of such a description. Therefore, it would be quite natural to

[4] A more detailed analysis of similar problems, which appear in the theory of interacting Bose gases, can be found on page 349 of [77].

obtain in its framework a qualitative explanation of the basic known facts concerning the thermodynamic properties of these physical objects. This is done in the second part of the book.

Thus, the main points of the Euclidean approach developed below are

(a) Constructing the local Euclidean Gibbs measures $\nu_{\beta,\Lambda}$ as measures on path spaces (spaces of continuous paths).

(b) Constructing and studying the conditional local Euclidean Gibbs measures and hence the local Gibbs specifications.

(c) Constructing the set $\mathcal{G}^{\mathrm{t}}_{\beta}$ of tempered Euclidean Gibbs measures, describing the whole infinite model as the set of probability measures which solve the DLR (equilibrium) equations defined by the local Gibbs specification.

(d) Studying the properties of $\mathcal{G}^{\mathrm{t}}_{\beta}$ and thereby describing phase transitions and quantum effects in the model considered.

This program is realized in the book as follows. Part I, as said above, is dedicated to the mathematical background. It consists of Chapters 1–3. In Chapter 1, we start by introducing the model and making natural stability assumptions regarding $J_{\ell\ell'}$ and V_{ℓ}. Then we introduce the state (6) and provide the essential facts concerning linear operators in Hilbert spaces, the Schrödinger operators of single harmonic and anharmonic oscillators, normal states, and von Neumann algebras. Afterwards, we prove the density theorem which allows for describing local Gibbs states by the Green functions corresponding to multiplication operators. Next, we give a complete proof of the multiple-time analyticity of the Green functions, which leads us to the Matsubara functions and then to the representation (13). In passing from the states (6) to the measures $\nu_{\beta,\Lambda}$, as a reference system we use the subsystem of noninteracting harmonic oscillators. Its Green and Matsubara functions are obtained explicitly. Thereby, we present and interpret a collection of concepts and tools from stochastic analysis, which will be used subsequently. This includes a number of facts from the theory of probability measures on complete separable metric spaces (called Polish spaces), in particular on separable Hilbert spaces. As we show, the Euclidean Gibbs measure of a single harmonic oscillator is the measure corresponding to the periodic Ornstein–Uhlenbeck velocity process (periodic oscillator process), which for the first time appeared in R. Høegh-Krohn's paper [156]. We call them Høegh-Krohn process and Høegh-Krohn measure respectively. The properties of the Høegh-Krohn measure play a significant role in our construction and are, therefore, analyzed in detail. We construct and study the local Euclidean Gibbs measures $\nu_{\beta,\Lambda}$ by using a version of the Feynman–Kac formula. Chapter 1, as all subsequent chapters, is concluded with comments and bibliographic notes.

In Chapter 2, for the local Euclidean Gibbs measures, we prove a number of correlation inequalities and similar useful facts. The proof is based on the 'lattice approximation' of the measures $\nu_{\beta,\Lambda}$, in which the approximating measures are local Gibbs

measures of classical models with 'unbounded spins'. The main point here is the approximation of the Høegh-Krohn process (which is a periodic Markov process) by a Markov chain. A similar approach is known in Euclidean quantum field theory. By means of this approximation we rederive the basic correlation inequalities known for classical spin models. Among the new results obtained here we mention the Lee–Yang property for a certain type of anharmonic potentials V_ℓ, scalar domination inequalities which allow for comparing scalar and vector versions of our model, and some new inequalities for Matsubara and Ursell functions.

Chapter 3 is dedicated to the construction and description of the Euclidean Gibbs states of the model (2) in complete generality. We start by discussing the thermodynamic limit and limiting Gibbs states. Then we introduce the spaces of all configurations Ω_β and tempered configurations $\Omega^{\mathrm{t}}_\beta$. The space Ω_β is constructed from the spaces of local configurations in a natural way. We equip Ω_β with the product topology that turns it into a Polish space. This fact is essential in view of the DLR techniques which we are going to use. The reason to introduce the space of tempered configurations $\Omega^{\mathrm{t}}_\beta$ is twofold. First, since the interaction intensities $J_{\ell\ell'}$ may have infinite range, we must impose some a priori restrictions on the L^2-norms $\|\xi_\ell\|_{L^2_\beta}$ of the components of configurations $\xi \in \Omega_\beta$. Otherwise, the local conditional Euclidean Gibbs measures $\nu_{\beta,\Lambda}(\cdot|\xi)$ cannot be defined. Second, even if $J_{\ell\ell'}$ had finite range, restrictions should be imposed to exclude measures which in a sense are 'improper'. By definition, *tempered* Euclidean Gibbs measures are to be supported by $\Omega^{\mathrm{t}}_\beta$. This is a usual procedure in the DLR theory of Gibbs measures of systems of 'unbounded spins'. However, as we show afterwards, the real support of the tempered Euclidean Gibbs measures is much smaller than $\Omega^{\mathrm{t}}_\beta$ and is independent of the way the latter set has been introduced. As to this way, the restrictions are imposed by means of weights, $\{w_\alpha\}_{\alpha\in\mathcal{I}}$, that among other properties have the one by which each function $-\log w_\alpha$, $\alpha \in \mathcal{I}$, is a metric on $\mathbb{L}$. We equip $\Omega^{\mathrm{t}}_\beta$ with a projective limit topology, so that it becomes a Polish space as well. In Section 3.2, we prove that the kernels $\pi_{\beta,\Lambda}$ obtained from the local conditional Gibbs measures obey certain exponential moment estimates, which play a key role in constructing and studying the tempered Euclidean Gibbs measures. In Section 3.3, we prove that the set of such measures $\mathcal{G}^{\mathrm{t}}_\beta$ is non-void and weakly compact. We also prove a number of statements characterizing $\mathcal{G}^{\mathrm{t}}_\beta$, among them the support property mentioned above. Next we develop an alternative approach to the construction of Euclidean Gibbs measures based on the Radon–Nikodym characterization. In this approach, $\mathcal{G}^{\mathrm{t}}_\beta$ is defined as the set of measures obeying an integration-by-parts formula. Subsequently, we present a more detailed study of the case of local interactions, where the intensities $J_{\ell\ell'}$ have finite range, and of the translation-invariant case, where $\mathbb{L} = \mathbb{Z}^d$, $V_\ell = V$, and $J_{\ell\ell'}$ are invariant with respect to the translations of $\mathbb{L}$. In the latter case, the set $\mathcal{G}^{\mathrm{t}}_\beta$ among others contains the so-called periodic elements. Finally, for $J_{\ell\ell'} \geq 0$ and $\nu = 1$, we introduce a stochastic order on $\mathcal{G}^{\mathrm{t}}_\beta$, with respect to which it has a minimal element, μ_-, and a maximal element, μ_+. This fact is then employed in further studying $\mathcal{G}^{\mathrm{t}}_\beta$. In particular, by means of these elements a uniqueness criterion is obtained.

Part II, comprising Chapters 4–7, is dedicated to the description of some physical properties of the model defined by (2) and (3). Here we concentrate on those related to phase transitions and critical points, as well as on quantum effects. In Chapter 4, we discuss in more detail which physical systems can be modeled by the Hamiltonians (2) and (3). Then we study the classical limit $m \to +\infty$, cf. (4), of the local Euclidean Gibbs measures and show that they coincide with those of the corresponding system of classical anharmonic oscillators. Next, we prove that $\mathscr{G}^{\mathrm{t}}_{\beta}$ is a singleton at high temperatures and/or weak interactions. Chapter 5 is dedicated to the study of the thermodynamic pressure, which up to a factor coincides with the free energy density. Here we suppose that $\mathbb{L}$ is a lattice and the model is translation-invariant. We begin by proving that the pressure exists and is the same for each state $\mu \in \mathscr{G}^{\mathrm{t}}_{\beta}$. Then we describe its dependence on the external field and formulate a uniqueness criterion in terms of differentiability of the pressure. Next, in Chapter 6, we study phase transitions. According to our definition, a phase transition occurs if the set of tempered Euclidean Gibbs measures contains more than one element that corresponds to the non-uniqueness of equilibrium phases. We also analyze the connection of this definition with the one based on an order parameter and with the definition of L. Landau. Then we prove that a number of versions of our model, including those with irregular $\mathbb{L}$, have a phase transition under certain conditions. The proof is based on the reflection positivity method, adapted here to the Euclidean approach, on correlation inequalities, and on appropriate analytic methods developed in the first part of the book. Next, we consider a hierarchical model of quantum anharmonic oscillators, which is a special case of the model (2), (3). For this model, we prove a statement describing its critical point.

The final Chapter 7 is dedicated to the theory of quantum effects in our model. Since the 1970s, understanding the influence of quantum effects on phase transitions is one of the main tasks in the theory of systems of this kind. As is commonly accepted, a ferroelectric phase transition in the KDP-type compounds is triggered by the ordering of protons on the hydrogen bounds and, therefore, the model (2) is quite appropriate to describe this class of physical objects. These ferroelectrics become less stable with respect to a structural phase transition if one replaces protons by deuterons[5]. On the other hand, high hydrostatic pressure, which increases tunneling of the particles by bringing minima of the wells closer to one another, decreases the transition temperature. We propose a theory, which qualitatively explains all these facts. It naturally comes from the results obtained above and is based on the following arguments. The key parameter here is $m\Delta^2$, where m is the mass parameter (4) and Δ is the least difference between the eigenvalues of the single-particle Hamiltonian H_ℓ (which depends on m). In the harmonic case, $m\Delta^2$ is merely the oscillator rigidity and the stability of the crystal corresponds to large values of this parameter. That is why we call $m\Delta^2$ *quantum rigidity*. If the tunneling between the wells gets more intensive (closer minima), or if the mass diminishes, $m\Delta^2$ gets bigger and the particle 'forgets' the details of the potential energy in the vicinity of the origin (including instability) and oscillates as if its equilibrium at zero were stable, as in the harmonic case. We provide a complete

[5]This amounts to altering the particle mass in (3) without changing any other parameters.

mathematical background for these arguments. First, we prove that the quantum rigidity is a continuous function of m and that $m\Delta^2 \to +\infty$ as $m \to 0$. Then we prove that the model has no phase transitions, at any temperature, if $m\Delta^2 < \hat{J}_0$, where $\hat{J}_0$ is the total energy of the interaction of the particle with the rest of the system. This can be considered as a further confirmation that our Euclidean approach is adequate to describe a large class of phenomena arising in solid state physics.