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Nikolai I. Lobachevsky

Pangeometry

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Foreword

Lobachevsky wrote his *Pangeometry* in 1855, the year before his death, at a time when he was completely blind. He dictated two versions of that work, a first one in Russian, and a second one in French. The *Pangeometry* is a résumé of Lobachevsky's work on non-Euclidean geometry and its applications, and can be considered as his clearest account of the subject. It is also the conclusion of his life's work, and his last attempt to acquire recognition.

In this memoir, after a survey of the foundations of hyperbolic geometry, taken from his earlier treatises, Lobachevsky presented the bases of a complete theory of differential and integral calculus in hyperbolic space, and then developed the applications of this theory to the computation of definite integrals. Computing definite integrals using techniques from hyperbolic geometry was one of Lobachevsky's favourite topics, on which he had worked from his initial writings. It was also the main subject of his memoir *Géométrie Imaginaire*, first published in Russian in 1835, and then in French, in *Crelle's Journal*, in 1837. Lobachevsky's integrals represent areas of surfaces and volumes of bodies in hyperbolic 3-space. Computing the area or the volume of the same object in different manners turns out to be an efficient way of finding attractive formulae for some definite integrals. But besides the intrinsic value of the results obtained, there are several reasons why Lobachevsky worked out these computations. First of all, using hyperbolic geometry for the computation of integrals was a way of showing the usefulness of hyperbolic geometry (which at that time was a controversial subject) in another branch of mathematics, namely analysis. At another level, drawing consequences of the new axiom system, like finding values of known integrals using these new methods was a way of checking that the new geometric system was not self-contradictory, which indeed was a major concern for Lobachevsky.

The *Pangeometry* has already been translated several times. Two German translations were published in 1858 and in 1902, an Italian translation appeared in 1867, and an English translation of a small part of this work appeared in 1929. See [102] for the exact references.¹ The present edition provides, to the best of my knowledge, the first complete English translation of the *Pangeometry* to appear in print. It is made from Lobachevsky's French version. It is accompanied by notes and followed by a commentary.

There are several reasons for producing the present edition. The first and obvious one is that Lobachevsky's *Pangeometry* is an exposition of its author's original and most profound ideas, and it deserves to be widely accessible to the mathematical world and to historians of mathematics and science. The second reason is that reading this memoir will also be most instructive for students in geometry. Although the text is more than 150 years old, it will give most readers a fresh point of view on the subject, namely, a model-free point of view which is not to be found in most of the more modern textbooks on hyperbolic geometry. Working with models is fine, and generally very useful, because it is efficient for making computations, namely, by using techniques

¹The bibliographical references are collected at the end of this volume.

of Euclidean geometry, including those of linear algebra. Furthermore, models of the hyperbolic space are also useful in other branches of mathematics. For instance, Poincaré's upper-half plane and disk models play essential roles in number theory, in complex analysis and in the theory of differential equations. Finally, working in models gives a systematic way of drawing pictures (which generally turn out to be nice pictures). Despite all these advantages, learning hyperbolic geometry through models is, in my opinion, intellectually and aesthetically less satisfying than in the model-free point of view, which requires more imagination and deep thought. I realised this fact only after having taught hyperbolic geometry for several years, using models as is usually done, and after I went through some of the ancient texts on the subject. In any case, the reader will form a personal opinion and choose between the model and the model-free points of view.

Students in geometry can use the *Pangeometry* at different levels. They can read this text at a superficial level, without going into the technical details. This will give them at least a flavour of the subject. But they can also try to go deeply into Lobachevsky's constructions and computations. This will need some investment of time and energy, but it is worth doing. Indeed, although it is relatively easy to follow the main stream of Lobachevsky's ideas, some of the details in the *Pangeometry* are rather obscure, and some of the computations are difficult to follow.

Let me recall Gauss's letter to the mathematician and astronomer C. L. Gerling, written on 8 February 1844, in which he compared Lobachevsky's writings to a "jungle through which it is difficult to find a passage and perspective, without first becoming acquainted with each tree that composes it."² Let me also quote Gauss's letter to the astronomer H. C. Schumacher, dated 28 November 1846: "In developing the subject, the author followed a road different from the one I took myself; Lobachevsky carried out the task in a masterly fashion and in a truly geometric spirit. I consider it a duty to call your attention to this book, since I have no doubt that it will give you a tremendous pleasure ..."³

Gauss's comments concern Lobachevsky's earlier writings. They show the respect that Gauss had for Lobachevsky's work. Concerning the *Pangeometry* itself, let me quote G. B. Halsted (1835–1922), a mathematician whose life was entirely dedicated to the defence and popularisation of non-Euclidean geometry, and in particular of Lobachevsky's writings, some of which he translated into English. In his biographical article *Lobachevsky* [62], Halsted wrote: "Though Lobachevsky's *Geometric Researches on the Theory of Parallels*, published in 1840, of which my English translation is now in its fourth edition and has been beautifully reproduced in Japan, remains even today the simplest introduction to the subject which has ever appeared; yet in it Lobachevsky has not reached that final breadth of view given first in John Bolyai's

²[die mehr einem verworrenen Walde gleichen, durch den es, ohne alle Bäume erst einzeln kennen gelernt zu haben, schwer ist, einen Durchgang und Übersicht zu finden.]

³[... materiell für mich Neues habe ich also im Lobatschewskyschen Werke nicht gefunden, aber die Entwicklung ist auf anderm Wege gemacht, als ich selbst eingeschlagen habe, und zwar von Lobatschewsky auf eine meisterhafte Art in ächt geometrischem Geiste. Ich glaube, Sie auf das Buch aufmerksam machen zu müssen, welches Ihnen gewiss ganz exquisiten Genuss gewähren wird.]

Science of Absolute Space, but also attained in Lobachevsky's last work *Pangéométrie*, which name is explicitly used as expressive of this final view."

Finally, let me quote J. Hoüel (1823–1886), another famous defender and translator (into French) of non-Euclidean geometry texts, who wrote in his *Notice sur la vie et les travaux de N.-I. Lobatchefsky* [76] that the *Pangeometry* is "one of Lobachevsky's most remarkable works regarding the clarity of exposition".

Reading the major texts of our mathematical ancestors is an expression of the esteem and gratitude that we owe them. This alone is a sufficient reason for publishing this English translation of the *Pangeometry*, and making it easily accessible to the mathematical public.

While I was working on this book, I not only thought about mathematics, but I also tried to enter into Lobachevsky's mysterious intimate world, and to understand his tragic life. This led me to include in this book, besides the mathematical material, historical and biographical elements. Beyond the particular case of Lobachevsky, I think that including historical comments in mathematical texts can be very helpful in making the reader feel the charm of the subject.

I would like to thank Manfred Karbe and Ernest B. Vinberg for their kind interest in this translation and the commentary, and for the invaluable information they provided on various editions of Lobachevsky's works. I am also grateful to Dmitry V. Millionshikov and Grigory M. Polotovskiy for help in the references, to A. S. Mishchenko who provided copies of the original versions of the *Pangeometry*, and to Evgeniy N. Sosov, Daniar H. Mushtari and Yi Zhang who kindly transmitted pictures from Kazan. The pictures on p. 212 and 216, and the upper picture on p. 214 belong to the museum of history of Kazan State University, and were published in the book *N. Lobachevsky*, in the series "Outstanding scientists and graduates of Kazan State University".

I am also grateful to Edwin Beschler and Guillaume Théret for their help in polishing the final manuscript and to Boris R. Frenkin who translated into Russian part of my commentary, see [116]. At the occasion of that translation, I discussed with Frenkin several points in that commentary. I am also grateful to Sergei S. Demidov who read my notes on Lobachevsky's biography, corrected inaccuracies, and sent information which was unknown to me. I am especially grateful to Jeremy Gray for an illuminating long conversation about this book. Gray also read an early version of the manuscript and suggested several valuable improvements.

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Athanase Papadopoulos
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