

# Introduction

Topological quantum field theories (TQFTs) produce topological invariants of manifolds using ideas suggested by quantum field theory; see [At], [Wi1]. For  $d \geq 0$ , a  $(d + 1)$ -dimensional TQFT over a commutative ring  $K$  assigns to every closed oriented  $d$ -dimensional manifold  $M$  a projective  $K$ -module of finite type  $A_M$  and assigns to every compact oriented  $(d + 1)$ -dimensional cobordism  $(W, M_0, M_1)$  a  $K$ -homomorphism  $\tau(W): A_{M_0} \rightarrow A_{M_1}$ . These modules and homomorphisms should satisfy several axioms including tensor multiplicativity with respect to disjoint union and functoriality with respect to gluing of cobordisms.

The study of TQFTs has been especially successful in low dimensions  $d = 0, 1, 2, 3$ . One-dimensional TQFTs ( $d = 0$ ) bijectively correspond to projective  $K$ -modules of finite type. Two-dimensional TQFTs ( $d = 1$ ) are fully classified in terms of commutative Frobenius algebras, see [Di], [Ab], [Kock]. Three-dimensional TQFTs ( $d = 2$ ) are closely related to quantum groups and braided categories; see [RT], [Tu2], [KRT], [BK]. Powerful four-dimensional TQFTs ( $d = 3$ ) arise from the Heegaard–Floer homology of 3-manifolds due to P. Ozsváth and Z. Szabó; see [OS1], [OS2]. Algebraic structures underlying four-dimensional TQFTs are yet to be unraveled; for work in this direction see [CKY], [CKS], [CJKLS], [Ma], [Oe].

In this monograph we apply the idea of a TQFT to maps from manifolds to topological spaces. This leads us to a notion of a  $(d + 1)$ -dimensional homotopy quantum field theory (HQFT) which may be described as a TQFT for closed oriented  $d$ -dimensional manifolds and compact oriented  $(d + 1)$ -dimensional cobordisms endowed with maps to a given space  $X$ . Such an HQFT yields numerical homotopy invariants of maps from closed oriented  $(d + 1)$ -dimensional manifolds to  $X$ . A TQFT may be interpreted in this language as an HQFT with target space consisting of one point. The general notion of a  $(d + 1)$ -dimensional HQFT was introduced in 1999 in my unpublished preprint [Tu3] and independently by M. Brightwell and P. Turner [BT1] for  $d = 1$  and simply connected target spaces.

If the ground ring  $K$  is a field, then the  $(0 + 1)$ -dimensional HQFTs with target  $X$  correspond bijectively to finite-dimensional representations of the fundamental group of  $X$  or, equivalently, to finite-dimensional flat  $K$ -vector bundles over  $X$ . This allows one to view HQFTs as high-dimensional generalizations of flat vector bundles.

We shall mainly study the case where  $X = K(G, 1)$  is the Eilenberg–MacLane space corresponding to a (discrete) group  $G$ . A manifold endowed with a homotopy class of maps to such  $X$  will be called a  $G$ -manifold. The maps to  $X = K(G, 1)$  classify principal  $G$ -bundles, and numerical invariants of principal  $G$ -bundles over closed oriented  $(d + 1)$ -dimensional manifolds provided by HQFTs with target  $X$  can be viewed as “quantum” characteristic numbers. From this perspective, the stan-

dard Witten–Reshetikhin–Turaev quantum invariants of 3-manifolds can be regarded as quantum characteristic numbers of the trivial bundles over 3-manifolds.

The main goal of this monograph is a construction of  $(d + 1)$ -dimensional HQFTs with target  $K(G, 1)$  for  $d = 1, 2$ . We focus on algebraic structures underlying such HQFTs. For  $d = 1$ , these structures are formulated in terms of  $G$ -graded algebras or, briefly,  $G$ -algebras. A  $G$ -algebra is an associative unital algebra  $L$  endowed with a decomposition  $L = \bigoplus_{\alpha \in G} L_{\alpha}$  such that  $L_{\alpha}L_{\beta} \subset L_{\alpha\beta}$  for any  $\alpha, \beta \in G$ . The  $G$ -algebras arising from 2-dimensional HQFTs have additional features including a natural inner product and an action of  $G$ . This leads us to a notion of a crossed Frobenius  $G$ -algebra. Our main result concerning 2-dimensional HQFTs with target  $K(G, 1)$  is a bijective correspondence between the isomorphism classes of such HQFTs and the isomorphism classes of crossed Frobenius  $G$ -algebras. This generalizes the standard equivalence between 2-dimensional TQFTs and commutative Frobenius algebras (the case  $G = 1$ ). Our second result is a classification of semisimple crossed Frobenius  $G$ -algebras in terms of 2-dimensional cohomology classes of the subgroups of  $G$  of finite index.

The study of 2-dimensional HQFTs has not yet brought to light new invariants of principal  $G$ -bundles over surfaces. The invariants arising from semisimple (crossed Frobenius)  $G$ -algebras are essentially homological. The invariants arising from non-semisimple  $G$ -algebras may in principle be new but are poorly understood. On the other hand, the study of 2-dimensional HQFTs finds interesting applications in certain enumeration problems. One of these problems concerns an arbitrary Serre fibration  $p: E \rightarrow W$  over a closed connected oriented surface  $W$  of positive genus. The bundle  $p$  may have sections, i.e., continuous mappings  $s: W \rightarrow E$  such that  $ps = \text{id}_W$  (we work in the pointed category so that  $E$  and  $W$  have base points preserved by  $p$  and  $s$ ). Assume that the fiber  $F$  of  $p$  is path-connected and the group  $\pi_1(F)$  is finite. Then the number of sections of  $p$ , considered up to homotopy and a natural action of  $\pi_2(F)$ , is finite (possibly zero). We give a formula expressing this number in terms of 2-dimensional cohomology classes associated with irreducible complex linear representations of  $\pi_1(F)$ . The definition of these cohomology classes and the statement of our formula are direct and do not involve HQFTs, while the proof heavily uses HQFTs. This yields, in particular, the following solution to the existence problem for sections:  $p$  has a section if and only if the number produced by our formula is non-zero. As a specific application, note the following theorem: in the case where the group  $\pi_1(F)$  is abelian (and finite), the bundle  $p: E \rightarrow W$  has a section if and only if the induced homomorphism  $p_*: H_2(E; \mathbb{Z}) \rightarrow H_2(W; \mathbb{Z})$  is surjective. A similar theorem holds for any finite group  $\pi_1(F)$  whose order is small with respect to the genus of  $W$ . Other topological applications concern principal bundles over  $W$  and non-abelian 1-cohomology of  $W$ . For group-theoretic applications (not discussed in the book), the reader is referred to [NT].

More generally, given a cohomology class  $\theta \in H^2(E; \mathbb{C}^*)$ , we can provide each section  $s$  of  $p$  with the weight  $\theta(s_*([W])) = s^*(\theta)([W]) \in \mathbb{C}^* = \mathbb{C} - \{0\}$ , where  $[W] \in H_2(W; \mathbb{Z})$  is the fundamental class of  $W$ . Counting sections of  $p$  with these

weights we obtain a complex number viewed as a  $\theta$ -weighted number of sections of  $p$ . We express this number in terms of 2-dimensional cohomology classes associated with irreducible complex projective representations of  $\pi_1(F)$ .

The enumeration problem for sections formulated above is equivalent to a special case of the following problem. Consider a group epimorphism  $G' \rightarrow G$  with finite kernel  $\Gamma$ . Consider a homomorphism  $g$  from the fundamental group of a closed connected oriented surface  $W$  to  $G$ . It is clear that  $g$  has only a finite number of lifts to  $G'$  (if any). How to compute this number? For the trivial homomorphism  $g = 1$ , a solution is given by the Frobenius–Mednykh formula; see [Fr], [Me]. We extend this formula to arbitrary  $g$ . Our formula computes the number of lifts of  $g$  in terms of 2-dimensional cohomology classes associated with irreducible complex linear representations of  $\Gamma$ . A more general formula counts the lifts of  $g$  with weights determined by an element of  $H^2(G'; \mathbb{C}^*)$ .

Our approach to 3-dimensional HQFTs is based on a connection between braided categories and knots. This connection is essential in the construction of topological invariants of knots and 3-manifolds from quantum groups. We extend this train of ideas to links  $\ell \subset S^3$  endowed with homomorphisms  $\pi_1(S^3 \setminus \ell) \rightarrow G$  and to 3-dimensional  $G$ -manifolds. To this end, we introduce crossed  $G$ -categories and study braidings and twists in such categories. This leads us to a notion of a modular crossed  $G$ -category.

We show that each modular crossed  $G$ -category gives rise to a three-dimensional HQFT with target  $K(G, 1)$ . This HQFT has two ingredients: a “homotopy modular functor”  $A$  assigning projective  $K$ -modules to  $G$ -surfaces and a functor  $\tau$  assigning  $K$ -homomorphisms to 3-dimensional  $G$ -cobordisms. In particular, the HQFT provides numerical invariants of closed oriented 3-dimensional  $G$ -manifolds. For  $G = 1$ , we recover the familiar construction of 3-dimensional TQFTs from modular categories; see [Tu2].

We discuss several algebraic methods producing crossed  $G$ -categories. In particular, we introduce quasitriangular Hopf  $G$ -coalgebras and show that they give rise to crossed  $G$ -categories. However, the problem of systematic finding of modular crossed  $G$ -categories is largely open.

This book is based on my papers [Tu3]–[Tu8]. Chapters I–IV cover [Tu3], though the exposition has been somewhat modified and Sections IV.1, IV.2 added. Chapter V covers [Tu5]–[Tu7]. Chapters VI, VII, and VIII cover [Tu4] and [Tu8]. These three chapters extend the first part of my monograph [Tu2]. Though techniques from [Tu2] are used in several proofs in Chapters VI and VII, the definitions and statements of theorems can be understood without knowledge of [Tu2]. The reader’s background is supposed only to include basics of algebra and topology and (starting from Chapter VI) basics of the theory of categories.

Here is a chapter-wise description of the book. In Chapter I we discuss a general setting of HQFTs. In particular, we show that  $(d + 1)$ -dimensional cohomology classes of the target space and of its finite-sheeted coverings give rise to  $(d + 1)$ -dimensional HQFTs. These HQFTs and their direct sums are called cohomological HQFTs. For  $d = 1$ , we introduce a wider class of semi-cohomological HQFTs.

In Chapter II we introduce and study  $G$ -algebras. We discuss various classes of  $G$ -algebras including Frobenius, crossed, semisimple, Hermitian, and unitary  $G$ -algebras. The main result of this chapter is a classification of semisimple crossed Frobenius  $G$ -algebras over a field of characteristic zero.

In Chapter III we associate with each 2-dimensional HQFT with target  $K(G, 1)$  an underlying crossed Frobenius  $G$ -algebra. This establishes a bijection between the isomorphism classes of 2-dimensional HQFTs with target  $K(G, 1)$  and the isomorphism classes of crossed Frobenius  $G$ -algebras. Under this bijection, the semi-cohomological HQFTs correspond to semisimple algebras at least when the ground ring is a field of characteristic zero. We also establish a Verlinde-type formula for the semi-cohomological HQFTs.

In Chapter IV we introduce biangular  $G$ -algebras. They give rise to lattice models for 2-dimensional HQFTs generalizing the lattice models for 2-dimensional TQFTs introduced in [BP], [FHK]. We prove that the lattice 2-dimensional HQFTs over algebraically closed fields of characteristic zero are semi-cohomological.

In Chapter V we discuss applications of HQFTs to enumeration problems. This chapter is almost entirely independent of the previous chapters except the proof of the main theorem given in Sections V.5 and V.6.

In Chapter VI we introduce crossed  $G$ -categories and various additional structures on them (braiding, twist, etc.). We also introduce  $G$ -links,  $G$ -tangles, and  $G$ -graphs in  $\mathbb{R}^3$  and define their colorings by objects and morphisms of a ribbon crossed  $G$ -category  $\mathcal{C}$ . The  $\mathcal{C}$ -colored  $G$ -graphs form a monoidal tensor category, and we define a canonical monoidal functor from this category to  $\mathcal{C}$ . The chapter ends with a study of dimensions of objects and traces of morphisms in  $\mathcal{C}$ .

In Chapter VII we introduce modular crossed  $G$ -categories. Each such category produces a 3-dimensional HQFT with target  $K(G, 1)$ .

In Chapter VIII we introduce Hopf  $G$ -algebras and discuss algebraic constructions of crossed  $G$ -categories and crossed  $G$ -algebras.

The book ends with seven Appendices. Appendix 1 is concerned with relative HQFTs generalizing the so-called open-closed TQFTs. In Appendix 2 we outline a state sum approach to invariants of 3-dimensional  $G$ -manifolds. In Appendix 3 we briefly discuss recent developments in the study of HQFTs and related areas. In Appendix 4 we formulate several open problems. Appendix 5 written by Michael Müger is concerned with his recent work on braided crossed  $G$ -categories. Appendices 6 and 7 written by Alexis Virelizier discuss algebraic properties of Hopf  $G$ -coalgebras and the 3-manifolds invariants derived from Hopf  $G$ -coalgebras.

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Throughout the book, the symbol  $K$  denotes a commutative ring with unit  $1_K$ . The multiplicative group of invertible elements of  $K$  is denoted  $K^*$ . The symbol  $G$  denotes a (discrete) group.