Contents

1 Introduction 1
- 1.1 Origins of $\mathcal{H}^2$-matrix methods . . . 1
- 1.2 Which kinds of matrices can be compressed? . . . 3
- 1.3 Which kinds of operations can be performed efficiently? . . . 4
- 1.4 Which problems can be solved efficiently? . . . 6
- 1.5 Organization of the book . . . 6

2 Model problem 9
- 2.1 One-dimensional integral operator . . . 9
- 2.2 Low-rank approximation . . . 10
- 2.3 Error estimate . . . 11
- 2.4 Local approximation . . . 15
- 2.5 Cluster tree and block cluster tree . . . 16
- 2.6 Hierarchical matrix . . . 20
- 2.7 Matrix approximation error . . . 22
- 2.8 $\mathcal{H}^2$-matrix . . . 23
- 2.9 Numerical experiment . . . 26

3 Hierarchical matrices 28
- 3.1 Cluster tree . . . 29
- 3.2 Block cluster tree . . . 34
- 3.3 Construction of cluster trees and block cluster trees . . . 37
- 3.4 Hierarchical matrices . . . 47
- 3.5 Cluster bases . . . 53
- 3.6 $\mathcal{H}^2$-matrices . . . 56
- 3.7 Matrix-vector multiplication . . . 59
- 3.8 Complexity estimates for bounded rank distributions . . . 63
- 3.9 Technical lemmas . . . 70

4 Application to integral operators 74
- 4.1 Integral operators . . . 76
- 4.2 Low-rank approximation . . . 76
- 4.3 Approximation by Taylor expansion . . . 79
- 4.4 Approximation by interpolation . . . 87
- 4.5 Approximation of derivatives . . . 103
- 4.6 Matrix approximation . . . 114
- 4.7 Variable-order approximation . . . 125
- 4.8 Technical lemmas . . . 149
- 4.9 Numerical experiments . . . 155

5 Orthogonal cluster bases and matrix projections 163
5.1 Orthogonal cluster bases . . . 164
5.2 Projections into $\mathcal{H}^2$-matrix spaces . . . 166
5.3 Cluster operators . . . 175
5.4 Orthogonalization . . . 180
5.5 Truncation . . . 185
5.6 Computation of the Frobenius norm of the projection error . . . 200
5.7 Numerical experiments . . . 202

6 Compression 211
6.1 Semi-uniform matrices . . . 212
6.2 Total cluster bases . . . 218
6.3 Approximation by semi-uniform matrices . . . 222
6.4 General compression algorithm . . . 227
6.5 Compression of hierarchical matrices . . . 234
6.6 Recompression of $\mathcal{H}^2$-matrices . . . 239
6.7 Unification and hierarchical compression . . . 248
6.8 Refined error control and variable-rank approximation . . . 259
6.9 Numerical experiments . . . 271

7 A priori matrix arithmetic 280
7.1 Matrix forward transformation . . . 281
7.2 Matrix backward transformation . . . 287
7.3 Matrix addition . . . 292
7.4 Projected matrix-matrix addition . . . 293
7.5 Exact matrix-matrix addition . . . 297
7.6 Matrix multiplication . . . 301
7.7 Projected matrix-matrix multiplication . . . 302
7.8 Exact matrix-matrix multiplication . . . 319
7.9 Numerical experiments . . . 328

8 A posteriori matrix arithmetic 332
8.1 Semi-uniform matrices . . . 334
8.2 Intermediate representation . . . 336
8.3 Coarsening . . . 347
8.4 Construction of adaptive cluster bases . . . 359
8.5 Numerical experiments . . . 360

9 Application to elliptic partial differential operators 363
9.1 Model problem . . . 364
9.2 Approximation of the solution operator . . . 366
9.3 Approximation of matrix blocks . . . 376
9.4 Compression of the discrete solution operator . . . 384

10 Applications 387
10.1 Indirect boundary integral equation 388
10.2 Direct boundary integral equation 395
10.3 Preconditioners for integral equations 401
10.4 Application to realistic geometries 411
10.5 Solution operators of elliptic partial differential equations 413

Bibliography 421

Algorithm index 427

Subject index 429