Introduction to Part I

Part I develops that aspect of nonabelian algebraic topology related to the Seifert–van Kampen Theorem (SvKT) in dimensions 1 and 2. The surprising fact is that in this part we are able in this way to obtain in homotopy theory many nonabelian calculations in dimension 2 which seem unavailable without this theory, and without any of the standard machinery of algebraic topology, such as simplicial complexes or simplicial sets, simplicial approximation, chain complexes, or homology theory.

We start in Chapter 1 by giving a historical background, and outline the proof of the Seifert–van Kampen Theorem in dimension 1. It was an analysis of this proof which suggested the higher dimensional possibilities.

We then explain in Chapter 2 the functor

$$\Pi_2 \colon \text{(pointed pairs of spaces)} \to \text{(crossed modules)}$$

in terms of second relative homotopy groups, state a 2-dimensional Seifert–van Kampen Theorem (2dSvKT) for this, and give applications.

Chapter 3 explains the basic algebra of crossed modules and their relations to other topics. The more standard structures of abelian groups or modules over a group are but pale shadows of the structure of a crossed module, as we see over the next two chapters.

Two important constructions for calculations with crossed modules, are *coproducts of crossed modules* on a fixed base group (Chapter 4) and *induced crossed modules* (Chapter 5). Both of these chapters of Part I illustrate how some nonabelian calculations in homotopy theory may be carried out using crossed modules. Induced crossed modules illustrate well the way in which low dimensional identifications in a space can influence higher dimensional homotopical information; they also include free crossed modules, which are important in applications to defining and determining identities among relations for presentations of groups. This last concept has a relation to the cohomology theory of groups, which will become clear in Chapters 10 and 12.

Finally in this part, Chapter 6 gives the proof of the Seifert–van Kampen Theorem for the functor Π_2, a theorem which gives precise situations where Π_2 preserves colimits. A major interest here is that this proof requires another structure, namely that of *double groupoid with connection*, which we abbreviate to *double groupoid*. We therefore construct in a simple way as suggested on p. xxii a functor

$$\rho_2 \colon \text{(triples of spaces)} \to \text{(double groupoids)},$$

and show that this is equivalent in a clear sense to a functor

$$\Pi_2 \colon \text{(triples of spaces)} \to \text{(crossed modules of groupoids)},$$

which is a natural generalisation of our earlier Π_2 functor. Here a triple of spaces is of the form (X, X_1, X_0), where $X_0 \subseteq X_1 \subseteq X$, and the pointed case is when X_0 is a singleton. In Part I we do not make much use of the many pointed case, but it becomes crucial in Part II. This final substantial chapter of Part I thus develops the 2-dimensional groupoid theory which is then used in the proof of Theorem 6.8.2.

Note that all the results contained in Chapters 2–5 are about crossed modules over groups, while in Chapter 6 we generalise to crossed modules over groupoids to prove the 2-dimensional Seifert–van Kampen Theorem. The fact that pushouts, and coequalisers, give the same results in these two contexts follows from the fact that these two types of colimit are defined by connected diagrams, and then applying Theorem B.1.7 of Appendix B.

All this theory generalises to higher dimensions, as we show in Parts II and III, but the ideas and basic intuitions are more easily explained and pictures drawn in dimension 2.