Preface

The field of dynamical systems originated in the difficult mathematical questions related to movements of the planets and the moon, questions like: Are there periodic orbits? Will the solar system keep its present beautiful form, also in the distant future, or could it happen that one of the planets, Jupiter for instance, leaves the system? Or could it come to a collision between planets, leading to a dramatic change of the solar system?

The mathematical theory of dynamical systems provides concepts, ideas and tools, in order to analyze and model dynamical processes in all fields of natural sciences, making use of nearly all branches of mathematics. On the other hand, already in the past, questions of dynamical systems in the real world have triggered new mathematical developments and led to whole new branches of mathematics. Here, typical questions would be: Knowing its present state, how will a dynamical system develop in the long run? Will it, for example, tend to an equilibrium state or will it come back to itself? What will happen to the long-time behavior if we change the initial conditions a little bit? And what will happen to the whole orbit structure of a system if we perturb the system itself?

The book addresses readers familiar with standard undergraduate mathematics. It is not a systematic monograph, but rather the lecture notes of an introductory course in the field of dynamical systems given in the academic year 2004/2005 at the ETH in Zürich for third year students in mathematics and physics. I selected relatively few topics, tried to keep the requirements of mathematical techniques minimal and provided detailed (sometimes excruciatingly detailed) proofs.

The introductory chapter discusses simple models of discrete dynamical systems, in which the dynamics is determined by the iteration of a map. There are examples for minimal, transitive, structurally stable and ergodic systems. Mappings that preserve the measure of a finite measure space have strong recurrence properties in view of a classical result due to H. Poincaré. In order to describe the statistical distribution of their orbits, the ergodic theorem of G. Birkhoff is proved.

Chapters II and III are devoted to unstable phenomena caused by a hyperbolic fixed point of a diffeomorphism. Such a point gives rise to two global invariant sets, the so-called stable, respectively unstable, invariant manifolds issuing from fixed point. These consist of points which tend to the fixed point under the iteration of the map and under the iteration of the inverse map, respectively. The transversal intersection of the stable and unstable manifolds in the so-called homoclinic points is one of the roads to chaos. The existence of homoclinic points, discovered by H. Poincaré in the 3-body problem of celestial mechanics, complicates the orbit structure considerably and gives rise to invariant hyperbolic sets. The chaotic structure of the orbits near such sets is analyzed by means of the Shadowing Lemma, which is also used to demonstrate S. Smale's theorem about the embeddings of Bernoulli systems near homoclinic orbits. The interpretation of unpredictable orbits, which are determined by random sequences, will be demonstrated in the simple system of a periodically perturbed mathematical pendulum.

In Chapter IV we deal with smooth flows generated by vector fields and with continuous flows on metric spaces. The concepts of limit set, attractor and Lyapunov function are introduced. The bounded solutions of a gradient-like flow tend to rest points in forward and in backward time. The rest points are found by mini-max principles. The intimate and fruitful relation between the dynamics of gradient flows and the topology of the underlying compact manifold is described by the Morse inequalities. The Morse theory is sketched at the end of the chapter.

Chapter V introduces the special class of Hamiltonian vector fields that are determined by a single function and defined on symplectic manifolds. These manifolds are even-dimensional and carry a symplectic structure. A symplectic structure is a 2-form that is closed and nondegenerate. In contrast to Riemannian structures which do exist on every manifold, not every even-dimensional manifold admits a symplectic structure. Symplectic manifolds of the same dimension are locally indistinguishable (Darboux). There are no local symplectic invariants. The Hamiltonian formalism is developed in the convenient language of the exterior calculus which will be briefly introduced. The very special integrable Hamiltonian systems are characterized by the property that they possess sufficiently many integrals of motion, so that the task of solving the Hamiltonian equations for all time becomes almost trivial. This will follow from the existence of action- and angle-variables established by V. Arnold and R. Jost.

Chapter VI motivates the study of global symplectic invariants different from the volume which will be introduced in Chapter VII. They are called symplectic capacities and go back to I. Ekeland and H. Hofer. In view of their monotonicity properties they represent, in particular, obstructions to symplectic embeddings. The Gromov non-squeezing phenomenon is an immediate consequence. A symplectic capacity of dynamical nature (Hofer–Zehnder capacity) measures the minimal oscillations of Hamiltonian functions needed to conclude the existence of a fast periodic solution of the associated Hamiltonian vector field. Its construction is based on a variational principle for the action functional of classical mechanics. The tools from the calculus of variations are developed from scratch.

Chapter VIII deals with applications of dynamical symplectic capacity to Hamiltonian systems. It turns out that a compact and regular energy surface gives rise to an abundance of periodic orbits nearby, if a neighborhood of the surface possesses a finite Hofer–Zehnder capacity. The existence of a periodic solution on the given energy surface (and not only nearby) necessarily requires additional properties of the surface. The contact type property, for example, immediately leads to the solution of the Weinstein conjecture due to C. Viterbo which generalizes the pioneering results of P. Rabinowitz and A. Weinstein. Finally, a classical result of H. Poincaré shows that a periodic solution of a Hamiltonian system is, in general, not isolated but belongs to a smooth family of periodic solutions parametrized by the energy and having similar periods.

The chapters devoted to Hamiltonian systems and their global periodic orbits related to the symplectic capacities rely heavily on the book *Symplectic Invariants and Hamiltonian Dynamics* [52] by H. Hofer and E. Zehnder and on the *Notes in Dynamical Systems* [74] by J. Moser and E. Zehnder.

Each chapter begins with a short survey of its contents and ends with a brief selection of references to literature giving an alternative view on the subjects or describing related and more advanced topics.

I would like to thank Marcel Nutz for setting the German text into LaTeX, for many improvements and for producing the beautiful figures. I also would like to thank Manfred Karbe and Irene Zimmermann for their encouragement, their patience, their careful editing and for many helpful suggestions. For his generous help in preparing and improving the text of these lectures, I am very grateful to Felix Schlenk. I thank Beata and Kris Wysocki for their patient assistance with technical problems. For her translation of the German text into English, I am very grateful to Jeannette Zehnder.

Zürich, March, 2010