Introduction

The theory we describe in this book was developed over a long period, starting about 1965, and always with the aim of developing groupoid methods in homotopy theory of dimension greater than 1. Algebraic work made substantial progress in the early 1970s, in work with Chris Spencer. A substantial step forward in 1974 by Brown and Higgins led us over the years into many fruitful areas of homotopy theory and what is now called 'higher dimensional algebra'[2]. We published detailed reports on all we found as the journey proceeded, but the overall picture of the theory is still not well known. So the aim of this book is to give a full, connected account of this work in one place, so that it can be more readily evaluated, used appropriately, and, we hope, developed.

Structure of the subject

There are several features of the theory and so of our exposition which divert from standard practice in algebraic topology, but are essential for the full success of our methods.

Sets of base points: Enter groupoids

The notion of a 'space with base point' is standard in algebraic topology and homotopy theory, but in many situations we are unsure which base point to choose. One example is if $p\colon Y \to X$ is a covering map of spaces. Then X may have a chosen base point x, but it is not clear which base point to choose in the discrete inverse image space $p^{-1}(x)$. It makes sense then to take $p^{-1}(x)$ as a set of base points.

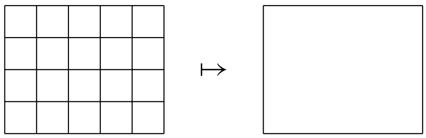
Choosing a set of base points according to the geometry of the situation has the implication that we deal with fundamental groupoids $\pi_1(X, X_0)$ on a *set* X_0 *of base points* rather than with the family of fundamental groups $\pi_1(X, x)$, $x \in X_0$. The intuitive idea is to consider X as a country with railway stations at the points of X_0; we then want to consider all the journeys *between* the stations and not just what is usually called 'change of base point', the somewhat bizarre concept of the set of return journeys from the individual stations, together with ways of moving from a return journey at one station to a return journey at another.

Sets of base points are used freely in what we call 'Seifert–van Kampen type situations' in [Bro06], when two connected open sets U, V have a disconnected intersection $U \cap V$. In such case it is sensible to choose a set X_0 of base points, say one point in each component of the intersection.[3]

The method is to use a Seifert–van Kampen type theorem to pass from topology to algebra by determining the fundamental groupoid $\pi_1(U \cup V, X_0)$ of a union, and then to compute a particular fundamental group $\pi_1(U \cup V, x)$ by using what we call 'combinatorial groupoid methods', i.e. using graphs and trees in combination with the groupoid theory. This follows the principle of keeping track of structure for as long as is reasonable.

Groupoids in 2-dimensional homotopy theory

The successful use of groupoids in 1-dimensional homotopy theory in [Bro68] suggested the desirability of investigating the use of groupoids in higher homotopy theory. One aspect was to find a mathematics which allowed 'algebraic inverse to subdivision', in the sense that it could represent multiple compositions as in the following diagram

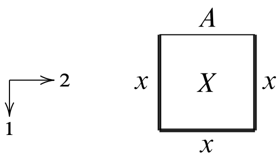


(multcomp)

in a manner analogous to the use of $(a_1, a_2, \dots, a_n) \mapsto a_1 a_2 \dots a_n$ in categories and groupoids, but in dimension 2. Note that going from right to left in the diagram is subdivision, a standard technique in mathematics.

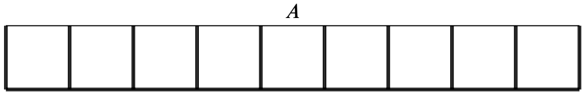
Traditional homotopy theory described the family $\pi_2(X, x)$ of homotopy groups, consisting of homotopy classes of maps $I^2 \to X$ which take the edges of the square I^2 to x, but this did not incorporate the groupoid idea, except under 'change of base point'.

Also considered were the relative homotopy groups $\pi_n(X, A, x)$ of a based pair (X, A, x) where $x \in A \subseteq X$. In dimension 2 the picture is as follows, where thick lines denote constant maps:

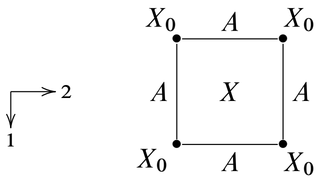


That is, we have homotopy classes of maps from the square I^2 to X which take the edge ∂_1^- to A, and the remaining three edges to the base point.

This definition involves choices, is unsymmetrical with respect to directions, and so is unaesthetic. The composition in $\pi_2(X, A, x)$ is the clear horizontal composition, and does give a group structure, but even large compositions are still 1-dimensional, i.e. in a line:



In 1974 Brown and Higgins found a new construction, finally published in [BH78a], which we called $\rho_2(X, A, X_0)$: it involves no such choices, and really does enable multiple compositions as wished for in Diagram (multcomp). We considered homotopy classes *rel vertices* of maps $[0, 1]^2 \to X$ which map edges to A and vertices to X_0:



Part of the geometric structure held by this construction is shown in the diagram:

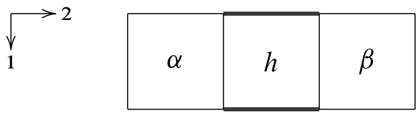
$$\rho_2(X, A, X_0) \ \begin{smallmatrix}\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow\end{smallmatrix}\ \pi_1(A, X_0) \rightrightarrows X_0$$

where the arrows denote boundary maps.

A horizontal composition in $\rho_2(X, A, X_0)$ is given by

$$\langle\!\langle \alpha \rangle\!\rangle +_2 \langle\!\langle \beta \rangle\!\rangle = \langle\!\langle \alpha +_2 h +_2 \beta \rangle\!\rangle$$

as shown in the following diagram, where h is a homotopy rel end points in A between an edge of α and an edge of β, and thick lines show constant paths.



The proof that this composition is well defined on homotopy classes is not entirely trivial and is given in Chapter 6. With a similar vertical composition, we obtain the structure of *double groupoid*, which enables multiple compositions as asked for in Diagram (multcomp).

There is still more structure which can be given to ρ_2, namely that of 'connections', which we describe in the section on cubical sets with connections on p. xxviii.

Crossed modules

A surprise was that the investigation of double groupoids led back to a concept due to Henry Whitehead when investigating the properties of second relative homotopy groups, that of crossed module. Analogous ideas were developed independently by Peiffer and Reidemeister in [Pei49], [Rei49], the war having led to zero contact between mathematicians in Germany and the UK[4]. It is interesting that Peiffer's paper was submitted in June, 1944. Work by Brown with C. B. Spencer in 1971–73 led to the discovery of a close relation between double groupoids and crossed modules. This, with the construction in the previous section, led to a 2-dimensional Seifert–van Kampen Theorem, making possible some new computations of nonabelian second relative homotopy groups which we give in detail in Chapters 4, 5.

A *crossed module* is a morphism

$$\mu \colon M \to P$$

of groups together with an action of the group P on the right of the group M, written $(m, p) \mapsto m^p$, satisfying the two rules:

CM1) $\mu(m^p) = p^{-1}(\mu m)p$;

CM2) $m^{-1}nm = n^{\mu m}$,

for all $p \in P, m, n \in M$. Algebraic examples of crossed modules include normal subgroups M of P; P-modules; the inner automorphism crossed module $M \to \mathrm{Aut}\, M$; and many others. There is the beginnings of a combinatorial, and also a related computational, crossed module theory.

The standard geometric example of crossed module is the boundary morphism of the second relative homotopy group

$$\partial \colon \pi_2(X, X_1, x) \to \pi_1(X_1, x)$$

where X_1 is a subspace of the topological space X and $x \in X_1$.

Our 2-dimensional Seifert–van Kampen Theorem, Theorem 2.3.1, yields computations of this crossed module in many useful conditions when X is a union of open sets, with special cases dealt with in Chapters 4 and 5. These results deal with nonabelian structures in dimension 2, and so are not available by the more standard methods of homology and covering spaces.

The traditional focus in homotopy theory has been on the second homotopy group, sometimes with its structure as a module over the fundamental group. However Mac Lane and Whitehead showed in [MLW50] that crossed modules model weak pointed homotopy 2-types; thus the 2-dimensional Seifert–van Kampen Theorem allowed new computations of some homotopy 2-types. It is not always straightforward to compute the second homotopy group from a description of the 2-type, but this can be done in some cases.

An aim to compute a second homotopy group is thus reached by computing a larger structure, the homotopy 2-type. This is not too surprising: a determination of the 2-type of a union should require information on the 2-types of the pieces and on the way these fit together. The 2-type also in principle determines the second homotopy group as a module over the fundamental group.

For all these reasons, crossed modules are commonly seen as good candidates for 2-*dimensional groups*. The algebra of crossed modules and their homotopical applications are the themes of Part I of this book.

In the proof of the 2-dimensional Seifert–van Kampen Theorem we use double groupoid structures which are related to *crossed modules of groupoids*; the latter are part of the structure of crossed complexes defined later.

Filtered spaces

Once the 2-dimensional theory had been developed it was easy to conjecture, particularly considering work of J. H. C. Whitehead in [Whi49b], that the theory in all dimensions should involve filtered spaces, a concept central to this book. An approach to algebraic topology via filtered spaces is unusual, so it is worth explaining here what is a filtered space and how this notion fits into algebraic topology.

A *filtered space* X_* is simply a topological space X and a sequence of subspaces:

$$X_*\colon\ X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \subseteq X_n \subseteq \cdots \subseteq X_\infty = X.$$

A standard example is the filtration of a geometric simplicial complex by its skeleta: X_n is the union of all the simplices in X of dimension $\leqslant n$. More generally, X would be a CW-complex, the generalisation of the finite cell complexes in [Bro06], and X_n is the union of all the cells of dimension $\leqslant n$. Here X_{n+1} is obtained from X_n by attaching cells of dimension $n+1$.

There are other simple examples, which are important for us. One is when (X, A, x) is a pointed pair of spaces, i.e. $x \in A \subseteq X$, and $n \geqslant 2$. Then we have a filtered space $X_*^{[n]}$ in which $X_i^{[n]}$ is $\{x\}$ for $i = 0$, A for $0 < i < n$ and is X for $i \geqslant n$. It may be asked: why go to this bother? Why not just stick to the pair (X, A, x)? The answer is that for $n \geqslant 3$ we want to use conditions such as $\pi_i(X, A, x) = 0,\ 1 < i < n$, and to this end we in some sense 'climb up' the above filtration $X_*^{[n]}$.

Another geometric example of filtered space is when X is a smooth manifold and $f : X \to \mathbb{R}$ is a smooth map. Morse theory shows that f may be deformed into a map g which induces what is called a handlebody decomposition of X, which is a filtration of X in which X_{n+1} is obtained from X_n by attaching 'handles' of type $n+1$. This area is explored by methods related to ours in Chapter VI of [Sha93]. A further refinement of filtered space is the notion of *topologically stratified space*, which occurs in singularity theory – see the entry in Wikipedia, for example, and also [Gro97], Section 5, which is especially interesting for Grothendieck's comments on the foundations of general topology. But the methods of this book have not yet been applied in that area.

It is of course standard to consider the simplicial singular complex SX of a topological space X, to obtain invariants from this, and then if X has a filtration to make further developments to get information on the filtered invariants. An example of this kind is when X is a CW-complex and we use the skeletal filtration. These ideas were developed by Blakers in [Bla48] for relating homology and homotopy groups, following work of Eilenberg in [Eil44] and Eilenberg–Mac Lane in [EML45b], and are related to the use of what are commonly called *Eilenberg subcomplexes*, see for example [Sch91].

In conclusion, we use filtered spaces because with them we can make this theory work, for understanding and for calculation.

Crossed complexes

Central to our work is the association to any filtered space X_* of its *fundamental crossed complex* ΠX_*. This is defined using the fundamental groupoid $\pi_1(X_1, X_0)$ and the family of relative homotopy groups $\pi_n(X_n, X_{n-1}, x)$ for all $x \in X_0$ and $n \geqslant 2$, and generalises the crossed module of a pointed pair of spaces.

A *crossed complex* C over C_1, where C_1 is a groupoid with object set C_0, is a sequence

$$\cdots \longrightarrow C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \cdots\cdots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1$$

of morphisms of groupoids over C_0 such that for $n \geqslant 2$ C_n is just a family of groups, abelian if $n \geqslant 3$; C_1 operates on C_n for $n \geqslant 2$; $\delta_{n-1}\delta_n = 0$ for $n \geqslant 3$; and other axioms hold which we give in full in Section 7.1.iii. The axioms are in fact those universally satisfied by ΠX_*, as we prove in Corollary 14.5.4.

One crucial point is that $\delta_2 \colon C_2 \to C_1$ is a crossed module (over the groupoid C_1). The whole structure has analogies to a chain complex with a groupoid of operators; this analogy is worked out in terms of a pair of adjoint functors in Section 7.4. However in passing from a crossed complex to its associated chain complex with operators some structure is lost. Crossed complexes have better realisation properties than these chain complexes: the crossed module part in dimensions 1 and 2 in crossed complexes allows the modelling of homotopy 2-types, unlike the chain complexes.

In the case X_0 is a singleton, which we call the *reduced* case, the construction of ΠX_* is longstanding, but the general case was defined by Brown and Higgins in [BH81], [BH81a].

Why crossed complexes?

• They generalise groupoids and crossed modules to all dimensions, and the functor Π is classical, involving relative homotopy groups.

• They are good for modelling CW-complexes.

• Free crossed resolutions enable calculations with small CW-models of $K(G,1)$s *and their maps* (Whitehead, Wall, Baues).

• Crossed complexes give a kind of 'linear model' of homotopy types which includes all 2-types. Thus although they are not the most general model by any means (they do no contain quadratic information such as Whitehead products), this simplicity makes them easier to handle and to relate to classical tools. The new methods and results obtained for crossed complexes can be used as a model for more complicated situations. This is how a general n-adic Hurewicz Theorem was found in [BL87a], [Bro89].

• They are convenient for some *calculations* generalising methods of computational group theory, e.g. trees in Cayley graphs. We explain some results of this kind in Chapter 10.

• They are close to the traditional chain complexes with a group(oid) of operators, as shown in MD6) on p. xxxii, and are related to some classical homological algebra (e.g. *identities among relations for groups*). Further, if SX is the simplicial singular complex of a space, with its skeletal filtration, then the crossed complex $\Pi(SX)$ can be considered as a slightly noncommutative version of the singular chains of a space. However crossed complexes have better realisation properties than the related chain complexes.

• The category of crossed complexes has a monoidal structure suggestive of further developments (e.g. *crossed differential algebras*).

• They have a good homotopy theory, with a *cylinder object, and homotopy colimits*. There are homotopy classification results (see Equation (MD9)) generalising a classical theorem of Eilenberg–Mac Lane.

• They have an interesting relation with the Moore complex of simplicial groups and of simplicial groupoids, [Ash88], [NT89a], [EP97].

• They are useful for calculations in situations where the operations of fundamental groups are involved. As an example, in Example 12.3.13 we consider the spaces $K = \mathbb{R}P^2 \times \mathbb{R}P^2$ and Z, the space $\mathbb{R}P^3$ with higher homotopy groups killed, and give a part calculation of the based homotopy classes of maps from $K \to Z$ which induce the morphism $(1,1)\colon \mathsf{C}_2 \times \mathsf{C}_2 \to \mathsf{C}_2$ on fundamental groups. This calculation uses most of the techniques developed here for crossed complexes.

Higher Homotopy Seifert–van Kampen Theorem

The reason why we deal with the filtered spaces defined in the section on p. xxv of this Introduction is the following. It is well known that many useful and geometrically interesting topological spaces are built by processes of gluing, or what we call colimits, from simpler spaces. Very often these simpler spaces have a natural, perhaps simple, filtration so that we often get an induced filtration on the colimit. One of our central results is a Higher Homotopy Seifert–van Kampen Theorem (HHSvKT), which involves the fundamental crossed complex functor Π of previous sections. The theorem shows that for a filtered space built as a 'nice' colimit of so called *connected* filtered spaces, not only is the colimit also connected but we can compute the homotopical invariant Π of the colimit as a colimit of the Π of the individual pieces from which the colimit is built, and the morphisms between them.

From this result we deduce, for example:

(i) the Brouwer Degree Theorem (the n-sphere S^n is $(n-1)$-connected and the homotopy classes of maps of S^n to itself are classified by an integer called the *degree* of the map);

(ii) the Relative Hurewicz Theorem, which is seen here as describing the morphism

$$\pi_n(X, A, x) \to \pi_n(X \cup CA, CA, x) \xrightarrow{\cong} \pi_n(X \cup CA, x)$$

when (X, A) is $(n-1)$-connected, and so does not require the usual involvement of homology groups;

(iii) Whitehead's theorem (1949) that $\pi_2(X \cup \{e^2_\lambda\}, X, x)$ is a free crossed $\pi_1(X, x)$-module;

(iv) a generalisation of that theorem to describe the crossed module

$$\pi_2(X \cup_f CA, X, x) \to \pi_1(X, x)$$

as induced by the morphism $f_* \colon \pi_1(A, a) \to \pi_1(X, x)$ from the identity crossed module $\pi_1(A, a) \to \pi_1(A, a)$; and

(v) a coproduct description of the crossed module $\pi_2(K \cup L, M, x) \to \pi_1(M, x)$ when $M = K \cap L$ is connected and (K, M), (L, M) are 1-connected and cofibred.

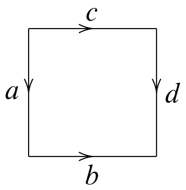
Note that (iii)–(v) are about nonabelian structures in dimensions 1 and 2. Of course proofs of the Brouwer Degree Theorem and Relative Hurewicz Theorem are standard in algebraic topology texts, and the theorem of Whitehead on free crossed modules is sometimes stated, but rarely proved. However it is not so well known that all of (i)–(v) are applications of colimit results for relative homotopy groups published before 1985. So one of our aims is to make such colimit arguments more familiar and accessible in algebraic topology, and so perhaps lead to wider applications.

We explain later other applications of crossed complexes in algebraic topology. However we are unable to prove our major results in the sole context of crossed complexes, and have to venture into new structures on *cubical sets*. The next section begins the explanation of the background which leads to cubical higher homotopy groupoids.

Cubical sets with connections

An extra structure which we needed for $\rho_2(X, A, X_0)$ in order to express the notion of cube with commutative boundary was what Chris Spencer and I called *connections*, because of a relation with path-connections in differential geometry. The background is as follows.

Even in ordinary category theory we need the 2-dimensional notion of commutative square:



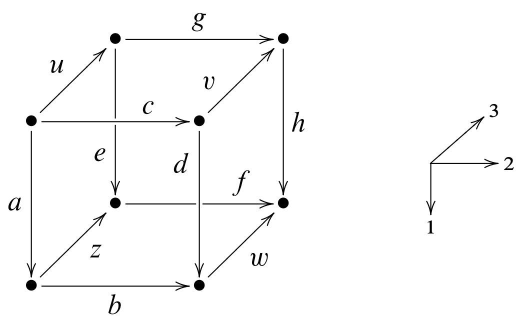
$ab = cd \quad (a = cdb^{-1}$ in the groupoid case).

An easy result is that any composition of commutative squares is commutative. For example, in ordinary equations:

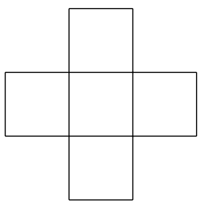
$$ab = cd,\ ef = bg \ \text{ implies } \ aef = abg = cdg.$$

The commutative squares in a category form a *double category*, and this fits with Diagram (multcomp).

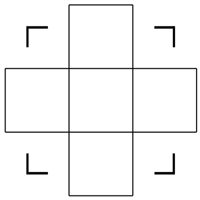
What is a commutative cube, or, more precisely, what is a cube with commutative boundary? Here is a diagram of a 3-cube with labelled and directed edges:



A prospective 'commutativity formula' involving just the edges is easy to write down. However, we want a 2-dimensional notion of the 'commutativity of the faces'. We want to say what it means for the *faces* to commute! We might try to say 'the top face is the composite of the other faces': so fold the other faces flat to give



which makes no sense as a composition! But notice that the two edges adjacent to a corner 'hole' are the same, since we have cut the cube to fold it. So we need canonical fillers to express this as in the diagram:



These extra kind of degeneracies were called *connections*, because of a relation with path connections in differential geometry, as explained in [BS76a]. They may also be thought of as 'turning left or right'. So we can obtain a formula which makes sense for a particular kind of double groupoid with this extra structure. These connections also need to satisfy enough axioms to ensure that composites of 'commutative cubes' in any of three directions are also commutative. It turns out that the axioms are sufficient for this and other purposes, including relating these kinds of double groupoids closely to a concept well established in the literature, that of *crossed module*. This led to the general concept of 'cubical set with connections', which is a key to the theory in all dimensions.

We also need sufficient axioms to be able to prove that any well-defined composition of commutative cubes is commutative. We give these axioms for this dimension in Chapter 6. The idea has then to be carried through in all dimensions. This is part of the work of Chapter 13, and clearly needs new ideas to avoid what might seem impossible complications. While cubical sets have been used since 1955, the use of cubical sets with connections and compositions is another departure from tradition.

Why cubical homotopy omega-groupoids with connections?

Standard algebraic topology uses a singular complex SX of a topological space, develops homology, and then if X has a filtration, needs to relate the algebraic topology of X to that of the filtered structure. Our approach is to take a singular complex which depends on the filtration; it is also necessary to work cubically.[5]

It was easy to conjecture that to generalise the construction $\rho_2(X, A, X_0)$ given above, we should consider a filtered space X_* and the family $R_n X_*$ of sets of maps $I^n \to X$ which map the r-skeleton of I^n into X_r, i.e. the filtered maps $I^n_* \to X_*$; and then take homotopy classes of such maps relative to the vertices of I^n, giving a quotient map $p\colon RX_* \to \rho X_*$. Both RX_* and ρX_* have easily the structure of cubical set, using well-known face and degeneracy maps. Cubical theory was initiated by D. M. Kan in 1955, but was abandoned for the simplicial theory, on which there is now an enormous literature. Nonetheless, multiple compositions are difficult simplicially, while the natural context for them is cubical. Such a cubical approach does move away from standard algebraic topology. Also it was necessary to introduce into the cubical theory the notion of connections in all dimensions.

It was not found easy to prove a central feature of our work that the easily defined multiple compositions in RX_* were inherited by ρX_*. A further difficulty was to relate the structure held by ρX_* to the crossed complex ΠX_* traditional in algebraic topology. These proofs needed new ideas and are stated and proved in Chapter 14.

Here are the basic elements of the construction.

I^n_*: the n-cube with its skeletal filtration.

Set $R_n X_* = \mathsf{FTop}(I^n_*, X_*)$. This is a *cubical set with compositions, connections, and inversions*.

For $i = 1, \ldots, n$ there are standard:

face maps $\partial_i^\pm \colon R_n X_* \to R_{n-1} X_*$;

degeneracy maps $\varepsilon_i \colon R_{n-1} X_* \to R_n X_*$;

connections $\Gamma_i^\pm \colon R_{n-1} X_* \to R_n X_*$;

compositions $a \circ_i b$ defined for $a, b \in R_n X_*$ such that $\partial_i^+ a = \partial_i^- b$;

inversions $-_i \colon R_n \to R_n$.

The connections are induced by $\gamma_i^\pm \colon I^n \to I^{n-1}$ defined using the monoid structures $\max, \min \colon I^2 \to I$. They are essential for many reasons, e.g. to discuss the notion of *commutative cube*.

These operations have certain algebraic properties which are easily derived from the geometry and which we do not itemise here – see for example [AABS02]. These were listed first in the Bangor thesis of Al-Agl [AA89]. (In the paper [BH81] the only basic connections needed are the Γ_i^+, from which the Γ_i^- are derived using the inverses of the groupoid structures.)

Here we explain why we need to introduce such new structures.

• The functor ρ gives a form of *higher homotopy groupoid*, thus confirming the visions of topologists of the early 20th century of higher dimensional nonabelian forms of the fundamental group.

• They are equivalent to crossed complexes, and this equivalence is a kind of cubical and nonabelian form of the Dold–Kan Theorem, relating chain complexes with simplicial abelian groups.

• They have a clear *monoidal closed structure*, and notion of homotopy, from which one can deduce analogous structures on crossed complexes, with detailed formulae, using the equivalence of categories.

• It is easy to relate the functor ρ to tensor products, but quite difficult to do this for Π.

• Cubical methods, unlike globular or simplicial methods, allow for a simple *algebraic inverse to subdivision*, involving multiple compositions in many directions, see p. xxii, and Remarks 6.3.2 and 13.1.11, which are crucial for the proof of our HHSvKT in Chapter 14; see also the arguments in the proof of say Theorem 6.4.10.

• The additional structure of 'connections', and the equivalence with crossed complexes, allows the notion of *thin cube*, Section 13.7, which subsumes the idea of commutative cube, and yields the proof that *multiple compositions of thin cubes are thin*. This last fact is another key component of the proof of the HHSvKT, see Theorem 14.2.9.

• The cubical theory gives a construction of a (*cubical*) *classifying space*

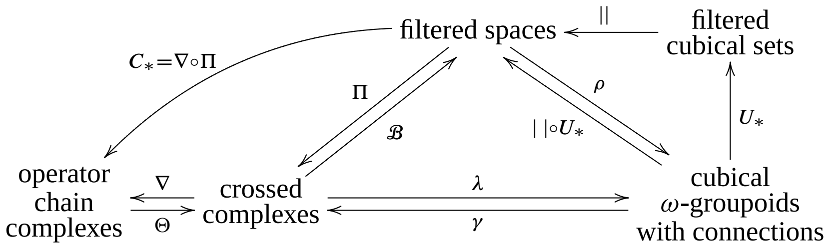
$$BC = (\mathcal{B}C)_\infty$$

of a crossed complex C, which generalises (cubical) versions of Eilenberg–Mac Lane spaces, including the local coefficient case.

• Many papers, including [BJT10], [BP02], [PRP09], [Mal09], [Gou03], [Koc10], [FMP11], [Živ06], [HW08] show a *resurgence of the use of cubes* in for example algebraic K-theory, algebraic topology, concurrency, differential geometry, combinatorics, and group theory.

Diagram of the relations between the main structures

The complete and intricate story has its main facts summarised in the following diagram and comments:



Main Diagram

in which

MD 1) the categories FTop of filtered spaces, Crs of crossed complexes and ω-Gpds of ω-groupoids, are monoidal closed, and have a notion of homotopy using $\otimes$ and unit interval objects;

MD 2) ρ, Π are homotopical functors (that is they are defined in terms of homotopy classes of certain maps), and preserve homotopies;

MD 3) λ, γ are inverse adjoint equivalences of monoidal closed categories, and λ is a kind of 'nerve' functor;

MD 4) there is a natural equivalence $\gamma\rho \simeq \Pi$, so that either ρ or Π can be used as appropriate;

MD 5) ρ preserves certain colimits and certain tensor products, and hence so also does Π;

MD 6) the category Chn of chain complexes with a groupoid of operators is monoidal closed, and ∇ is a monoidal functor which has a right adjoint Θ;

MD 7) by definition, the *cubical filtered classifying space* is $\mathscr{B} = |\ | \circ U_* \circ \lambda$ where U_* is the forgetful functor to filtered cubical sets using the filtration of an ω-groupoid by skeleta, and $|\ |$ is geometric realisation of a cubical set;

MD 8) there is a natural equivalence $\Pi \circ \mathscr{B} \simeq 1$;

MD 9) if C is a crossed complex and its cubical classifying space is defined as $BC = (\mathscr{B}C)_\infty$, then for a CW-complex X, and using homotopy as in MD1) for crossed complexes, there is a natural bijection of sets of homotopy classes

$$[X, BC] \cong [\Pi X_*, C]. \qquad \text{(MD9)}$$

Structure of the book

Because of the complications set out above in the Main Diagram, and in order to communicate the basic intuitions, we divide our account into three parts, each with an introduction giving the chapter structure of that part.

Part I is on the history and proofs of the 1- and 2-dimensional Seifert–van Kampen Theorems, and the applications of the 2-dimensional theorem to crossed modules of groups. This part covers the main nonabelian colimit results and is intended to convey the context and intuitions in a case where one can easily draw pictures.

Part II is on the theory and applications of crossed complexes over groupoids, using the fundamental crossed complex Π of a filtered space, and giving a full account of applications. The principal tools are: the Higher Homotopy Seifert–van Kampen Theorem for Π; the monoidal closed structure on the category of crossed complexes, which gives a full context for homotopies and higher homotopies; and the cubical classifying space of a crossed complex. A recurring theme is the relation of crossed complexes with chain complexes with a groupoid of operators, which thus relates the material to more classical considerations. An aim of the theory is Chapter 12, which deals with cohomology and the homotopy classification of maps, and the relations of crossed complexes with group and groupoid cohomology.

Part III justifies the theorems on crossed complexes by proving an equivalence between crossed complexes and cubical ω-groupoids, and then proving the main results in the latter context. These main theorems were essentially, and maybe only have been, conjectured in the latter context. Thus this part realises the intuitions behind the main results.

Part III ends with a chapter on ‘Further directions?’ suggesting a number of open areas and questions.

There are also three Appendices giving accounts of various aspects of category theory which are helpful for understanding of the topics, and to give wider context. This account of category theory does not claim to be complete but hopefully gives a useful and somewhat different emphasis from other texts. There is an extended account of fibrations and cofibrations of categories, to give background to the general use of pushouts and pullbacks, and as more examples of ‘categories for the working mathematician’, in showing analogies between different areas of mathematics.

Notes

2 p. xxi The paper [Bro87], p. 124, suggested that “n-dimensional phenomena require for their description n-dimensional algebra”, and this led to the term ‘higher dimensional algebra’, which widens the term ‘higher dimensional group theory’ used in [Bro82].

3 p. xxi Here we give some history on this theorem. The first result describing the fundamental group of a union was that of Seifert in [Sei31], for the union of two connected subcomplexes, with connected intersection, of a simplicial complex. The next result was that of van Kampen in [Kam33]. He also gives a formula for the case of nonconnected intersection. His proofs are difficult to follow. Some further history of the subject is given in [Gra92].

The start of the modern approach is the paper of Crowell [Cro59], based on lectures of R.H. Fox, which used the term colimit and the proof was by verification of the universal property. The paper deals with arbitrary unions.

Olum in [Olu58] gave a proof for the case of a union of two sets with connected intersection using nonabelian cohomology with coefficients in a group, and he also carefully analyses Seifert–van Kampen’s local conditions. The Mayer–Vietoris type sequence given by Olum was extended in [Bro65a], so that the fundamental group of the circle, or a wedge of circles, could be computed.

It was then found that a more powerful result with simpler proof could be obtained using groupoids, [Bro67]; this gave the fundamental groupoid on a set of base points for the case of nonconnected intersection of two open sets. This result was suggested by the use by Higgins in [Hig64] of free product with amalgamation of groupoids. Thus an aim to compute a fundamental group was reached by first computing a larger structure, a fundamental groupoid on a set of base points, and then giving methods of a combinatorial character for computing the group from the larger structure.

It was also noticed that this possibility ran contrary to the general scope of methods in homological algebra and algebraic topology, which often used exact sequences which did not give such complete results, since an invariant relating close dimensions could often be described immediately only up to extension.

A generalisation to unions of families was given in [BRS84]. A general result for the nonconnected case but still only for groups is in [Wei61], using the notion of the nerve of the cover to describe graph theoretic properties of the components of the intersections of the open sets. A combination of the method of Olum with the use of groupoids is given in [BHK83].

All these insights have been important for the generalisations to higher dimensions. Thus we find it convenient to refer to theorems of these types as *Seifert–van Kampen Theorems*.

We say more later on other extensions and analogues of the 1-dimensional theorem.

We note that the basic results here are referred to in the literature either as Seifert–van Kampen Theorems or as van Kampen Theorems.

We feel it is important to recognise the great contribution of Seifert and the German school of topology. The classic book, [ST80], first published in 1934, had an influence well into the 1950s, and is still worth consulting for the geometric background. It is also worth stating that Seifert was politically in opposition to the Nazi regime in Germany, and was never officially nominated as a full Professor at Heidelberg during the Nazi period. He was nominated after the war, and was then the only scientist at Heidelberg University the American administration would accept to become the Dean of the newly introduced Faculty of Natural Sciences, see [Pup99], [Pup97].

4 p. xxiv Reidemeister, like Seifert, was in opposition to the Nazi ideology and lost his Professorship in Königsberg in 1933, but did, however, become Professor at Marburg, [Art72], [Seg99]. By contrast, the British topologists M. H. A. Newman, J. H. C. Whitehead, and S. Wylie were all working at Bletchley Park during the war, along with many other mathematicians.

5 p. xxx This work progressed in the 1970s when we abandoned the attempt to define a 'higher homotopy groupoid' for a space and instead worked with pairs of spaces and for higher dimensions with filtered spaces. This enabled us to construct the cubical homotopy ω-groupoid $\rho(X_*)$ which is at the heart of this work. Nowadays this would be called a 'strict' ω-groupoid. There is a tendency to call the simplicial complex SX the 'fundamental ∞-groupoid' of the space X, and even to label it ΠX, see for example [Lur09]. Our notation ΠX_* is intended to reflect the close relation to traditional concepts in homotopy theory, the relative homotopy groups. In a similar manner, the notation $\Pi\mathbf{X}$ is used in [BL87] to denote the strict structure of what is there termed the fundamental cat^n-group of an n-cube of spaces $\mathbf{X}$.