Introduction

Summary

Chapter 1 starts with elementary examples (§A), the first being the one that is depicted on the cover of the book of Kemeny and Snell [K-S]. This is followed by an informal description ("What is a Markov chain?", "The graph of a Markov chain") and then (§B) the axiomatic definition as well as the construction of the trajectory space as the standard model for a probability space on which a Markov chain can be defined. This quite immediate first impact of measure theory might be skipped at first reading or when teaching at an elementary level. After that we are back to basic transition probabilities and passage times (§C). In the last section (§D), the first encounter with generating functions takes place, and their basic properties are derived. There is also a short explanation of transition probabilities and the associated generating functions in purely combinatorial terms of paths and their weights.

Chapter 2 contains basic material regarding irreducible classes (§A) and periodicity (§B), interwoven with examples. It ends with a brief section (§C) on the spectral radius, which is the inverse of the radius of convergence of the Green function (the generating function of n-step transition probabilities).

Chapter 3 deals with recurrence vs. transience (§A & §B) and the fundamental convergence theorem for positive recurrent chains (§C & §E). In the study of positive recurrence and existence and uniqueness of stationary probability distributions (§B), a mild use of generating functions and de l'Hospital's rule as the most "difficult" tools turn out to be quite efficient. The convergence theorem for positive recurrent, aperiodic chains appears so important to me that I give two different proofs. The first (§C) applies primarily (but not only) to finite Markov chains and uses Doeblin's condition and the associated contraction coefficient. This is pure matrix analysis which leads to crucial probabilistic interpretations. In this context, one can understand the convergence theorem for finite Markov chains as a special case of the famous Perron–Frobenius theorem for non-negative matrices. Here (§D), I make an additional detour into matrix analysis by reversing this viewpoint: the convergence theorem is considered as a main first step towards the proof of the Perron–Frobenius theorem, which is then deduced. I do not claim that this proof is overall shorter than the typical one that one finds in books such as the one of Seneta [Se]; the main point is that I want to work out how one can proceed by extending the lines of thought of the preceding section. What follows (§E) is another, elegant and much more probabilistic proof of the convergence theorem for general positive recurrent, aperiodic Markov chains. It uses the coupling method, see Lindvall [Li]. In the original Italian text, I had instead presented the proof of the convergence theorem that is due to Erdös, Feller and Pollard [20], a breathtaking piece of "elementary" analysis of sequences; see e.g, [Se, §5.2]. It is certainly not obsolete, but I do not think I should have included a third proof here, too. The second important convergence theorem, namely, the ergodic theorem for Markov chains, is featured in §F. The chapter ends with a short section (§G) about ρ-recurrence.

Chapter 4. The chapter (most of whose material is not contained in [W1]) starts with the network interpretation of a reversible Markov chain (§A). Then (§B) the interplay between the spectrum of the transition matrix and the speed of convergence to equilibrium (= the stationary probability) for finite reversible chains is studied, with some specific emphasis on the special case of symmetric random walks on finite groups. This is followed by a very small introductory glimpse (§C) at the very impressive work on geometric eigenvalue bounds that has been promoted in the last two decades via the work of Diaconis, Saloff-Coste and others; see [SC] and the references therein, in particular, the basic paper by Diaconis and Stroock [15] on which the material here is based. Then I consider recurrence and transience criteria for infinite reversible chains, featuring in particular the *flow criterion* (§D). Some very basic knowledge of Hilbert spaces is required here. While being close to [W2, §2.B], the presentation is slightly different and "slower". The last section (§E) is about recurrence and transience of random walks on integer lattices. Those Markov chains are not always reversible, but I figured this was the best place to include that material, since it starts by applying the flow criterion to symmetric random walks. It should be clear that this is just a very small set of examples from the huge world of random walks on lattices, where the classical source is Spitzer's famous book [Sp]; see also (for example) Révész [Ré], Lawler [La] and Fayolle, Malyshev and Men'shikov [F-M-M], as well as of course the basic material in Feller's books [F1], [F2].

Chapter 5 first deals with two specific classes of examples, starting with birth-and-death chains on the non-negative integers or a finite interval of integers (§A). The Markov chains are nearest neighbour random walks on the underlying graph, which is a half-line or line segment. Amongst other things, the link with analytic continued fractions is explained. Then (§B) the classical analysis of the Galton–Watson process is presented. This serves also as a prelude of the next section (§C), which is devoted to an outline of some basic features of branching Markov chains (BMCs, §C). The latter combine Markov chains with the evolution of a "population" according to a Galton–Watson process. BMCs themselves go beyond the theme of this book, Markov chains. One of their nice properties is that certain probabilistic quantities associated with BMC are expressed in terms of the generating functions of the underlying Markov chain. In particular, ρ-recurrence of the chain has such an interpretation via criticality of an embedded Galton–Watson process. In view of my insisting on the utility of generating functions, this is a very appealing propaganda instrument regarding their probabilistic nature.

In the sections on the Galton–Watson process and BMC, I pay some extra attention to the rigorous construction of a probability space on which the processes can be defined completely and with all their features; see my remarks about a certain *nonchalance* regarding the existence of the "probabilistic heaven" further below which appear to be particularly appropriate here. (I do not claim that the proposed model probability spaces are the only good ones.)

Of this material, only the part of §A dealing with continued fractions was already present in [W1].

Chapter 6 displays basic notions, terminology and results of potential theory in the discrete context of transient Markov chains. The discrete Laplacian is $P - I$, where P is the transition matrix and I the identity matrix. The starting point (§A) is the finite case, where we declare a part of the state space to be the *boundary* and its complement to be the *interior*. We look for functions that have preassigned value on the boundary and are harmonic in the interior. This discrete *Dirichlet problem* is solved in probabilistic terms.

We then move on to the infinite, transient case and (in §B) consider basic features of harmonic and superharmonic functions and their duals in terms of measures on the state space. Here, functions are thought of as column vectors on which the transition matrix acts from the left, while measures are row vectors on which the matrix acts from the right. In particular, transience is linked with the existence of non-constant positive superharmonic functions. Then (§C) induced Markov chains and their interplay with superharmonic functions and excessive measures are displayed, after which (§D) classical results such as the Riesz decomposition theorem and the approximation theorem for positive superharmonic functions are proved. The chapter ends (§E) with an explanation of "balayage" in terms of first entrance and last exit probabilities, concluding with the domination principle for superharmonic functions.

Chapter 7 is an attempt to give a careful exposition of Martin boundary theory for transient Markov chains. I do not aim at the highest level of sophistication but at the broadest level of comprehensibility. As a mild but natural restriction, only irreducible chains are considered (i.e., all states communicate), but substochastic transition matrices are admitted since this is needed anyway in some of the proofs. The starting point (§A) is the definition and first study of the extreme elements in the convex cone of positive superharmonic functions, in particular, the minimal harmonic functions. The construction/definition of the Martin boundary (§B) is preceded by a preamble on compactifications in general. This section concludes with the statement of one of the two main theorems of that theory, namely convergence to the boundary. Before the proof, martingale theory is needed (§C), and we examine the relation of supermartingales with superharmonic functions and, more subtle and important here, with excessive measures. Then (§D) we derive the Poisson–Martin integral representation of positive harmonic functions and show that it is unique over the minimal boundary. Finally (§E) we study the integral representation of bounded harmonic functions (the Poisson boundary), its interpretation via terminal random variables, and the probabilistic Fatou convergence theorem. At the end, the alternative approach to the Poisson–Martin integral representation via the approximation theorem is outlined.

Chapter 8 is very short and explains the rather algebraic procedure of finding all minimal harmonic functions for random walks on integer grids.

Chapter 9, on the contrary, is the longest one and dedicated to nearest neighbour random walks on trees (mostly infinite). Here we can harvest in a concrete class of examples from the seed of methods and results of the preceding chapters. First (§A), the fundamental equations for first passage time generating functions on trees are exhibited, and some basic methods for finite trees are outlined. Then we turn to infinite trees and their boundary. The geometric boundary is described via the end compactification (§B), convergence to the boundary of transient random walks is proved directly, and the Martin boundary is shown to coincide with the space of ends (§C). This is also the minimal boundary, and the limit distribution on the boundary is computed. The structural simplicity of trees allows us to provide also an integral representation of *all* harmonic functions, not only positive ones (§D). Next (§E) we examine in detail the Dirichlet problem at infinity and the regular boundary points, as well as a simple variant of the radial Fatou convergence theorem. A good part of these first sections owes much to the seminal long paper by CARTIER [Ca], but one of the innovations is that many results do not require local finiteness of the tree. There is a short *intermezzo* (§F) about how a transient random walk on a tree approaches its limiting boundary point. After that, we go back to transience/recurrence and consider a few criteria that are specific to trees, with a special eye on trees with *finitely many cone types* (§G). Finally (§H), we study in some detail two intertwined subjects: rate of escape (i.e., variants of the law of large numbers for the distance to the starting point) and spectral radius. Throughout the chapter, explicit computations are carried out for various examples via different methods.

Examples are present throughout all chapters.

Exercises are not accumulated at the end of each section or chapter but "built in" the text, of which they are considered an integral part. Quite often they are used in the subsequent text and proofs. The imaginary ideal reader is one who solves those exercises in real time while reading.

Solutions of all exercises are given after the last chapter.

The bibliography is subdivided into two parts, the first containing textbooks and other general references, which are recognizable by citations in letters. These are also intended for further reading. The second part consists of research-specific references, cited by numbers, and I do not pretend that these are complete. I tried to have them reasonably complete as far as material is concerned that is relatively recent, but going back in time, I rely more on the belief that what I'm using has already reached a confirmed status of public knowledge.

Raison d'être

Why another book about Markov chains? As a matter of fact, there is a great number and variety of textbooks on Markov chains on the market, and the older ones have by no means lost their validity just because so many new ones have appeared in the last decade. So rather than just praising in detail my own *opus*, let me display an incomplete subset of the mentioned variety.

For me, the all-time classic is CHUNG's *Markov chains with stationary transition probabilities* [Ch], along with KEMENY and SNELL, *Finite Markov chains* [K-S], whose first editions are both from 1960. My own learning of the subject, years ago, owes most to *Denumerable Markov chains* by KEMENY, SNELL and KNAPP [K-S-K], for which the title of this book is thought as an expression of reverence (without claiming to reach a comparable amplitude). Besides this, I have a very high esteem of SENETA's *Non-negative matrices and Markov chains* [Se] (first edition from 1973), where of course a reader who is looking for stochastic adventures will need previous motivation to appreciate the matrix theory view.

Among the older books, one definitely should not forget FREEDMAN [Fr]; the one of ISAACSON and MADSEN [I-M] has been very useful for preparing some of my lectures (in particular on non time-homogeneous chains, which are not featured here), and REVUZ' [Re] profound French style treatment is an important source permanently present on my shelf.

Coming back to the last 10–12 years, my personal favourites are the monograph by BRÉMAUD [Br] which displays a very broad range of topics with a permanent eye on applications in all areas (this is the book that I suggest to young mathematicians who want to use Markov chains in their future work), and in particular the very nicely written textbook by NORRIS [No], which provides a delightful itinerary into the world of stochastics for a probabilist-to-be. Quite recently, D. STROOCK enriched the selection of introductory texts on Markov processes by [St2], written in his masterly style.

Other recent, maybe more focused texts are due to BEHRENDS [Be] and HÄGGSTRÖM [Hä], as well as the St. Flour lecture notes by SALOFF-COSTE [SC]. All this is complemented by the high level exercise selection of BALDI, MAZLIAK and PRIOURET [B-M-P].

In Italy, my lecture notes (the first in Italian dedicated exclusively to this topic) were followed by the densely written paperback by PINTACUDA [Pi]. In this short review, I have omitted most of the monographs about Markov chains on non-discrete state spaces, such as Nummelin [Nu] or Hernández-Lerma and Lasserre [H-L] (to name just two besides [Re]) as well as continuous-time processes.

So in view of all this, this text needs indeed some additional reason of being. This lies in the three subtitle topics *generating functions, boundary theory, random walks on trees,* which are featured with some extra emphasis among all the material.

Generating functions. Some decades ago, as an apprentice of mathematics, I learnt from my PhD advisor Peter Gerl at Salzburg how useful it was to use generating functions for analyzing random walks. Already a small amount of basic knowledge about power series with non-negative coefficients, as it is taught in first or second year calculus, can be used efficiently in the basic analysis of Markov chains, such as irreducible classes, transience, null and positive recurrence, existence and uniqueness of stationary measures, and so on. Beyond that, more subtle methods from complex analysis can be used to derive refined asymptotics of transition probabilities and other limit theorems. (See [53] for a partial overview.) However, in most texts on Markov chains, generating functions play a marginal role or no role at all. I have the impression that quite a few of nowadays' probabilists consider this too analytically-combinatorially flavoured. As a matter of fact, the three Italian reviewers of [W1] criticised the use of generating functions as being too heavy to be introduced at such an early stage in those lecture notes. With all my students throughout different courses on Markov chains and random walks, I never noticed any such difficulties.

With humble admiration, I sympathise very much with the vibrant preface of D. Stroock's masterpiece *Probability theory: an analytic view* [St1]: (quote) "I have never been able to develop sufficient sensitivity to the distinction between a *proof* and a *probabilistic proof*". So, confirming hereby that I'm not a (quote) "dyed-in-the-wool probabilist", I'm stubborn enough to insist that the systematic use of generating functions at an early stage of developing Markov chain basics is very useful. This is one of the specific *raisons d'être* of this book. In any case, their use here is very very mild. My original intention was to include a whole chapter on the application of tools from complex analysis to generating functions associated with Markov chains, but as the material grew under my hands, this had to be abandoned in order to limit the size of the book. The masters of these methods come from analytic combinatorics; see the very comprehensive monograph by Flajolet and Sedgewick [F-S].

Boundary theory and elements of discrete potential theory. These topics are elaborated at a high level of sophistication by Kemeny, Snell and Knapp [K-S-K] and Revuz [Re], besides the literature from the 1960s and '70s in the spirit of abstract potential theory. While [K-S-K] gives a very complete account, it is not at all easy reading. My aim here is to give an introduction to the language and basics of the potential theory of (transient) denumerable Markov chains, and, in particular, a rather complete picture of the associated topological boundary theory that may be accessible for good students as well as interested colleagues coming from other fields of mathematics. As a matter of fact, even advanced non-experts have been tending to mix up the concepts of Poisson and Martin boundaries as well as the Dirichlet problem at infinity (whose solution with respect to some geometric boundary does *not* imply that one has identified the Martin boundary, as one finds stated). In the exposition of this material, my most important source was a rather old one, which still is, according to my opinion, the best readable presentation of Martin boundary theory of Markov chains: the expository article by Dynkin [Dy] from 1969.

Potential and boundary theory is a point of encounter between probability and analysis. While classical potential theory was already well established when its intrinsic connection with Brownian motion was revealed, the probabilistic theory of denumerable Markov chains and the associated potential theory were developed hand in hand by the same protagonists: to their mutual benefit, the two sides were never really separated. This is worth mentioning, because there are not only probabilists but also analysts who distinguish between a *proof* and a *probabilistic proof* – in a different spirit, however, which may suggest that if an analytic result (such as the solution of the Dirichlet problem at infinity) is deduced by probabilistic reasoning, then that result is true only *almost surely* before an *analytic proof* has been found.

What is not included here is the potential and boundary theory of recurrent chains. The former plays a prominent role mainly in relation with random walks on two-dimensional grids, and Spitzer's classic [Sp] is still a prominent source on this; I also like to look up some of those things in Lawler [La]. Also, not much is included here about the ℓ^2-potential theory associated with reversible Markov chains (networks); the reader can consult the delightful little book by Doyle and Snell [D-S] and the lecture notes volume by Soardi [So].

Nearest neighbour random walk on trees is the third item in the subtitle. Trees provide an excellent playground for working out the potential and boundary theory associated with Markov chains. Although the relation with the classical theory is not touched here, the analogy with potential theory and Brownian motion on the open unit disk, or rather, on the hyperbolic plane, is striking and obvious. The combinatorial structure of trees is simple enough to allow a presentation of a selection of methods and results which are well accessible for a sufficiently ambitious beginner. The resulting, rather long final chapter takes up and elaborates upon various topics from the preceding chapters. It can serve as a link with [W2], where not as much space has been dedicated to this specific theme, and, in particular, the basics are not developed as broadly as here.

In order to avoid the impact of additional structure-theoretic subtleties, I insist on dealing only with nearest neighbour random walks. Also, this chapter is certainly far from being comprehensive. Nevertheless, I think that a good part of this material appears here in book form for the first time. There are also a few new results and/or proofs.

Additional material can be found in [W2], and also in the ever forthcoming, quite differently flavoured wonderful book by Lyons with Peres [L-P].

At last, I want to say a few words about

the role of measure theory. If one wants to avoid measure theory, and in particular the extension machinery in the construction of the trajectory space of a Markov chain, then one can carry out a good amount of the theory by considering the Markov chain in a finite time interval $\{0, \dots, n\}$. The trajectory space is then countable and the underlying probability measure is atomic. For deriving limit theorems, one may first consider that time interval and then let $n \to \infty$. In this spirit, one can use a rather large part of the initial material in this book for teaching Markov chains at an elementary level, and I have done so on various occasions.

However, it is my opinion that it has been a great achievement that probability has been put on the solid theoretical fundament of measure theory, and that students of mathematics (as well as physics) should be exposed to that theoretical fundament, as opposed to fake attempts to make their curricula more "soft" or "applied" by giving up an important part of the mathematical edifice.

Furthermore, advanced probabilists are quite often – and with very good reason – somewhat *nonchalant* when referring to the spaces on which their random processes are defined. The attitude often becomes one where we are confident that there always is some big probability space somewhere up in the clouds, a kind of *probabilistic heaven,* on which all the random variables and processes that we are working with are defined and comply with all the properties that we postulate, but we do not always care to see what makes it sure that this probabilistic heaven is solid. Apart from the suspicion that this attitude may be one of the causes of the vague distrust of some analysts to which I alluded above, this is fine with me. But I believe this should not be a guideline of the education of master or PhD students; they should first see how to set up the edifice rigorously before passing to nonchalance that is based on firm knowledge.

What is not contained about Markov chains is of course much more than what *is* contained in this book. I could have easily doubled its size, thereby also changing its scope and intentions. I already mentioned recurrent potential and boundary theory, there is a lot more that one could have said about recurrence and transience, one could have included more details about geometric eigenvalue bounds, the Galton–Watson process, and so on. I have not included any hint at continuous-time Markov processes, and there is no random environment, in spite of the fact that this is currently very much *en vogue* and may have a much more *probabilistic* taste than Markov chains that evolve on a deterministic space. (Again, I'm stubborn enough to believe that there is a lot of interesting things to do and to say about the situation where randomness is restricted to the transition probabilities themselves.) So, as I also said elsewhere, I'm sure that every reader will be able to single out her or his favourite among those topics that are *not* included here. In any case, I do hope that the selected material and presentation may provide some stimulus and usefulness.