1 Introduction

In this chapter[1], we give a brief historical introduction to the $\bar{\partial}$-Neumann problem, combined with an outline of the organization of this monograph.

The $\bar{\partial}$-Neumann problem was formulated in the fifties by D. C. Spencer as a means to generalize the theory of harmonic integrals, i.e., Hodge theory, to non-compact complex manifolds. Apart from antecedents (such as [120], [146], [147]), the only written record of this introduction appears to be a set of notes [279] of lectures given at the Collège de France in 1955 ([199], p. 19). But then 'the early work on the $\bar{\partial}$-Neumann problem owes much more to D. C. Spencer than is documented in print' ([173], p. 330). For domains in $\mathbb{C}^n$, which is the context we restrict ourselves to in this monograph, the problem can be formulated as follows. Denote by Ω a pseudoconvex domain in $\mathbb{C}^n$, and by $\mathcal{L}^2_{(0,q)}(\Omega)$ the space of $(0,q)$-forms on Ω with square integrable coefficients. Each such form u can be written uniquely as a sum

$$u = {\sum_J}' u_J \, d\bar{z}_J, \tag{1.1}$$

where $J = (j_1, \ldots, j_q)$ is a multi-index with $j_1 < j_2 < \cdots < j_q$, $d\bar{z}_J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, and the $'$ indicates summation over increasing multi-indices. The inner product

$$(u, v) = \Big({\sum_J}' u_J \, d\bar{z}_J, {\sum_J}' v_J \, d\bar{z}_J \Big) = {\sum_J}' \int_\Omega u_J \overline{v_J} \, dV \tag{1.2}$$

turns $\mathcal{L}^2_{(0,q)}(\Omega)$ into a Hilbert space. Set

$$\bar{\partial}\Big({\sum_J}' u_J \, d\bar{z}_J \Big) = \sum_{j=1}^{n} {\sum_J}' \frac{\partial u_J}{\partial \bar{z}_j} d\bar{z}_j \wedge d\bar{z}_J, \tag{1.3}$$

where the derivatives are computed as distributions, and the domain of $\bar{\partial}$ is defined to consist of those $u \in \mathcal{L}^2_{(0,q)}(\Omega)$ where the result is a $(0, q+1)$-form with square integrable coefficients. Then $\bar{\partial} = \bar{\partial}_q$ is a closed, densely defined operator from $\mathcal{L}^2_{(0,q)}(\Omega)$ to $\mathcal{L}^2_{(0,q+1)}(\Omega)$, and as such has a Hilbert space adjoint. This adjoint is denoted by $\bar{\partial}^*_q$. (We will not use the subscripts when the form level at which the operators act is clear or not an issue.) One can check that $\bar{\partial}\bar{\partial} = 0$, so that we arrive at a complex, the $\bar{\partial}$ (or Dolbeault)-complex:

$$\mathcal{L}^2(\Omega) \xrightarrow{\bar{\partial}} \mathcal{L}^2_{(0,1)}(\Omega) \xrightarrow{\bar{\partial}} \mathcal{L}^2_{(0,2)}(\Omega) \xrightarrow{\bar{\partial}} \cdots \xrightarrow{\bar{\partial}} \mathcal{L}^2_{(0,n)}(\Omega) \xrightarrow{\bar{\partial}} 0.$$

In analogy to the Laplace–Beltrami operator associated to the de Rham complex on a Riemannian manifold, one forms the complex Laplacian

$$\Box_q = \bar{\partial}_{q-1}\bar{\partial}^*_{q-1} + \bar{\partial}^*_q \bar{\partial}_q, \tag{1.4}$$

[1]This chapter is a modified and expanded version of the introduction to my survey [286]. I am grateful to K. Diederich and J. Kohn for comments regarding that introduction.

with domain so that the compositions are defined. Alternatively, $\Box_q$ can be defined as the (unique) self adjoint operator on $\mathcal{L}^2_{(0,q)}(\Omega)$ associated to the quadratic form $Q(u,u) = \|\bar{\partial}u\|^2 + \|\bar{\partial}^*u\|^2$. The $\bar{\partial}$-Neumann problem is the problem of inverting $\Box_q$; that is, given $v \in \mathcal{L}^2_{(0,q)}(\Omega)$, find $u \in \mathrm{Dom}(\Box_q)$ such that $\Box_q u = v$. Note that $\mathrm{Dom}(\Box_q)$ involves the two boundary conditions $u \in \mathrm{Dom}(\bar{\partial}^*)$ and $\bar{\partial}u \in \mathrm{Dom}(\bar{\partial}^*)$; these are the $\bar{\partial}$-Neumann boundary conditions. The condition $u \in \mathrm{Dom}(\bar{\partial}^*)$ is equivalent to a Dirichlet condition for the (complex) normal component of u. Similarly, the condition $\bar{\partial}u \in \mathrm{Dom}(\bar{\partial}^*)$ is equivalent to a Dirichlet condition on the normal component of $\bar{\partial}u$, that is, a complex Neumann condition for u. The two conditions together are referred to as the $\bar{\partial}$-Neumann boundary conditions.

From the point of view of partial differential equations, the $\bar{\partial}$-Neumann problem represents the prototype of a problem where the operator is elliptic, but the boundary conditions are not coercive (so that the classical elliptic theory does not apply). From the point of view of several complex variables, the importance of the problem stems from the fact that its solution provides a Hodge decomposition in the context of the $\bar{\partial}$-complex, together with the attendant elegant machinery (as envisioned by Spencer). For example, such a decomposition readily produces a solution to the inhomogeneous $\bar{\partial}$ equation, as follows. Assume for the moment that $\Box_q$ has a (bounded) inverse in $\mathcal{L}^2_{(0,q)}(\Omega)$, say N_q. Then we have the orthogonal decomposition

$$u = \bar{\partial}\bar{\partial}^* N_q u + \bar{\partial}^*\bar{\partial} N_q u, \quad u \in L^2_{(0,q)}(\Omega). \tag{1.5}$$

If $\bar{\partial}u = 0$, then $\bar{\partial}^*\bar{\partial}N_q u$ is $\bar{\partial}$-closed as well (from (1.5)). Consequently, $\bar{\partial}^*\bar{\partial}N_q u = 0$ (since it is also orthogonal to $\ker(\bar{\partial})$), and

$$u = \bar{\partial}(\bar{\partial}^* N_q u), \tag{1.6}$$

with $\|\bar{\partial}^* N_q u\|^2 = (\bar{\partial}\bar{\partial}^* N_q u, N_q u) \leq C\|u\|^2$. Thus the operator $\bar{\partial}^* N_q$ provides an $\mathcal{L}^2$-bounded solution operator to $\bar{\partial}$. In fact, this operator gives the (unique) solution orthogonal to $\ker(\bar{\partial})$ (equivalently: the solution with minimal norm). This solution is called the canonical solution.

That $\Box_q$ *does* have a bounded inverse N_q was known for strictly pseudoconvex domains by the early 1960s. Kohn ([187], [189], [188], [190]), starting from his generalization of an estimate discovered by Morrey ([228]), showed that in this case, not only is there an $\mathcal{L}^2$-bounded inverse, but N_q exhibits a subelliptic gain of one derivative as measured in the $\mathcal{L}^2$-Sobolev scale. Another interesting approach was given by the second author in [229], [230]. Shortly after Kohn's work, Hörmander ([170], [172], see also Andreotti–Vesentini [2] for similar techniques), combining ideas from [228], [188], and [4] with the use of weighted norms, proved certain Carleman type estimates which in the case of bounded pseudoconvex domains imply the existence of N_q as a bounded self-adjoint operator on $\mathcal{L}^2_{(0,q)}(\Omega)$. The weights are such that these techniques are also applicable in the $\mathcal{L}^2_{\mathrm{loc}}(\Omega)$-category. Interior elliptic regularity, applied to the weighted canonical solution, then gives (a new proof of) solvability of $\bar{\partial}$ in $C^\infty(\Omega)$ as well[2]. Other early applications of the new ideas included the real analytic embedding of compact real analytic manifolds ([228], [229], [230])[3], a new solution ([188], [170]) of the Levi problem[4], a new proof ([188]) of the Newlander–Nirenberg theorem on integrable almost complex structures ([234]), and, in general, an approach to several complex variables which takes advantage of $\bar{\partial}$-methods ([170], [172], [125], [243]). Interesting 'eyewitness' accounts of this foundational period by two of the principals appear in [173] and [199], respectively.

We reverse the historical order and discuss the $\mathcal{L}^2$-results on general pseudoconvex domains in Chapter 2 and the subelliptic estimates on strictly pseudoconvex domains in Chapter 3. The Carleman type estimates of Hörmander and Andreotti–Vesentini arise from considering the $\bar{\partial}$-complex in weighted $\mathcal{L}^2$-norms. This point of view results in a very useful extra term in the so called Kohn–Morrey formula (the basis for the results in the strictly pseudoconvex case) when the weights have certain plurisubharmonicity properties. A little over twenty years later, Ohsawa and Takegoshi ([244]) discovered that also introducing a 'twisting' factor into the $\bar{\partial}$-complex results in yet another additional new term which allows to compensate, in certain situations, for the lack of plurisubharmonicity in the weights. Their work was simplified and extended in the mid nineties by Berndtsson ([35]), McNeal ([220]), and Siu ([275]). Boas and the author then noted ([52]) that it is advantageous to base the $\mathcal{L}^2$-existence theory on the resulting 'twisted' version of the Kohn–Morrey–Hörmander formula as well. We take this approach in Chapter 2. The chapter closes with an application to extension, with $\mathcal{L}^2$-bounds, of holomorphic functions from affine submanifolds: we prove (the most basic version of) the Ohsawa–Takegoshi extension theorem.

It is not hard to see that Kohn's results for strictly pseudoconvex domains are optimal: N can never gain more than one derivative, and it can gain one derivative only when the domain is strictly pseudoconvex. However, under what circumstances subellipticity with a fractional gain of less than one derivative holds was not understood until the early eighties. Kohn gave sufficient conditions in [193] and noted that work of Diederich and Fornæss ([107]) implies that these conditions are satisfied when the boundary is real analytic. Kohn's students Catlin ([66], [67], [70]) and D'Angelo ([87], [88], [89]) resolved the issue: on a smooth bounded pseudoconvex domain in $\mathbb{C}^n$, the

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[2] For domains of holomorphy, this existence theorem was obtained in the early fifties via sheaf theoretic methods (Cartan [61], Serre [267], Dolbeault [118]). The solution of the Levi problem (the fact that pseudoconvex domains are domains of holomorphy, see below), also accomplished by the early fifties, then implies solvability on pseudoconvex domains. The remarks following the proof of Theorem 2.14 contain further details.

[3] Shortly after [228] was circulated, this result was generalized by Grauert ([149]), using sheaf theoretic methods, to manifolds with countable topology. Moreover, [228] contains a gap (fixed by the author in [229], [230], see also [189], [188], [170]) related to density in the graph norm of forms smooth up to the boundary (see Proposition 2.3; for historical details, see [173]). Our discussion shows that nevertheless, [228] was quite influential.

[4] The Levi problem, that is, the question whether pseudoconvex domains are domains of holomorphy, was one of the main problems in several complex variables during the first half of the last century. It was solved in the affirmative independently by Bremermann ([58]), Norguet ([238]), and Oka ([246], [247]). See Remarks (ii) and (iii) following the proof of Theorem 2.16 and Remark (i) following the proof of Theorem 3.7 for details and references, both historical and mathematical.

$\bar{\partial}$-Neumann problem is subelliptic if and only if each boundary point is of finite type, that is, the order of contact, at the point, of complex varieties with the boundary is bounded above. This elegant characterization not withstanding, the question of how to determine the exact range of subellipticity remains open. One of the main tools in [193] is provided by the notion of subelliptic multipliers and their ideals. Although subellipticity is established in [70] by a different method, these ideas raised algebro-geometric questions of independent interest (see [236], [278] for recent work), and they turned out to be influential in later developments in complex and algebraic geometry ([232], [99], [276], [277], [198]).

We take up the subelliptic estimates for strictly pseudoconvex domains in Chapter 3. Because $\Box_q$ acts coefficientwise as (a constant multiple of) the real Laplacian, and in view of the boundary term in the Kohn–Morrey formula, the estimates at the ground level are a consequence of the corresponding Sobolev estimates in the Dirichlet problem for the real Laplacian. Lifting these estimates to higher Sobolev norms leads to a situation common in partial differential equations: one can prove certain Sobolev estimates, *assuming* that the Sobolev norms in question are finite (because one has to absorb these norms). But that these norms are finite is precisely what one wants to prove. The classical method to deal with this problem consists in obtaining uniform estimates for difference quotients (which *are* in $\mathcal{L}^2$ if the function/form is), and then letting the difference parameter tend to zero, rather than estimating derivatives directly. A method better suited for the $\bar{\partial}$-Neumann problem is elliptic regularization, developed in the context of operators defined by certain quadratic forms by Kohn and Nirenberg in the mid sixties ([201]). We give a careful discussion of the method in the case of the $\bar{\partial}$-Neumann problem and prove in particular that the regularized operators do have the claimed elliptic properties. Thus Chapter 3 gives an essentially self-contained proof of the subelliptic estimates in the strictly pseudoconvex case. By contrast, the general subelliptic estimates on domains of finite type are only briefly described, and subelliptic multipliers are omitted entirely; a detailed treatment of each of these topics would warrant a monograph in its own right.

When N_q does not gain derivatives, but is still compact (as an operator on $\mathcal{L}^2_{(0,q)}(\Omega)$), it follows from the already quoted work of Kohn and Nirenberg ([201]) in the mid sixties that N_q preserves the Sobolev spaces $W^s_{(0,q)}(\Omega)$ for all $s \geq 0$. In particular, N_q preserves $C^\infty_{(0,q)}(\overline{\Omega})$ (it is globally regular). These two authors did not, however, investigate when the compactness condition is actually satisfied. But work of Catlin ([68], compare also Takegoshi [293]) and Sibony ([272]) in the eighties shows that compactness provides indeed a viable route to global regularity: the compactness condition can be verified on large classes of domains. This verification was achieved via a potential theoretic condition called property(P), introduced and shown to imply compactness in [68]. In [272], property(P) is studied in detail using tools from Choquet theory. One striking result is that even when the set of boundary points of infinite type is large, for example has positive measure, property(P), and hence compactness, may still hold. We prove these results in Chapter 4.

The most blatant violation of property(P) is an analytic disc in the boundary. The obvious question whether such a disc is necessarily an obstruction to compactness received an affirmative answer quickly for the case of domains in $\mathbb{C}^2$ (commonly attributed to unpublished work of Catlin which became folklore). When the domain is in $\mathbb{C}^n$ with $n \geq 3$, the answer is not known. Şahutoğlu and the author recently obtained a partial answer which does generalize the $\mathbb{C}^2$ result to higher dimensions ([265]): when the disc contains a point at which the domain is strictly pseudoconvex in the directions transverse to the disc (a condition void in $\mathbb{C}^2$), then this disc is indeed an obstruction to compactness. On the other hand, Christ's student Matheos was able to show in his dissertation ([217]) in the mid nineties that there are obstructions to compactness more subtle than discs in the boundary. Shortly afterwards, Fu and the author discovered that there is however a large class of domains where the analysis, the potential theory, and the geometry mesh perfectly ([141]). They proved that on a locally convexifiable domain, the following three conditions are equivalent: the $\bar{\partial}$-Neumann operator is compact; the boundary satisfies property(P); the boundary contains no analytic discs.

According to a recent result of Christ and Fu ([86]), compactness and property(P) are equivalent also on smooth bounded pseudoconvex Hartogs domains in $\mathbb{C}^2$. (By what was said above, on these domains, they imply the absence of analytic discs from the boundary, but are not equivalent to it.) In general, however, it is not understood how much room there is between compactness and property(P). In fact, until about five years ago, the only way to obtain compactness of the $\bar{\partial}$-Neumann operator was via verifying P. Then, in [285], [231], the authors developed a new method for verifying compactness in certain cases. But while the method does not proceed via property(P), it is not clear whether among the domains where it applies, there are ones without property(P). A little earlier, McNeal had introduced a modification of property(P) that is formally weaker and that still implies compactness ([221]). To what extent it is strictly weaker is not understood at present. These results (with the exception of [86]) are also proved in Chapter 4.

Studying regularity properties of a differential operator is natural from a partial differential equations perspective. When the $\bar{\partial}$-Neumann operator N_q is globally regular, that is, when it preserves $C^{\infty}_{(0,q)}(\overline{\Omega})$, the canonical solution operator $\bar{\partial}^* N_q$ gives a solution operator for $\bar{\partial}$ which preserves $C^{\infty}(\overline{\Omega})$ as well. There are other important implications of global regularity for several complex variables; chief among these is the relevance for boundary behavior of biholomorphic or proper holomorphic maps. Work of Bell, Catlin, Diederich, Fornæss, and Ligocka ([32], [23], [30], [108]) shows that if Ω_1 and Ω_2 are two bounded pseudoconvex domains in $\mathbb{C}^n$ with smooth boundaries, such that the $\bar{\partial}$-Neumann operator on $(0,1)$-forms on Ω_1 is globally regular[5], then any proper holomorphic map from Ω_1 to Ω_2 extends smoothly to the boundary of Ω_1.

[5]Actually, the regularity property that is needed in these results is global regularity of the Bergman projection, which in the pseudoconvex case is a consequence of global regularity of the $\bar{\partial}$-Neumann operator (more can be said; see Theorem 5.5 for the precise relationship). [32], which deals with biholomorphic maps, does not require pseudoconvexity; for a generalization in the case of proper maps to the nonpseudoconvex setting, see [25].

This result represents a vast generalization (as well as a simplification) of the celebrated mapping theorem of Fefferman ([123]), which covered strictly pseudoconvex domains and biholomorphic maps. It is highly nontrivial: in contrast to the one variable situation, where at least the biholomorphic case is classical (going as far back as Painlevé ([249], [250]; see [31] for the history), the general case in higher dimensions, even for biholomorphic maps, is open. Further exposition of the ideas and issues involved here can be found in [113], [16], [90], [31], [27], [206], [131], [81], [139].

In the early seventies, Kohn ([191]) noticed that by choosing suitable weights in Hörmander's method, one can obtain a weighted $\bar{\partial}$-Neumann operator that is continuous in Sobolev norms up to a certain level. More precisely, for every $k \in \mathbb{N}$, there is a weight so that the weighted $\bar{\partial}$-Neumann operator is continuous on $W^s_{(0,q)}(\Omega)$ for $0 \leq s \leq k$. The associated canonical solution operator is also continuous in $W^s(\Omega)$, $0 \leq s \leq k$. When combined with a Mittag-Leffler argument (credited in [192] to Hörmander), these solution operators yield a solution of $\bar{\partial}$ in the $C^\infty(\overline{\Omega})$-category on any smooth bounded pseudoconvex domain. The weighted theory also allows one to determine the exact relationship, discovered in the late eighties ([47]), between regularity properties of the $\bar{\partial}$-Neumann operators and those of the Bergman projections. Chapter 5 begins with these results.

Global regularity may hold when compactness fails. Throughout the eighties and into the early nineties, there appeared a series of results on global regularity that concerned domains with transverse symmetries ([28], [8], [10], [281], [79]), domains with partially transverse symmetries that allow the normal to be well approximated by holomorphic fields on the rest of the boundary ([46], [49]), or that combined these techniques on a portion of the boundary with subellipticity or compactness arguments on the rest of the boundary ([78], [44]). These methods apply in particular to Reinhardt domains and to many circular domains.

In the early nineties, Boas and the author proved in [48] that if Ω admits a defining function whose complex Hessian is positive semi-definite at points of the boundary (a condition slightly more restrictive than pseudoconvexity), then the $\bar{\partial}$-Neumann problem is globally regular (for all q). This class of domains includes in particular all (smooth) convex domains. (For convex domains in $\mathbb{C}^2$, the result was obtained independently by Chen, [80].) The proof is based on the existence of certain families of vector fields which have good approximate commutator properties with $\bar{\partial}$ (different treatments were given recently in [169], [287]). This method also covers (and was inspired by) the results mentioned earlier based on transverse symmetries and holomorphic vector fields. When computing the relevant commutators, there is a one-form, introduced into the literature by D'Angelo ([91], [92]), that comes up naturally. The existence of the required families of vector fields is equivalent to this one form being 'approximately exact'. From this point of view, the case when the domain admits a defining function that is plurisubharmonic at the boundary is the 'trivial' case: the form vanishes (in the directions that matter), so it is trivially approximately exact.

The same authors then studied the situation when the boundary points of infinite type form a complex submanifold (with boundary) of the boundary of the domain

([51]). The form mentioned in the previous paragraph defines a de Rham cohomology class on such a submanifold. This cohomology class is the obstruction to the existence of the vector fields needed. In particular, a simply connected complex manifold in the boundary is benign for global regularity of the $\bar{\partial}$-Neumann problem. The obvious question whether the cohomology class is also necessarily an obstruction to global regularity is still open. It is noteworthy that this class also plays a role in deciding whether or not the closure of the domain admits a Stein neighborhood basis; in this role, it had appeared already in the late seventies in work of Bedford and Fornæss ([17]).

A natural next step was taken in [289], where the authors considered the case where the boundary is finite type except for a Levi-flat piece which is 'nicely' foliated by complex hypersurfaces. Whether or not the families of vector fields with good approximate commutator properties with $\bar{\partial}$ exist turns out to be equivalent to a property of the Levi foliation much studied in foliation theory, namely whether or not the foliation can be defined *globally* by a closed one-form. These connections, while of interest in their own right, also allow one to bring tools from foliation theory to bear on the problem of finding the required families of vector fields and thus obtaining global regularity of the $\bar{\partial}$-Neumann operator ([289], [132]).

The question how to unify the two main approaches to global regularity, via compactness or via vector fields with good commutator properties with $\bar{\partial}$, arose as soon as [48], [51] were completed. Chapter 5 closes with a recent result of the author that proposes such a unified treatment of global regularity.

So far we have only discussed positive results. Whether global regularity holds on general pseudoconvex domains turned out to be a very difficult question that was resolved only in the mid nineties. Barrett ([11], see also [9] and [183] for predecessors) showed that on the worm domains of Diederich and Fornæss ([105]), N_1 does not preserve $W^s_{(0,1)}(\Omega)$ for s sufficiently large, depending on the winding (that is, exact regularity fails). Christ ([83], see also [84], [85]) resolved the question by proving certain a priori estimates for N_1 on these domains that would imply exact regularity in Sobolev spaces (and thus would contradict Barrett's result) if N_1 were to preserve the space of forms smooth up to the boundary. While worm domains are discussed in Chapter 5, the reader is referred to the original sources for the proofs of the Barrett-Christ results.

In addition to these proofs, and a detailed treatment of subelliptic estimates already mentioned, there are other important topics in, or closely related to, the $\mathcal{L}^2$-Sobolev theory on bounded domains in $\mathbb{C}^n$ that are not treated in this monograph. The spectral theory of the $\bar{\partial}$-Neumann operator studies connections between the spectrum and the boundary geometry; compare [137], [138], [140] and the references there. One can also ask what happens with regard to Sobolev estimates when the domain is not assumed to be C^∞-smooth. For results on Lipschitz domains, we refer the reader to Shaw's survey [270]. It is furthermore useful to consider (a version of) the $\bar{\partial}$-Neumann problem on nonpseudoconvex domains. When $q > 1$, an assumption weaker than pseudoconvexity suffices to make the $\mathcal{L}^2$-Hilbert space machine run, starting from the (twisted) Kohn–Morrey–Hörmander formula. While we do discuss this case, there are variants of the

Kohn–Morrey–Hörmander formula still more general that we do not include. In all these cases, results typically hold for a restricted set of form levels q, the restrictions depending on properties of the boundary related to pseudoconvexity (see for example [171], [125], [268], [305], [169], [1], [255], [271]). Another topic closely related to the subject of this monograph, but not treated here, is the $\mathcal{L}^2$-Sobolev theory of the boundary complex, or, more generally, of the $\bar{\partial}_b$-complex on CR-manifolds ([269], [45], [195], [81], [197], [185], [235], [200], [255], [254]). Finally, we mention recent activity towards creating an $\mathcal{L}^2$-theory of $\bar{\partial}$ on singular complex spaces ([109], [129], [248], [262] and their references).