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Steffen Börm

Efficient Numerical Methods for Non-local Operators

$\mathcal{H}^2$-Matrix Compression, Algorithms and Analysis

European Mathematical Society

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2010 Mathematical Subject Classification: 65-02; 65F05, 65F30, 65N22, 65N38, 65R20

Key words: Hierarchical matrix, data-sparse approximation, boundary element method, preconditioner

ISBN 978-3-03719-091-3
Corrected 2nd printing, 2013

The Swiss National Library lists this publication in The Swiss Book, the Swiss national bibliography, and the detailed bibliographic data are available on the Internet at http://www.helveticat.ch.

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CH-8092 Zürich
Switzerland

Phone: +41 (0)44 632 34 36
Email: info@ems-ph.org
Homepage: www.ems-ph.org

Typeset using the author's TEX files: I. Zimmermann, Freiburg
Printing and binding: Beltz Bad Langensalza GmbH, Bad Langensalza, Germany
∞ Printed on acid free paper
9 8 7 6 5 4 3 2

Foreword

Non-local operators appear naturally in the field of scientific computing: non-local forces govern the movement of objects in gravitational or electromagnetic fields, non-local density functions describe jump processes used, e.g., to investigate stock prices, and non-local kernel functions play an important role when studying population dynamics.

Applying standard discretization schemes to non-local operators yields matrices that consist mostly of non-zero entries ("dense matrices") and therefore cannot be treated efficiently by standard sparse matrix techniques. They can, however, be approximated by *data-sparse* representations that significantly reduce the storage requirements.

Hierarchical matrices ($\mathcal{H}$-matrices) [62] are one of these data-sparse representations: $\mathcal{H}$-matrices not only approximate matrices arising in many important applications very well, they also offer a set of matrix arithmetic operations like evaluation, multiplication, factorization and inversion that can be used to construct efficient preconditioners or solve matrix equations. $\mathcal{H}^2$-matrices [70], [64] introduce an additional hierarchical structure to reduce the storage requirements and computational complexity of $\mathcal{H}$-matrices.

In this book, I focus on presenting an overview of theoretical results and practical algorithms for working with $\mathcal{H}^2$-matrices. I assume that the reader is familiar with basic techniques of numerical linear algebra, e.g., norm estimates, orthogonal transformations and factorizations. The error analysis of integral operators, particularly Section 4.7, requires some results from polynomial approximation theory, while the error analysis of differential operators, particularly Section 9.2, is aimed at readers familiar with standard finite element techniques and makes use of a number of fundamental properties of Sobolev spaces.

Different audiences will probably read the book in different ways. I would like to offer the following suggestions: Chapters 1–3 provide the basic concepts and definitions used in this book and any reader should at least be familiar with the terms $\mathcal{H}$-matrix, $\mathcal{H}^2$-matrix, cluster tree, block cluster tree, admissible and inadmissible blocks and cluster bases. After this introduction, different courses are possible:

- If you are a student of numerical mathematics, you should read Sections 4.1–4.4 on integral operators, Sections 5.1–5.5 on orthogonalization and truncation, Sections 6.1–6.4 on matrix compression, and maybe Sections 7.1, 7.2, 7.6 and 7.7 to get acquainted with the concepts of matrix arithmetic operations.

- If you are interested in using $\mathcal{H}^2$-matrices to treat integral equations, you should read Chapter 4 on basic approximation techniques and Chapters 5 and 6 in order to understand how the storage requirements can be reduced as far as possible. Remarks on practical applications can be found in Sections 10.1–10.4.

- If you are interested in applying $\mathcal{H}^2$-matrices to elliptic partial differential equations, you should consider Sections 5.1–5.5 on truncation, Chapter 6 on compression, Chapter 8 on adaptive matrix arithmetic operations and Sections 10.5 and 10.3. Convergence estimates can be found in Chapter 9.

If you would like to try the algorithms presented in this book, you can get the HLib software package that I have used to provide the numerical experiments described in Chapters 4–10. Information on this package is available at http://www.hlib.org and it is provided free of charge for research purposes.

This book would not exist without the help and support of Wolfgang Hackbusch, whom I wish to thank for many fruitful discussions and insights and for the chance to work at the Max Planck Institute for Mathematics in the Sciences in Leipzig.

I am also indebted to my colleagues Stefan A. Sauter, Lars Grasedyck and J. Markus Melenk, who have helped me find answers to many questions arising during the course of this work.

Last, but not least, I thank Maike Löhndorf, Kai Helms and Jelena Djokić for their help with proofreading the (already quite extensive) drafts of this book, Dirk Boysen for his contributions to the corrected reprint, and Irene Zimmermann for preparing the final version for publication.

Kiel, November 2010 and July 2013 Steffen Börm