Preface

The aim of this book is to present in a concise and accessible way, as well as in a common setting, various tools and methods arising from spectral theory, ergodic theory and probability theory, which contribute interactively to the current research on almost everywhere convergence problems. The recent developments in the study of these questions are often obtained by combining either methods of spectral theory with principles of ergodic theory or methods from probability theory with tools and principles from spectral theory and ergodic theory. The spectral criterion of Gaposhkin, and later, following a remarkable metric entropy inequality of Talagrand, the spectral regularization developed in the setting of the study of square functions and oscillation functions in ergodic theory, are typical examples of this fruitful interaction. Another example of thorough interaction is certainly the work of Bourgain and notably his famous entropy criterion, at the basis of which lies the continuity principle of Stein.

It was not our aim to write a complete treatise in ergodic theory, assuming such enterprise to be conceivable. The development of this theory during the last twenty years was indeed considerable. A similar remark can be made for the part concerning the study of the regularity of stochastic processes. The work is also not a synthesis of most significant results, complete with sketched proofs and references. We chose the intermediate route to writing a book in the spirit of lectures oriented towards research. The book provides an easy access to many tools, methods and results used in current research, presenting each of them in as wide a setting as possible. The proofs of these results are often given with full details.

This book is divided in four parts, which came more or less naturally while writing it. Part I is devoted to spectral results and is followed by Part II, in which tools and results from ergodic theory are presented. In the third part, in connection with the description of two main methods, namely the metric entropy method and the majorizing measure method, recent applications to ergodic theory are given via the study of some maximal inequalities of Gál–Koksma type and the L^p norm, $1 \leq p \leq \infty$, of important classes of polynomials. Finally, in the last part of the book we recollect classical results, as well as recent advances concerning Riemann sums and Khintchin sums, and the value distribution of divisors of Bernoulli or Rademacher sums, used in the study of Riemann sums.

In Part I we begin elementarily with the spectral inequality. Chapter 1 concerns von Neumann's theorem, which forms with Birkhoff's ergodic theorem the basis of ergodic theory. It seems natural to include in this chapter Talagrand's metric entropy estimate for the set $\{A_n^T f, n \geq 1\}$ where A_n^T is the average operator $\frac{I+T+\cdots+T^{n-1}}{n}$ of a contraction T in a Hilbert space, thus completing naturally the von Neumann theorem. Recently discovered, remarkably efficient, spectral regularization inequalities analysing other structural properties of the set $\{A_n^T f, n \geq 1\}$, followed by Weyl's criterion and the van der Corput principle, complete this chapter. Chapter 2 starts with presenting the arguments leading to the representation of a weakly stationary process as Fourier transform of a random measure with orthogonal increments. Next we study Gaposhkin's spectral criterion.

In Part II, we first review in Chapter 3 classical ergodic and mixing properties of measurable dynamical systems. We also study several standard examples. Chapter 4 is devoted to Birkhoff's pointwise theorem, to dominated ergodic theorems in L^p and to BMO spaces of associated maximal operators. This is continued with a discussion around spectral characterizations of the speed of convergence in Birkhoff's pointwise theorem. Next we examine oscillation functions of ergodic averages. The transference principle and Wiener–Wintner theorems are discussed. A study of weighted ergodic averages concludes this chapter. In Chapter 5, some basic tools from ergodic theory, the Banach principle, the continuity principle and the conjugacy lemma are studied in detail. Chapter 6 concerns entropy criteria of Bourgain. Several functional inequalities linking the studied sequence of L^2-operators with the canonical Gaussian process on L^2 are established, from which the criteria are then easily deduced.

Study of the statistic of the ergodic averages naturally leads to investigating the question of the existence of some $f \in L^2$ such that the related ergodic averages satisfy a central limit theorem, the invariance principle or the almost sure central limit theorem. Chapter 7 is devoted to this study. A detailed proof of the theorem of Burton–Denker on the existence, in any aperiodic dynamical system, of the central limit theorem is given. The method of proof relies upon Kakutani–Rochlin's lemma and imitates the analogous result for irrational rotations of the unit circle which is obtained by using Fourier series. A fundamental fact in the background of the entire construction is provided by using Rochlin's result on a factor space of Lebesgue space. The case of irrational rotations involving various remarkably efficient methods is more closely investigated. The existence of L^2 elements of the torus satisfying the central limit theorem (CLT) is established for various types of means: nonlinear ergodic means, weighted ergodic means, and ergodic means along the squares. For the latter case, the circle method is used. The chapter concludes with a recent study of a kind of achieved form of the CLT, the convergence in variation implying the convergence of related density distributions in the spaces $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, in the symptomatic case of lacunary random Fourier series.

Two rather general methods are investigated in Part III: the metric entropy method and the majorizing measure method. In Chapter 8, a useful criterion for almost everywhere convergence involving covering numbers is proved, and then used to prove in a unified setting several classical results, such as Stechkin's theorem, Gál–Koksma theorems and quantitative Borel–Cantelli lemmas. The metric entropy method is next applied to establish quite useful estimates of the supremum of random polynomials, notably random Dirichlet polynomials, and to study almost sure convergence properties of weighted series of contractions and random perturbation of some intersective sets in ergodic theory. Chapter 9 concerns an important tool: the majorizing measure method. A general criterion for almost sure convergence of averages is proved by means of this method. We continue with recent applications of the majorizing measure method to the study of the supremum of random polynomials, including a strictly stronger form of the well-known Salem–Zygmund estimate. Some remarkable classes of examples are studied. Chapter 10 is a succinct study of Gaussian processes presented in the form of a toolbox. Various fundamental results from the theory are discussed, sometimes with historical comments and proofs. Much importance is given to very handy correlation inequalities.

Part IV is devoted to three studies: the study of Riemann sums, the study of convergence properties of the system $\{f(n_k x), k \geq 1\}$ and a probabilistic approach concerning divisors with applications.

Chapters 1 to 6 and partially Chapters 8 to 10 are based on lectures given at the Mathematical Institute of the University of Strasbourg. Chapters 11 to 13 are mainly based on research articles, as well as some parts of Chapters 1, 4, 7, 8, 9. In writing this book, we followed a general principle: where the proofs in our source readings were only sketched, we fill in the gaps in as much detail as possible. Further, we give quasi-systematically complete references with page numbers and/or precise numeration of cited results. We always keep in mind the wish to help, as much as we can, the researcher but also the teacher and the graduate student in their work in these beautiful areas of mathematics, trying also to spare their time and to let them share our passion for research at the interfaces of related problems.

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I devote this book to my wife Marie-Christine. She always provided a favourable atmosphere for mathematical work.