Contents

Preface v

Part I Spectral theorems and convergence in mean 1

1 The von Neumann theorem and spectral regularization 3
1.1 Bochner–Herglotz lemma . . . 3
1.2 The spectral inequality . . . 8
1.3 The von Neumann theorem . . . 10
1.4 The spectral regularization inequality . . . 26
1.5 Moving averages . . . 44
1.6 Uniform distribution mod a – the Weyl criterion . . . 51
1.7 The van der Corput principle . . . 55

2 Spectral representation of weakly stationary processes 61
2.1 Weakly stationary processes . . . 61
2.2 Spectral representation of unitary operators . . . 64
2.3 Elements of stochastic integration . . . 76
2.4 Spectral representation of weakly stationary processes . . . 78
2.5 Weakly stationary sequences and orthogonal series . . . 80
2.6 Gaposhkin's spectral criterion . . . 85

Part II Ergodic Theorems 91

3 Dynamical systems – ergodicity and mixing 93
3.1 Measurable dynamical systems – topological dynamical systems . . . 93
3.2 Ergodicity of a dynamical system . . . 101
3.3 Weak mixing, strong mixing, continuous spectrum . . . 103
3.4 Spectral mixing theorem . . . 110
3.5 Other equivalences and other forms of mixing . . . 114
3.6 Examples . . . 121

4 Pointwise ergodic theorems 129
4.1 Birkhoff's pointwise theorem . . . 129
4.2 Dominated ergodic theorems . . . 139
4.3 Classes $L \log^m L$ . . . 144
4.4 A converse . . . 145

4.5 Speed of convergence . . . 148
4.6 Oscillation functions of ergodic averages . . . 152
4.7 Wiener–Wintner theorem . . . 165
4.8 Weighted ergodic averages . . . 168
4.9 Subsequence averages . . . 193

5 Banach principle and continuity principle 200
5.1 Banach principle . . . 200
5.2 Continuity principle . . . 206
5.3 Applications . . . 217
5.4 A principle of domination – conjugacy lemma . . . 226

6 Maximal operators and Gaussian processes 230
6.1 Some liaison theorems . . . 230
6.2 Two preliminary lemmas . . . 242
6.3 Proof of Theorem 6.1.1 . . . 247
6.4 Proof of Theorem 6.1.6 . . . 249
6.5 The case L^p, $1 < p < 2$ . . . 254
6.6 A remarkable GB set property . . . 259

7 The central limit theorem for dynamical systems 267
7.1 Introduction and preliminaries . . . 267
7.2 A theorem of Burton and Denker . . . 269
7.3 The central limit theorem for orbits . . . 284
7.4 A theorem of Volný . . . 289
7.5 CLT for rotations . . . 291
7.6 Lacunary series and convergence in variation . . . 315

Part III Methods arising from the theory of stochastic processes 339

8 The metric entropy method 341
8.1 Introduction and general results . . . 341
8.2 A theorem of Stechkin . . . 349
8.3 An application to the quantitative Borel–Cantelli lemma . . . 353
8.4 Application to Gál–Koksma's theorems . . . 364
8.5 An application to the supremum of random polynomials . . . 369
8.6 Application to a.s. convergence of weighted series of contractions . . . 387
8.7 An application to random perturbation of intersective sets . . . 403
8.8 An application to the discrepancy of some random sequences . . . 409
8.9 An application to random Dirichlet polynomials . . . 415

9 The majorizing measure method 433
9.1 Introduction – the exponential case . . . 433

9.2 A general approach . . . 438
9.3 A useful criterion . . . 447
9.4 Proof of Theorem 9.3.3 . . . 457
9.5 Proof of Theorems 9.3.10 and 9.3.11 . . . 469
9.6 Proof of Theorem 9.3.12 and some examples . . . 471
9.7 A stronger form of Salem–Zygmund's estimate . . . 475
9.8 Some examples and discussion . . . 478
9.9 Uniform convergence of random Fourier series . . . 488

10 Gaussian processes 491
10.1 Gaussian variables and correlation estimates . . . 491
10.2 0-1 laws, integrability and comparison lemmas . . . 504
10.3 Regularity and irregularity of Gaussian processes . . . 510
10.4 Gaussian suprema . . . 517
10.5 Oscillations of Gaussian Stein's elements . . . 529
10.6 Tightness of Gaussian Stein's elements . . . 537

Part IV Three studies 547

11 Riemann sums 549
11.1 Introduction . . . 549
11.2 The results of Jessen and Rudin . . . 551
11.3 Individual theorems of spectral type . . . 554
11.4 Breadth and dimension . . . 557
11.5 Bourgain's results . . . 562
11.6 Connection with number theory . . . 565
11.7 Riemann sums and the randomly sampled trigonometric system . . . 573
11.8 Almost sure convergence and square functions of Riemann sums . . . 587

12 A study of the system $(f(nx))$ 601
12.1 Introduction and mean convergence . . . 601
12.2 Almost sure convergence – sufficient conditions . . . 611
12.3 Almost sure convergence – necessary conditions . . . 634
12.4 Random sequences . . . 642

13 Divisors and random walks 659
13.1 Introduction . . . 659
13.2 Value distribution and small divisors of Bernoulli sums . . . 661
13.3 An LIL for arithmetic functions . . . 675
13.4 On the order of magnitude of the divisor functions . . . 685
13.5 Value distribution of the divisors of $n^2 + 1$ . . . 691
13.6 Value distribution of the divisors of Rademacher sums . . . 699
13.7 The functional equation and the Lindelöf Hypothesis . . . 701

13.8 An extremal divisor case 711

Bibliography 729

Index 759