Introduction

Let us begin with the following (elementary) problem.

(S-$\mathscr{C}$) We are given two domains $D \subset \mathbb{R}^p$, $G \subset \mathbb{R}^q$ and a function

$$f : D \times G \to \mathbb{R}$$

that is *separately continuous* on $D \times G$, i.e.,

- $f(a, \cdot)$ is continuous on G for arbitrary $a \in D$,
- $f(\cdot, b)$ is continuous on D for arbitrary $b \in G$.

We ask whether the above conditions imply that f is continuous on $D \times G$.

It is well known that the answer is negative. However, recall that the answer was not known for instance to A. Cauchy, who in 1821 in his *Cours d'Analyse* claimed that f must be continuous.

> **I.$^{\text{er}}$ Théorème.** *Si les variables $x, y, z, \dots$ ont pour limites respectives les quantités fixes et déterminées $X, Y, Z, \dots$, et que la fonction $f(x, y, z, \dots)$ soit continue par rapport à chacune des variables $x, y, z, \dots$ dans le voisinage du systéme des valeurs particuliéres $x = X$, $y = Y, z = Z, \dots$, $f(x, y, z, \dots)$ aura pour limite $f(X, Y, Z, \dots)$.* A. Cauchy [Cau 1821], p. 39

See also [Pio 1985-86], [Pio 1996], [Pio 2000].

According to C. J. Thomae (cf. [Tho 1870], p. 13, [Tho 1873], p. 15, see also [Rose 1955]), the first counterexample had been discovered by E. Heine:

$$f(x, y) := \begin{cases} \sin(4 \arctan \frac{x}{y}) & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

A simpler counterexample is the function

$$f(x, y) := \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases} \tag{$*$}$$

which was already known to G. Peano in 1884 (cf. [Gen 1884], p. 173).

Since the answer is in general negative, one can ask how big is the set $\mathcal{S}_{\mathscr{C}}(f)$ of discontinuity points $(a, b) \in D \times G$ of a separately continuous function f. A partial answer was first given in 1899 by R. Baire ([Bai 1899], see also [Rud 1981]), who proved that every separately continuous function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is of the *first Baire class*, i.e. there exists a sequence $(f_k)_{k=1}^{\infty} \subset \mathscr{C}(\mathbb{R}^2, \mathbb{R})$ such that $f_k \to f$ pointwise on $\mathbb{R}^2$. Consequently, if $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is separately continuous, then f is Borel measurable.

> Several years ago I used to pose this question to randomly selected analysts. The typical answer was something like this: "Hmm – well – probably not – why should it be ?" The only group that did a little better were the probabilists. And there was just one person who said: "Let's see, yes, it is – and it is of Baire class 1 – and …". He knew.
>
> W. Rudin [Rud 1981]

Moreover, $\mathcal{S}_{\mathcal{C}}(f)$ must be of the *first Baire category*, i.e. $\mathcal{S}_{\mathcal{C}}(f)\subset\bigcup_{k=1}^{\infty}F_k$, where $\operatorname{int}\bar{F}_k=\varnothing$, $k\in\mathbb{N}$. Baire also proved that if $f\colon[0,1]\times[0,1]\to\mathbb{R}$ is separately continuous, then $\mathcal{S}_{\mathcal{C}}(f)$ is an $\mathcal{F}_\sigma$-set (i.e. a countable union of closed sets) whose projections are of the first Baire category. Conversely, if $S\subset[0,1]\times[0,1]$ is an $\mathcal{F}_\sigma$-set whose projections are of the first Baire category, then there exists a separately continuous function $f\colon[0,1]\times[0,1]\to\mathbb{R}$ with $\mathcal{S}_{\mathcal{C}}(f)=S$ (cf. [Ker 1943], [Mas-Mik 2000]). Moreover, if $S\subset[0,1]\times[0,1]$ is an $\mathcal{F}_\sigma$-set whose projections are nowhere dense, then there exists a separately $\mathcal{C}^\infty$ function $f\colon[0,1]\times[0,1]\to\mathbb{R}$ with $\mathcal{S}_{\mathcal{C}}(f)=S$ (cf. [Ker 1943]). Summarizing, the singularity sets $\mathcal{S}_{\mathcal{C}}(f)$ are small in the topological sense. However, G. P. Tolstov ([Tol 1949]) showed that for any $\varepsilon\in(0,1)$ there exists a separately $\mathcal{C}^\infty$ function $f\colon[0,1]\times[0,1]\to\mathbb{R}$ such that the measure of $\mathcal{S}_{\mathcal{C}}(f)$ is larger than ε.

It is natural to ask whether the above results may be generalized to the case of separately continuous functions $f\colon\mathbb{R}^n\to\mathbb{R}$, $n\geq 3$, i.e. those functions f for which $f(x_1,\dots,x_{j-1},\cdot,x_{j+1},\dots,x_n)\in\mathcal{C}(\mathbb{R})$ for arbitrary $(x_1,\dots,x_n)\in\mathbb{R}^n$ and $j\in\{1,\dots,n\}$. H. Lebesgue proved ([Leb 1905]) that every such a function is of the $(n-1)$ *Baire class*, i.e. there exists a sequence $(f_k)_{k=1}^{\infty}$ of functions of the $(n-2)$ Baire class such that $f_k\to f$ pointwise on $\mathbb{R}^n$. In particular, every separately continuous function $f\colon\mathbb{R}^n\to\mathbb{R}$ is Borel measurable. Moreover, H. Lebesgue proved that the above result is exact, i.e. for $n\geq 3$ there exists a separately continuous function $f\colon\mathbb{R}^n\to\mathbb{R}$ that is not of the $(n-2)$ Baire class.

It is clear that one may formulate similar problems substituting the class $\mathcal{C}$ of continuous functions by other classes $\mathcal{F}$, e.g.:

- $\mathcal{F}=\mathcal{C}^k=$ the class of $\mathcal{C}^k$*-functions*, $k\in\mathbb{N}\cup\{\infty,\ \omega\}$, where $\mathcal{C}^\omega$ means the class of *real analytic functions*,
- $\mathcal{F}=\mathcal{H}=$ the class of *harmonic functions*,
- $\mathcal{F}=\mathcal{SH}=$ the class of *subharmonic functions* (in this case we allow that $f\colon D\times G\to[-\infty,+\infty)$).

Thus our more general problem is the following one.

(S-$\mathcal{F}$) We are given two domains $D\subset\mathbb{R}^p$, $G\subset\mathbb{R}^q$ and a function

$$f\colon D\times G\to\mathbb{R}$$

that is *separately of class* $\mathcal{F}$ on $D \times G$, i.e.,
- $f(a, \cdot) \in \mathcal{F}(G)$ for arbitrary $a \in D$,
- $f(\cdot, b) \in \mathcal{F}(D)$ for arbitrary $b \in G$.

We ask whether $f \in \mathcal{F}(D \times G)$.

Moreover, in the case where the answer is negative, one may study the set

$$\mathcal{S}_{\mathcal{F}}(f) := \{(a, b) \in D \times G : f \notin \mathcal{F}(U) \text{ for every neighborhood } U \text{ of } (a, b)\}.$$

Observe that the Peano function (∗) is separately real analytic. Consequently, our problem has a negative solution for $\mathcal{F} = \mathcal{C}^k$ with arbitrary $k \in \mathbb{N} \cup \{\infty, \omega\}$ and, therefore, one may be interested in the structure of $\mathcal{S}_{\mathcal{C}^k}(f)$. The structure of $\mathcal{S}_{\mathcal{C}^\omega}(f)$ was completely characterized in [StR 1990], [Sic 1990], and [Bło 1992] (cf. Theorem 5.8.2). In particular, in contrast to Tolstov's result, the set $\mathcal{S}_{\mathcal{C}^\omega}(f)$ must be of zero measure.

Surprisingly, in the case of harmonic functions the answer is positive – *every separately harmonic function is harmonic*, cf. [Lel 1961] (Theorem 5.6.5).

In the case of separately subharmonic functions the answer is once again negative, cf. § 5.8.2.

Analogous problems may be formulated in the case where $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ are domains and $f : D \times G \to \mathbb{C}$ is a function that is separately of class $\mathcal{F}$ with:
- $\mathcal{F} = \mathcal{O} =$ the class of all *holomorphic functions*,
- $\mathcal{F} = \mathcal{M} =$ the class of all *meromorphic functions*.

In the case of holomorphic functions the answer is positive – *every separately holomorphic function is holomorphic* (Theorem 1.1.7) – this is the famous *Hartogs theorem* (cf. [Har 1906]). In the sequel we will be mostly concentrated on the holomorphic case. We would like to point out that investigations of separately holomorphic functions began in 1899 ([Osg 1899]), that is almost at the same time as Baire's first results on separately continuous functions ([Bai 1899]).

Since the answer to the main question (S-$\mathcal{O}$) is positive, we may consider the following strengthened problem.

(S-$\mathcal{O}_{\mathrm{H}}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a non-empty *test set* $B \subset G$, and a function $f : D \times G \to \mathbb{C}$ such that:
- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in D$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$ (only in B).

We ask whether $f \in \mathcal{O}(D \times G)$.

The problem has a long history that began with M. Hukuhara [Huk 1942] (Theorem 1.4.2). (We have to mention that there was a misunderstanding related to the year of publication of [Huk 1942]. Many papers (including ours) name 1930, but the reader may verify (pp. 281–283) that, in fact, it was 1942.) The problem has been continued in [Ter 1967], [Ter 1972], see Theorems 4.2.2 and 4.2.5; compare also the survey article

[Pfl 2003]. Terada was the first to use the pluripotential theory – the newest tool at that time. Roughly speaking, the final result says that the answer is positive iff the set B is not *pluripolar* (i.e. B is not thin from the point of view of the pluricomplex potential theory, cf. Definition 2.3.19).

The problem (S-$\mathcal{O}_{\mathrm{H}}$) leads to the following general question.

(S-$\mathcal{O}_{\mathrm{C}}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, two non-empty *test sets* $A \subset D$, $B \subset G$. We ask whether there exists an open neighborhood $\widehat{\boldsymbol{X}} \subset D \times G$ of the *cross* $\boldsymbol{X} := (A \times G) \cup (D \times B)$ such that every *separately holomorphic function* $f : (A \times G) \cup (D \times B) \to \mathbb{C}$, i.e.,

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in A$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$,

extends holomorphically to $\widehat{\boldsymbol{X}}$.

Note that (S-$\mathcal{O}_{\mathrm{H}}$) is just the case where $A = D$ (and, consequently, $\boldsymbol{X} = \widehat{\boldsymbol{X}} = D \times G$).

Investigations of (S-$\mathcal{O}_{\mathrm{C}}$) began with [Ber 1912] and have been continued for instance in [Cam-Sto 1966], [Sic 1968], [Sic 1969a], [Sic 1969b], [Akh-Ron 1973], [Zah 1976], [Sic 1981a], [Shi 1989], [NTV-Sic 1991], [NTV-Zer 1991], [NTV-Zer 1995], [NTV 1997], [Ale-Zer 2001], [Zer 2002]. It turned out (Theorem 5.4.1) that if the sets A, B are *regular* (i.e. every point of A (resp. B) is a density point of A (resp. B) in the sense of the pluricomplex potential theory, cf. Definition 3.2.8), then such a neighborhood $\widehat{\boldsymbol{X}}$ exists.

Similar questions as above may be formulated for a *boundary cross*. To be more precise:

(S-$\mathcal{O}_{\mathrm{B}}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$ and two non-empty sets $A \subset \partial D$, $B \subset \partial G$. We ask whether there exists an open subset $\widehat{\boldsymbol{X}}$ of $D \times G$ with $\boldsymbol{X} \subset \overline{\widehat{\boldsymbol{X}}}$ such that every function $f : \big(A \times (G \cup B)\big) \cup \big((D \cup A) \times B\big) \to \mathbb{C}$ for which

- $f(a, \cdot) \in \mathcal{O}(G)$ for every $a \in A$,
- $f(\cdot, b) \in \mathcal{O}(D)$ for every $b \in B$,
- $f(a, b) = \lim\limits_{D \ni z \to a} f(z, b) = \lim\limits_{G \ni w \to b} f(a, w)$, $(a, b) \in A \times B$ (the limits are taken in a certain sense, e.g. non-tangential),

extends to an $\hat{f} \in \mathcal{O}(\widehat{\boldsymbol{X}})$ and

$$f(a, b) = \lim_{\widehat{\boldsymbol{X}} \ni (z,w) \to (a,b)} \hat{f}(z, w), \quad (a, b) \in \boldsymbol{X},$$

cf. e.g. [Dru 1980], [Gon 1985], [Pfl-NVA 2004], [Pfl-NVA 2007], see Chapter 8.

So far our separately holomorphic functions $f : \boldsymbol{X} \to \mathbb{C}$ had no singularities on $\boldsymbol{X}$. The fundamental paper by E. M. Chirka and A. Sadullaev ([Chi-Sad 1987]) and next some applications to mathematical tomography ([Ökt 1998], [Ökt 1999]) showed that the following problem seems to be important.

(S-$\mathcal{O}_S$) Suppose that $\widehat{X}$ is a solution of (S-$\mathcal{O}_C$). We are given a relatively closed "thin" set (in a certain sense, e.g. pluripolar) $M \subset X := (A \times G) \cup (D \times B)$. We ask whether there exists a "thin" relatively closed set $\widehat{M} \subset \widehat{X}$ such that every *separately holomorphic* function $f: X \setminus M \to \mathbb{C}$, i.e.,

- $f(a, \cdot)$ is holomorphic in $\{w \in G : (a, w) \notin M\}$ for every $a \in A$,
- $f(\cdot, b)$ is holomorphic in $\{z \in D : (z, b) \notin M\}$ for every $b \in B$,

extends holomorphically to $\widehat{X} \setminus \widehat{M}$.

The problem (S-$\mathcal{O}_S$) has been studied for example in [Sic 2001], [Jar-Pfl 2001], [Jar-Pfl 2003a], [Jar-Pfl 2003b], [Jar-Pfl 2003c], [Jar-Pfl 2007], [Jar-Pfl 2010a], [Jar-Pfl 2010b], [Jar-Pfl 2011], see Chapter 10. Observe that the case where $M = \varnothing$ reduces to (S-$\mathcal{O}_C$).

Analogous problems may also be stated for separately meromorphic functions, but their solutions are essentially more difficult. For example, the Hartogs problem corresponds to the following question for separately meromorphic functions.

(S-$\mathcal{M}$) We are given two domains $D \subset \mathbb{C}^p$, $G \subset \mathbb{C}^q$, a "thin" (in a certain sense) relatively closed set $S \subset D \times G$, and a function $f: D \times G \setminus S \to \mathbb{C}$ that is *separately meromorphic* on $D \times G$, i.e.,

- $f(a, \cdot)$ extends meromorphically to G for "almost all" (in a certain sense) $a \in D$,
- $f(\cdot, b)$ extends meromorphically to D for "almost all" $b \in B$.

We ask under what assumptions on S the function f extends meromorphically to $D \times G$.

The problem has been studied for instance in [Kaz 1976], [Kaz 1978], [Kaz 1984], [Shi 1989], [Shi 1991], [Jar-Pfl 2003c], [Pfl-NVA 2003], see Chapter 11.

All the above problems may be formulated also for more general objects than crosses and in the category of Riemann domains over $\mathbb{C}^n$ and/or complex manifolds.

Notice that, instead of complex-valued functions $f: X \to \mathbb{C}$, we may discuss mappings $f: X \to Z$ with values in a complex manifold or even a complex space Z. We will not go in this direction. Let us only mention that in this general case results may be essentially different than for complex-valued functions. For example ([Bar 1975]), let $f: \mathbb{C} \times \mathbb{C} \to \overline{\mathbb{C}}$ be given by

$$f(z_1, z_2) := \begin{cases} \frac{(z_1+z_2)^2}{z_1-z_2} & \text{if } z_1 \neq z_2, \\ \infty & \text{if } z_1 = z_2 \neq 0, \\ 0 & \text{if } z_1 = z_2 = 0. \end{cases}$$

Then f is separately holomorphic but nevertheless, it is not continuous at the origin.

On the other hand we have the following positive result (cf. [Gau-Zer 2009]). Let $\mathbb{P}^m$ be an m-dimensional complex projective space, let $D \subset \mathbb{C}^n$ be a domain, and let $f : D \to \mathbb{P}^m$ be such that for any $(a_1, \dots, a_n) \in D$ and $j, k \in \{1, \dots, n\}$, $j < k$, the function

$$(z_j, z_k) \mapsto f(a_1, \dots, a_{j-1}, z_j, a_{j+1}, \dots, a_{k-1}, z_k, a_{k+1}, \dots, a_n)$$

is holomorphic in an open neighborhood of (a_j, a_k). Then f is holomorphic on D.

See also [Shi 1990], [Shi 1991], [LMH-NVK 2005] for characterizations of those complex spaces Z for which Hartogs' theorem on separately holomorphic mappings $f : D \to Z$ hold.