Preface

Ché se la voce tua sarà molesta
nel primo gusto, vital nutrimento
lascerà poi, quando sarà digesta.

(Dante, Paradiso, XVII, 130–132)

This book consists of lectures on classical algebraic geometry, that is, the methods and results created by the great geometers of the late nineteenth and early twentieth centuries.

This book is aimed at students of the last two years of an undergraduate program in mathematics: it contains some rather advanced topics that could form material for specialized courses and which are suitable for the final two years of undergraduate study, as well as interesting topics for a senior thesis. The book will be welcomed by teachers and students of algebraic geometry who are seeking a clear and panoramic path leading from the basic facts about linear subspaces, conics and quadrics, learned in courses on linear algebra and advanced calculus to a systematic discussion of classical algebraic varieties and the tools needed to study them.

The topics chosen throw light on the intuitive concepts that were the starting point for much contemporary research, and should therefore, in our opinion, make up part of the cultural baggage of any young student intending to work in algebraic geometry. Our hope is that this text, which can be a first step in recovering an important and fascinating patrimony of mathematical ideas, will stimulate in some readers the desire to look into the original works of the great geometers of the past, and perhaps even to find therein motivation for significant new research.

Another reason which induced us to write this book is the observation that many young researchers, though able to obtain significant results by using the sophisticated techniques presently available, can also encounter notable difficulty when faced with questions for which classical methods are particularly indicated. This book combines the more classical and intuitive approach with the more formally rigorous and modern approach, and so contributes to filling a gap in the literature.

This book, which we consider new and certainly different from texts published in the last fifty years, is the text we would have liked on our desk when we began our studies; it is our hope that it will serve as a useful introduction to Algebraic Geometry along classical lines.

The ideal use for this text could well be to provide a solid preliminary course to be mastered before approaching more advanced and abstract books. Thus we lay a firm classical foundation for understanding modern expositions such as Hartshorne [51], Mumford [69], Liu [66], or also Dolgachev's forthcoming treatise [34]. Our text can also be considered as a more modern version of Walker's classic book [116], but greatly enriched with respect to the latter by the discussion of important classes of higher dimensional varieties as mentioned above.

Prerequisites. We suppose that the reader knows the foundational elements of Projective Geometry, and the geometry of projective space and its subspaces. These are topics ordinarily encountered in the first two years of undergraduate programs in mathematics. The basic references for these topics are the classic treatment of Cremona [31] and the texts of Berger [13] and Hodge and Pedoe [53, Vols. 1, 2]. The introductory text [10] by the authors of the present volume is also useful. For the convenience of the reader in the purely introductory Chapter 1 we have given a concise review of those facts that will be most frequently used in the sequel.

Moreover to understand the book, in addition to a few elementary facts from Analysis, the reader should also be familiar with the basic structures of Algebra (groups, rings, polynomial rings, ideals, prime and maximal ideals, integral domains and fields, the characteristic of a ring), as well as extensions of fields (algebraic and transcendental elements, minimal polynomials, algebraically closed fields) as found in the texts of [35] or [76].

Possible "Itineraries". The book contains several "itineraries" that could suggest or constitute topics for different advanced undergraduate courses in mathematics, and also for graduate level courses. Here are some more precise indications, which also offer a view of the topics treated here.

• Chapters 2 and 3 can be the introduction to any course in algebraic geometry. They contain the essential notions regarding algebraic and projective sets: the Hilbert Nullstellensatz, morphisms and rational maps, dimension, simple points and singular points of an algebraic set, tangent spaces and tangent cones, the order of a projective variety. If one then adds the brief comments on elimination theory in Chapter 4, one has enough material for a semester course.

• Chapter 5 is dedicated to hypersurfaces in $\mathbb{P}^n$ with particular attention to algebraic plane curves and surfaces in $\mathbb{P}^3$. It assumes only the rudiments of the geometry of projective space and a thorough familiarity with projective coordinates. The topics covered in this chapter, suitably amplified and accompanied by the exercises given in Sections 5.7, 5.8, can in themselves form the program for a course that probably requires more than a semester, especially if one adds the first two paragraphs of Chapter 9 which are dedicated to quadratic transformations between planes and their most important applications, for example the proof of the existence of a plane model with only ordinary singularities for any algebraic curve.

• Chapter 6, which deals with linear systems of algebraic hypersurfaces in $\mathbb{P}^n$, contains topics necessary for the subsequent chapters. Veronese varieties and mappings are introduced, as well as the notion of the blowing up of $\mathbb{P}^n$ with center a subvariety of codimension ≥ 2.

• The program for a specialized one semester course for advanced undergraduates could be furnished by Chapter 7 and the first two sections of Chapter 9, which are dedicated respectively to algebraic curves in $\mathbb{P}^n$ (with particular attention to rational curves and the curves on a quadric in $\mathbb{P}^3$) and to quadratic transformations between planes. The genus of a curve is introduced, and its birational nature is placed in evidence. Adding the remaining results discussed in Chapter 9, which has some originality with respect to the existing literature on Cremona transformations, would give rise to a full year course.

• Chapter 8 is the natural continuation and completion of Chapter 7, and also makes use of some results from the first two sections of Chapter 9. It deals with the theory of linear series on an algebraic curve, including an extensive discussion on the Riemann–Roch theorem, and an approach to the classification of algebraic curves in $\mathbb{P}^n$ in terms of properties of the canonical series and the canonical curve. This chapter was largely inspired by Severi's classic text [102], where the so called "quick method" for studying the geometry of algebraic curves is expounded. The content of this chapter would give rise to a one semester course.

• Chapter 10 can furnish material for a one semester course for students who already have a good mastery of the geometry of hyperspaces ([53, Vol. 1, Chapter V]), of plane projective curves (Chapter 5, Section 5.7) and of Cremona transformations between planes (Chapter 9, Sections 9.1, 9.3). Thus this chapter is well adapted for an upper-undergraduate level course in mathematics or also for graduate level courses. Nevertheless, the methods used are rather elementary. Among the topics to which the most space is dedicated, we mention the rational normal ruled surfaces, the Veronese surface, and the Steiner surface. Some of the surfaces already described in the last section of Chapter 5 are here rediscovered and seen in a new light. They are studied together with other surfaces that occupy an important place in algebraic-projective geometry.

• Veronese varieties, Segre varieties, and Grassmann varieties are discussed in Chapter 6, Section 6.7, Chapter 11 and Chapter 12 respectively. They constitute examples of special varieties that every student of geometry should know. These topics too could be part of an advanced course or graduate course. Among other things, they might well suggest topics for research projects or a senior thesis.

• The numerous exercises of the text are in part distributed throughout the various chapters, and in part collected in Chapter 13. They can be quite useful to young graduates who are preparing for admission to a doctoral program or a position as research assistant, or also to high school teachers preparing to qualify for promotion. The easier exercises are merely stated; others, almost always new to this text, offer various levels of difficulty. Most of them are accompanied by a complete solution, but in some cases the method of solution is merely suggested.

Sources. In addition to the already cited text [10] to which the present volume may be seen as the natural successor, classical references and sources of inspiration for part of the material contained here are the books by Bertini [14], [15], Castelnuovo [25], Comessatti [27], Enriques and Chisini [36], Fano and Terracini [38], Hodge and Pedoe [53], B. Segre [82], Semple and Roth [93], and C. Segre's memoir [84]. We have also been influenced by more modern texts like Shafarevich [106] and Harris [49], and, with particular reference to the topics regarding algebraic sets, rational regular functions, and rational maps developed in the second chapter, by Reid's text [75].

Besides the texts mentioned above, in our opinion the very nice introductory texts of Musili [70] and Kunz [61] as well as Kempf's more advanced book [60] merit special mention. We also call the reader's attention to the charming "bibliographie commentée" in Dieudonné's text [33] which offers a panoramic view of the basic and advanced texts and the fundamental articles which have constituted the history and development of Algebraic Geometry, from the origins of Greek mathematics up to the late 1960s. The bibliography is rendered even more valuable, topic by topic, through interesting comments and historical notes illustrating an "excursus" that starts from Heath's interpretation of certain algebraic methods in Diophantus and arrives at Mumford's construction of the space of moduli for curves of a given genus.

Changes and improvements with respect to the Italian version. The present text offers some substantial changes and improvements with respect to the original Italian version [9]. Among the major changes are an entirely new chapter, Chapter 8, devoted to linear series on algebraic curves, a major revision to Chapter 2, the new Section 4.3, giving greater detail regarding intersection multiplicities in Chapter 4. Moreover a number of new exercises have been added throughout the book, including, in particular, a new final section of Chapter 13.

Among the minor changes there is a new final paragraph in Chapter 10 dealing with birational Cremona transformations between projective spaces of dimension 3. There are also numerous corrections of minor typographical and mathematical errors.

We thank the many colleagues and students who have had occasion to read parts of the Italian version of the book, thus contributing to the correction of errors and improving the exposition of the material. In particular, we wish to thank our friends and colleagues L. Bădescu, E. Catalisano, A. Del Padrone, T. Fatemi, A. Geramita, P. Ionescu, R. Pardini for their comments. We would like to thank I. Dolgachev who first encouraged us to consider the possibility of a translation of the original version of the book. We would also like to thank our friend and colleague A. Languasco for his invaluable assistance in resolving various problems involving the use of LaTeX. We also wish to thank D. B. Leep for his careful reading of portions of the text and the useful suggestions for improvements in the presentation that he gave. We thank our students, who read parts of the book, for their stimulating questions. Special thanks are also due to F. Sullivan for the translation and for helping to make that task a truly friendly interaction.

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Finally, let us mention the web page

http://www.dima.unige.it/~beltrame/book.pdf (dvi)

where data and updates regarding the book will be collected and an "errata corrige" placed online.

Mauro C. Beltrametti
Ettore Carletti
Dionisio Gallarati
Giacomo Monti Bragadin

The authors and their translator
(from left to right: D. G., M. C. B., E. C., G. M. B., and F. S.)